

Which AdS_3 configurations contribute to the SCFT_2 elliptic genus?

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ABSTRACT: According to the AdS/CFT duality, the superconformal index of a superconformal field theory should have an AdS interpretation as a Euclidean functional integral with periodic boundary conditions on the fermions. Unlike the thermal case, the Euclidean continuation of the supersymmetric AdS black hole does not smoothly fill in these boundary conditions, leading us to ask the title question. In the context of $\text{AdS}_3/\text{CFT}_2$, we show using supersymmetric localization that the gravitational functional integral for the elliptic genus localizes onto asymptotically AdS_3 configurations that are annihilated by a certain supercharge, in the relevant off-shell supergravity theory. For $(0, 4)$ superconformal field theories, we find such a localizing configuration in the 5d $\mathcal{N} = 2$ off-shell supergravity theory that is asymptotically $\text{AdS}_3 \times S^2$. This configuration interpolates smoothly between the supersymmetric BTZ black hole in the interior and a constant gauge field configuration at the boundary, thus smoothly filling in the $(++)$ boundary conditions for spinors on the boundary torus. It has an action equal to the supersymmetric BTZ black hole, holomorphic in the complex structure τ of the boundary torus. Our results have interesting implications for the black hole Farey tail in AdS_3 , as well as for higher dimensional AdS theories.

KEYWORDS: [AdS/CFT](#), [Elliptic Genus](#), [Localization](#), [BTZ black hole](#).

Contents

1. Introduction and summary	1
1.1 A more detailed consideration of the puzzle	3
1.2 Strategy and results	5
2. Partition functions on AdS_3	6
2.1 Gravitational theories on AdS_3	6
2.2 The partition function of the thermal AdS_3 and the BTZ black hole	9
2.3 The elliptic genus in supergravity	12
3. Localization of the AdS functional integral	13
3.1 AdS_3 superalgebra	13
3.2 Localization	15
4. $5d$ off-shell supergravity and the $\text{AdS}_3 \times S^2$ solution	16
4.1 Superconformal multiplets and superconformal action	16
4.2 The maximally supersymmetric $\text{AdS}_3 \times S^2$ solution	19
5. Localizing BPS configurations	20
6. Analysis of the localizing solution	24
7. Discussion and future directions	27
A. Conventions	29
B. Details of the Killing spinor analysis	31
C. Off-shell BPS solutions in AdS_3	33
D. Evaluation of the action on the localizing solutions	36

1. Introduction and summary

One of the most interesting applications of the AdS/CFT correspondence is to use calculations in the boundary field theory to gain insights about the dual quantum theory of gravity. Faced with the strong/weak coupling nature of the correspondence, a good strategy is to consider quantities that are protected by supersymmetry, and therefore do not change under a continuous change of the coupling constant. One computes such a quantity in a weakly coupled field theory regime and then attempts to interpret it in terms of the

gravitational variables which are more natural in the opposite regime of strongly coupled field theory. Since we have quite a few exact finite charge results for the weakly coupled field theory computations in various dimensions, a successful application of this strategy can give detailed information about the quantum gravitational theory. Recently, this has been applied to the case of the $\text{AdS}_2/\text{CFT}_1$ case with a good degree of success [1, 2], and it would be very interesting to extend this to the higher dimensional $\text{AdS}_{d+1}/\text{CFT}_d$ cases.

One such quantity of interest, which will be the focus of this paper, is the supersymmetric index. At an inverse temperature β , the thermal partition function has the form $\text{Tr} e^{-\beta H}$, where H is the Hamiltonian, and the trace is over the Hilbert space of the theory. The supersymmetric index $\text{Tr} (-1)^F e^{-\beta H}$ is the partition function weighted by the fermion number. Depending on the details of the theory, it can also be weighted by chemical potentials which couple to conserved charges.

The thermal partition function has an interesting interpretation in terms of the gravitational variables – at low temperatures, the spectrum can be understood as a gas of gravitons, while at high temperatures, the dominant contribution to the partition function is from a black hole in AdS space [3]. It is natural to ask if a similar interpretation exists for the supersymmetric index, where one could hope to make exact statements, this is the question posed in the title. In particular, does the SCFT_d superconformal index have a gravitational interpretation as containing a supersymmetric black hole¹ in AdS_{d+1} for some range of chemical potentials (similar to the high temperature thermal partition function)?

In order to make a more precise investigation, we shall use the Euclidean functional integral formalism. We fix the boundary conditions to the classical AdS_{d+1} configuration at the boundary, and we would like to integrate over all fluctuations in the interior². Here we run into a puzzle: the Euclidean AdS black hole has a contractible time circle in the interior, and hence the only allowed smooth boundary condition for the fermions is the anti-periodic ($-$) one. This is consistent with the interpretation that the black hole dominates the thermal functional integral which also has ($-$) boundary conditions. On the other hand, the supersymmetric index is computed as a functional integral with periodic ($+$) boundary conditions for the fermions. We are thus led to conclude that:

the supersymmetric Euclidean AdS black hole cannot contribute to the AdS gravitational functional integral dual to the supersymmetric index.

One may try to remedy this by turning on flat gauge fields around the time circle, but this does not work, since one cannot smoothly turn on a flat gauge field around a contractible circle. This conclusion seems to be at odds with a general belief based on some data (which we review below), that the SCFT_2 index, interpreted as a bulk AdS_3 object, generically has a black hole like behavior. In this paper, we shall sharpen and resolve this puzzle.

¹By a supersymmetric black hole, we mean the supersymmetric limit of the corresponding non-extremal configuration, this allows us to cleanly compute its contribution to the functional integral [4].

²Since the theory includes gravity, there is, of course, the usual problem with the ultraviolet behavior of the theory – the fluctuations grow too fast at high energies and the naive functional integral is not convergent. Nevertheless, by assuming that there is some theory, like string theory, which cuts off the high energy fluctuations in an appropriate way, we can try to make sense of the semiclassical limit of the functional integral.

The resolution of this puzzle is part of our more general program to exactly evaluate the supersymmetric functional integral of a gravitational theory on AdS_3 . We will consider the elliptic genus of a SCFT_2 [5, 6, 7, 8, 9, 10, 11], which needs at least $(0, 2)$ supersymmetry³. Euclidean AdS_3 is topologically a solid torus, with one of the circles being contractible in the interior. In the functional integral formalism, the elliptic genus computation is done with $(++)$ boundary conditions for the spinors around the two circles at the boundary.

The idea of an exact evaluation of the AdS_3 functional integral dual to the elliptic genus of SCFT_2 was proposed in the seminal paper [13], and progress on the idea was made in [14] (see [15] for a nice review). The bulk configurations in [13] had flat gauge fields turned on around this contractible circle, and the induced delta function singularity at the center of AdS_3 was interpreted as coming from point-like sources. Considering that the interpretation of the thermal partition function does not need any such sources, this situation is somewhat unsatisfactory, and it is interesting to ask if there are any gravitational configurations which smoothly fill in the $(++)$ boundary conditions.

A related problem which was left open in [13] and its follow-ups is the following: the elliptic genus of a $(0, 2)$ SCFT_2 is known to be holomorphic in the complex structure τ of the torus on which it is defined, since the fermion number $(-1)^F$ pairs the massive right-moving modes. What is the mechanism in the dual AdS_3 problem by which the bulk functional integral becomes holomorphic in the complex structure τ of the boundary torus? This question can be reformulated in the following manner⁴ – what is saddle point of the AdS_3 Euclidean functional integral with the boundary torus having a generic complex structure τ ? Such a torus has $\bar{\tau} = \tau^*$, and so the answer cannot be the Euclidean continuation of the extremal BTZ black hole [16] which has $\tau = M + iJ$ finite and $\bar{\tau} = M - iJ = 0$.

In this paper, we shall resolve both these puzzles. In the rest of this introduction, we shall expand on these related puzzles, sharpen them by including a consideration of gauge fields, and explain our strategy to solve the problem.

1.1 A more detailed consideration of the puzzle

Entropy considerations in SCFT_2

We first review some data that seem to point to a black hole interpretation of the AdS_3 index. The elliptic genus of the SCFT_2 is a generating function for the number of (bosonic – fermionic) states in the Ramond sector annihilated by the right-moving supercharges. The right-movers are therefore in the Ramond sector ground state ($\tilde{L}_0 - \tilde{c}/24 = 0$), and there are non-trivial excitations in the left-moving sector with energy L_0 . Their degeneracy can be estimated by the Cardy formula to have an exponential growth of states $\Omega \approx \exp(2\pi\sqrt{cL_0/6})$, where c is the central charge of the SCFT_2 .

In the dual AdS_3 , the supersymmetric BTZ black hole preserves the right-moving supersymmetry of the original AdS_3 theory, and has non-negative mass M ($= \text{spin } |J|$), so that $L_0 - \frac{c}{24} = M$, $\tilde{L}_0 - \frac{\tilde{c}}{24} = 0$. Further, it has an entropy given by $S = k_B \log \Omega$. It is thus natural to think that the black hole represents the collection of supersymmetric states

³ $\mathcal{N} = 1$ supergravity on AdS_3 has been discussed in [12].

⁴This was emphasized to us by Ashoke Sen, for which we thank him.

in the boundary theory counted by the elliptic genus, *i.e.* states in the R sector vacuum⁵ [17]. As mentioned above, a more refined version of this statement has been envisioned in the *black hole Farey tail* [13], wherein the exact elliptic genus would be accounted for by the BTZ black hole and fluctuations around it, summed over the $SL(2, \mathbb{Z})$ family of black holes [18] in the Euclidean theory.

By contrast, in AdS_4 and AdS_5 , the most general supersymmetric index defined using the superalgebra of the theory grows too slowly [19, 20, 21] to accommodate the density of states of the corresponding supersymmetric AdS black holes [22, 23, 24]. Our conclusion that black holes do not contribute to the index seems to be consistent⁶ with the entropy data of higher dimensions, but not three dimensions. The case of the lower dimensional AdS_2 is special since, as we review below, the gravitational entropy and index are in fact equal [25, 26]. Indeed, there has indeed been impressive agreement of the AdS_2 partition function with the microscopic index, well beyond the infinite charge expansion ([27, 28, 29], see [4] for a review, and very recently, [1, 30, 31, 32, 2]).

Gauge fields in the functional integral

One may try to save the black hole interpretation in AdS_3 by turning on a gauge field under which the fermions of the theory are charged, in the background of the black hole. A gauge field configuration which has a non-zero integral around the time circle at infinity will indeed change the effective periodicity of the fermions which couple to it, but such a configuration will necessarily enclose flux due to the contractibility of the circle. In the Lorentzian theory, one can turn on a smooth flat gauge field configuration [33], but its Euclidean continuation will lead to a singularity at the origin [34]. If we demand that the gauge field configuration be smooth, it will have a non-zero field strength, and it will backreact to destroy the black hole geometry.

Since we would eventually like to go beyond the saddle point approximation and evaluate exact functional integrals, it is important to fix the ensemble in which we work. This is specified by the choice of boundary conditions for the AdS functional integral. Near the boundary, where the fields can be taken to be approximately free, the Maxwell field has two independent solutions – the gauge potential and the electric field.

In AdS_2 the electric field mode which carries the charge dominates the gauge potential mode near the boundary. This implies that the gauge potential mode is integrated over, and hence the value of $\oint A$ around the time circle at infinity can be changed by a shift of the integration variable in the functional integral⁷. In practice therefore, one can work with the smooth AdS_2 solution as a saddle point. In contrast, in $AdS_{d>3}$, the dominant mode

⁵For (2, 2) theories, the RR vacuum would be the “zero mass” ($L_0 - c/24 = 0$) BTZ black hole.

⁶In general, entropy considerations like the above are not conclusive since one is comparing the microscopic index which counts the difference in the number of bosons and fermions, with the macroscopic black hole entropy which should represent a count of the absolute number of states. However, the entropy is never less than the index for a given set of charges, and our puzzle for AdS_3 remains.

⁷We thank Ashoke Sen for explaining this point to us in detail. The argument can also be understood quite simply in the Hamiltonian formalism [25, 26]. One naturally has the *microcanonical ensemble* where all the charges, including the one associated to the fermion number current F are held fixed. This implies that $(-1)^F$ has a constant value over all the states of the ensemble and so $\text{Tr}(-1)^F$ is proportional to $\text{Tr} 1$. Thus, upto a constant, the supersymmetric index equals the entropy of the system.

is the gauge potential, while the electric field mode is sub-dominant. Fixing the gauge potential brings us to the *canonical ensemble* where the charge is allowed to fluctuate. The effective periodicity of the fermions cannot be changed by a normalizable deformation.

The case of AdS_3 is subtle since the long distance theory is described by Chern-Simons theory with a first order equation of motion. If one fixes the radial component of the gauge field to be zero by a gauge choice, the other two components are canonical conjugates, and one should not fix both their values at the boundary. Instead, the prescription [35, 36] is to fix one of these legs of the gauge field and allow the other to fluctuate in the interior. For the elliptic genus computation, one must fix the mode of the gauge field which couples to the holomorphic chemical potential [13].

If we now demand that gauge fields at infinity have the standard reality condition imposed on them, namely that the conjugate chemical potentials obey $\bar{z} = z^*$, this fixes the values of both the legs of the gauge field at infinity. This can be restated in the boundary theory as follows. In order to go from the NS to the R sector in the boundary SCFT_2 (this operation is often referred to as spectral flow), one has to change the chemical potential z , which is clearly part of the fixed data of the boundary theory. The bottom line is that in AdS_3 , the gauge fields, and therefore the periodicity of the fermions, are fixed at infinity and cannot be changed by a fluctuation in the quantum theory. Our results can also be thought of as finding a smooth mechanism for spectral flow in the bulk.

1.2 Strategy and results

Supersymmetric localization

Our strategy to answer the questions we posed above is based on the technique of supersymmetric localization [37, 38, 39, 40]. We shall argue that the AdS_3 functional integral can be localized to a set of configurations which are annihilated by a certain supercharge of the theory. These configurations will, in general, not solve the equations of motion of the theory, and we shall need to use an off-shell version of supergravity including the auxiliary fields in which the supersymmetry variations close without using the equations of motion of the theory⁸.

We investigate the case of a SCFT_2 with $(0, 4)$ supersymmetry, which has an $\text{SU}(2)$ R -symmetry. Accordingly, we shall place ourselves in the context of 5d supergravity with $\text{AdS}_3 \times S^2$ boundary conditions with $(++)$ fermion boundary conditions on the boundary torus. In the purely gravitational sector, we find a smooth solution to the localization equations which has the following features. Near the boundary, the configuration is AdS_3 with a constant gauge field potential which couples to the \tilde{J}_3 component of the $\text{SU}(2)$ R -symmetry. Near the origin, the configuration is a supersymmetric BTZ black hole whose geometry is determined by the equations of motion of the theory. The localizing solution has a non-trivial fibration of the S^2 over the AdS_3 , and smoothly interpolates between the supersymmetric black hole near the origin and the torus at infinity, but it does not obey the equations of motion in the intermediate region. The action of this configuration is exactly equal to that of the supersymmetric black hole. We consider this to be the correct

⁸On-shell supergravities on AdS_3 have been explored earlier in [41] and [42].

universal starting point for the index calculation in AdS₃ supergravity, analogous to how the black hole saddle point is a good starting point for the thermal partition function.

Our configurations are essentially a lift of the off-shell BPS configurations in AdS₃ found in [43], to five dimensions. These configurations did not have a physical meaning in the three dimensional theory of [43], since one needed the equation of motion to close the superalgebra [41]. Here, they find a genuine life as supersymmetric configurations of the 5d off-shell $\mathcal{N} = 2$ supergravity with non-trivial auxiliary fields.

Plan of the paper

In §2, we review facts about gravitational theories on AdS₃, and the interpretation of the gravitational partition function as that of a boundary CFT₂. We also sharpen the problem with the spectral flow mechanism in the bulk. In §3, we review the superalgebra on AdS₃, and the supersymmetric index built using this algebra. We then briefly review the technique of supersymmetric localization, and apply it to a theory on AdS₃ space. In §4, we briefly review the $\mathcal{N} = 2$ off-shell supergravity in five dimensions coupled to an arbitrary number of vector multiplets, and present the maximally supersymmetric AdS₃ × S² solution of this theory. In §5, we analyze the localizing equations and find the off-shell BPS localizing configurations which have the (++) spinor boundary conditions. In §6, we analyze these solutions to get a physical picture. In §7, we end with a discussion of the implications of our solutions and interesting future directions to take. In four appendices, we discuss various technical details of our calculations.

2. Partition functions on AdS₃

In this section, we review some facts about gravitational theories on AdS₃. In §2.1, we shall present the effective action and the solutions corresponding to pure AdS₃ and the BTZ black hole. We shall then discuss the asymptotic structure and symmetries of the space. In §2.2, we shall discuss the partition function of the theory, interpreted as a gravitational functional integral on AdS₃, including a brief discussion of the relation between the thermal AdS₃ and the BTZ black hole. In §2.3, we shall discuss the puzzles that arise while considering the elliptic genus and spectral flow. In the first two subsections, we shall mostly follow the presentation of the review [15], and the related papers [14, 44].

2.1 Gravitational theories on AdS₃

We shall consider theories of the metric, a collection of gauge fields, scalar fields, and fermions. The three-dimensional bosonic effective action for the metric is

$$\mathcal{S}_{\text{grav}} = \frac{1}{16\pi G_3} \int_{\text{AdS}_3} d^3x \sqrt{g} \left(R - \frac{2}{\ell^2} \right) - \frac{1}{8\pi G_3} \int_{\partial \text{AdS}_3} d^2x \sqrt{h} \left(K - \frac{1}{\ell} \right) + \dots, \quad (2.1)$$

We have explicitly written the two derivative action containing the Einstein-Hilbert term with cosmological constant and the Gibbons-Hawking boundary term. The ellipses indicate possible higher derivative terms, but we shall not consider them in this paper.

The two derivative action has the AdS₃ solution:

$$ds^2 = - \left(1 + r^2/\ell^2 \right) dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2, \quad (2.2)$$

a space of constant negative curvature which we have written in global coordinates. The radial coordinate r runs from 0 to ∞ , $r = \infty$ being the boundary of the space, and the angular coordinate ϕ is periodic with period 2π . This global AdS_3 configuration plays the role of the vacuum of the theory. In the quantum theory, we should consider fluctuations around this space which preserve the asymptotic AdS_3 boundary conditions. More precisely, expressing the metric in the Fefferman-Graham form [45]:

$$g_{ij} = r^2 g_{ij}^{(0)} + g_{ij}^{(2)} + \mathcal{O}\left(\frac{1}{r}\right), \quad (2.3)$$

we should keep $g_{ij}^{(0)}$ fixed, and allow the subleading terms to fluctuate. $g_{ij}^{(0)}$ is the conformal boundary metric, which shall be identified with the metric on the space on which the boundary CFT_2 lives.

A two-parameter family of solutions of the effective action (2.1) which obey the boundary conditions (2.3) is given by the rotating BTZ black hole solutions [16]. These solutions are labeled by the radii r_{\pm} of the inner and outer horizon

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 \ell^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{\ell r^2} dt \right)^2. \quad (2.4)$$

The mass and angular momentum of the black hole are related to the horizon radii as:

$$M = \frac{r_+^2 + r_-^2}{8G_3 \ell^2}, \quad J = \frac{r_+ r_-}{4G_3 \ell}. \quad (2.5)$$

The extremal BTZ black hole has $r_+ = r_-$, or, equivalently, $M\ell = J$.

Virasoro algebra

As was discovered by Brown and Henneaux [46], gravitational theories on AdS_3 have an infinite dimensional symmetry algebra, equivalent to two copies of the Virasoro algebra with central charges (c, \tilde{c}) . This algebra can be identified with the left- and right-moving Virasoro algebra of the boundary CFT_2 . For two derivative theories, the central charges are equal and can be expressed in terms of the AdS radius and the 3d Newton constant as $c = \tilde{c} = 3\ell/2G_3$.

The global charges $(L_{0,\pm 1}, \tilde{L}_{0,\pm 1})$ are the generators of the finite dimensional algebra $\text{SO}(2, 2) = \text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$ which are the isometries of the AdS_3 geometry (2.2). The mass and angular momentum in AdS_3 are related to the Virasoro charges as:

$$L_0 - \frac{c}{24} = \frac{1}{2}(M\ell + J), \quad \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{1}{2}(M\ell - J). \quad (2.6)$$

One can write these charge generators in terms of the geometric variables, as:

$$L_0 - \frac{c}{24} = \frac{c}{48\pi} \left(\frac{d}{dt} - \frac{1}{\ell} \frac{d}{d\phi} \right) + \mathcal{O}\left(\frac{1}{r^4}\right), \quad \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{c}{48\pi} \left(\frac{d}{dt} + \frac{1}{\ell} \frac{d}{d\phi} \right) + \mathcal{O}\left(\frac{1}{r^4}\right). \quad (2.7)$$

One can also write explicit expressions for the full set of geometric Brown-Henneaux Virasoro generators [46], but we will not need them here.

For an on-shell theory of gravity, the boundary stress tensor can be defined as the on-shell variation of the action with respect to the boundary metric. For a metric configuration of the form (2.3), one has [47]:

$$T_{\alpha\beta}^{\text{grav}} = \frac{1}{8\pi G\ell} \left(g_{\alpha\beta}^{(2)} - (\text{Tr}^{(0)} g^{(2)}) g_{\alpha\beta}^{(0)} \right) + \text{higher derivative} . \quad (2.8)$$

In the following, we will need the Virasoro charges of the classical solutions (2.2) and (2.4), which we can compute using the above considerations. We have that the pure AdS₃ metric has $L_0 = \tilde{L}_0 = 0$, and for the extremal BTZ black hole with $M\ell = J$ we have $\tilde{L}_0 = \tilde{c}/24$, $L_0 = M\ell + c/24$.

Gauge fields

In this paper, we shall deal with left-moving U(1) gauge fields A^I , and a right-moving U(1) gauge field \tilde{A} (which is the third component of an SU(2) gauge field). A generic gauge field A on AdS₃ admits an expansion analogous to (2.3)

$$A = A^{(0)} + \frac{1}{r^2} A^{(2)} + \mathcal{O}\left(\frac{1}{r^3}\right) , \quad (2.9)$$

we shall choose the gauge $A_r = 0$. The leading long-distance action for the gauge fields is given by the Chern-Simons term

$$\mathcal{S}_{\text{gauge}} = \frac{i}{8\pi} \int_{\text{AdS}_3} d^3x \left(k^{IJ} A_I dA_J - \tilde{k} \tilde{A} d\tilde{A} \right) - \frac{1}{16\pi} \int_{\partial\text{AdS}_3} d^2x \sqrt{g} g^{\alpha\beta} \left(k^{IJ} A_{I\alpha} A_{J\beta} + \tilde{k} \tilde{A}_\alpha \tilde{A}_\beta \right) . \quad (2.10)$$

The boundary term is obtained by demanding a consistent variational principle [14]. The quantities \tilde{k} and \tilde{c} are related by supersymmetry as $\tilde{c} = 6\tilde{k}$.

Since the boundary term in (2.10) depends on the metric, it contributes to the stress-energy tensor as:

$$T_{\alpha\beta}^{\text{gauge}} = \frac{k^{IJ}}{8\pi} \left(A_{I\alpha}^{(0)} A_{J\beta}^{(0)} - \frac{1}{2} A_I^{(0)\gamma} A_{J\gamma}^{(0)} g_{\alpha\beta} \right) + \frac{\tilde{k}}{8\pi} \left(\tilde{A}_\alpha^{(0)} \tilde{A}_\beta^{(0)} - \frac{1}{2} \tilde{A}^{(0)\gamma} \tilde{A}_\gamma^{(0)} g_{\alpha\beta} \right) . \quad (2.11)$$

Boundary currents which couple to the gauge fields are defined as the variation of the action with respect to the boundary value of the gauge fields. For the above action (2.10), we get:

$$J_\alpha^I = \frac{ik}{4} \left(A_{I\alpha}^{(0)} - i\epsilon_a^\beta A_{I\beta}^{(0)} \right) , \quad \tilde{J}_\alpha = \frac{i\tilde{k}}{4} \left(\tilde{A}_\alpha^{(0)} - i\epsilon_a^\beta \tilde{A}_\beta^{(0)} \right) . \quad (2.12)$$

In the later sections of the paper, we shall consider 5d supergravity whose bosonic two derivative action is of the form

$$\mathcal{S}^{5d} = \frac{1}{4\pi} \int d^5x \sqrt{g} \left(\mathcal{R} + \frac{1}{4} G_{IJ} F_{\mu\nu}^I F^{J\mu\nu} \right) + \frac{i}{24\pi} \int C_{IJK} W^I \wedge F^J \wedge F^K , \quad (2.13)$$

where C_{IJK} is a symmetric tensor. On compactification of this theory on S^2 , one obtains a theory on AdS₃. This theory has left-moving gauge fields A^I coming from the 5d gauge fields W^I , and a right-moving gauge field \tilde{A} from the Kaluza-Klein reduction, which couples to the R -symmetry current of the dual boundary SCFT₂. The Chern-Simon term for the gauge field \tilde{A} has a coefficient \tilde{k} which is determined by the tensor C_{IJK} and the magnetic charges p^I as $\tilde{k} = \frac{2}{3} C_{IJK} p^I p^J p^K$.

2.2 The partition function of the thermal AdS₃ and the BTZ black hole

The AdS₃/CFT₂ dictionary states that the partition function of the AdS₃ is equal to the partition function of the dual CFT₂. The CFT₂ is defined on the boundary of AdS₃, which we take to be a flat torus of modular parameter τ . We introduce the Euclidean time coordinate $t_E = -it$, and identify the boundary torus T_{bdry}^2 coordinate as $w = \phi + it_E$. The metric on this torus T_{bdry}^2 has a line element $ds^2 = dw d\bar{w}$, where the complex coordinate w has a periodicity

$$w \sim w + 2\pi \sim w + 2\pi\tau . \quad (2.14)$$

This boundary metric is identified with the leading value $g_{ij}^{(0)}$ of the bulk metric in (2.3). In this manner, the complex structure τ sets the boundary condition for the AdS₃ metric.

The CFT₂ partition function can be written in the Hamiltonian form as:

$$Z_{\text{CFT}_2} = \text{Tr} \left[e^{2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\bar{\tau}(\tilde{L}_0 - \frac{\tilde{c}}{24})} \right] . \quad (2.15)$$

If the theory contains fermions, we need to specify their periodicities around the circles of the torus. The periodicities around the space circle are imposed by restricting to the R (periodic, +) or NS (antiperiodic, -) sector. As written, (2.15) implies anti-periodic boundary conditions around the time circle. In order to impose a periodic boundary condition, we need to insert a $(-1)^F$ operator in the trace, where F is the fermion number. By using the relation (2.6), we see that $\text{Im}(\tau)$ plays the role of inverse temperature, while $\text{Re}(\tau)$ is a chemical potential for the AdS₃ angular momentum.

We would like to interpret Z_{CFT_2} as the partition function of the gravitational theory on AdS₃. The conventional understanding of the Euclidean functional integral takes the form

$$Z_{\text{AdS}_3}(\tau, \bar{\tau}) = \sum e^{-\mathcal{S}} , \quad (2.16)$$

where the summation runs over each saddle point of the action, and \mathcal{S} is the local effective action around the saddle point including the quantum fluctuations of massive modes.

The simplest saddle point is simply the thermal AdS₃, which is obtained by identifying the imaginary time direction $t \sim t + i\beta$ in the Lorentzian solution (2.2). The Euclidean solution is topologically equivalent to a solid torus, whose boundary is the torus T_{bdry}^2 with complex structure τ as above. A second saddle point is the Euclidean continuation of the BTZ black hole (2.4). The Euclidean continuation is performed by analytically continuing both the time coordinate as above, and the parameter $r_- \rightarrow ir_-^E$, and then letting t_E, r_-^E be real. Regularity of the Euclidean section at r_+ requires identifying

$$(t, \phi) \sim \left(t + \frac{i}{\mathfrak{I}}, \phi + \frac{i\mathfrak{J}}{\mathfrak{I}} \right) , \quad \mathfrak{I} = \frac{r_+^2 - r_-^2}{2\pi\ell r_+} , \quad \mathfrak{J} = \frac{r_-}{r_+} . \quad (2.17)$$

The Euclidean section is a solid torus filled with a three-dimensional hyperbolic metric. The A-cycle $(t(s), \phi(s)) = (t_0 + is/\mathfrak{I}, \phi_0 + i\mathfrak{J}s/\mathfrak{I})$ with $0 \leq s < 1$ is contractible in the full geometry, and hence identified as the thermal circle, while the B-cycle $(t(s), \phi(s)) = (t_0, \phi_0 + 2\pi s)$ is non-contractible. The complex structure of the torus T_{bdry}^2 generated by

$\partial_{t_E}, \partial_\phi$ at fixed radial distance r is parametrized by the modulus [48]

$$\tau_+ = \frac{i}{\ell} \left(r_- + r_+ \sqrt{\frac{r^2 - r_-^2}{r^2 - r_+^2}} \right). \quad (2.18)$$

We define

$$\tau_- = \frac{i}{\ell} \left(r_- - r_+ \sqrt{\frac{r^2 - r_-^2}{r^2 - r_+^2}} \right), \quad (2.19)$$

such that τ_+ and τ_- are complex conjugate to each other when r_- (and hence the angular momentum) is imaginary. At large radius, the complex structure of the induced metric on the torus goes to a constant,

$$\tau_\pm^\infty = \frac{i}{\ell} (r_- \pm r_+). \quad (2.20)$$

In the Euclidean theory, one identifies τ_\pm^∞ with $(\tau, \bar{\tau})$, which are complex conjugates of each other, after rotating r_- to the imaginary axis.

The above Euclidean BTZ solution is related to the thermal AdS₃ by a modular transformation $\tau \rightarrow -1/\tau$ [18]. In fact, there is an $\text{SL}(2, \mathbb{Z})$ family of Euclidean solutions which can be obtained by performing modular transformations $\tau \rightarrow \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}$ on the complex structure τ , where $\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\Gamma \equiv \text{SL}(2, \mathbb{Z})$. These geometries are in one-to-one correspondence with the left coset $\Gamma_\infty \backslash \Gamma$ where the group Γ_∞ is the parabolic subgroup of translations and is generated by $\tau \rightarrow \tau + 1$. All the known regular and black hole geometries arising as solutions of the Einstein equations with a negative cosmological constant in three dimensions can be obtained in this way. It follows that the path integral in the leading order approximation can be written as the sum over all saddle points

$$Z_{\text{grav}} \sim \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{-\mathcal{S}(\gamma \cdot \tau)}. \quad (2.21)$$

We shall now present a quantitative discussion in a slightly more general setting, by including a set of left-moving U(1) charges, one right-moving U(1) charge (which will be the R -charge \tilde{J}_0^3 in the $(0, 4)$ setting), and their corresponding chemical potentials. The CFT₂ partition function with left- and right-moving chemical potentials (z^I, \tilde{z}) is:

$$Z_{\text{CFT}_2} = \text{Tr} \left[e^{2\pi i \tau (L_0 - \frac{c}{24}) - 2\pi i \bar{\tau} (\tilde{L}_0 - \frac{\tilde{c}}{24})} e^{2\pi i z_I q^I} e^{-2\pi i \tilde{z} \tilde{q}} \right]. \quad (2.22)$$

The dual gravitational theory now has gauge fields A_I, \tilde{A} whose boundary values are set by the chemical potentials. As mentioned in the introduction, we cannot fix the boundary values of both $A_w, A_{\bar{w}}$ due to the first order nature of the Chern-Simons action which governs the physics at long distances.

The gravitational path integral should have the form

$$Z_{\text{grav}} = \int [\mathcal{D}\Phi] e^{-\mathcal{S} - \frac{i}{2\pi} \int dw d\bar{w} (A_I^w J_w^I + \tilde{A}^{\bar{w}} \tilde{J}_{\bar{w}})} \equiv \int [\mathcal{D}\Phi] e^{-\mathcal{S}^{\text{eff}}}. \quad (2.23)$$

As discussed in the previous subsection, this is achieved by introducing a boundary term in addition to the Chern-Simons action (2.10). In the classical theory, the currents are

simply given by the variation of the action with respects to the gauge fields, and one gets (2.12).

To compute the bulk functional integral, we need to evaluate the action for the solutions that contribute, including boundary counterterms. For an on-shell solution around the AdS₃ vacuum, one can evaluate the action by the following trick [47]. One first evaluates the variation of the action with respect to the boundary metric $g^{(0)}$ and the gauge fields $A^{(0)}, \tilde{A}^{(0)}$:

$$\delta\mathcal{S}^{\text{eff}} = \int d^2x \sqrt{g^{(0)}} \left[\frac{1}{2} T^{ij} \delta g_{ij}^{(0)} + \frac{i}{2\pi} J^{Ii} \delta A_{Ii}^{(0)} + \frac{i}{2\pi} \tilde{J}^i \delta \tilde{A}_i^{(0)} \right]. \quad (2.24)$$

Using the expressions (2.8), (2.11), (2.12), reexpressing in the complex coordinates of the boundary metric, and integrating the above equation (2.24), one obtains:

$$\begin{aligned} \mathcal{S}^{\text{eff}}(\tau) = & -2\pi i\tau \left(L_0^{\text{grav}} - \frac{c}{24} \right) + 2\pi i\bar{\tau} \left(\tilde{L}_0^{\text{grav}} - \frac{\tilde{c}}{24} \right) \\ & - \frac{i\pi}{2} k (\tau A_w^2 + \bar{\tau} A_{\bar{w}}^2 + 2\bar{\tau} A_w A_{\bar{w}}) + \frac{i\pi}{2} \tilde{k} (\tau \tilde{A}_w^2 + \bar{\tau} \tilde{A}_{\bar{w}}^2 + 2\bar{\tau} \tilde{A}_w \tilde{A}_{\bar{w}}). \end{aligned} \quad (2.25)$$

The first line in the above expression comes from the gravitational part of the action. The above trick is, in fact, not necessary and one can actually evaluate the two-derivative action with appropriate counterterms on the solution to get exactly the same result. In our analysis in the following sections, we will go to an off-shell theory, and we will evaluate the action on our solutions which have non-trivial auxiliary fields turned on. The second line in (2.25), on the other hand, comes purely from the boundary piece of the gauge field action (2.10), which is universal since it is the boundary term accompanying the universal Chern-Simons action. Since our off-shell solutions will asymptote to the on-shell solutions that we have been considering so far, we can use the contribution of the boundary terms of the gauge fields exactly as above.

One can use the above expression (2.25) to find the contribution of the various solutions. For pure AdS₃ and fluctuations around it, one uses (2.25) directly, while for the black hole and fluctuations around it, one needs to implement the modular transform $\tau \rightarrow -1/\tau$ on the various fields of the solution. Following this procedure, and correctly identifying [14] the boundary gauge fields A_I in (2.23), and the chemical potentials z_I in (2.22), one finds the leading term of the action in the high temperature regime as

$$\mathcal{S}^{\text{eff}} = -\frac{i\pi k}{2\tau} + \frac{i\pi \tilde{k}}{2\bar{\tau}} + \frac{2\pi i k z^2}{\tau} - \frac{2\pi i \tilde{k} \tilde{z}^2}{\bar{\tau}} - \frac{\pi}{\tau_2} (k z^2 + \tilde{k} \tilde{z}^2). \quad (2.26)$$

Using this expression and a saddle point evaluation of the functional integral, we find the black hole entropy which is consistent with the Cardy formula. To compare with the CFT₂ thermal partition function, one needs to take into account a subtlety about the relation between the Lagrangian and Hamiltonian formulations [14], which amounts to a shift of the final two quadratic terms in (2.25). Bearing this fact in mind, one finds that, indeed the black hole contribution is precisely the leading term in the thermal CFT₂ partition function.

2.3 The elliptic genus in supergravity

The elliptic genus is a superconformal index of a $(0, 2)$ SCFT₂ defined as

$$\chi(\tau, z_I) = \text{Tr}_R \left[(-1)^F e^{2\pi i \tau (L_0 - c/24)} e^{-2\pi i \bar{\tau} (\tilde{L}_0 - \tilde{c}/24)} e^{2\pi i z_I q^I} \right], \quad (2.27)$$

where the trace is taken in the R sector of the SCFT₂ living on a torus T_{bdry}^2 of modular parameter τ . The SCFT₂ has a set of left-moving U(1) charges q^I with corresponding chemical potentials z_I . Due to the pairing of massive modes due to $(-1)^F$, only the ground states, with $\tilde{L}_0 - \tilde{c}/24 = 0$, contribute in the right-moving sector, so the elliptic genus does not depend on $\bar{\tau}$. On the other hand, all left-moving states can contribute. The elliptic genus is invariant under smooth deformations of the SCFT₂. This follows from the quantization of the charges and of $L_0 - \tilde{L}_0$, together with the fact that only right-moving ground states contribute.

We are interested in the contributions to the elliptic genus written as a bulk functional integral. The trace over the R sector with $(-1)^F$ insertion dictates that the fermions must be periodic around both the circles. The elliptic genus can thus be expressed as the functional integral

$$\chi(\tau, z_I) = \left\langle \exp \left[-\frac{i}{2\pi} z_I \int J^I \right] \right\rangle_{++}. \quad (2.28)$$

Naively, it may seem that the elliptic genus receives contributions from the extremal BTZ black hole similar to how the thermal BTZ black hole contributes to the thermal partition function discussed in the previous section. Indeed, the extremal BTZ black hole has $\tilde{L}_0 - \tilde{c}/24 = 0$, as mentioned below (2.8).

However, as mentioned in the introduction, the extremal BTZ black hole solution does not extremize the variational problem with a regular boundary torus T_{bdry}^2 of complex structure τ (with $\bar{\tau} = \tau^*$ as we have). The extremal BTZ black hole has a boundary complex structure with τ finite and $\bar{\tau} = 0$, as one sees from (2.20). For the regular torus, the solution of the equations of motion which fills it in is the maximally non-extremal BTZ black hole with $J = 0$, $\Rightarrow L_0 = \tilde{L}_0$ – even if the action is supersymmetric. The other problem with the extremal black hole geometry is, as discussed in the introduction, that the contractible circle forces the fermions to have anti-periodic boundary conditions.

Both these problems can be resolved if we turn on a chemical potential $\tilde{z} = 1/2$ on top of the black hole geometry. The corresponding bulk operation is to turn on a constant gauge field at the boundary. Since all the fermions are charged under this gauge field, the periodicity is effectively changed. Moreover, we see from the discussion around (2.26) that the appropriate partition function becomes purely holomorphic, and it becomes exactly what we expect from the $\tau \rightarrow 0$ expansion of the elliptic genus. This operation is well-known in the boundary theory, which is called spectral flow. This is what we turn to next.

Spectral flow and an associated problem

The global vacuum of the theory is the NS sector ground state with anti-periodic boundary conditions, whose bulk dual geometry is the global AdS₃ (2.2). Our real interest

is to study the elliptic genus which is a trace in the R sector. To go from the NS to the R sector, one has to turn on a constant boundary gauge field (or, equivalently, the potential z), in the boundary theory. This operation is known as *spectral flow*.

If we find a bulk dual to the spectral flow operation, we would have effectively have solved the problem of finding bulk geometries which contribute to the elliptic genus – we start from the NS sector vacuum (or the more generally, a chiral primary configuration), and flow to the corresponding configuration in the R sector. We can then do a modular transformation to find the analog of the high temperature partition function. The bulk dual of this operation clearly involves turning on constant gauge fields at infinity.

The problem, as mentioned in the introduction, is that a flat gauge field configuration with a non-zero winding around the torus, which is well-defined in the boundary theory, cannot be extended into the bulk smoothly. By Stokes theorem, the boundary value of $\oint \tilde{A}$ around a contractible circle measures the flux enclosed within this circle. If we insist that the angular component of the gauge field be constant everywhere, then we encounter a delta function singularity at the origin. One can imagine deforming the singular flux string configuration to a lump of flux concentrated at the center of AdS_3 , such that the gauge field asymptotes to the required value at the boundary. This will solve the smoothness problem, but the non-zero flux in the bulk of AdS_3 will backreact and destroy the background geometry. With no further change in the metric, this will not obey the equations of motion, and will therefore no longer be a saddle point.

In our investigations presented below, we find that, in fact, there are no supersymmetric configurations of this type that solve the equations of motion. This seems to point to a dead end. However, there could be *off-shell configurations* which contribute to the functional integral. In the absence of supersymmetry, this would be an impossible problem to solve, since we do not have a guiding principle for which class of configurations to consider. In our case, we use the technique of supersymmetric localization, which tells us to look for the (measure zero) subspace of gravitational configurations which are annihilated by a certain supercharge in the off-shell theory. This is what we shall do this in the following sections.

3. Localization of the AdS functional integral

In this section, we shall discuss the supersymmetry algebra on AdS_3 . We shall then discuss the formalism of supersymmetric localization. Applying this formalism to the AdS_3 functional integral, we shall set up the localization equations that we need to solve.

3.1 AdS_3 superalgebra

The bosonic isometry algebra of an AdS_{d+1} space is the conformal algebra $\text{SO}(d, 2)$ in d dimensions. For $d = 2$, the conformal algebra is an infinite dimensional algebra, which splits up into separate holomorphic and anti-holomorphic pieces. One has a number \mathcal{N} of holomorphic spin $3/2$ supercurrents and holomorphic spin one R -symmetry currents in addition to the spin two stress tensors. The number of currents can be $\mathcal{N} = 1, 2$,

or 4 in order to preserve the linear local superconformal symmetry⁹. For the $\mathcal{N} = 2$ superconformal algebra, the R -symmetry group is $U(1)$, and for the $\mathcal{N} = 4$ algebra, the R -symmetry group is $SU(2)$. There is a corresponding structure on the anti-holomorphic side with $\overline{\mathcal{N}}$ supercurrents.

In this paper, we shall discuss theories with $(\mathcal{N}, \overline{\mathcal{N}}) = (0, 4)$ supersymmetry. These could arise as the near-horizon limits of M5-branes wrapping a Calabi-Yau geometry in M-theory [51], but this will not be important for us here. There are four supercurrents $\tilde{G}^{i\alpha}(\bar{z})$ where the index i is in a doublet of the $SU(2)$ R -symmetry (generated by the currents $\tilde{J}^a(\bar{z})$, $a = 1, 2, 3$), and the index α is in a doublet of an outer automorphism $SU(2)'$ symmetry, which shall later be identified with the R -symmetry of the $\mathcal{N} = 2$ supergravity in five dimensions. The algebra is as follows:

$$\begin{aligned}
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{\tilde{c}}{12}m(m^2-1)\delta_{m+n,0}, \\
[\tilde{L}_n, \tilde{G}_r^{i\alpha}] &= (\frac{n}{2}-r)\tilde{G}_{r+n}^{i\alpha}, \\
\{\tilde{G}_r^{i+}, \tilde{G}_s^{j-}\} &= 2\delta^{ij}\tilde{L}_{r+s} + (r-s)\sigma_{ij}^a\tilde{J}_{r+s}^a + \frac{\tilde{c}}{3}(r^2-\frac{1}{4})\delta_{r+s,0}\delta^{ij}, \\
\{\tilde{G}_r^{i+}, \tilde{G}_s^{j+}\} &= \{\tilde{G}_r^{i-}, \tilde{G}_s^{j-}\} = 0, \\
[\tilde{L}_n, \tilde{J}_m^a] &= -m\tilde{J}_{m+n}^a, \\
[\tilde{J}_n^a, \tilde{G}_r^{i+}] &= \sigma_{ij}^a\tilde{G}_{r+n}^{j+}, \quad [\tilde{J}_n^a, \tilde{G}_r^{i-}] = -\tilde{G}_{r+n}^{j-}\sigma_{ji}^a, \\
[\tilde{J}_m^a, \tilde{J}_n^b] &= -2i\epsilon^{abc}\tilde{J}_{m+n}^c + \frac{\tilde{c}}{3}m\delta_{m+n,0}\delta^{ab}.
\end{aligned} \tag{3.1}$$

In the NS sector, the supercharges are half-integer moded, and the eight supercharges $\tilde{G}_r^{i\alpha}$, $r = \pm\frac{1}{2}$, along with $\tilde{L}_{0,\pm 1}$ and the $SU(2)$ charges \tilde{J}_0^a , $a = 1, 2, 3$ make up the supergroup $SU(1, 1|2)$. The $SU(2)$ R -symmetry algebra is represented geometrically in a minimal way as the rotations of an S^2 . In the following sections, we shall consider 5d supergravity theories with asymptotically $AdS_3 \times S^2$ boundary conditions. The global (NS) vacuum will be pure $AdS_3 \times S^2$, this is annihilated by all the eight supercharges of the $SU(1, 1|2)$, this solution will be presented in §4.2.

In theories with $\mathcal{N} = (0, 4)$ supersymmetry, the \tilde{J}^3 component of the right-moving $SU(2)$ R -symmetry algebra is related to the fermion number as $F = 2\tilde{J}_0^3$. By turning on a constant gauge field dual to this charge, we can spectrally flow from the NS to the R sector. The spectral flow relates the \tilde{L}_0 generators in the two sectors as

$$\tilde{L}_0^R = \tilde{L}_0^{NS} - \tilde{J}_0^{3NS}/2. \tag{3.2}$$

In order to apply the formalism of localization, we will need to find a subalgebra generated by one real supercharge Q which obeys

$$Q^2 = \tilde{L}_0^R. \tag{3.3}$$

Since the supercharges are integer moded in the R sector, it is clear from the algebra (3.1), and the spectral flow (3.2), that all the zero modes $\tilde{G}_0^{i\alpha}$ obey the equation (3.3). The right

⁹Superconformal theories with $\mathcal{N} = 8$ supersymmetry have been discussed in the AdS_3 context in [49, 50]. In this case, the global supersymmetry can be extended to a current algebra at the cost of introducing non-linearity and non-unitarity.

hand side of (3.3) is manifestly compact, since \tilde{L}_0^{NS} is a compact generator of the Virasoro algebra acting on AdS_3 (2.7), and \tilde{J}_0^{3NS} is the generator of rotation on S^2 .

3.2 Localization

Let us recall a few facts about the localization of integrals over supermanifolds [52, 37, 39, 38, 40, 53]. Consider a supermanifold \mathcal{M} with an integration measure $d\mu$. Let Q be an odd (fermionic) vector field on this manifold under which the measure is invariant and which squares to a compact bosonic symmetry H . Consider the integral

$$I := \int_{\mathcal{M}} d\mu e^{-\mathcal{S}}. \quad (3.4)$$

where \mathcal{S} is a Q -invariant action. To evaluate this integral using localization, one first deforms the integral to

$$I(\lambda) = \int_{\mathcal{M}} d\mu e^{-\mathcal{S} - \lambda QV}, \quad (3.5)$$

where V is a fermionic, H -invariant function which means $Q^2V = 0$ and QV is Q -exact. It is easy to see that the derivative of $I(\lambda)$ with respect to λ vanishes using the Q -invariance of h , \mathcal{S} , and the $d\mu$. One can thus perform the integral $I(\lambda)$ in the limit of large λ instead of at $\lambda = 0$. In this limit, the semiclassical evaluation of the functional integral becomes exact and in particular it localizes onto the critical points of the functional $\mathcal{S}^Q := QV$. One can choose

$$V = (Q\Psi, \Psi), \quad (3.6)$$

where Ψ are the fermionic coordinates with some positive definite inner product defined on the fermions. In this case, the bosonic part of \mathcal{S}^Q can be written as a perfect square $(Q\Psi, Q\Psi)$, and hence critical points of \mathcal{S}^Q are the same as the critical points of Q which we refer to as the localizing solutions. Let us denote this set of critical points of Q by \mathcal{M}_Q . The reasoning above shows that the integral over the supermanifold \mathcal{M} localizes to an integral over the submanifold \mathcal{M}_Q . In the large λ limit, the integration for directions transverse can be performed exactly in the saddle point evaluation. One is then left with an integral over the submanifold \mathcal{M}_Q

$$I = \int_{\mathcal{M}_Q} d\mu_Q e^{-\mathcal{S}}, \quad (3.7)$$

with a measure $d\mu_Q$ induced on the submanifold by the original measure.

In our case, \mathcal{M} is the field space of off-shell supergravity, \mathcal{S} is the off-shell supergravity action with appropriate boundary terms. We need to pick a subalgebra of the full supersymmetry algebra discussed above, whose bosonic generator is compact. As discussed above, any Q which is a zero mode of one of the supercurrents $\tilde{G}^{i\alpha}$ squares to the compact generator $\tilde{L}_0 - \tilde{J}_0^3$. The localizing Lagrangian is then defined by

$$\mathcal{L}^Q := QV \quad \text{with} \quad V := (Q\Psi, \Psi), \quad (3.8)$$

where Ψ refers to all fermions in the theory. The localizing action is then defined by

$$\mathcal{S}^Q = \int d^4x \sqrt{g} \mathcal{L}^Q. \quad (3.9)$$

The localization equations that follow from this action are

$$Q\Psi = 0. \tag{3.10}$$

These are the equations that we need to solve subject to the AdS_3 boundary conditions, we shall find Q as a particular linear combination of $\tilde{G}_0^{i\alpha}$ in the following sections.

4. 5d off-shell supergravity and the $\text{AdS}_3 \times S^2$ solution

Since localization is employed at the level of the functional integral and not just at the level of a classical action, it is important to use an off-shell formulation of supergravity. A convenient method for dealing with off-shell formulations of supergravity theories is provided by the superconformal multiplet calculus. This calculus was originally constructed for the 4d $\mathcal{N} = 2$ supergravity [54, 55, 56, 57]. For 5d supergravity, the conformal supergravity approach was developed relatively recently by several groups [58, 59, 60, 61, 62], and these results were exploited further in [63, 64, 65, 66]. Since this section is meant to set the stage for our computations in the following sections, we shall give a very brief summary of the subject. The interested reader is referred to [62] for a more detailed recent treatment, whose notations and conventions we shall follow here.

The main idea is that the Poincaré algebra is extended to the superconformal algebra to obtain an off-shell version of the Poincaré supergravity. For the 5d $\mathcal{N} = 2$ supergravity, the superconformal algebra is given by the exceptional superalgebra $F^2(4)$ whose bosonic part is $\text{SO}(2, 5) \times \text{SU}(2)'$. Extending conformal supergravity to a gauge theory of this superalgebra provides an irreducible off-shell realization of the gravity and matter multiplets. Then, by imposing constraints, the gauge theory is identified as a gravity theory. Upon gauge fixing the extra superconformal symmetries, one obtains the Poincaré supergravity. In this formalism, the supersymmetry transformation laws do not depend on the form of the action, and are completely fixed by the superconformal algebra.

In §4.1, we list the multiplets of the 5d $\mathcal{N} = 2$ superconformal theory that will enter the theories we consider, and discuss the supersymmetry transformations and the invariant action of the theory. In §4.2, we review the maximally supersymmetric pure $\text{AdS}_3 \times S^2$ configuration. This will serve as a guide to the new off-shell solutions that we shall find in the next section. The conventions are spelled out in appendix §A. More details of the various multiplets including the full supersymmetry transformation rules can be found in [62], here we shall only reproduce the formulas that are relevant for the calculations in the following sections.

4.1 Superconformal multiplets and superconformal action

We consider the 5d $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets. We need one other compensating multiplet to eliminate the extra degree of freedom, we shall choose this to be a hypermultiplet. The theory has an R -symmetry $\text{SU}(2)'$, under which all the fermionic fields are doublets. We use the notation that greek indices (μ, ν, \dots) indicate the curved spacetime, latin indices (a, b, \dots) indicate the flat tangent space indices, and

(i, j, \dots) denote the $SU(2)'$ indices. The $SU(2)'$ indices are raised and lowered by complex conjugation. All the fermionic fields are represented by symplectic-Majorana spinors.

We now describe the field content of the various multiplets. We shall use the term ‘‘auxiliary field’’ below to mean a field which becomes an auxiliary field in the gauge fixed theory.

- Weyl multiplet

$$\mathbf{W} = (e_\mu^a, \omega_\mu^{ab}, b_\mu, f_\mu^a, \psi_\mu^i, \phi_\mu^i, V_\mu^{ij}, T_{ab}, D, \chi^i). \quad (4.1)$$

The fields $(e_\mu^a, \omega_\mu^{ab})$ are the gauge fields for translations (vielbein) and Lorentz transformations (spin connection); ψ_μ^i, ϕ_μ^i are the gauge fields for Q -supersymmetries and the conformal S -supersymmetries; (b_μ, f_μ^a) are the gauge fields for dilatations and the special conformal transformations; and V_μ^{ij} are the gauge fields for the $SU(2)'$ R -symmetries. The $SU(2)'$ doublet of spinors χ^i , the antisymmetric two-form field T_{ab} and the real scalar field D are auxiliary fields, some of these will play a non-trivial role later.

- Vector multiplet

$$\mathbf{V} = (\sigma^I, W_\mu^I, Y_{ij}^I, \Omega_i^I). \quad (4.2)$$

Here, W_μ^I are the gauge fields, and σ^I are real scalar fields, Ω_i^I are the gaugini, and Y_{ij}^I is a triplet of auxiliary fields. The index I labels the generators of the gauge group G . Here, we consider G to be $n_V + 1$ copies of $U(1)$. The field strength is given by

$$\hat{F}_{\mu\nu}^I = \partial_\mu W_\nu^I - \partial_\nu W_\mu^I - \bar{\Omega}_i^I \gamma_{[\mu} \psi_{\nu]}^i + \frac{1}{2} i \sigma^I \bar{\psi}_{[\mu i} \psi_{\nu]}^i. \quad (4.3)$$

- Hypermultiplet

The components of \mathbf{H} , hypermultiplets, are

$$\mathbf{H} = (\mathcal{A}_i^\alpha, \zeta^\alpha, \mathcal{F}_\alpha^i), \quad (4.4)$$

where the indices $\alpha = 1 \dots 2r$ (in our case $r = 1$) label the fundamental representation of $USp(2r)$. The hypermultiplet scalars $\mathcal{A}_i^\alpha(\phi)$ can be realized as a section of $USp(2r) \times SU(2)$ bundle over the hyper-Kähler cone whose coordinates are locally written in terms of ϕ . The information on the target-space metric is contained in the so-called hyper-Kähler potential,

$$\varepsilon_{ij} \chi = \Omega_{\alpha\beta} \mathcal{A}_i^\alpha \mathcal{A}_j^\beta, \quad (4.5)$$

where Ω is the skew-symmetric symplectic $USp(2N)$ invariant tensor.

Below, we shall need the covariant derivative \mathcal{D}_μ which is covariant with respect to all the bosonic gauge symmetries with the exception of the conformal boosts. Acting on the spinors, it has the form:

$$\mathcal{D}_\mu \epsilon^i = \left(\partial_\mu - \frac{1}{4} \omega_\mu^{cd} \gamma_{cd} + \frac{1}{2} b_\mu \right) \epsilon^i + \frac{1}{2} V_{\mu j}^i \epsilon^j. \quad (4.6)$$

On the hypermultiplets, the covariant derivative takes the form

$$\mathcal{D}_\mu \mathcal{A}_i^\alpha = \partial_\mu \mathcal{A}_i^\alpha - \frac{3}{2} b_\mu \mathcal{A}_i^\alpha + \frac{1}{2} V_\mu^j \mathcal{A}_j^\alpha + \partial_\mu \phi^A \Gamma_A^\alpha{}_\beta \mathcal{A}_i^\beta, \quad (4.7)$$

where $\Gamma_A^\alpha{}_\beta$ is the $\text{USp}(2r)$ connection associated with rotations of the fermions.

Before solving the BPS equations, we will fix a number of gauge conditions. We first fix the K-gauge by setting the dilatational gauge field to zero, $b_\mu = 0$. Then, we identify $\mathcal{A}_\alpha^i = \text{const} \times \delta_\alpha^i$ to gauge-fix the $\text{SU}(2)'$ symmetry. In addition, we set the fields V_μ^{ij} to their on-shell values $V_\mu^{ij} = 0$, and all the fermion backgrounds to zero. Now we can fix the S -supersymmetry generated by η_i by solving the purely bosonic part of the BPS condition for the hypermultiplet fermion

$$\delta\zeta^\alpha = -\frac{1}{2} i \not{D} \mathcal{A}_i^\alpha \epsilon^i + \frac{3}{2} \mathcal{A}_i^\alpha \eta^i = 0. \quad (4.8)$$

For the field configurations $b_\mu = 0$, $V_\mu^{ij} = 0$, the covariant derivative (4.6) becomes a regular partial derivative, which vanishes for the configuration $\mathcal{A}_\alpha^i = \text{const} \times \delta_\alpha^i$. We therefore have $\eta^i = 0$ for the solution of (4.8).

With this gauge choice, and considering configurations where there are no fermion bilinears in the background, the supersymmetry variations of the fermionic fields are:

$$\delta\psi_\mu^i = \left[\mathcal{D}_\mu + \frac{1}{4} i T_{ab} (3\gamma^{ab} \gamma_\mu - \gamma_\mu \gamma^{ab}) \right] \epsilon^i = 0, \quad (4.9)$$

$$\delta\Omega^{Ii} = -\frac{1}{4} (\hat{F}_{ab}^I - 4\sigma^I T_{ab}) \gamma^{ab} \epsilon^i - \frac{1}{2} i \not{D} \sigma^I \epsilon^i - \varepsilon_{jk} Y^{Iij} \epsilon^k = 0, \quad (4.10)$$

$$\delta\chi^i = \frac{1}{4} \epsilon^i D + \frac{3}{128} i (3\gamma^{ab} \not{D} + \not{D} \gamma^{ab}) T_{ab} \epsilon^i - \frac{3}{32} T_{ab} T_{cd} \gamma^{abcd} \epsilon^i = 0, \quad (4.11)$$

where we have the covariant derivative $\mathcal{D}_\mu = \partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}$ due to the gauge-fixing. These are the equations we will now solve in the rest of the paper.

The two-derivative bosonic Lagrangian is:

$$\mathcal{L} = \mathcal{L}_{\text{VVV}} + \mathcal{L}_{\text{hyper}}. \quad (4.12)$$

The first piece is the Lagrangian cubic in vector multiplets:

$$\begin{aligned} 8\pi^2 \mathcal{L}_{\text{VVV}} = & 3 C_{IJK} \sigma^I \left[\frac{1}{2} \mathcal{D}_\mu \sigma^J \mathcal{D}^\mu \sigma^K + \frac{1}{4} F_{\mu\nu}^J F^{\mu\nu K} - Y_{ij}^J Y^{ijK} - 3\sigma^J F_{\mu\nu}^K T^{\mu\nu} \right] \\ & + \frac{1}{8} C_{IJK} e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} W_\mu^I F_{\nu\rho}^J F_{\sigma\tau}^K - C(\sigma) \left[\frac{1}{8} \mathcal{R} - 4D - \frac{39}{2} T^2 \right], \end{aligned} \quad (4.13)$$

where $C(\sigma) = C_{IJK} \sigma^I \sigma^J \sigma^K$. The second piece is the Lagrangian for the hypermultiplets (recall that we have only one hypermultiplet which acts as the compensating multiplet):

$$8\pi^2 \mathcal{L}_{\text{hyper}} = -\frac{1}{2} \Omega_{\alpha\beta} \varepsilon^{ij} \mathcal{D}_\mu \mathcal{A}_i^\alpha \mathcal{D}^\mu \mathcal{A}_j^\beta + \chi \left[\frac{3}{16} \mathcal{R} + 2D + \frac{3}{4} T^2 \right]. \quad (4.14)$$

The constant in the gauge fixing condition for $\mathcal{A}_\alpha^i = \text{const} \times \delta_\alpha^i$ is fixed in such a way that the hyper-Kähler potential χ satisfies $\chi = -2C(\sigma)$, to ensure that there is no tadpole for the scalar D .

We note that we have gauge fixed all the extra symmetries of the superconformal supergravity, compared to the Poincaré supergravity *except* the local scaling or dilatation symmetry. Keeping this symmetry unfixed has the advantage of keeping the symplectic symmetry acting on the vector multiplets manifest. After finding solutions of the BPS equations, we will need to fix this remaining dilatation symmetry.

4.2 The maximally supersymmetric $\text{AdS}_3 \times S^2$ solution

The above theory has a maximally supersymmetric pure $\text{AdS}_3 \times S^2$ on-shell solution which we shall now review. This configuration can be interpreted as the magnetic attractor solution since $\text{AdS}_3 \times S^2$ is the near horizon geometry of a magnetic string [67, 64, 65, 66]. This solution is important because it is the bulk dual of the NS sector ground state of the boundary SCFT₂, our review will also serve to illustrate the workings of the theory as a warmup for the next section.

The maximally supersymmetric configurations also satisfy the equations of motion of the theory, and all the auxiliary fields are set to their on-shell values. Since we have maximal supersymmetry, there are no projection conditions on the Killing spinor ϵ , and therefore terms in the BPS equations with different structures of the gamma matrices should independently vanish. The gaugino variation equation (4.10) then leads to

$$\sigma^I = \text{constant}, \quad F^I = 4\sigma^I T, \quad Y_{ij}^I = 0. \quad (4.15)$$

Here we can see the characteristic feature of the attractor solutions that the scalar field takes constant values in terms of the gauge field charges. With this identification, we see that the gravitino variation equation (4.9) becomes the usual Killing spinor equation in on-shell treatments.

Next, let us look at the equation (4.11). Imposing vanishing of the independent tensor structure of the γ -matrices gives:

$$D = 0, \quad (4.16)$$

and equations which the auxiliary two-form T must satisfy:

$$\varepsilon^{abcde} \mathcal{D}_a T_{bc} = 0, \quad (4.17)$$

$$i\mathcal{D}^b T_{ba} - \varepsilon_{abcde} T^{bc} T^{de} = 0. \quad (4.18)$$

The first equation (4.17) is the Bianchi identity and the second (4.18) is equivalent to the equation of motion for the gauge fields, as consistent with the identification of the two as above (4.15).

Finally, we proceed to consider the Killing spinor equations (4.9). The geometries which admit full Killing spinors are classified in [67]. Among those, we are interested in $\text{AdS}_3 \times S^2$ which is interpreted as the near horizon geometry of a magnetic string. The metric has the form

$$ds^2 = ds_{\text{AdS}_3}^2 + ds_{S^2}^2. \quad (4.19)$$

The gauge field strength F^I , (and therefore the auxiliary two-form T), is proportional to the volume form of S^2 , so that the fluxes of the gauge fields through the S^2 are fixed in terms of the magnetic charges p_I :

$$p^I = \frac{1}{8\pi} \int_{S^2} F^I. \quad (4.20)$$

We are then left with the task of computing the ratio of the radii of AdS_3 to the one of S^2 , and the relation of the constant value of the scalar field σ^I to the magnetic charge p^I .

The relation between the two radii is determined by an integrability condition for the commutator of covariant derivatives acting on the Killing spinor $\ell_{\text{AdS}_3} = 2\ell_{S^2} \equiv \ell$. We now choose coordinates so that the metric is

$$\begin{aligned} ds_{\text{AdS}_3}^2 &= \cosh^2 \rho dt^2 + \sinh^2 \rho \ell^2 d\phi^2 + \ell^2 d\rho^2, \\ ds_{S^2}^2 &= \frac{\ell^2}{4} [d\theta^2 + \sin^2 \theta d\psi^2]. \end{aligned} \quad (4.21)$$

The auxiliary two-form T then takes the form:

$$T = \frac{\ell}{4} \sin \theta d\theta \wedge d\psi = \frac{1}{\ell} \text{vol}_{S^2}, \quad (4.22)$$

using which, one finds the relation $p^I = \ell \sigma^I / 2$.

With the metric (4.21), the Killing spinor equations (4.9) are also decomposed into the AdS₃ and S^2 part

$$\begin{aligned} \text{AdS}_3 : \quad D_m \epsilon_{\text{AdS}_3} &= \frac{1}{2\ell} (\sigma_3 \otimes \mathbb{1}) \gamma_m \epsilon_{\text{AdS}_3}, \\ S^2 : \quad D_j \epsilon_{S^2} &= \frac{1}{\ell} (\sigma_3 \otimes \mathbb{1}) \gamma_j \epsilon_{S^2}. \end{aligned} \quad (4.23)$$

The solutions to these Killing spinor equations can be written as:

$$\begin{aligned} \epsilon_{\text{AdS}_3} &= \exp \left[\frac{1}{2} \rho (\sigma_3 \otimes \mathbb{1}) \gamma_3 \right] \exp \left[-\frac{i}{2} (\phi + i \frac{t}{\ell}) (\sigma_3 \otimes \mathbb{1}) \gamma_1 \right] \mathbb{1} \otimes \epsilon_{\text{AdS}_3}^0, \\ \epsilon_{S^2} &= \exp \left[\frac{1}{2} \theta \gamma_5 \right] \exp \left[\frac{1}{2} \psi (\sigma_3 \otimes \mathbb{1}) \right] \epsilon_{S^2}^0 \otimes \mathbb{1}, \end{aligned} \quad (4.24)$$

where ϵ_0 is a constant spinor [68, 69] and we choose the gamma matrices as in (A.11). To see that the Killing spinor (4.24) solves the equations (4.23), we use the Baker-Campbell-Hausdorff formula (A.12).

5. Localizing BPS configurations

In this section, we shall achieve our stated goal, that is to find smooth BPS configurations in the off-shell theory, with $(++)$ boundary condition on fermionic fields. As reviewed in §2, we need non-zero gauge field holonomies around both the asymptotic circles of the geometry. To this end, we shall consider a geometry which asymptotically is an S^2 fibration over AdS₃. We do not *a priori* fix conditions on the interior, except that it be everywhere smooth. To implement the fibration, we introduce Kaluza-Klein gauge fields \tilde{A}_μ . As discussed in §2, the KK gauge fields $\tilde{A}_\mu(r)$ will have a non-trivial dependence on the radial coordinate to preserve smoothness of the configurations¹⁰.

We thus reach the following ansatz. We want an asymptotically AdS₃ space with an S^2 fibration so that we have fixed non-zero $\oint \tilde{A}$ around the two circles of the asymptotic AdS₃. We would like that this configuration satisfies the BPS equations of the off-shell supergravity of §4 with respect to at least one supercharge, and without necessarily imposing the equations of motion of the theory.

The BPS analysis is not trivial since we do not know the form of the metric *a priori*. This type of problem can be systematically solved by using the methods of [67], but here

¹⁰This is, of course, not true in the Lorentzian theory, wherein sphere fibrations with constant KK gauge fields have been discussed in the context of AdS₃ × S^2 [34] and AdS₃ × S^3 [70, 71, 33, 72].

we shall make use of the solutions solved in [43] instead. It will be very interesting to fully classify the solution spectrum of the off-shell BPS equations, and we intend to return to this problem in the future. The authors of [43] have found 1/2-BPS solutions to a supergravity coupled to Chern-Simons theory on asymptotically AdS₃ spaces. We shall lift the solutions in [43] to supersymmetric solutions of the 5d off-shell supergravity. As in [43], we begin with general ansatz of the 5d metric

$$ds^2 = \left[f(r)^2 + \left(\frac{u(r)}{r} \right)^2 \right] dt^2 + 2u(r)dt d\phi + h(r)^2 dr^2 + r^2 d\phi^2 + \frac{\ell^2}{4} \left[d\theta^2 + \sin^2 \theta \left\{ d\psi + \tilde{A}_t(r)dt + \tilde{A}_\phi(r)d\phi \right\}^2 \right]. \quad (5.1)$$

The first line represents a three dimensional base space which is asymptotically AdS₃, and the second line expresses an S^2 fibration over the base space with $(\tilde{A}_t(r), \tilde{A}_\phi(r))$ being the KK gauge fields. Note that details like the choice of vielbeins and the corresponding spin connections are presented in appendix §B. We would like to solve the supersymmetry variation equations (4.9), (4.10), (4.11) with this metric.

Choice of the auxiliary two-form T

The gravitino variation equations (4.9) are:

$$\left[\partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{i}{4} T_{ab} (3 \gamma^{ab} \gamma_\mu - \gamma_\mu \gamma^{ab}) \right] \epsilon = 0. \quad (5.2)$$

The relevant Killing spinor equations in [43] are truncated versions of (5.2), obtained by setting the derivative terms \tilde{A}'_μ of the KK gauge fields to zero. In this case, the variation of the gravitino is supported (in the language of our set up) by the auxiliary two-form T of the form:

$$T^{(0)} = \frac{\ell}{4} \sin \theta d\theta \wedge \left[d\psi + \tilde{A}_t(r)dt + \tilde{A}_\phi(r)d\phi \right]. \quad (5.3)$$

So far, this is as in the maximally BPS solution (4.22). If we want to lift the 1/2 BPS solutions of [43], we need to kill the terms proportional to the derivatives of the KK gauge fields \tilde{A}'_μ , which appear in the above equation due to their presence in the spin connections ω_μ^{ab} (B.2). This will involve a deviation δT of the auxiliary two-form from its value $T^{(0)}$ proportional to \tilde{A}'_μ .

Here, we shall make educated guesses to find the deviation δT . Since we are interested in localizing solutions preserving even one supercharge, we are allowed to impose further projection conditions on the spinors, which allows us to make progress. Hence, we begin with imposing the projection condition

$$(1 - \gamma_5) \epsilon = 0. \quad (5.4)$$

This projection condition is inspired by the form of metric (5.1), similar projections have been chosen based on the isometries of the metric in [64]. Using this projection condition, one can find the deviations

$$\begin{aligned} \delta T^{31} &= 2\delta T^{24} = i(u\tilde{A}'_\phi - r^2\tilde{A}'_t) \frac{\ell \sin \theta}{12r^2 f h}, \\ \delta T^{23} &= 2\delta T^{14} = \frac{i\ell\tilde{A}'_\phi \sin \theta}{12rh}, \\ \delta T^{ab} &= 0 \text{ otherwise,} \end{aligned} \quad (5.5)$$

such that no term proportional to \tilde{A}' survives in the Killing spinor equations (5.2). We present the method to obtain this configuration for the auxiliary two-form in appendix §B.

Gravitino variations

With this choice of the auxiliary two-form T , the Killing spinor equations (5.2) can be written as

$$\left[\partial_\mu + iB_\mu^a \gamma_a (\sigma_3 \otimes \mathbb{1}) - \frac{1}{2} \tilde{A}_\mu (\cos \theta \gamma^4 - i \sin \theta \gamma^5) \right] \epsilon = 0, \quad \mu = (t, \phi, r), \quad (5.6)$$

$$\left[D_j - \frac{1}{\ell} (\sigma_3 \otimes \mathbb{1}) \gamma_j \right] \epsilon = 0, \quad j = (\theta, \psi), \quad (5.7)$$

where $B^a = -\frac{1}{4} \varepsilon^{abc} \omega_{bc} + \frac{i}{2\ell} e^a$, ($a, b, c \in \{1, 2, 3\}$) are

$$\begin{aligned} B^1 &= \left(\frac{u'}{4rh} + \frac{if}{2\ell} \right) dt + \frac{1}{2h} d\phi, \\ B^2 &= \left[\frac{1}{4h} \left(\frac{2u^2}{r^3 f} - \frac{uu'}{r^2 f} - 2f' \right) + \frac{i u}{2\ell r} \right] dt + \left[\frac{1}{4fh} \left(\frac{2u}{r} - u' \right) + \frac{ir}{2\ell} \right] d\phi, \\ B^3 &= \left[\frac{1}{4rf} \left(u' - \frac{2u}{r} \right) + \frac{ih}{2\ell} \right] dr. \end{aligned} \quad (5.8)$$

Since the Killing spinor equations are factorized into the AdS₃ (base) part (5.6), and S² (fiber) part (5.7), we factorize the Killing spinor ϵ in the same manner:

$$\epsilon = \exp \left[\frac{i}{2} \theta \gamma_5 \right] \exp \left[\frac{i}{2} \psi (\sigma_3 \otimes \mathbb{1}) \right] \zeta. \quad (5.9)$$

With this ansatz, ϵ satisfies the S² components of the Killing spinor equations (5.7) as in the maximally supersymmetric case (4.24). In this case, we are left to solve the 3d base components (5.6) of the Killing spinor equations:

$$\left[\partial_\mu + iB_\mu^a \gamma_a (\sigma_3 \otimes \mathbb{1}) - \frac{i}{2} \tilde{A}_\mu (\sigma_3 \otimes \mathbb{1}) \right] \zeta = 0. \quad (5.10)$$

This equation (5.10) is almost the same as the Killing spinor equations solved in [43] as we aimed. (See Eq. (2.3) in [43].) We present the method to obtain solutions of these equations and detailed calculations in appendix §C, here we present the results. The Killing spinors are of the form:

$$\zeta = a(r) e^{(i\frac{n}{2}\phi + i\frac{m}{2}t)(\sigma_3 \otimes \mathbb{1})} (1 + \Gamma^a b_a) \zeta_0, \quad (5.11)$$

where n, m are integers, Γ_a are the 3d gamma matrices defined in (C.7), and ζ_0 is a constant spinor. The real function $a(r)$ and the complex functions $b_a(r)$ (written explicitly in (C.25), (C.22)) are determined in terms of one real function $u(r)$. The solutions (5.11) obeys the projection condition

$$(1 - \Gamma^a b_a) \zeta = 0. \quad (5.12)$$

Since in total we impose two projection conditions, (5.4) and (5.12), the Killing spinors (5.9) with (5.11) are 1/4 BPS solutions.

The metric admitting the Killing spinors (5.11) is¹¹:

$$ds^2 = \left[\left(\frac{r}{\ell} \right)^2 + \frac{2u}{\ell} + 1 \right] dt^2 + 2iudtd\phi + \frac{\left(1 + \frac{\ell u'}{2r} \right)^2}{\left(\frac{r}{\ell} + \frac{u}{r} \right)^2 + 1} dr^2 + r^2 d\phi^2 + \frac{\ell^2}{4} \left[d\theta^2 + \sin^2 \theta \left\{ d\psi + \tilde{A}_t dt + \tilde{A}_\phi d\phi \right\}^2 \right], \quad (5.13)$$

where the KK gauge fields are given by

$$\tilde{A}_t = \frac{i u'}{2r + \ell u'} + \frac{i}{\ell} + \frac{m}{\ell}, \quad \tilde{A}_\phi = \frac{2r}{2r + \ell u'} + n. \quad (5.14)$$

The Killing spinors (5.11) are periodic (+) for even integers m, n . We will pick $n = m = 0$ to get the spinor zero modes as required. If we pick the function $u(r)$ to approach a constant asymptotically, we will get constant KK gauge fields at infinity as required. We shall analyze the solutions in more detail in the next section.

Gaugino variations

Next, we need to solve the gaugino variation equations:

$$\left(-\frac{1}{4} F^I \cdot \gamma + \sigma^I T \cdot \gamma - \frac{1}{2} i \not{D} \sigma^I \right) \epsilon^i - \varepsilon_{jk} Y^{Iij} \epsilon^k = 0. \quad (5.15)$$

The auxiliary two-form T is given by (5.3), (B.7). The field strength of the gauge field W^I given by

$$F^I = \ell \sigma^I \sin \theta d\theta \wedge \left[d\psi + \tilde{A}_t dt + \tilde{A}_\phi d\phi \right] - \ell \sigma^I \cos \theta \left[\tilde{A}'_t dr \wedge dt + \tilde{A}'_\phi dr \wedge d\phi \right] \quad (5.16)$$

solves the BPS condition (5.15) with the auxiliary field taking the value

$$Y^{Ii}{}_j = \pm e^{-i\theta} \frac{\ell \sigma^I \tilde{A}'_\phi}{4r f h} \delta^i{}_j, \quad (5.17)$$

and constant scalar fields σ^I .

One can check that the field strength (5.16) satisfies the Bianchi identity $dF^I = 0$. Actually, there is a very natural construction of this field strength which has been discussed in [44]. If one writes the metric (5.1) as

$$ds^2 = ds^2_{\text{AdS}_3} + \frac{\ell^2}{4} (dy^i - \tilde{A}^{ij} y^j)(dy^i - \tilde{A}^{ik} y^k), \quad (5.18)$$

where $\sum_{i=1}^3 (y^i)^2 = 1$ and the one-forms \tilde{A}^{ik} are the SO(3) KK gauge fields, then the form (5.16) of the field strength F^I can be written in terms of this coordinate:

$$F^I = \frac{\ell \sigma^I}{2} \epsilon_{ijk} (Dy^i Dy^j - \tilde{F}^{ij}) y^k, \quad (5.19)$$

where the one form and modified field strength are defined as

$$Dy^i = dy^i - \tilde{A}^{ij} y^j, \quad \tilde{F}^{ij} = d\tilde{A}^{ij} - \tilde{A}^{ik} \tilde{A}^{kj}. \quad (5.20)$$

¹¹Note that we have substituted $\alpha = i/\ell$, $\beta = 1$ and redefined $u \rightarrow iu$ in (C.29), so that the function $u(r)$ now takes real values.

We find that the third BPS equation (4.11) is also solved, if the field D takes on a non-trivial dictated by the two-form T_{ab} that we have. Since the field D drops out of the action (6.10), the actual value is not important for our purposes.

One can ask if a first projection condition like (5.4) was even necessary, *i.e.* can one directly lift the solutions (5.11) to 1/2 BPS solutions in five dimensions. One can check by substitution that this is not the case, and we really have 1/4-BPS solutions.

6. Analysis of the localizing solution

In the previous section, we have found smooth Killing spinors (5.11) which have $(++)$ boundary conditions around both the circles of the boundary torus T_{bdry}^2 . These spinors live on the asymptotically AdS_3 (5.13) with the KK gauge fields (5.14). They depend on one function $u(r)$ which supersymmetry does not fix. In this section, we shall analyze the properties of this solution. We first analyze how smoothness restricts the form of the geometry, and then compute the action of the smooth configuration that we find.

Firstly, since $u(r)$ is a normalizable deformation, we should have (with constant u_∞):

$$u(r) \rightarrow \ell u_\infty + \mathcal{O}\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty. \quad (6.1)$$

This means that the values of the gauge fields at infinity are:

$$\tilde{A}_\phi(r) \xrightarrow{r \rightarrow \infty} 1, \quad \tilde{A}_t(r) \xrightarrow{r \rightarrow \infty} i/\ell. \quad (6.2)$$

Smoothness of the configurations requires the angular component of a gauge field to vanish at the origin of a contractible circle. The form of the metric dictates that $r = 0$ should be the origin. Choosing ϕ to be the contractible circle, we should impose that $\tilde{A}_\phi(r)$ vanishes at the origin. This is achieved by

$$u'(r) \xrightarrow{r \rightarrow 0} \mathcal{O}(1), \quad (6.3)$$

which gives

$$\tilde{A}_\phi(r) \xrightarrow{r \rightarrow 0} 0, \quad \tilde{A}_t(r) \xrightarrow{r \rightarrow 0} 2i/\ell. \quad (6.4)$$

It is convenient to make the combinations

$$\tilde{A}_w \equiv \frac{1}{2}(\tilde{A}_\phi + i\ell\tilde{A}_t), \quad \tilde{A}_{\bar{w}} \equiv \frac{1}{2}(\tilde{A}_\phi - i\ell\tilde{A}_t), \quad (6.5)$$

which have the limiting values:

$$\begin{aligned} \tilde{A}_w(r) &\xrightarrow{r \rightarrow \infty} 0, & \tilde{A}_{\bar{w}}(r) &\xrightarrow{r \rightarrow \infty} 1; \\ \tilde{A}_w(r) &\xrightarrow{r \rightarrow 0} -1, & \tilde{A}_{\bar{w}}(r) &\xrightarrow{r \rightarrow 0} 1. \end{aligned} \quad (6.6)$$

The field $\tilde{A}_{\bar{w}}$ which everywhere takes the value 1 is identified with the (appropriately normalized) right-moving chemical potential \tilde{z} of the boundary SCFT_2 which we would like to “turn on” to have a value 1/2.

We will now analyze the near-horizon region $r \rightarrow 0$. In order to zoom in near this region, we use the change of coordinates

$$r = \lambda\rho, \quad \lambda \rightarrow 0, \quad (6.7)$$

and keep only the terms which survive in the limit [4]. The near-horizon metric takes the form:

$$ds^2 = (2\tilde{u} + 1) dt^2 + 2i\tilde{u} dt d\phi + \frac{\tilde{u}'^2}{4\tilde{u}^2} d\rho^2 + \frac{1}{4} [d\theta^2 + \sin^2 \theta (d\psi + 2i dt)^2]. \quad (6.8)$$

The function $\tilde{u}(\rho) \equiv u(\lambda\rho)$ can be thought of as the effective radial coordinate, and we get a non-singular geometry at the origin if we choose $\tilde{u}(\rho) = \rho/2$, in which case we get:

$$ds^2 = (\rho+1) \left[dt + \frac{i\rho d\phi}{2(\rho+1)} \right]^2 + \frac{1}{4} \left[\frac{d\rho^2}{\rho^2} + \frac{\rho^2}{\rho+1} d\phi^2 \right] + \frac{1}{4} [d\theta^2 + \sin^2 \theta (d\psi + 2i dt)^2]. \quad (6.9)$$

Near $\rho \rightarrow 0$, we recognize the familiar form of the AdS_2 in the (ρ, ϕ) coordinates, with a circle t and a two-sphere (θ, ψ) fibered over it. In fact, the three dimensional space (6.8) spanned by (ρ, ϕ, t) is a locally AdS_3 space, as can be checked by computing the Ricci scalar. The near-horizon region is indeed an $\text{AdS}_3 \times S^2$, with a constant KK gauge field. In the very-near-horizon region [17], we see that the geometry (6.9) is actually that of the supersymmetric Euclidean BTZ black hole.

It is easy to check that, in the near-horizon limit, the field strengths F^I take their constant on-shell values determined by the magnetic charges as in §4.2, and that the auxiliary two-form δT_{ab} (5.5) and Y_{ij}^I (5.17) vanish in the near-horizon region. Note that the function $u(r)$ is fixed to a certain value both near the boundary and near the horizon. Since our configuration is supersymmetric, this implies that the solution (6.9) is actually on-shell. This can also be explicitly checked by verifying that (6.9) solves the equations of the 5d $\mathcal{N} = 2$ Poincaré supergravity.

The action of the configurations

Now that we have found smooth configurations that contribute to the functional integral, we want to know how much they contribute. In general, this can be answered by evaluating the action on the solutions. For on-shell configurations, we can actually simplify this step and deduce the contribution by the methods as discussed in §2. Here, since we have configurations that do not solve the equations of motion, we shall actually have to evaluate the action on the configurations, which is what we turn to next.

We recall from §4 that the fields of the compensating hypermultiplet take the values $\mathcal{A}_\alpha^i = \text{const} \times \delta_\alpha^i$ and $\chi = -2C(\sigma)$. With these values, the two-derivative Lagrangian (4.12) is

$$8\pi^2 \mathcal{L} = 3C_{IJK} \sigma^I \left[\frac{1}{4} F_{\mu\nu}^J F^{\mu\nu K} - Y_{ij}^J Y^{ijK} - 3\sigma^J F_{\mu\nu}^K T^{\mu\nu} \right] + \frac{i}{8} C_{IJK} e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} W_\mu^I F_{\nu\rho}^J F_{\sigma\tau}^K - C(\sigma) \left[\frac{1}{2} \mathcal{R} - 18T^2 \right]. \quad (6.10)$$

It is straightforward to compute the various quantities in the Lagrangian evaluated on the localizing solution given by the metric (5.13), the auxiliary two-form field T_{ab} (B.8), the gauge fields W^I (5.16) and the auxiliary fields Y_{ij}^I (5.17).

The bulk action is obtained by integrating this Lagrangian over the five dimensional space. This integral is divergent due to the infinite volume of the asymptotically AdS₃ space. We shall regulate this divergence by imposing a cutoff at large $r = \Lambda$. Doing so, one finds (the details of this calculation are presented in appendix §D):

$$\mathcal{S}_{\text{bulk}} = -\frac{C(\sigma)}{8\pi} [\Lambda^2 + \ell^2 u_\infty] \text{Area}(T_{\text{bdry}}^2), \quad (6.11)$$

where T_{bdry}^2 denotes the boundary torus with coordinates (2.14) on which the (0, 4) SCFT₂ lives.

As explained in §2, we need to include the Gibbons-Hawking term written in terms of the extrinsic curvature at the boundary, as well as the boundary counterterm to remove the divergence. The details of the evaluation of the boundary terms is also presented in appendix §D. The results are:

$$\mathcal{S}_{\text{GH}} = \frac{C(\sigma)}{8\pi^2} \int_{r=\Lambda} d^4x \sqrt{h} K = \frac{C(\sigma)}{8\pi} \text{Area}(T_{\text{bdry}}^2) [2\Lambda^2 + \ell^2(1 + 2u_\infty)] + \mathcal{O}(\frac{1}{\Lambda^2}), \quad (6.12)$$

and

$$\mathcal{S}_{\text{ct}} = -\frac{C(\sigma)}{8\pi^2} \int_{r=\Lambda} d^4x \sqrt{h} \frac{1}{\ell} = -\frac{C(\sigma)}{8\pi} \text{Area}(T_{\text{bdry}}^2) \left[\Lambda^2 + \frac{\ell^2}{2}(1 + 2u_\infty) \right] + \mathcal{O}(\frac{1}{\Lambda^2}). \quad (6.13)$$

In order to express the final result in a useful manner, we need to evaluate the overall factor $C(\sigma)$ is proportional to the inverse of the 5d Newton's constant G_5 , which in turn can be written in terms of the 3d Newton's constant G_3 and the volume of the S^2 . This in turn is related to the central charge of the AdS₃. These relations can be summarize as follows:

$$c = \tilde{c} = 6\tilde{k} = \frac{3\ell^3 C(\sigma)}{2} = \frac{3\pi\ell^3}{2G_5} = \frac{3\ell}{2G_3}. \quad (6.14)$$

Putting all these calculations together, and taking the cutoff $\Lambda \rightarrow \infty$, we obtain

$$\mathcal{S}_{\text{reg}} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{GH}} + \mathcal{S}_{\text{ct}} = -\frac{i\pi c}{12\ell} (\tau - \bar{\tau}). \quad (6.15)$$

In the on-shell case reviewed in §2, the analogous action, *i.e.* the first line of (2.25) was computed without much calculation. Here, we go through a slightly laborious process, but eventually get the simple answer (6.15).

Finally, we need to consider the boundary term coming from the Chern-Simons term, which is subtle, but standard. The careful treatment explained in §2 yields the second line of (2.25).

$$\mathcal{S}_{\text{gauge}}^{\text{bdry}} = -\frac{i\pi}{2} k (\tau A_w^2 + \bar{\tau} A_{\bar{w}}^2 + 2\bar{\tau} A_w A_{\bar{w}}) + \frac{i\pi}{2} \tilde{k} (\tau \tilde{A}_w^2 + \bar{\tau} \tilde{A}_{\bar{w}}^2 + 2\tau \tilde{A}_w \tilde{A}_{\bar{w}}). \quad (6.16)$$

The only Chern-Simons term that we consider here is for the right-moving gauge field \tilde{A} , whose boundary values are given by (6.6). Adding this in, we find that the $\bar{\tau}$ dependence

completely drops out and we are left with

$$\mathcal{S}^{\text{eff}} = \mathcal{S}_{\text{reg}} + \mathcal{S}_{\text{gauge}}^{\text{bdry}} = -\frac{i\pi c \tau}{12\ell}, \quad (6.17)$$

which is holomorphic in τ .

We would now like to make a few comments. Firstly, recalling that the full effective action can be interpreted in the boundary theory as

$$\mathcal{S}^{\text{eff}}(\tau) = -2\pi i\tau \left(L_0 - \frac{c}{24} \right) + 2\pi i\bar{\tau} \left(\tilde{L}_0 - \frac{\tilde{c}}{24} \right), \quad (6.18)$$

we deduce that the action of our localizing configuration is equal to that of the R sector ground state, with $\tilde{L}_0 = \tilde{c}/24$. In our solution, the left-movers seem to be in their ground state, this is due to the form of the gauge field strength (5.16). More generally, we will have purely holomorphic excitations coming from turning on left-moving potentials z^I . According to our discussion in §2, after modular transformation, this is the same as the action of the supersymmetric BTZ black hole. When we do not turn on any z^I , we get the action of the zero mass BTZ black hole.

Secondly, from computing the Ricci curvature of the pieces of the near-horizon region (6.9), we see that it is indeed of the form $\text{AdS}_3 \times S^2$. The value of the Ricci scalar (D.1) approaches a constant near the origin, and the contribution of the auxiliary fields vanish in this region, making the region on-shell. The boundary region is, of course, also on-shell, but in order to connect these two smoothly, the solution is forced to go off-shell, as can be seen from the fact that, in the intermediate, finite r region, the auxiliary fields are non-zero.

Finally, we see that although the values of the Ricci scalar, the field strengths, the auxiliary two-form T_{ab} (B.8) and Y_{ij}^I (5.17) all depend on the function $u(r)$ in the intermediate region, the value of the total action is independent of $u(r)$, indicating the presence of a gauge symmetry. Recalling that in our analysis of the BPS equations in §4, we gauge fixed all the gauge symmetries of the conformal supergravity *except* for the local dilatation symmetry. It is very likely that the (spurious) function $u(r)$ is an indication of this symmetry. We can verify this by perturbatively fixing $u(r)$ near the horizon and near the boundary, we leave a detailed analysis of this issue for the future. We already note that the function $u(r)$ factors out of the functional integral by canceling the volume of the gauge group of local dilatations. We thus understand the configuration (5.13) as the unique localizing solution in the gravitational sector, replacing the role of the BTZ black hole solution in the gravitational computations of supersymmetric functional integrals.

7. Discussion and future directions

The main motivation of this work is to compute the exact gravitational functional integral dual to the elliptic genus, or more generally, the superconformal index of a field theory. We summarize here the main findings of our work:

The bosonic configuration (5.13) represents an everywhere smooth, asymptotically AdS_3 space with an S^2 fibered over the three-dimensional geometry such that the KK gauge

fields (5.14) carry a non-trivial flux. The flux is concentrated near the origin and dies away towards infinity such that the gauge fields have a constant winding around the torus. The near-horizon configuration is that of a supersymmetric BTZ black hole with constant flux. The near-horizon region and the asymptotic region are both solutions to the equations of motion of the theory. The intermediate region has non-zero auxiliary fields and is therefore off-shell, but nevertheless, it is a solution of the off-shell BPS equations.

We would like to reemphasize that our approach is Euclidean, and our localizing solution is complex. We do not have anything to say here about which intermediate states propagate in loops in the physical theory. This situation is not unusual in Euclidean functional integrals. In the full evaluation, we will of course have to do the usual Gibbons-Hawking analytic continuation, similar to [1].

This is, however, only the first step in this direction, and in order to really compute the elliptic genus from gravity, there are many more steps to take on this path: firstly, although we have argued that our localizing configurations are special, we have not shown that they are the only BPS configurations ($++$) boundary conditions, and we need to work out the full set of supersymmetric gravitational fluctuations. Perhaps the methods of [67] would be useful in this regard. It would also be nice to have a more complete treatment of the gauge freedom $u(r)$.

Further, we need to find the most general solution in the vector multiplet sector. These solutions should provide a holomorphic action to the system. One may envision the AdS₃ analog of [1, 2], where on integrating these fluctuations, one finds the polar part of the elliptic genus. In this paper, we only included the two-derivative action of the 5d supergravity, but in general, of course, we need to include the full effective action which may contain higher derivative terms. Taking all these into account, one can now put the black hole Farey tail [13] on a much more rigorous footing. One imagines starting with our solution in this paper, including the full set of fluctuations obeying the localization equations, and then summing over the modular transforms to get the full elliptic genus¹². These issues are currently under investigation.

Another much bigger issue is what is the full off-shell space of gravity, and whether the 5d off-shell supergravity (and in particular, its field space) comes from a more fundamental theory like string theory. Our attitude in this paper has been that we have an example of a theory which tells us which off-shell configurations to integrate over to compute the supersymmetric functional integral. It seems to be able to capture a large class of $\mathcal{N} = 2$ compactifications in five dimensions related to M theory on a Calabi-Yau three-fold [51]. It seems like we can generalize the approach to other dimensions as well. For example, one can also try to extend our analysis to the (4, 4) theory, dual to AdS₃ \times S^3 . This needs a six dimensional off-shell formalism.

A particularly interesting direction is the analogous story in higher dimensions. In this regard, note that, from our point of view of the Euclidean functional integral, it is not a puzzle, and in fact completely natural, that for AdS _{$d \geq 4$} , the most general superconformal

¹²Here we assume that there is no wall-crossing while going from weak to strong coupling. In general, there will be other gravitational configurations. One example where these can be dealt with exactly is in [73].

index does not grow fast enough to accommodate the density of states of a black hole [19, 20, 21]. It will of course, be interesting, if there is a Lorentzian interpretation of this statement, perhaps an existence of a pair of fermionic zero modes in the background of the supersymmetric AdS $_{d \geq 4}$ black hole. Even within the Euclidean functional integral formalism, it would be interesting if one can compute the known exact expressions for the superconformal indices in $d \geq 3$ SCFTs using localization in the dual AdS $_{d+1}$ theories.

We would like to comment that our localized solutions in AdS $_3$ are really (off-shell) fluctuations around the vacuum, and there may be similar localized solutions in AdS $_{d \geq 4}$. Integrating over these may reproduce the relevant (slowly-growing) index in the boundary theory. The reason we have come close to reproducing the (exponentially-growing) elliptic genus here is that the modular transforms of our solutions have the action of a supersymmetric black hole. The known boundary calculations in the higher dimensional case suggest that there may be no analog of the modular transformation between the ground state and highly excited states.

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A. Conventions

We summarize our notational conventions in this appendix. Firstly, we work with the Euclidean signature $(+\cdots+)$. The curved metric is written by $g_{\mu\nu}$ and the flat metric is denoted by δ_{ab} . Greek letters μ, ν, \dots express coordinate indices, and Latin letters a, b, \dots represent orthonormal indices where they take values $1, \dots, 5$. All the fermionic fields are represented by symplectic-Majorana spinors.

The fünfbein is denoted by e_μ^a and we write its inverse by e_a^μ . They satisfy $e_\mu^a e_a^\nu = \delta_\mu^\nu$. The curved and flat metrics are related by

$$g_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}, \quad \delta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu}. \quad (\text{A.1})$$

Curved and flat indices are converted by

$$V_a = e_a{}^\mu V_\mu, \quad V_\mu = e_\mu{}^a V_a. \quad (\text{A.2})$$

Curved and flat indices are moved up and down with $g_{\mu\nu}$ and η_{ab} and their inverses $g^{\mu\nu}$ and η^{ab} , respectively. We define

$$e = \det(e_\mu{}^a) = \sqrt{\det(g_{\mu\nu})}. \quad (\text{A.3})$$

The spin connections¹³ $\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu$ are defined by

$$\begin{aligned} \omega^a{}_b &= -\omega^b{}_a, \\ de^a &= \omega^a{}_b \wedge e^b. \end{aligned} \quad (\text{A.4})$$

The Riemann tensors are defined by

$$\mathcal{R}_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac} \omega_{\nu c}{}^b + \omega_\nu{}^{ac} \omega_{\mu c}{}^b. \quad (\text{A.5})$$

By contraction, one obtains the Ricci tensors

$$\mathcal{R}_\mu{}^a = \mathcal{R}_{\mu\nu}{}^{ab} e_b{}^\nu, \quad (\text{A.6})$$

and the Ricci scalar

$$\mathcal{R} = \mathcal{R}_\mu{}^a e_a{}^\mu. \quad (\text{A.7})$$

Next, we shall fix the convention of spinors. The gamma matrices obey

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab} \mathbb{1}. \quad (\text{A.8})$$

We define the products of the gamma matrices

$$\gamma^{\mu_1 \dots \mu_p} = \gamma^{[\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_p]} = \frac{1}{p!} (\gamma^{\mu_1} \dots \gamma^{\mu_p} \pm \text{cyclic}). \quad (\text{A.9})$$

Then the products of the gamma matrices satisfy

$$\begin{aligned} \gamma^{ab} \gamma^{cd} &= -(\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}) - (\gamma^{ac} \delta^{bd} - \gamma^{bc} \delta^{ad} + \gamma^{bd} \delta^{ac} - \gamma^{ad} \delta^{bc}) + \gamma^{abcd}, \\ \gamma^a \gamma^{bc} &= \delta^{ab} \gamma^c - \delta^{ac} \gamma^b + \gamma^{abc}, \\ \gamma_{abcde} &= -\mathbb{1} \varepsilon_{abcde}, \end{aligned} \quad (\text{A.10})$$

where $\varepsilon_{12345} = \varepsilon^{12345} = +1$. To solve the Killing spinor equations explicitly in §4.2 and §5, we choose the 5d gamma matrices to be

$$\text{AdS}_3 \quad \begin{cases} \gamma_1 = \sigma_3 \otimes \sigma_1, \\ \gamma_2 = \sigma_3 \otimes \sigma_2, \\ \gamma_3 = \sigma_3 \otimes \sigma_3, \end{cases}$$

¹³Note that our convention for the spin connections is different from the most common one in the literature. Hence, the curvature tensors are also different from the conventional definitions so that the Ricci scalar curvatures for AdS spaces are positive.

$$S^2 \begin{cases} \gamma_4 = \sigma_1 \otimes \mathbb{1} , \\ \gamma_5 = \sigma_2 \otimes \mathbb{1} . \end{cases} \quad (\text{A.11})$$

Note that, with this base choice of the gamma matrices, they satisfy $(\gamma^a)^\dagger = \gamma^a$.

In the manipulations of the gamma matrices, we use the Baker-Campbell-Hausdorff formula

$$e^{\frac{i}{2}\theta X} Y e^{-\frac{i}{2}\theta X} = Y \cos \theta - Z \sin \theta , \quad (\text{A.12})$$

for matrices X, Y and Z , such that $[X, Y] = 2iZ$, and $[X, Z] = -2iY$.

To fix the convention of spinors, we introduce the charge conjugation matrix C which satisfies

$$C\gamma_a C^{-1} = \gamma_a^T, \quad C^T = -C, \quad C^\dagger = C^{-1} .$$

For an $SU(2)'$ doublet λ^i , the symplectic Majorana condition is defined by

$$(\bar{\lambda}_i)^T = C \varepsilon_{ij} \lambda^j . \quad (\text{A.13})$$

Here, ε_{ij} is an antisymmetric two-by-two matrix of the form

$$(\varepsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (\text{A.14})$$

The indices $i, j, \dots = 1, 2$ are raised and lowered according to the convention: $\lambda_i := \lambda^j \varepsilon_{ji}$ and $\lambda^i = \varepsilon^{ij} \lambda_j$, where $\varepsilon^{ij} = \varepsilon_{ij}$ and therefore $\varepsilon^{ik} \varepsilon_{kj} = -\delta_j^i$.

For a spinor ζ^α , ($\alpha = 1, \dots, 2r$) in the fundamental representation of the $USp(2r)$ R -symmetry group, the symplectic-Majorana condition reads as

$$C^{-1} \bar{\zeta}_\alpha^T = \Omega_{\alpha\beta} \zeta^\beta , \quad (\text{A.15})$$

where Ω is the symplectic $USp(2N)$ invariant tensor. For a more general spinor $\chi^{\alpha\beta\dots\rho\sigma\dots}$, the symplectic Majorana constraint would read

$$C^{-1} (\bar{\chi}_{\alpha\beta\dots\rho\sigma\dots})^T = \Omega_{\alpha\gamma} \Omega_{\beta\delta} \Omega^{\rho\xi} \Omega^{\sigma\zeta} \dots \chi^{\gamma\delta\dots\xi\zeta\dots} . \quad (\text{A.16})$$

B. Details of the Killing spinor analysis

In this appendix, we fill in some of the details of the Killing spinor analysis in the main text §5.

One can take an orthonormal frame for the metric (5.1) as

$$\begin{aligned} e^1 &= f(r) dt , \\ e^2 &= \frac{u(r)}{r} dt + r d\phi , \\ e^3 &= h(r) dr , \\ e^4 &= \frac{\ell}{2} d\theta , \\ e^5 &= \frac{\ell}{2} \sin \theta \left[d\psi + \tilde{A}_t(r) dt + \tilde{A}_\phi(r) d\phi \right] . \end{aligned} \quad (\text{B.1})$$

With respect to this orthonormal frame, the spin connections are:

$$\begin{aligned}
\omega^1_2 &= \frac{1}{4rf} \left(\frac{2u}{r} - u' \right) dr, \\
\omega^1_3 &= \left[\frac{1}{2h} \left(\frac{2u^2}{r^3f} - \frac{uu'}{r^2f} - 2f' \right) + (u\tilde{A}'_\phi - r^2\tilde{A}'_t) \frac{\ell^2 \tilde{A}_t \sin^2 \theta}{8r^2fh} \right] dt, \\
&\quad + \left[\frac{1}{2fh} \left(\frac{2u}{r} - u' \right) + (u\tilde{A}'_\phi - r^2\tilde{A}'_t) \frac{\ell^2 \tilde{A}_\phi \sin^2 \theta}{8r^2fh} \right] d\phi + (u\tilde{A}'_\phi - r^2\tilde{A}'_t) \frac{\ell^2 \sin^2 \theta}{8r^2fh} d\psi, \\
\omega^2_3 &= - \left[\frac{u'}{2rh} + \frac{\ell^2 \tilde{A}_t \tilde{A}'_\phi \sin^2 \theta}{8rh} \right] dt - \left[\frac{1}{h} + \frac{\ell^2 \tilde{A}_\phi \tilde{A}'_\phi \sin^2 \theta}{8rh} \right] d\phi - \frac{\ell^2 \tilde{A}'_\phi(r) \sin^2 \theta}{8rh} d\psi, \\
\omega^1_5 &= (u\tilde{A}'_\phi - r^2\tilde{A}'_t) \frac{\ell \sin \theta}{4r^2f} dr, \\
\omega^2_5 &= - \frac{\ell \tilde{A}'_\phi \sin \theta}{4r} dr, \\
\omega^3_5 &= \frac{\ell \sin \theta}{4h} (\tilde{A}'_t dt + \tilde{A}'_\phi d\phi), \\
\omega^4_5 &= \cos \theta \left[d\psi + \tilde{A}_t dt + \tilde{A}_\phi d\phi \right]. \tag{B.2}
\end{aligned}$$

The gravitino variation equations (4.9) are:

$$(\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + \Xi_\mu)\epsilon = 0, \tag{B.3}$$

where

$$\Xi_\mu = \frac{i}{4} T_{ab} (3\gamma^{ab}\gamma_\mu - \gamma_\mu\gamma^{ab}) = \frac{i}{2} T^{ab} e_\mu^c (\gamma_{abc} - 4\delta_{ac}\gamma_b), \tag{B.4}$$

the spin connections ω_μ^{ab} (B.2) contain the derivative terms \tilde{A}'_μ of the KK gauge fields. To make use of the solutions in [43], we will need to cancel the derivative terms, which we shall do by changing the auxiliary two-form by δT away from the ambient value (5.3).

Since an S^2 is fibered by rotating the ψ -direction, there is no change in the θ -direction in the metric (5.1). Imposing the projection condition (5.4), we have the following constraints on the auxiliary two-form T_{ab} from $\Xi_\theta\epsilon = 0$.

$$\begin{aligned}
\delta T^{15} &= \delta T^{25} = \delta T^{35} = \delta T^{45} = 0, \\
\delta T^{32} + 2\delta T^{14} &= 0, \\
\delta T^{13} + 2\delta T^{24} &= 0, \\
\delta T^{21} + 2\delta T^{34} &= 0. \tag{B.5}
\end{aligned}$$

With these identities, one can write

$$\Xi_\psi = \frac{3i\ell}{4} \sin \theta \left[\delta T^{12}\gamma_{12} + \delta T^{13}\gamma_{13} + \delta T^{23}\gamma_{23} \right]. \tag{B.6}$$

One can choose the auxiliary two-form such that it cancels the \tilde{A}' pieces of the spin connections ω_ψ^{13} and ω_ψ^{23} :

$$\begin{aligned}
\delta T^{12} &= \delta T^{34} = 0, \\
\delta T^{31} &= 2\delta T^{24} = i(u\tilde{A}'_\phi - r^2\tilde{A}'_t) \frac{\ell \sin \theta}{12r^2fh}, \\
\delta T^{23} &= 2\delta T^{14} = \frac{i\ell \tilde{A}'_\phi \sin \theta}{12rh}. \tag{B.7}
\end{aligned}$$

With this choice of the auxiliary two-form, one can check, that no term proportional to \tilde{A}' survives in the gravitino variation (B.3).

The full value of the auxiliary two-form T is obtained by collecting the configurations (5.3) and (5.5):

$$T = \frac{\ell}{4} \sin \theta d\theta \wedge \left[d\psi + \tilde{A}_t dt + \tilde{A}_\phi d\phi \right] - \frac{i\ell}{6} \sin \theta \left[\tilde{A}'_t dr \wedge dt + \tilde{A}'_\phi dr \wedge d\phi \right] + \frac{i\ell^2 \tilde{A}'_\phi \sin \theta}{24r fh} \left(1 + \frac{r}{\ell} \sqrt{f^2 - 1} \right) dt \wedge d\theta - \frac{\ell^2 \tilde{A}'_\phi \sin \theta}{24fh} \sqrt{f^2 - 1} d\phi \wedge d\theta. \quad (\text{B.8})$$

C. Off-shell BPS solutions in AdS₃

In this appendix, we will solve the Killing spinor equations (5.10)

$$\left[\partial_\mu + iB_\mu^a \gamma_a (\sigma_3 \otimes \mathbb{1}) - \frac{i}{2} \tilde{A}_\mu (\sigma_3 \otimes \mathbb{1}) \right] \zeta = 0, \quad (\text{C.1})$$

by following the methods in [43]. We only need a minor change of their strategy to construct solutions for the equations (C.1). The difference comes from the fact that we work in the Euclidean signature instead of the Minkowski signature and we have the extra factor $\sigma_3 \otimes \mathbb{1}$ in the equations due to the 5d lift.

The equations (C.1) have the integrability condition

$$[S^\mu{}_a \gamma^a (\sigma_3 \otimes \mathbb{1}) - G^\mu (\sigma_3 \otimes \mathbb{1})] \zeta = 0, \quad (\text{C.2})$$

where

$$\begin{aligned} S^\mu{}_a &\equiv \varepsilon^{\mu\nu\rho} (\partial_\nu B_{\rho a} - \varepsilon_{abc} B_\nu{}^b B_\rho{}^c), \\ G^\mu &\equiv \frac{1}{2} \varepsilon^{\mu\nu\rho} \partial_\nu \tilde{A}_\rho. \end{aligned} \quad (\text{C.3})$$

With the metric ansatz (5.1), one finds that

$$S^r{}_1 = S^r{}_2 = S^t{}_3 = S^\phi{}_3 = 0, \quad G^r = 0. \quad (\text{C.4})$$

As in [43], we impose a projection condition

$$(1 - \Gamma^a b_a) \zeta = 0, \quad (\text{C.5})$$

for some complex functions $b_a(r)$ satisfying

$$b \cdot b = 1. \quad (\text{C.6})$$

Here we take the 3d gamma matrices Γ^a ($a = 1, 2, 3$) as

$$\Gamma^1 = \sigma^3 \otimes \sigma^1, \quad \Gamma^2 = \sigma^3 \otimes \sigma^2, \quad \Gamma^3 = \mathbb{1} \otimes \sigma^3. \quad (\text{C.7})$$

The projection condition (C.5) on the Killing spinor ζ can be solved by writing

$$\zeta = N(1 + \Gamma^a b_a) \zeta_0, \quad (\text{C.8})$$

for an arbitrary complex function $N(t, \phi, r)$ and a constant spinor ζ_0 . When (C.8) is substituted into the integrability condition (C.2), one finds that

$$\begin{aligned} & S^\mu_3 b^3 + (S^\mu_1 b^1 + S^\mu_2 b^2 - G^\mu)(\sigma_3 \otimes \mathbb{1}) + \left[S^\mu_3 \gamma^3 (\sigma_3 \otimes \mathbb{1}) + (i\varepsilon_{3bc} S^\mu b^b b^c - G^\mu b_3) \gamma^3 \right] \\ & + (S^\mu_1 + i\varepsilon_{1bc} S^\mu b^b b^c - G^\mu b_1) \gamma^1 (\sigma_3 \otimes \mathbb{1}) + (S^\mu_2 + i\varepsilon_{2bc} S^\mu b^b b^c - G^\mu b_2) \gamma^2 (\sigma_3 \otimes \mathbb{1}) = 0, \end{aligned} \quad (\text{C.9})$$

which amounts to

$$S^\mu_3 = 0, \quad S^\mu_a + i\varepsilon_{abc} S^\mu b^b b^c - G^\mu b_a = 0. \quad (\text{C.10})$$

This leads to

$$\begin{aligned} & B_{\phi 2} B_{t 1} = B_{\phi 1} B_{t 2}, \\ & (1 - (b_1)^2) S^{t 1} + (i b_3 - b_2 b_1) S^{t 2} = 0, \\ & (1 - (b_1)^2) S^{\phi 1} + (i b_3 - b_2 b_1) S^{\phi 2} = 0. \end{aligned} \quad (\text{C.11})$$

We now return to the Killing spinor equations (C.1). Given that the Killing spinor ζ takes the form of (C.8), we find by substitution into (C.1) that

$$\begin{aligned} & \partial_{t, \phi}(\ln N) + i b_a B_{t, \phi}^a (\sigma_3 \otimes \mathbb{1}) - \frac{i}{2} \tilde{A}_{t, \phi} (\sigma_3 \otimes \mathbb{1}) = 0, \\ & \partial_r(\ln N) + i b_a B_r^a = 0, \end{aligned} \quad (\text{C.12})$$

and

$$B_\mu^a - i\varepsilon^{abc} b_b B_{\mu c} - i\partial_\mu b^a - b^a b_c B_\mu^c = 0. \quad (\text{C.13})$$

The r component of (C.13) yields

$$\begin{aligned} & i b'_1 = -B_r^3 (i b_2 + b_1 b_3), \\ & i b'_2 = B_r^3 (i b_1 - b_2 b_3), \\ & i b'_3 = B_r^3 (1 - (b_3)^2). \end{aligned} \quad (\text{C.14})$$

The t and ϕ components of (C.13) reduce to the algebraic equations

$$\begin{aligned} & B_\phi^1 (1 - (b_1)^2) + B_\phi^2 (-b_1 b_2 + i b_3) = 0, \\ & B_\phi^1 (-i b_3 - b_1 b_2) + B_\phi^2 (1 - (b_2)^2) = 0, \\ & B_\phi^1 (i b_2 - b_3 b_1) + B_\phi^2 (-b_3 b_2 - i b_1) = 0. \end{aligned} \quad (\text{C.15})$$

Solving the second of these equations for b_3 and substituting it into the other two equations, we find that

$$b_1 = -R b_2 \pm \sqrt{1 + R^2}, \quad b_3 = -i[R \mp b_2 \sqrt{1 + R^2}], \quad (\text{C.16})$$

where

$$R = \frac{B_\phi^2}{B_\phi^1} = -\frac{1}{2f} \left(u' - \frac{2u}{r} \right) + \frac{i h r}{\ell}. \quad (\text{C.17})$$

Substituting this result into the equations (C.14), one discovers that R and b_2 satisfy the ordinary differential equations

$$R' - 2(1 + R^2) B_r^3 = 0, \quad (\text{C.18})$$

and

$$b_2' = \pm B_r^3 \sqrt{1 + R^2} (1 - b_2^2) . \quad (\text{C.19})$$

The equation for R together with the first equation in (C.11) is equivalent to the following two equations that determine the functions f and h in terms of u and two constants α and β :

$$\frac{2hf}{\ell} = -i \left(2\alpha + \frac{u'}{r} \right) , \quad f^2 = - \left(\alpha r + \frac{u}{r} \right)^2 + \beta^2 . \quad (\text{C.20})$$

where we assume that $u(r)$ and α are purely imaginary. The constant α must be purely imaginary and cannot vanish if the spacetime is to be asymptotic to anti-de Sitter space (or the black hole vacuum).

One can now deduce the following useful formulae:

$$R = \frac{i\sqrt{f^2 - \beta^2}}{f} , \quad \sqrt{1 + R^2} = \frac{\beta}{f} , \quad B_r^3 = \frac{1}{2f} \left(\alpha r + \frac{u}{r} \right)' . \quad (\text{C.21})$$

Using these formulae one can easily solve (C.19) for b_2 and simplify the expressions for the other components of b_a . The result is

$$\begin{aligned} b_1 &= -\frac{i\sqrt{f^2 - \beta^2}}{f} b_2 + \frac{\beta}{f} , \\ b_2 &= i \frac{k_+ \sqrt{f + \beta} + k_- \sqrt{f - \beta}}{k_- \sqrt{f + \beta} - k_+ \sqrt{f - \beta}} , \\ b_3 &= \frac{\sqrt{f^2 - \beta^2}}{f} + \frac{i\beta}{f} b_2 , \end{aligned} \quad (\text{C.22})$$

where k_{\pm} are arbitrary constants.

We can bring the function N to the form

$$N = a(r, \phi) e^{(\frac{in}{2}\phi + \frac{im}{2}\frac{t}{\ell})(\sigma_3 \otimes \mathbb{1})} , \quad (\text{C.23})$$

where m, n are integers and $a(r, \phi)$ is a real function. Taking real and imaginary parts of (C.12) now determines the one-form \tilde{A} and the function $a(r, \phi)$. It is at this point that one discovers that β must be purely real rather than purely imaginary because $a(r, \phi)$ would otherwise be a non-periodic function of ϕ . For real β one finds that

$$\tilde{A} = \tilde{A}_t dt + \tilde{A}_\phi d\phi = \left(\frac{i\beta u'}{\ell(2\alpha r + u')} + \frac{i\beta}{\ell} + \frac{m}{\ell} \right) dt + \left(\frac{2i\beta r}{\ell(2\alpha r + u')} + n \right) d\phi , \quad (\text{C.24})$$

and

$$a = a(r) \equiv k_+ \sqrt{f - \beta} - k_- \sqrt{f + \beta} . \quad (\text{C.25})$$

Combining (C.8) with (C.22) now yields the following result for the Killing spinor ζ :

$$\begin{aligned} \zeta &= e^{(\frac{in}{2}\phi + \frac{im}{2}\frac{t}{\ell})(\sigma_3 \otimes \mathbb{1})} [k_+ \sqrt{f - \beta} - k_- \sqrt{f + \beta}] \\ &\times \left\{ \left[1 + \frac{1}{f} (\beta \Gamma^1 + \sqrt{f^2 - \beta^2} \Gamma^3) \right] + b_2 \Gamma^2 \left[1 - \frac{1}{f} (\beta \Gamma^1 + \sqrt{f^2 - \beta^2} \Gamma^3) \right] \right\} \zeta_0 . \end{aligned} \quad (\text{C.26})$$

It is convenient to normalize the constant spinor ζ_0 such that

$$\zeta_0^\dagger \zeta_0 = 1. \quad (\text{C.27})$$

In addition, we may require without loss of generality that ζ_0 satisfy $P\zeta_0 = \zeta_0$ for some constant projection matrix, P , that projects out two components of ζ_0 , since (for fixed k_\pm) the real dimension of the space of Killing spinors ϵ is two, not four, despite the fact that ζ_0 has two complex, and hence four real,

$$\left[1 + \frac{(k_+^2 - k_-^2)\Gamma^1 + 2k_+k_- \Gamma^3}{k_+^2 + k_-^2}\right] \zeta_0 = 0. \quad (\text{C.28})$$

The metric admitting the Killing spinors (C.26) is give by

$$ds^2 = [-(\alpha r)^2 - 2\alpha u + \beta^2] dt^2 + 2udtd\phi + \frac{\ell^2 \left(\alpha + \frac{u'}{2r}\right)^2}{\left(\alpha r + \frac{u}{r}\right)^2 - \beta^2} dr^2 + r^2 d\phi^2 \\ + \frac{\ell^2}{4} \left[d\theta^2 + \sin^2 \theta \left\{ d\psi + \left(\frac{i\beta u'}{\ell(2\alpha r + u')} + \frac{i\beta}{\ell} + \frac{m}{\ell} \right) dt + x \left(\frac{2i\beta r}{\ell(2\alpha r + u')} + n \right) d\phi \right\}^2 \right]. \quad (\text{C.29})$$

Notice that the requirement that there exist Killing spinors leaves undetermined the purely imaginary function $u(r)$ and the constants α, β .

We shall now summarize some useful identities which are relevant to the content of §5. After changing $u \rightarrow iu$ as done in (5.13) and (5.14), we find $-i\ell\tilde{A}_t + \tilde{A}_\phi = \text{const}$ which implies $\tilde{A}'_\phi = i\ell\tilde{A}'_t$. The identity $\tilde{A}'_\phi = i\ell\tilde{A}'_t$ is used for the evaluation of the 3d CS term in (D.3). In addition, by using (C.20), the field configuration in the middle of (5.5) can be rewritten as

$$\frac{r^2 \tilde{A}'_t - iu\tilde{A}'_\phi}{r^2 fh} = -\frac{i\tilde{A}'_\phi \sqrt{-1 + f^2}}{rhf}. \quad (\text{C.30})$$

From this result, one can show

$$\tilde{A}'_t dr \wedge dt + \tilde{A}'_\phi dr \wedge d\phi = -\frac{r^2 \tilde{A}'_t - iu\tilde{A}'_\phi}{r^2 fh} e^1 \wedge e^3 - \frac{\tilde{A}'_\phi}{rh} e^2 \wedge e^3 \\ = \frac{\tilde{A}'_\phi}{rh} \left[i \frac{\sqrt{-1 + f^2}}{f} e^1 \wedge e^3 - e^2 \wedge e^3 \right], \quad (\text{C.31})$$

which is relevant to (5.16) and (B.8).

D. Evaluation of the action on the localizing solutions

In this appendix, we shall explicitly evaluate the action (6.10) on the BPS configurations found in §5.

The Ricci scalar of the metric (5.13) is

$$\mathcal{R} = -\frac{2}{\ell^2} + \frac{\ell^2 (\tilde{A}'_\phi)^2 \sin^2 \theta}{2(2r + \ell u')^2} + \frac{8\ell(u' - ru'')}{(2r + \ell u')^3}. \quad (\text{D.1})$$

Note that we use the convention of Wess and Bagger for curvature tensors where the Ricci scalar \mathcal{R} of a AdS space is positive (see appendix §A for details). The gauge field and auxiliary fields combine to give

$$3 C_{IJK} \sigma^I \left[\frac{1}{4} F_{\mu\nu}^J F^{\mu\nu K} - Y_{ij}^J Y^{ijK} - 3 \sigma^J F_{\mu\nu}^K T^{\mu\nu} \right] + C(\sigma) 18 T^2 = C(\sigma) \left[-\frac{3}{\ell^2} + \frac{\ell^2 (\tilde{A}'_\phi)^2 \sin^2 \theta}{4(2r + \ell u')^2} \right]. \quad (\text{D.2})$$

The 5d Chern-Simons term WFF in (6.10) generates a 3d Chern-Simons term

$$\begin{aligned} -\frac{i}{64\pi^2} C_{IJK} \int W^I \wedge F^J \wedge F^K &= \frac{i\ell^3 C(\sigma)}{16\pi} \int \tilde{A} d\tilde{A} = \frac{i\tilde{k}}{4\pi} \int \tilde{A} d\tilde{A} \\ &= \frac{i\ell^3 C(\sigma)}{8\pi} \int \tilde{A}'_\phi dt \wedge d\phi \wedge dr. \end{aligned} \quad (\text{D.3})$$

Using these results, one can evaluate the bulk action integral

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \frac{1}{8\pi^2} \int d^5 x \sqrt{g} \left[3 C_{IJK} \sigma^I \left(\frac{1}{4} F_{\mu\nu}^J F^{\mu\nu K} - Y_{ij}^J Y^{ijK} - 3 \sigma^J F_{\mu\nu}^K T^{\mu\nu} \right) \right. \\ &\quad \left. - C(\sigma) \left(\frac{1}{2} \mathcal{R} - 18 T^2 \right) \right] - \frac{i}{64\pi^2} C_{IJK} \int W^I \wedge F^J \wedge F^K \\ &= \frac{C(\sigma)}{8\pi^2} \int d^5 x \frac{\ell^2}{8} (2r + \ell u') \sin \theta \left[-\frac{1}{2} \frac{4}{\ell^2} \right] \\ &= -\frac{C(\sigma)}{8\pi} [r^2 + \ell u(r)]_0^\Lambda \text{Area}(T_{\text{bdry}}^2) \\ &= -\frac{C(\sigma)}{8\pi} [\Lambda^2 + \ell^2 u_\infty] \text{Area}(T_{\text{bdry}}^2). \end{aligned} \quad (\text{D.4})$$

where T_{bdry}^2 denotes the boundary torus with coordinates (2.14) on which the $(0, 4)$ SCFT₂ lives. In the second line, we use the identities $\sqrt{g} = \frac{\ell^2}{8} (2r + \ell u') \sin \theta$ and $\frac{u' - r u''}{(2r + \ell u')^2} = \frac{\tilde{A}'_\phi}{2\ell}$. Since the action integral (D.4) will diverge due to the infinite volume of AdS₃, we regulate it by imposing a cutoff $r = \Lambda$. In the last line, we substitute the limits at infinity and the value at the origin $r = 0$ of the function $u(r)$

$$u(r) \xrightarrow{r \rightarrow \infty} \ell u_\infty, \quad u(r) \xrightarrow{r \rightarrow 0} 0, \quad (\text{D.5})$$

where u_∞ is a constant. These limits are compatible with the discussion in §6.

The boundary action has three pieces, which were discussed in §2. Firstly, we have the Gibbons-Hawking term which ensures that, upon variation with fixed metric at the boundary, the action yields the Einstein equations

$$\begin{aligned} \mathcal{S}_{\text{GH}} &= \frac{C(\sigma)}{8\pi^2} \int_{r=\Lambda} d^4 x \sqrt{h} K \\ &= \frac{C(\sigma)}{8\pi^2} \int d^4 x \sin \theta \left[\frac{\Lambda^2}{2} + \frac{\ell^2}{4} (1 + 2u_\infty) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \right] \\ &= \frac{C(\sigma)}{8\pi} \text{Area}(T_{\text{bdry}}^2) [2\Lambda^2 + \ell^2 (1 + 2u_\infty)] + \mathcal{O}\left(\frac{1}{\Lambda^2}\right), \end{aligned} \quad (\text{D.6})$$

where $h_{\mu\nu}$ is the induced metric on the boundary and K is the trace of the extrinsic curvature on the boundary, which are given by

$$\sqrt{h} = \frac{\ell}{4} \sin \theta [r^2 \ell^2 + (r^2 + \ell u(r))^2]^{1/2} \Big|_{r=\Lambda} = \frac{\ell}{4} \sin \theta [\Lambda^2 \ell^2 + (\Lambda^2 + \ell^2 u_\infty)^2]^{1/2},$$

$$\begin{aligned}
K &= \left. \frac{2r(\ell^2 + 2r^2 + 2\ell u) + 2\ell(r^2 + \ell u)u'}{\ell(2r + \ell u')\sqrt{r^2\ell^2 + (r^2 + \ell u)^2}} \right|_{r=\Lambda} \\
&= [\Lambda^2\ell^2 + (\Lambda^2 + \ell^2 u_\infty)^2]^{-1/2} \frac{1}{\ell} [2\Lambda^2 + \ell^2(1 + 2u_\infty)] + \mathcal{O}(\frac{1}{\Lambda^3}). \tag{D.7}
\end{aligned}$$

To produce a finite result, one also needs to add the boundary cosmological constant (this term and the Gibbons-Hawking term were already indicated in (2.1)):

$$\begin{aligned}
\mathcal{S}_{\text{ct}} &= -\frac{C(\sigma)}{8\pi^2} \int_{r=\Lambda} d^4x \sqrt{h} \frac{1}{\ell} \\
&= -\frac{C(\sigma)}{8\pi^2} \int d^4x \frac{1}{4} \sin \theta [\Lambda^2\ell^2 + (\Lambda^2 + \ell^2 u_\infty)^2]^{1/2} \\
&= -\frac{C(\sigma)}{8\pi^2} \int d^4x \sin \theta \left[\frac{\Lambda^2}{4} + \frac{\ell^2}{8}(1 + 2u_\infty) + \mathcal{O}(\frac{1}{\Lambda^2}) \right] \\
&= -\frac{C(\sigma)}{8\pi} \text{Area}(T^2) \left[\Lambda^2 + \frac{\ell^2}{2}(1 + 2u_\infty) \right] + \mathcal{O}(\frac{1}{\Lambda^2}). \tag{D.8}
\end{aligned}$$

So far, the evaluation of the action reduces to

$$\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{GH}} + \mathcal{S}_{\text{ct}} = \frac{\ell^2 C(\sigma)}{16\pi} \text{Area}(T_{\text{bdry}}^2). \tag{D.9}$$

Substituting the various constants (6.14), we get:

$$\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{GH}} + \mathcal{S}_{\text{ct}} = -\frac{i\pi c}{12\ell} (\tau - \bar{\tau}). \tag{D.10}$$

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