

Sigma Models for Bundles on Calabi-Yau: A Proposal for Matrix String Compactifications

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We describe a class of supersymmetric gauged linear sigma-model, whose target space is the infinite dimensional space of bundles on a Calabi-Yau 3- or 2-fold. This target space can be considered the configuration space of D-branes wrapped around the Calabi-Yau. We propose that this model can be used to define matrix string theory compactifications. In the infrared limit the model flows to a superconformal non-linear sigma-model whose target space is the moduli space of BPS configurations of branes on the compact space, containing the moduli space of semi-stable bundles. We argue that the bulk degrees of freedom decouple in the infrared limit if semi-stability implies stability. We study topological versions of the model on Calabi-Yau 3-folds. The resulting B -model is argued to be equivalent to the holomorphic Chern-Simons theory proposed by Witten. The A -model and half-twisted model define the quantum cohomology ring and the elliptic genus, respectively, of the moduli space of stable bundles on a Calabi-Yau 3-fold.

1. Introduction

It is by now well established that non-perturbative string theory fits into a greater scheme, involving also 11 dimensional supergravity, which we call M-theory [1]. A full microscopic foundation for this theory is however still lacking. In the matrix theory proposal of [2], the full dynamics of uncompactified M-theory was proposed to be captured by a certain large N limit of supersymmetric matrix quantum mechanics. This matrix quantum mechanics arises as the quantum theory of many partons, which are the only degrees of freedom left in the infinite momentum frame in the uncompactified situation. The partons can be identified with the D-particles in the corresponding type IIA string theory. When the theory is compactified on a circle, this leads to matrix string theory [3], which is described by the maximal $\mathcal{N}_{ws} = (8, 8)$ supersymmetric $U(N)$ Yang-Mills theory in 1+1 dimensions. In the infrared, this theory describes the ordinary string moving on a symmetric product of the transverse target space \mathbb{R}^8 .

A full description of non-perturbative string theory crucially involves D-branes [4]. D-branes are effectively described by (supersymmetric) gauge theories, living on the world-volume. Even general configurations of D-branes are described by gauge theories involving non-trivial gauge configurations; the different branes are described by the fluxes in the gauge theory [5]. Configurations of D-branes may therefore be viewed as a stringy description of vector bundles (or more generally sheaves) [6]. Matrix (string) theory compactified on a non-trivial manifold should involve also the degrees of freedom for branes wrapped around cycles in the compactification manifold. Indeed, it was already shown in [2] that membranes could be described in the original M(atrrix) theory. The simplest compactification manifolds are tori. The compactified M(atrrix) theory is described by a supersymmetric gauge theory living on the dual torus [2][7]. For the circle, this leads to the matrix string theory. For compactifications on a higher dimensional torus $S^1 \times T^n$, we may view the full gauge theory on the dual torus $\widehat{S}^1 \times \widehat{T}^n$ from a matrix string perspective as a gauged linear sigma model whose target space is the infinite dimensional space provided by the gauge theory on the torus \widehat{T}^n . The covariant derivatives on the dual torus are identified with (some of) the adjoint scalars that live on the world-sheet, and which are now infinite dimensional matrices. In this sense, it is a sector of the large N matrix string theory. These gauge theories automatically describe the dynamics of the wrapped D-branes on the torus, which are represented by the fluxes in this gauge theory. We may even view the gauge theory on the dual torus as the configuration space of the wrapped D-branes.

In this paper we propose to describe compactifications of the matrix string related to more general Calabi-Yau manifolds by gauged linear sigma models whose target space is the infinite dimensional space of gauge bundles. As for the torus, we may view this linear target space as the fibre of an infinite rank gauge bundle on the worldsheet. Also the gauge group is infinite dimensional, and is formed by the gauge transformations in the bundle on the Calabi-Yau manifold. In general, not all the supersymmetry will be preserved. For example for compactifications on $K3$ and CY_3 , we should have $\mathcal{N}_{ws} = (4, 4)$ and $\mathcal{N}_{ws} = (2, 2)$ supersymmetry respectively. Note that in general the target space not only is described by the pure gauge bundles, but also include certain (adjoint) scalars, which describe the movement of branes in the bulk. We should remark that such a model is not directly related to matrix string theory compactified on the Calabi-Yau manifold under consideration. Indeed, for the torus we know that the gauge theory lives on the dual torus. Therefore, this model should more appropriately be considered as the matrix string compactified on some dual manifold. This dual manifold should be a certain moduli space of bundles. For example, the dual torus can be considered the moduli space flat bundles on the torus, while also for the compactification on $K3$ surfaces the compactification manifold of the matrix string is a dual $K3$, which is identified as a certain moduli space of bundles on the $K3$ space where the gauge bundles live.

Formally the infrared limit of the matrix string corresponds to the limit where the bulk string coupling constant becomes zero. The theory then flows to a superconformal non-linear sigma model, whose target space is the locus of vanishing potential; this is the moduli space of the gauge bundles describing the linear sigma model. This target space should then be identified with the space on which the fundamental string lives – or rather a symmetric product of it, as the matrix string described second quantized string theory. Indeed, this was found for the uncompactified matrix string [3]. Also for compactifications on four dimensional Calabi-Yau manifolds $K3$ and T^4 , it is known that the appropriate moduli space of bundles is related to the symmetric product of the manifold. This symmetric product is in general smoothed out, as the resolution of the quotient singularity is a marginal deformation of the matrix string theory. For higher dimensional compactifications, the interaction of the strings corresponds to an irrelevant operator. Therefore, we do not expect the infrared target space to be given by a symmetric product in any limit in parameter space. For Calabi-Yau 3-folds, we do not know of any relation in general between the infrared target space of our proposed model and a symmetric product (it may however be that asymptotically for well separated strings this

space looks like a symmetric product). For certain special cases however where the Calabi-Yau manifold is a $K3$ -fibration and the dimension of the infrared target space is exactly 6, it is known that this space is a Calabi-Yau manifold [8]. But certainly, we should at least find a finite dimensional infrared target space, if we want to make sense out of this theory. This already puts very strong conditions on the model, and seems to imply that we can not define our model beyond the Calabi-Yau case. Even if the infrared target space is not a symmetric product, we may still identify our model as a compactification of matrix string theory, but in a more generalized sense. The only thing we can not do is the identification of the usual string theory in a geometrical way.

Apart from a proposal for matrix string compactification on Calabi-Yau manifolds, the model we describe in this paper can also be considered as a natural scheme to study topological properties of stable holomorphic bundles on Calabi-Yau manifolds. Because of the relation between these bundles and BPS configurations of branes, we could also physically view this as a model studying these BPS states. Natural elements to study are counting formulae, which have natural physical interpretations as black hole entropies, and (quantum) intersection rings of these configuration spaces. These calculations will unfortunately be outside the scope of this paper, although we make a start by studying certain properties of the topologically twisted models. We hope to come back to these interesting properties in the future.

The paper is organized as follows. In the section 2, we argue why gauged linear sigma models are the natural setting to study matrix string theory compactifications. We then give an overview of the general gauged linear sigma approach that we will be using. We will use for this the language of equivariant cohomology, rather than the more standard superspace approach, so this part can also be considered as an introduction of our notations. we also comment on possible relations with the non-linear sigma model approach proposed by Douglas et al. [9][10].

In section 3, we introduce the actual model describing bundles on a Calabi-Yau 3-fold. we start with describing some properties of the space of bundles which we need for the formulation of the model. After that, the construction of the model will be straightforward. We then study the infrared limit of the theory, described by a non-linear sigma model. Then we study the case where the Calabi-Yau 3-fold is of the form $K3 \times T^2$. This relates the model to the matrix string description of the five-brane [11]. We conclude this section by studying the decoupling from bulk degrees of freedom.

In section 4, we consider topologically twisted versions of the model, along the lines of [12][13]. The localization and observables in the A - and B -model are studied.

2. Preliminaries

In this section we review some salient features of $\mathcal{N}_{ws} = (2, 2)$ gauged linear sigma models (GLSM) [14][15], in a language suitable for our purpose. This also involves an infinite dimensional generalization of the usual GLSM, which can be described in the language of equivariant cohomology. We also briefly compare our proposal the one of Douglas et al., in terms of non linear sigma models.

2.1. $\mathcal{N}_{ws} = (2, 2)$ Gauged Linear Sigma Model and Equivariant Dolbeault Cohomology

We shall now describe the gauged linear sigma models (GLSM) in some more detail, but still in a quite general sense. The GLSM's we describe are slightly generalized, as we allow for an equivariant extension of the supersymmetry; that is we allow the supersymmetry algebra to be closed up to certain gauge transformations. Also, we want to generalize to allow for infinite dimensional target spaces. We will concentrate on theories which have $\mathcal{N}_{ws} = (2, 2)$ worldsheet supersymmetry, as this is the amount of supersymmetry expected for Calabi-Yau 3-fold compactifications. This amount of supersymmetry generally requires a target space which allows for a Kähler structure. Furthermore, as we want to get a linear sigma model, the target space will generally be flat. To have an anomaly free theory, we will also require that the first Chern class of the Kähler manifold vanishes. So we consider a flat Kähler manifold, which we denote \mathcal{A} . The path integral of the sigma model on Σ with target space \mathcal{A} involves the space of all maps $A : \Sigma \rightarrow \mathcal{A}$. Because of the Kähler structure, we can split up these coordinates into complex coordinates A^i and their complex conjugates $A^{\bar{i}}$. The left and right super-partners ψ_{\pm}^i of the two dimensional scalars A^i are spinors on the worldsheet and holomorphic tangent vectors in the target space. The \pm indices will denote worldsheet spinor indices. When we restrict to a point in Σ , we see from this and the supersymmetry commutation relations (which are trivial on a point) that the left and right supercharges act as two copies d_{\pm} of the exterior derivative d on \mathcal{A} . In terms of field theory, we may also state this as the relation that the supersymmetry restricted to a point on Σ reduces to a BRST symmetry on the target space \mathcal{A} , as $d^2 = 0$. The Kähler structure on the target space \mathcal{A} implies that we have $\mathcal{N}_{ws} = (2, 2)$ supersymmetry, due to the decomposition $d = \partial + \bar{\partial}$ of the exterior derivative.

Now we consider the case that a group \mathcal{G} acts on \mathcal{A} preserving the complex and Kähler structures. One may attempt to define a sigma model for the quotient space \mathcal{A}/\mathcal{G} by gauging the symmetry \mathcal{G} . The problems one may encounter is that we rarely have a

good quotient and the Kähler structure may not descend to it. The $\mathcal{N}_{ws} = (2, 2)$ gauged sigma model however resolves these problems in whole sale! To describe it, we use the relation between the supersymmetry on the worldsheet and the BRST symmetry (exterior derivative) in the target space noted above. The BRST cohomology is naturally generalized in the gauged situation to so called equivariant cohomology. The hart of the construction is an *automatic* equivariant extension of the space \mathcal{A} to $\mathcal{A}_{\mathcal{G}} = E\mathcal{G} \times_{\mathcal{G}} \mathcal{A}$ where \mathcal{G} acts *freely*. Here $E\mathcal{G}$ denotes the universal \mathcal{G} -bundle.¹ The left and right supercharges now act on \mathcal{A} as two copies of \mathcal{G} -equivariant exterior derivatives (the exterior derivatives in $\mathcal{A}_{\mathcal{G}}$), which satisfy the modified commutation relations

$$d_{\mathcal{G}} = d - i\phi^a i_a, \quad d_{\mathcal{G}}^2 = -i\phi^a \mathcal{L}_a \quad (2.1)$$

where ϕ^a denotes the generator of the \mathcal{G} -action, i_a denotes the contraction with the vector field V^a associated with the \mathcal{G} -action and \mathcal{L}_a is the Lie-derivative with respect to this vector field. The \mathcal{G} -equivariant cohomology of \mathcal{A} is the ordinary cohomology of the extended space $\mathcal{A}_{\mathcal{G}}$. The super-partners of A become equivariant differential one forms on \mathcal{A} . As for the ordinary derivative d we have a decomposition of the equivariant derivative $d_{\mathcal{G}}$ as $d_{\mathcal{G}} = \partial_{\mathcal{G}} + \bar{\partial}_{\mathcal{G}}$, such that

$$\partial_{\mathcal{G}}^2 = 0, \quad \{\partial_{\mathcal{G}}, \bar{\partial}_{\mathcal{G}}\} = -i\phi^a \mathcal{L}_a, \quad \bar{\partial}_{\mathcal{G}}^2 = 0, \quad (2.2)$$

which defines equivariant Dolbeault cohomology [18].

Such a decomposition again implies an extension to extended $\mathcal{N}_{ws} = (2, 2)$ worldsheet supersymmetry, using the equivariant derivatives above to construct the worldsheet supersymmetry generators. We denote the $\mathcal{N}_{ws} = (1, 1)$ supercharges by $Q_{\pm} = \mathbf{s}_{\pm} + \bar{\mathbf{s}}_{\pm}$, where \pm denotes the left and right spinor indices on the worldsheet Σ . If we reduce Σ to a point then Q_{\pm} are two copies of the equivariant exterior derivative $d_{\mathcal{G}}$ and \pm denote the charges under an internal symmetry of a graded equivariant cohomology. Such graded equivariant differentials first appeared as twisted supercharges of four-dimensional $\mathcal{N} = 4$ SYM [19], and in general are called balanced equivariant differentials [20]. The further decomposition $Q_{\pm} \rightarrow \mathbf{s}_{\pm} \oplus \bar{\mathbf{s}}_{\pm}$ in the Kähler case gives rise to differentials of a balanced

¹ For details on equivariant cohomology the reader is referred to the papers [16][17]. We will always use the Cartan model.

equivariant Dolbeault cohomology [21] on the target space. The supercharges satisfy the following commutation relations following (2.2)

$$\begin{aligned} \{\mathbf{s}_+, \bar{\mathbf{s}}_+\} &= i\nabla_{++}, & \{\mathbf{s}_+, \bar{\mathbf{s}}_-\} &= -ig_s^{-1}\sigma^a \mathcal{L}_a, & \{\mathbf{s}_+, \mathbf{s}_-\} &= 0, & \mathbf{s}_\pm^2 &= 0, \\ \{\mathbf{s}_-, \bar{\mathbf{s}}_-\} &= i\nabla_{--}, & \{\bar{\mathbf{s}}_+, \mathbf{s}_-\} &= -ig_s^{-1}\bar{\sigma}^a \mathcal{L}_a, & \{\bar{\mathbf{s}}_+, \bar{\mathbf{s}}_-\} &= 0, & \bar{\mathbf{s}}_\pm^2 &= 0, \end{aligned} \quad (2.3)$$

where $\nabla_{\pm\pm} = \partial_{\pm\pm} - v_{\pm\pm}^a \mathcal{L}_a$ are the covariant derivatives on the worldsheet Σ and g_s is the string coupling constant, which has scaling dimension one on the worldsheet Σ . Here we have introduced gauge fields $v_{\pm\pm}$ on Σ and the group \mathcal{G} is extended to a group of local gauge transformations on Σ . Note that σ is an adjoint scalar for this gauge group.² We see that in effect the equivariant extension leads on the worldsheet to a gauging of the symmetry by \mathcal{G} . We should note that the above supersymmetry algebra is the dimensional reduction of the $\mathcal{N} = 1$ supersymmetry in 4 dimensions, where the fields σ and $\bar{\sigma}$ are the reduced components of the gauge field [14][15]. We may interpret these supercharges as differentials of a balanced $\mathcal{G} \times P_\Sigma$ -equivariant cohomology, where P_Σ denotes the group of translations along Σ . The internal consistency of the commutation relations (2.3) determines a $\mathcal{N}_{ws} = (2, 2)$ vector multiplet, transforming according to the diagram

$$\begin{array}{ccccc} \bar{\sigma} & \xrightarrow{\mathbf{s}_+} & \eta_+ & \xleftarrow{\bar{\mathbf{s}}_-} & v_{++} \\ \downarrow \bar{\mathbf{s}}_- & & \downarrow \bar{\mathbf{s}}_- & & \downarrow \bar{\mathbf{s}}_- \\ \bar{\eta}_- & \xrightarrow{\mathbf{s}_+} & D & \xleftarrow{\bar{\mathbf{s}}_-} & \bar{\eta}_+ \\ \uparrow \bar{\mathbf{s}}_+ & & \uparrow \bar{\mathbf{s}}_+ & & \uparrow \bar{\mathbf{s}}_+ \\ v_{--} & \xrightarrow{\mathbf{s}_+} & \eta_- & \xleftarrow{\bar{\mathbf{s}}_-} & \sigma \end{array} \quad (2.4)$$

The supersymmetry above and this vector multiplet can be found also from dimensional reduction of the four dimensional $\mathcal{N} = 1$ supersymmetry and vector multiplet. The complex scalar $(\sigma, \bar{\sigma})$ then corresponds to the components along the compactified directions. Physically, the extension of \mathcal{A} to $\mathcal{A}_{\mathcal{G}}$ is just gauging of the global symmetry \mathcal{G} of the target space \mathcal{A} . The supermultiplet associated with the bosonic field $A^I(x)$ is completely determined by the complex structure on \mathcal{A} . Decomposing $A^I = A^i + A^{\bar{i}}$ as earlier the A^i should extend to a chiral (holomorphic) multiplet, i.e. $\bar{\mathbf{s}}_\pm A^i = 0$

$$\begin{array}{ccccc} \psi_-^i & \xleftarrow{\bar{\mathbf{s}}_-} & A^i & \xrightarrow{\mathbf{s}_+} & \psi_+^i \\ & \searrow \bar{\mathbf{s}}_+ & & \swarrow \bar{\mathbf{s}}_- & \\ & & H^i & & \end{array} \quad (2.5)$$

² Note that the gauge field $v_{\pm\pm}$ is anti-hermitian in this convention, while also $\bar{\sigma} = -\sigma^\dagger$.

An important property of the $\mathcal{N}_{ws} = (2, 2)$ supersymmetric model is the $U(1)$ \mathcal{R} -symmetry. The left and right \mathcal{R} -charges (J_L, J_R) of the supercharges are set to the following values

$$\begin{aligned} \mathbf{s}_+ &: (+1, 0), & \bar{\mathbf{s}}_+ &: (-1, 0), \\ \mathbf{s}_- &: (0, +1), & \bar{\mathbf{s}}_- &: (0, -1). \end{aligned} \tag{2.6}$$

The \mathcal{R} -charges of the fields in the vector multiplet are then determined by the assignment of zero charges to the gauge field; this gives

Table 1

| | v_{++} | v_{--} | σ | $\bar{\sigma}$ | D | η_+ | η_- | $\bar{\eta}_+$ | $\bar{\eta}_-$ |
|-------|----------|----------|----------|----------------|-----|----------|----------|----------------|----------------|
| J_L | 0 | 0 | +1 | -1 | 0 | 0 | +1 | 0 | -1 |
| J_R | 0 | 0 | -1 | +1 | 0 | +1 | 0 | -1 | 0 |

Together with the obvious left and right spin charges they determine the graded form degrees of balanced equivariant Dolbeault cohomology.

The action functional of the theory is defined by

$$\begin{aligned} S(r, \bar{r}) &= \mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \left(-\text{Tr}(\sigma \bar{\sigma}) + \mathcal{K}(A^i, A^{\bar{i}}) \right) \\ &+ \frac{r}{g_s} \bar{\mathbf{s}}_+ \mathbf{s}_- \int d^2x \text{Tr} \sigma + \frac{\bar{r}}{g_s} \mathbf{s}_+ \bar{\mathbf{s}}_- \int d^2x \text{Tr} \bar{\sigma} \\ &+ \frac{1}{g_s} \mathbf{s}_+ \mathbf{s}_- \int d^2x \mathcal{W}(A^i) + \frac{1}{g_s} \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int d^2x \bar{\mathcal{W}}(A^{\bar{i}}), \end{aligned} \tag{2.7}$$

where $\mathcal{K}(A^i, A^{\bar{i}})$ denotes the Kähler potential of the flat space \mathcal{A} and $r = i\zeta + \theta/2\pi$ belongs to the center of $\text{Lie}(\mathcal{G}) \approx \text{Lie}(\mathcal{G})^*$. The trace is some suitable trace in a representation of the gauge group \mathcal{G} . The action functional S is obviously invariant under the $\mathcal{N}_{ws} = (2, 2)$ the supersymmetry, since $(\bar{\mathbf{s}}_+ \oplus \bar{\mathbf{s}}_-)A^i = (\mathbf{s}_+ \oplus \bar{\mathbf{s}}_-)\sigma = 0$.

From (2.4) we see that the generators of balanced equivariant Dolbeault cohomology consist of the bosonic fields in a $\mathcal{N}_{ws} = (2, 2)$ vector multiplet. The remaining bosonic auxiliary fields D and H_i form a crucial ingredient of the theory. Imposing the *algebraic* equation of motions for these fields one always has

$$\begin{aligned} D &= \frac{1}{g_s^2} (\mu - \zeta), \\ H_i &= \frac{1}{g_s} \left(\frac{\partial \mathcal{W}}{\partial A^i} \right), \end{aligned} \tag{2.8}$$

where μ is the equivariant momentum map $\mu : \mathcal{A} \rightarrow \text{Lie}(\mathcal{G})^*$ for the action of \mathcal{G} on \mathcal{A} . The potential energy V in the sigma model above is given by

$$V = g_s^2 \|D\|^2 + \sum_i \|H_i\|^2 + \frac{1}{g_s^2} \sum_i (\|\sigma^a \mathcal{L}_a A^i\|^2 + \|\bar{\sigma}^a \mathcal{L}_a A^i\|^2) + \frac{1}{2g_s^2} \|[\sigma, \bar{\sigma}]\|^2. \quad (2.9)$$

Here for the auxiliary fields H_i and D the on-shell values (2.8) should be substituted. In the infrared limit $g_s \rightarrow 0$ the dominant contributions to the path integral come from maps $A : \Sigma \rightarrow \mathcal{M}_\zeta$ to the locus of vanishing potential, modulo gauge transformations. We see from (2.9) that this is described by a symplectic quotient at level ζ :

$$\mathcal{M}_\zeta = (H_i^{-1}(0) \cap \mu^{-1}(\zeta)) / \mathcal{G}. \quad (2.10)$$

The group action preserves the condition $H_i = 0$ and the subvariety $H_i^{-1}(0) \subset \mathcal{A}$ inherits the complex and Kähler structures by restriction. The quotient space \mathcal{M}_ζ inherits the Kähler structure from the ambient space \mathcal{A} by the restrictions and the reduction. If ζ takes on a generic value, the group \mathcal{G} acts freely and \mathcal{M}_ζ is a smooth Kähler manifold. For such a case the infrared limit of the theory can be identified with the non-linear sigma-model whose target space is \mathcal{M}_ζ . For non-generic ζ the quotient space develops singularities or even may not exist at all. The infrared theory however should make sense also in these situations. For such cases however one always has some extra degrees of freedom not described by the moduli space, due to the extension of \mathcal{A} to $\mathcal{A}_\mathcal{G}$. If we vary ζ within the set of regular values the target space in general undergoes birational transformations. This is a physical realization of the variation of symplectic quotients. The well-known relation between the symplectic and geometrical invariant theory (GIT) quotients also is an important part of the story [22]. The essential point is that the condition $H_i = 0$ is preserved by the complexified group action $\mathcal{G}^\mathbb{C}$, while the condition $D = 0$ is only preserved by the real group action. The complex gauge group in general does not act freely on the submanifold $H_i^{-1}(0)$, so that taking the quotient directly would lead to unwanted singularities. The GIT quotient considers the complex quotient by restricting to some *stable* subset $H_i^{-1}(0)_s \subset H_i^{-1}(0)$, on which the complexified gauge group acts freely, and sets

$$H_i^{-1}(0) // \mathcal{G}^\mathbb{C} := H_i^{-1}(0)_s / \mathcal{G}^\mathbb{C}.$$

A proper condition for the stability should give rise to the equivalence $H_i^{-1}(0) // \mathcal{G}^\mathbb{C} = \mathcal{M}_\zeta$ for generic and regular ζ .

If $c_1(\mathcal{M}_\zeta) = 0$ the theory in the infrared limit is expected to flow to a $\mathcal{N}_{ws} = (2, 2)$ superconformal theory. The chiral operators of such a conformal theory by definition correspond to elements of \mathcal{G} -equivariant de Rham or Dolbeault cohomology on the space \mathcal{A} , carrying a suitable grading. The equivariant cohomology is a powerful mathematical tool. Note that if the moduli space is smooth, the equivariant cohomology on \mathcal{A} is equivalent to the ordinary cohomology on the moduli space. When the moduli space develops singularities, the ordinary cohomology is not well defined, while there is in general no problem with the equivariant cohomology on \mathcal{A} . Thus we may even see the equivariant cohomology as a string-inspired generalization of ordinary cohomology. From the viewpoint of the gauged linear sigma model this cohomology corresponds to the classical part of the story. The quantum properties of the theory are even more striking and beautiful, as exploited in many papers such as [14][15][23].

Some Finite Dimensional Examples

We now consider some examples, mainly to indicate the comparison of our notation with the standard supersymmetry approach. We may also see this example as a sector of uncompactified matrix string theory, in the presence of extra branes. We start with $U(N)$ super-Yang-Mills in two dimensions. The theory contains a vector multiplet, containing the $U(N)$ connection one-forms (on the worldsheet) $v_{\pm\pm}$, the adjoint complex scalar σ and fermions, as in (2.4). The action functional of the theory is defined by the formula

$$S(r, \bar{r}) = -\mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \operatorname{Tr} \sigma \bar{\sigma} + \frac{r}{g_s} \bar{\mathbf{s}}_+ \mathbf{s}_- \int d^2x \operatorname{Tr} \sigma + \frac{\bar{r}}{g_s} \mathbf{s}_+ \bar{\mathbf{s}}_- \int d^2x \operatorname{Tr} \bar{\sigma}.$$

The expression is similar to the superspace expression, where we integrate over the fermionic coordinates, and replace the scalar σ by the full vector superfield. This is equivalent, because the Berezin integral over the fermionic coordinates picks out exactly the supersymmetry transforms of the scalars, as in the expression above. We now generalize to the model for the Grassmannian considered in [15]. So we introduce k complex scalars in the fundamental representation of $\mathcal{G} = U(N)$ and combine them into a $N \times k$ matrix q . In the space of all such matrices we introduce a complex structure such that $\bar{\mathbf{s}}_{\pm} q = 0$. The \mathcal{G} -action on such a space is given by $q \rightarrow gq$ where $g \in \mathcal{G}$. The above condition determines a chiral and an anti-chiral multiplet and the supersymmetry transformation laws. On the space of matrices q we have a natural Hermitian structure given by $\int d^2x \operatorname{Tr} qq^*$. The corresponding action functional for this GLSM is then given by

$$S(r, \bar{r}) = \mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \operatorname{Tr} (-\sigma \bar{\sigma} + qq^*) + \frac{r}{g_s} \bar{\mathbf{s}}_+ \mathbf{s}_- \int d^2x \operatorname{Tr} \sigma + \frac{\bar{r}}{g_s} \mathbf{s}_+ \bar{\mathbf{s}}_- \int d^2x \operatorname{Tr} \bar{\sigma}.$$

The above action functional defines a GLSM for the Grassmannian $G(N, k)$ – the space of N complex planes in \mathbb{C}^k . After turning on the FI term ζ the model flows to a $\mathcal{N}_{ws} = (2, 2)$ non-linear sigma model whose target space is $G(N, k)$, as can be seen from the localization equations [15].

2.2. Digression: Comments on Gauged Non-Linear Sigma Models

In this subsection we briefly comment on the non-linear generalization of the gauged sigma-model and its possible applications. The main motivation for this section is to compare the proposal of Douglas et al. on matrix string theory on Calabi-Yau [9][10] with our proposal.

To begin with we consider an example of an $U(1)$ theory. Besides from the $\mathcal{N}_{ws} = (2, 2)$ vector multiplet we introduce three complex scalars ϕ^i , $i = 1, 2, 3$, representing the holomorphic coordinates of a complex 3-dimensional Kähler manifold X , i.e. $\bar{s}_\pm \phi^i = 0$. We assume that the ϕ^i are not charged under the $U(1)$ symmetry. These conditions lead to three chiral multiplets and determine the supersymmetry transformation laws. Let $K(\phi^i, \phi^{\bar{i}})$ be a Kähler potential for X . Then the action functional is defined by

$$S = \mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \left(-\sigma \bar{\sigma} + K(\phi^i, \phi^{\bar{i}}) \right) \quad (2.11)$$

The resulting theory has two decoupled sectors; one is the $U(1)$ SYM theory, where the ϕ^i are all zero, and the other is the supersymmetric non-linear sigma model.³

The above model can easily be generalized to the non-abelian version. We simply replace the gauge group by $U(N)$ and the complex scalars ϕ^i by $U(N)$ adjoint-valued complex scalars, and consider the following action functional

$$S = \mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \operatorname{Tr} \left(-\sigma \bar{\sigma} + \mathbf{K}(\phi^i, \phi^{\bar{i}}) \right) + \frac{1}{g_s} \mathbf{s}_+ \mathbf{s}_- \int d^2x \operatorname{Tr} \mathcal{W}(\phi^i) + \frac{1}{g_s} \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int d^2x \operatorname{Tr} \overline{\mathcal{W}}(\phi^{\bar{i}}), \quad (2.12)$$

where \mathbf{K} is a gauge covariant real functional and $\mathcal{W}(\phi^i)$ is a gauge covariant holomorphic functional of the ϕ^i . The above action functional has manifest $\mathcal{N}_{ws} = (2, 2)$ supersymmetry. This type of model has several interesting mathematical structures – matrix versions of

³ After adding the topological term defined by the pull-back of the Kähler form on M , we have the standard action functional for a $\mathcal{N}_{ws} = (2, 2)$ non-linear sigma model.

Kähler metrics, Christoffel symbols, Riemann tensor etc. However it is unclear what the conditions are for having a consistent quantum theory.

As we assume that string theory is a consistent theory, at least the action for D-branes moving on curved space should be consistent. Furthermore, when this curved space is Kähler, we expect them to be described by a gauged non-linear sigma model as above, as argued by Douglas [9]. So the criteria for the gauged non-linear sigma model for describing D-branes on Kähler manifolds should then be sufficient. Such criteria, called the axioms of D-brane geometry, were formulated by Douglas, and were suggested to be used as a starting point for defining matrix theory on a curved space X [9]. One of these axioms is the requirement that the moduli space of vanishing potential (modulo gauge symmetry) is the N th symmetric product of X . Comparing to our point of view, this moduli space should be identified with the quotient space (2.10), so that we would need

$$(H_i^{-1}(0) \cap \mu^{-1}(0)) / U(N) = S^N X. \quad (2.13)$$

Here X is the base manifold represented by the center of mass of the matrix coordinates ϕ^i , i.e., $\{\frac{1}{N} \text{Tr} \phi^i\}$. Another important axiom is the mass condition, which states that the off-diagonal matrix elements have masses proportional to the geodesic distance between the points on the diagonal. It is shown that the axioms require M to be Ricci-flat, and fix the holomorphic potential $\mathcal{W}(\phi^i)$ to the following minimal form

$$\mathcal{W}(\phi^i) = \phi^1[\phi^2, \phi^3]. \quad (2.14)$$

It is also shown that those axioms can be used to determine the matrix version of a Kähler potential \mathbf{K} in terms of the Kähler potential of base manifold X [10].

However it was demonstrated that such a model can be constructed only for Ricci-flat manifolds X with vanishing six-dimensional Euler density [24].⁴ This result implies that matrix string theory compactifications on Calabi-Yau 3-folds based on a $\mathcal{N}_{ws} = (2, 2)$ non-linear matrix sigma-model is not satisfactory so far.

In this paper we take an alternative approach. Instead of a $3N^2$ complex dimensional configuration space (described by the matrices ϕ^i) we consider the infinite dimensional

⁴ An example of such a manifold is the direct product $S \times C$ where S is a hyper-Kähler surface. Then we actually expect to have a $\mathcal{N}_{ws} = (4, 4)$ theory. It is not even clear if a non-linear choice for \mathbf{K} always allows for a suitable $\mathcal{W}(\phi^i)$ maintaining $\mathcal{N}_{ws} = (4, 4)$ supersymmetry. Clearly the choice (2.14) is compatible only with flat S .

linear space of all bundles on a Calabi-Yau 3-fold. The infrared target space will then be the moduli space of stable bundles representing BPS configuration of D-branes on this Calabi-Yau. Also the gauge group will be infinite dimensional, consisting of gauge transformations in the bundles. Our model will be defined only on Calabi-Yau manifolds as we will see shortly. It would be conceivable that we could relate to a non-linear sigma model by integrating out massive degrees of freedom in the infrared theory. These massive modes would be higher modes on the Calabi-Yau where the bundles are defined. This would also reduce the gauge group to the finite dimensional $U(N)$ gauge group found in the approach of Douglas. In this way, the non-linear sigma model would turn up as an effective theory related to our linear sigma model.

3. Sigma Model for Bundles on Calabi-Yau 3-Folds

In this section we construct a $\mathcal{N}_{ws} = (2, 2)$ gauged linear sigma model whose target space is the infinite dimensional space of bundles on a Calabi-Yau 3-fold.

3.1. The Basic Settings

We now come to the explicit construction of the model. Consider a Calabi-Yau 3-fold X with Kähler form ω and holomorphic 3-form $\omega^{3,0}$. We fix a rank N C^∞ -bundle E over X , endowed with a Hermitian structure. We fix the topological type of the bundle, by specifying its Chern character $\text{ch}(E)$, or rather the Mukai vector $\text{ch}(E)\sqrt{\widehat{A}(X)}$. For a Calabi-Yau 3-fold, the Mukai vector is given by

$$Q = \left(\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E) - \frac{p_1(X)}{48} \text{ch}_0(E), \text{ch}_3(E) - \frac{p_1(X)}{48} \text{ch}_1(E) \right),$$

where $p_1(X)$ is the first Pontryagin class of the Calabi-Yau manifold. We may sum over different topological types later. The bundles may be seen as describing D-branes wrapped around the Calabi-Yau manifold X . The D-brane charges are precisely given by the components of the Mukai vector [6][25]. We will denote these D-branes by their part wrapped around the Calabi-Yau. For example, the rank $N = \text{ch}_0(E)$ corresponds to the number of D_6 -branes wrapped around X and more generally the charges $Q_{3-n}(E) \sim \text{ch}_{3-n}(E)$ correspond to D_{2n} -branes wrapped around cycles in X [26]. In a type IIB setting, these branes do not exist in the total ten-dimensional space-time. To get a type IIB brane, one should wrap the brane around another direction; this will of course be the spatial direction

of the matrix string. The D_6 -brane (in our notation) then corresponds in the full type IIB string theory to a $D7$ -brane.

We denote by $Lie(G)$ the Lie algebra of $G = U(N)$ and by $\text{End}(E) = E \otimes E^*$ the bundle of endomorphisms. Let \mathcal{A} be the infinite dimensional space of all connections, and \mathcal{G} the infinite dimensional group of gauge transformations $g : X \rightarrow G$. As usual \mathcal{A} is an affine space, and a tangent vector is represented by $\delta A \in \Omega^1(X, \text{End}(E))$. We want to use this (infinite dimensional) linear space \mathcal{A} and the group \mathcal{G} as the target space and gauge group respectively for a GLSM. To fit the above data in the framework described in the previous section we need some preparations – complex structure, Kähler potential, Dolbeault equivariant cohomology and a holomorphic potential leading to integrability.

Given the complex structure on X , we may introduce a complex structure on the space of connections \mathcal{A} as follows. Let A denote a connection one-form, which is decomposed into its holomorphic and antiholomorphic components $A = A^{1,0} + A^{0,1}$. One introduces a complex structure \mathcal{A} by declaring $\delta A^{0,1} \in \Omega^{0,1}(X, \text{End}(E))$ to be a holomorphic tangent vector. Endowed with this complex structure \mathcal{A} becomes an infinite dimensional flat Kähler manifold with Kähler form ϖ given by

$$\varpi(\delta A^{1,0}, \delta A^{0,1}) = \frac{i}{8\pi^2} \int_X \text{Tr}(\delta A^{1,0} \wedge \delta A^{0,1}) \wedge \omega \wedge \omega. \quad (3.1)$$

The group of gauge transformations \mathcal{G} acts with isometries on this space. The Kähler potential for ϖ is given by

$$\frac{1}{4\pi^2} \mathcal{K}(A^{1,0}, A^{0,1}) = \frac{i}{8\pi^2} \int_X \kappa \text{Tr}(F \wedge F) \wedge \omega, \quad (3.2)$$

where κ is a Kähler potential for ω . Thus both the complex structure and Kähler moduli in \mathcal{A} depend on those in the base space X .

Now we consider \mathcal{G} -equivariant differentials \mathbf{s} and $\bar{\mathbf{s}}$ on \mathcal{A} (they constitute the operators $\partial_{\mathcal{G}}$ and $\bar{\partial}_{\mathcal{G}}$ in (2.2)) such that

$$\begin{aligned} \mathbf{s}A^{0,1} &= i\psi^{0,1}, & \mathbf{s}\psi^{0,1} &= 0, \\ \bar{\mathbf{s}}A^{0,1} &= 0, & \bar{\mathbf{s}}\psi^{0,1} &= -\bar{\partial}_A\phi, & \mathbf{s}\phi &= 0, \\ \mathbf{s}A^{1,0} &= 0, & \mathbf{s}\bar{\psi}^{1,0} &= -\partial_A\phi, & \bar{\mathbf{s}}\phi &= 0, \\ \bar{\mathbf{s}}A^{1,0} &= i\bar{\psi}^{1,0}, & \bar{\mathbf{s}}\bar{\psi}^{1,0} &= 0, \end{aligned} \quad (3.3)$$

where $\psi^{0,1} \in \Omega^{0,1}(X, \text{End}(E))$ represents a holomorphic (co)-tangent vector on \mathcal{A} and the adjoint scalar $\phi \in \Omega^0(X, \text{End}(E))$ is the generator of an infinitesimal \mathcal{G} -action on \mathcal{A} . We

have $\{\mathbf{s}, \bar{\mathbf{s}}\}A = -id_A\phi$ satisfying (2.2). Using these equivariant differentials, we have an equivariant Kähler identity

$$\begin{aligned}\widetilde{\varpi} &= \frac{i}{4\pi^2} \mathbf{s} \bar{\mathbf{s}} \mathcal{K}(A^{1,0}, A^{0,1}) \\ &= \frac{i}{4\pi^2} \int_X \text{Tr}(i\phi F) \wedge \omega \wedge \omega + \frac{i}{4\pi^2} \int_X \text{Tr}(\psi^{0,1} \wedge \bar{\psi}^{1,0}) \wedge \omega \wedge \omega,\end{aligned}\tag{3.4}$$

where the second term can be identified with the Kähler form ϖ and the first term is the moment map $\phi^a \mu_a, \mu : \mathcal{A} \rightarrow \text{Lie}(\mathcal{G})^* = \Omega^6(X, \text{End}(E))$ for the action of \mathcal{G} on \mathcal{A} ,

$$\mu(A) = \frac{1}{4\pi^2} F \wedge \omega \wedge \omega = \frac{1}{12\pi^2} (\Lambda F) \omega \wedge \omega \wedge \omega,\tag{3.5}$$

where Λ is the adjoint of wedge multiplication by ω . $\widetilde{\varpi}$ is known as an equivariant Kähler form.

The Kähler structure on the space of bundles does not give enough structure for our purpose. The moduli space of bundles, which in the end will be identified with the infrared target space of our model, should be a finite dimensional space. The space of gauge equivalence classes of bundles however can never be finite dimensional. This can easily be seen as follows. Using the Kähler structure on X , we can decompose the curvature two-form of the bundle E into type according to $F = F^{2,0} + F^{1,1} + F^{0,2}$. Using the moment map μ , we can restrict only the $F^{1,1}$ part of the curvature. The $F^{0,2}$ part however will not be restricted, thus leading to an infinite dimensional space of deformations. There is a natural way to further restrict the set of gauge bundles. To this end, we consider the infinite dimensional subvariety $\mathcal{A}^{1,1}$ of all connections for which the curvature is of type $(1,1)$, so

$$F_A^{0,2} = 0.\tag{3.6}$$

Thus $\bar{\partial}_A^2 = 0$ for $A \in \mathcal{A}^{1,1}$. This condition endows the bundle E with a holomorphic structure. The moduli space of holomorphic bundles is the set of bundle isomorphism classes. It can be given by the following complex quotient

$$\mathcal{A}^{1,1}/\mathcal{G}^{\mathbb{C}},\tag{3.7}$$

where $\mathcal{G}^{\mathbb{C}}$ is the complexification of \mathcal{G} . As discussed in the last section, we can restrict the model by adding a holomorphic potential \mathcal{W} as in (2.7) to the model. Indeed, we are even forced to do so, as the moduli space (2.10) would otherwise not be finite dimensional. The holomorphic potential should be a holomorphic functional of the coordinates $A^{0,1}$ on \mathcal{A} .

Furthermore, it should be gauge invariant at least for gauge transformations connected to the identity. This essentially fixes the holomorphic potential to the holomorphic Chern-Simons functional, which is given by

$$\mathcal{W}(A^{0,1}) = \int_X \omega^{3,0} \wedge \text{Tr} \left(\frac{1}{2} A^{0,1} \wedge \bar{\partial} A^{0,1} + \frac{1}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right). \quad (3.8)$$

Note that this potential through (2.8) gives rise to exactly the condition (3.7) in the infrared,

$$\frac{\delta \mathcal{W}}{\delta A^{0,1}} = 0 \longrightarrow F_A^{0,2} = 0. \quad (3.9)$$

Note that the construction of this holomorphic potential is only possible on a Calabi-Yau manifold, as it makes use of the holomorphic 3-form $\omega^{3,0}$. This functional was first considered by Witten [27], but was interpreted there as the action functional rather than a superpotential for his effective open string theory. We come back to the relation of his model to ours later in this paper.

We have now defined all the data needed for the construction of a $\mathcal{N}_{ws} = (2, 2)$ gauged linear sigma model associated with the infinite dimensional pair $(\mathcal{A}, \mathcal{G})$.

3.2. The $\mathcal{N}_{ws} = (2, 2)$ GLSM

To construct the GLSM explicitly, we consider a vector bundle \tilde{E} over $X \times \Sigma$, with structure group $U(N)$. The group of all gauge transformations in this vector bundle will be denoted $\tilde{\mathcal{G}}$, and $\tilde{\mathcal{A}}$ is the space of all connections on \tilde{E} . We denote by E , \mathcal{G} and \mathcal{A} the restrictions of \tilde{E} , $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{A}}$ respectively to $X \times \{pt\}$. We will use spinor notation for the world-sheet Σ and differential form notation for the Calabi-Yau X . The supercharges evaluated at a point $\{pt\} \in \Sigma$ are differentials of balanced \mathcal{G} -equivariant Dolbeault cohomology on \mathcal{A} . The $\mathcal{N}_{ws} = (2, 2)$ supersymmetry algebra is defined by the commutation relations (2.3). In terms of infinitesimal generator ϵ_- , $\bar{\epsilon}_-$, which are sections of $K_\Sigma^{-1/2}$, and ϵ_+ , $\bar{\epsilon}_+$, which are sections of $\overline{K}_\Sigma^{-1/2}$, we denote $\delta = \bar{\epsilon}_- \mathbf{s}_+ + \bar{\epsilon}_+ \mathbf{s}_- + \epsilon_+ \bar{\mathbf{s}}_- + \epsilon_- \bar{\mathbf{s}}_+$. The supercharges transform as scalars on the Calabi-Yau 3-fold X .

The GLSM related to gauge bundles on the CY has the following field content. First, there is the $\mathcal{N}_{ws} = (2, 2)$ vector multiplet as in the diagram (2.4). The world-sheet vector multiplet transforms as an adjoint valued scalar on X . The explicit transformation laws are given in Appendix A. The covariant derivatives on the worldsheet are defined as $\nabla_{\pm\pm} = \partial_{\pm\pm} + v_{\pm\pm}$, and its curvature is $f_\Sigma = [\nabla_{++}, \nabla_{--}]$. The fields in the vector multiplet $(v_{\pm\pm}, \eta_{\pm}, D, \sigma)$ will have worldsheet scaling dimensions $(1, 3/2, 2, 0)$. Secondly we have

chiral and anti-chiral multiplets from the connection one-form $A = A^{1,0} + A^{0,1}$ on X . According to our choice of complex structure on \mathcal{A} , we build up chiral ($\bar{\mathbf{s}}_{\pm} A^{0,1} = 0$) multiplets from $A^{0,1}$. The transformations in the chiral multiplet are as in the following diagram

$$\begin{array}{ccccc}
\psi_-^{0,1} & \xleftarrow{\mathbf{s}_-} & A^{0,1} & \xrightarrow{\mathbf{s}_+} & \psi_+^{0,1} \\
& & \searrow_{\mathbf{s}_+} & & \swarrow_{\bar{\mathbf{s}}_-} \\
& & H^{0,1} & &
\end{array} \tag{3.10}$$

The explicit transformation rules for a chiral multiplet can be found in Appendix A.

Similarly, the fields in the $A^{1,0}$ multiplet form anti-chiral ($\mathbf{s}_{\pm} A^{1,0} = 0$) multiplets

$$\begin{array}{ccccc}
\bar{\psi}_-^{1,0} & \xleftarrow{\bar{\mathbf{s}}_-} & A^{1,0} & \xrightarrow{\bar{\mathbf{s}}_+} & \bar{\psi}_+^{1,0} \\
& & \searrow_{\bar{\mathbf{s}}_+} & & \swarrow_{\mathbf{s}_-} \\
& & H^{1,0} & &
\end{array} \tag{3.11}$$

Note that, as A is a gauge field, any commutator with A should be replaced by a covariant derivative. We will write the covariant Dolbeault operator related to $A^{0,1}$ and $A^{1,0}$ as $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ and $\partial_A = \partial + A^{1,0}$ respectively.

The left and right $U(1)$ \mathcal{R} -charges (J_L, J_R) for the chiral and anti-chiral matter multiplets are given in Table 2.

Table 2

| | A | $\bar{\psi}_+^{1,0}$ | $\bar{\psi}_-^{1,0}$ | $\psi_+^{0,1}$ | $\psi_-^{0,1}$ | $H^{1,0}$ | $H^{0,1}$ |
|-------|-----|----------------------|----------------------|----------------|----------------|-----------|-----------|
| J_L | 0 | -1 | 0 | +1 | 0 | -1 | +1 |
| J_R | 0 | 0 | -1 | 0 | +1 | -1 | +1 |

The action functional can be given as in the general formula (2.7). Using the particular form of the Kähler and super potential as given above, this can be written

$$\begin{aligned}
S(r, \bar{r}) = & -\mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \int_X d\mu_X \text{Tr}(\sigma \bar{\sigma}) + \mathbf{s}_+ \mathbf{s}_- \bar{\mathbf{s}}_+ \bar{\mathbf{s}}_- \int_{\Sigma} d^2x \mathcal{K}(A^{1,0}, A^{0,1}) \\
& + \frac{r}{g_s} \bar{\mathbf{s}}_+ \mathbf{s}_- \int_{\Sigma} d^2x \int_X d\mu_X \text{Tr}(\sigma) + c.c. \\
& + \frac{1}{g_s} \mathbf{s}_+ \mathbf{s}_- \int_{\Sigma} d^2x \mathcal{W}(A^{0,1}) + c.c.,
\end{aligned} \tag{3.12}$$

where $d\mu_X$ denotes the volume form on X . Since $\mathbf{s}_+ \mathbf{s}_-$ has $U(1)$ \mathcal{R} -charge $(1, -1)$, $\mathcal{W}(A^{0,1})$ should have charges $(-1, 1)$ to preserve the \mathcal{R} -symmetry. Since A naturally has charges $(0, 0)$, the only choice left is to assign charges $(-1, 1)$ to $\omega^{3,0}$.

The model is realized as an eight-dimensional $U(N)$ gauge theory on the product manifold $\Sigma \times X$. Some of the supercharges in this theory are broken due to the nontrivial background. The surviving supercharges should be covariantly constant on X , while spinors on the worldsheet Σ . They can then be identified with scalars on X , by a trivial twist.⁵ These are our supercharges $\mathbf{s}_\pm, \bar{\mathbf{s}}_\pm$. As explained, we regard our model as a linear sigma model in two dimensions with infinite dimensional target space \mathcal{A} and gauged isometry group \mathcal{G} . We may regard the Calabi-Yau 3-fold as a parameter space describing a continuous family of pairs $(\mathcal{A}, \mathcal{G})$. As we discussed earlier \mathcal{A} inherited both its complex and Kähler structure from X . There is no inconsistency here since the supercharges are topological when restricted to X . The path integral is then independent of the size of X and we can take the limit $vol(X) \rightarrow 0$ to recover the two-dimensional sigma-model on Σ . The number of bose and fermi fields coincides with those of $\mathcal{N} = 1$ SYM theory in ten dimensions.

The six-dimensional model was also considered in [29] in terms of a $\mathcal{N}_T = 1$ cohomological field theory, as a special example of a more general construction of cohomological theories for moduli spaces of bundles.

3.3. The Infrared Limit

For finite string coupling constant g_s arbitrary bundles on the Calabi-Yau contribute to the path integral. In the infrared limit $g_s \rightarrow 0$ the dominant contributions to the path integral come from the space of all maps from the worldsheet Σ to the vanishing locus of the potential V (given in (2.9)) modulo the \mathcal{G} -action. With our choice of Kähler and super potential, these are determined by the following conditions

$$\begin{aligned} F^{0,2} &= 0, \\ \Lambda F - \zeta I &= 0, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} d_A \sigma &= 0, \\ [\sigma, \bar{\sigma}] &= 0. \end{aligned} \tag{3.14}$$

⁵ The model can thus be identified with a twisted version of eight-dimensional super Yang-Mills, similar to the approach of [28] in four dimensions.

The connections solving the first two equations are called *Einstein-Hermitian* (EH) connections [30]. They correspond to Einstein-Hermitian vector bundles. The moduli space of EH connections is the symplectic quotient

$$\mathcal{M}_{EH} = (\mathcal{A}^{1,1} \cap \mu^{-1}(\zeta)) / \mathcal{G}. \quad (3.15)$$

We denote by \mathcal{M}_{EH}^* the moduli space of irreducible EH connections. By the Donaldson-Uhlenbeck-Yau theorem the moduli space \mathcal{M}_{EH}^* is diffeomorphic to the moduli space of ω -stable holomorphic vector bundles defined by the GIT quotient [31][32]

$$\mathcal{A}^{1,1} // \mathcal{G}^{\mathbb{C}} = \mathcal{A}_s^{1,1} / \mathcal{G}^{\mathbb{C}} \quad (3.16)$$

If the connection is irreducible, the condition $d_A \sigma = 0$ implies that $\sigma = 0$. The two equations in (3.14) have non-trivial solutions if an EH connection is reducible. Typically a reducible connection gives rise to a singularity in \mathcal{M}_{EH} . The equations (3.13) do not guarantee that we always have irreducible connections. From the equations in (3.14) we see that such a reducible connection also gives rise to a non-compact direction in the localization manifold. These non-compact directions are not specially related to these singularities; the moduli space $\mathcal{M}_{EH}(X)$ is non-compact even if there are no reducible connections.⁶

A fundamental result of Witten for $\mathcal{N}_{ws} = (2, 2)$ gauged linear sigma models states that the physics of the infrared super conformal theory is smooth even if the target space develops singularities [14]. In many respects the string theory compactifies the target space and we may constructively identify the infrared target space as the moduli space of semistable torsion free sheaves on X . Note that the notion of (semi-)stability is variable depending on the polarization. If one changes the polarization the moduli space may undergo a sequence of birational transformations. Witten's analysis implies that the physics is independent of the polarization. In our case the Kähler form ω on X determines the polarization of stability (ω -stability). The (semi-)stability plays an important role in the model, and is also related to stability of bound states of wrapped D-branes [6].

⁶ According to Gieseker the moduli space of stable bundles is an open subset of the moduli space of semi-stable bundle. This provides the Gieseker compactification for the moduli space of stable bundles by taking the closure. The definition of stable bundles involves torsion free sheaves. One may consult a nice book [30] for details.

We briefly discuss the role of the FI term ζ . In general, the second equation in (3.13) is called the weak Einstein condition. The parameter ζ is constant along the worldsheet Σ . However, ζ can be a real function on X . If it is constant the second equation in (3.13) is called the Einstein condition with factor ζ . In the more general case one can relate the weak Einstein condition to the Einstein condition by a conformal change of the Hermitian metric on the bundle E . We will here take ζ to be constant. The Einstein condition then directly implies that ζ is given by

$$\zeta = \left(\int_X \text{ch}_1(E) \wedge \omega \wedge \omega \right) / \left(\frac{N}{6\pi} \int_X \omega \wedge \omega \wedge \omega \right) \quad (3.17)$$

thus depends only on the cohomology classes of ω and $c_1(E)$.

We may now conclude that our model flows to a non-linear sigma model for a Calabi-Yau with semi-stable bundles. We expect that the resulting sigma-model is superconformal since \mathcal{M}_{EH} inherits a Calabi-Yau structure. The case of rank N corresponds to a Calabi-Yau 3-fold with N D_6 -branes bounded with D_{2n} -branes classified by the Chern characters $\text{ch}_{3-n}(E)$. For example the equation (3.17) implies that if the volume of the D_4 -brane collapses to zero we should have $\zeta = 0$. The condition to preserve supersymmetry translates to stability. EH bundles can only exist when the following topological condition is met

$$\int_X (2N \text{ch}_2(E) - \text{ch}_1(E) \wedge \text{ch}_1(E)) \wedge \omega \geq 0, \quad (3.18)$$

where the equality holds if and only if E is projectively flat. If we do not have any D_2 - and D_4 -branes the bundles are flat. For $\text{ch}_1(E) = 0$, that is when there are no D_4 -branes, the above condition reduces to

$$\int_X \text{ch}_2(E) \wedge \omega \geq 0. \quad (3.19)$$

This is a direct generalization of the condition in four dimensions that only ASD connections survive. The more general condition (3.18) is just a slight modification of this restriction.

3.4. Reduction to Matrix String Theory of Five-Branes Compactified on a K3 Surface

Here we briefly comment on relation with matrix string theory of five-branes, whose world-volume is compactified on a $K3$ surface.

We consider the case that the Calabi-Yau 3-fold X is a product manifold $X = K3 \times T^2$. We will consider the limit of vanishing T^2 . Then we can T -dualize along the

T^2 -direction to reduce our model to a gauged linear sigma model for bundles on $K3$. This amounts to the simple dimensional reduction along the T^2 direction. The vector Q of D-brane charges reduces to the Mukai vector for the bundle on $K3$. The connection $(0, 1)$ -form in six dimensions then decomposes into $A^{0,1} \oplus \tau$, where $A^{0,1}$ is the component along the $K3$ and τ the component along the torus, which becomes a complex adjoint scalar on the $K3$ surface. More generally, the chiral multiplet (3.11) decomposes into two chiral multiplets; one including the connection $(0, 1)$ -form on the $K3$ surface,

$$\begin{array}{ccc} \psi_-^{0,1} & \xleftarrow{s_-} & A^{0,1} & \xrightarrow{s_+} & \psi_+^{0,1} \\ & \searrow s_+ & & \swarrow s_- & \\ & & H^{0,1} & & \end{array}, \quad (3.20)$$

and the other with the adjoint complex scalar τ ,

$$\begin{array}{ccc} \lambda_- & \xleftarrow{s_-} & \tau & \xrightarrow{s_+} & \lambda_+ \\ & \searrow s_+ & & \swarrow s_- & \\ & & H & & \end{array}. \quad (3.21)$$

After the above reduction the holomorphic superpotential \mathcal{W} in (3.9) reduces to

$$\mathcal{W}_4 = \int_{K3} \omega^{2,0} \wedge \text{Tr} \tau F^{0,2}, \quad (3.22)$$

where $\omega^{2,0}$ denotes the holomorphic symplectic form on the $K3$ surface. Since $\omega^{2,0}$ is a nowhere vanishing non-degenerated 2-form we may regard τ as a holomorphic two-form $\tau^{2,0} := \tau \omega^{2,0}$. This should of course be extended to the full chiral multiplet (3.21). Similarly the Kähler potential \mathcal{K} in (3.2) decomposes into

$$\mathcal{K}_4 = \int_{K3} \text{Tr} \left(i\kappa F \wedge F - \tau^{2,0} \wedge \bar{\tau}^{0,2} \right), \quad (3.23)$$

where κ is a Kähler potential for the $K3$. The action functional is given by the same formula (3.12), where \mathcal{W} and \mathcal{K} are replaced by their respective expressions given above.

The worldsheet supersymmetry of the resulting model enhances to $\mathcal{N}_{ws} = (4, 4)$ supersymmetry. The adjoint chiral multiplet with bosonic component τ in (3.21) combines with the $\mathcal{N}_{ws} = (2, 2)$ vector multiplet in (2.4) into a $\mathcal{N}_{ws} = (4, 4)$ vector multiplet. In this correspondence, the scalars τ and σ combine into a self-dual 2-form B^+ and a real scalar C , as follows

$$B^+ = \tau \omega^{2,0} + \bar{\tau} \omega^{0,2} + \text{Im} \sigma \omega, \quad C = \text{Re} \sigma. \quad (3.24)$$

This gives exactly the field content of the twisted $\mathcal{N} = 4$ SYM on $K3$ studied by Vafa and Witten [19].

In the infrared limit the theory reduces to a $\mathcal{N}_{ws} = (4, 4)$ non-linear sigma model. The target space is given by the solutions of the following equations, modulo the gauge transformations

$$\begin{aligned} F^{0,2} &= 0, \\ \Lambda F - \zeta I &= 0, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} d_A \sigma &= d_A \tau = 0, \\ [\sigma, \bar{\sigma}] &= [\tau, \bar{\tau}] = [\sigma, \tau] = [\bar{\sigma}, \tau] = 0. \end{aligned} \tag{3.26}$$

Note that the EH condition reduces to the condition of ASD connections on $K3$.⁷ If the EH connection is irreducible, the equations can only be solved by $\sigma = \tau = 0$. Then the target space of this model is the moduli space of stable bundles on $K3$, which can be identified with the moduli space \mathcal{M}_{ASD}^* of irreducible anti-self-dual (ASD) connections on the $K3$ surface. Our model in the infrared limit can be identified with the matrix string theory of the five-brane compactified on $K3$ discussed in [34][11][33], which was based on orbifold conformal field theory.⁸ Our reduced model describes the matrix string theory of the five-brane as a gauged linear sigma model in accordance with our general philosophy.⁹ This moduli space is known to be birational to a symmetric product of a (dual) $K3$ surface $\mathcal{M}_{ASD}^* = S^N \widetilde{K3}$ in general. In this way, we can identify the infrared limit of the model on $K3$ as a system of (weakly coupled) fundamental strings on $\widetilde{K3}$, in accordance also with the axioms of D-brane geometry [9]. The model in the infrared limit describes the Higgs branch of a $D1 - D5$ system where the $D5$ -brane is wrapped around $K3$. By

⁷ This is deformed by the FI parameter ζ , which also is a natural deformation in the $K3$ situation [33].

⁸ Our gauge theoretic description has an obvious problem due to the non-compactness of the moduli space of instantons. We better regard the infrared target space as the moduli space of torsion-free coherent sheaves, as emphasized in [6]. If the moduli space contains strictly semi-stable sheaves, which is inevitable in certain cases, the identification with orbifold conformal theory may be problematic. The torsion free sheaves are also relevant to matrix string theory in the presence of k five-branes [35]. The infrared target space is then the moduli space of torsion-free sheaves on \mathbb{R}^4 via the ADHM description.

⁹ The $\mathcal{N}_{ws} = (4, 4)$ world-sheet supersymmetry evaluated at a point on the worldsheet defines a balanced \mathcal{G} -equivariant hyper-Kähler cohomology [36].

applying T and S duality the model describes the matrix string theory of the five-brane wrapped around $K3$. The model describes six-dimensional interacting micro matrix strings and can be regarded as a microscopic definition of IIB string theory on $AdS_3 \times S^3 \times K3$ due to a celebrated conjecture of Maldacena [37].

The above identification is evidence that our model for bundles on Calabi-Yau can be regarded as the matrix string theory of Calabi-Yau compactifications. The model was already suggested in [21] based on a similar approach. If we perform dimensional reduction along the world-sheet our reduced model becomes the Vafa-Witten model of $\mathcal{N} = 4$ super-Yang-Mills theory on $K3$. The partition function of this topological field theory computes the Euler characteristic of the moduli space of instantons [19].

3.5. Decoupling the Bulk Degrees of Freedom

Stable bundles appear naturally in the context of non-perturbative string theory [6]. They correspond to the stable BPS configurations of branes wrapped around non-trivial cycles in the compactified part X of the bulk space time Z . These are also naturally associated with extremal black-hole solutions of the low energy effective supergravity. A suitable counting of the number of stable orbits corresponds to counting the microscopic degrees of freedom of these black holes. The asymptotic growth of the degeneracy then gives the black-hole entropy. In our context the natural object to study is the elliptic genus directly relevant to the four-dimensional black-hole. The semi-stable orbits which are not stable correspond to marginally stable brane configuration. They correspond to branes wrapped around certain vanishing cycles in X . Physically such states are new massless (tensionless) states free to escape to the bulk Z . Indeed, in the strictly semi-stable case the equations (3.14) (or (3.26) in the case of $K3$) allow for nontrivial solutions for σ (and τ), which describe the degrees of freedom outside the space X . Such an orbit also introduces singularities in the moduli space, indicating that the degrees of freedom of the bundle do not contain all the information necessary to describe the system.

Now we examine the above properties in the context of our models for $X = CY_3, K3$. We note that the equations for the infrared target space (3.13)(3.14) or (3.25)(3.26) are precisely the equations for BPS configurations for D-branes wrapped around X [6].¹⁰ Formally, from the viewpoint of the string world-sheet, the infrared limit corresponds to

¹⁰ As was remarked in [6], this BPS condition should be valid only in the limit of vanishing string coupling. This is consistent with our description, as we find these equations only at the infrared limit.

the limit where the bulk string coupling constant becomes zero. The string theory then flows to a superconformal non-linear sigma-model whose target space is the moduli space of semi-stable bundles together with the linear space spanned by the zero-modes of the equations in (3.14) or (3.26) for the adjoint complex scalars (σ for CY_3 and (σ, τ) for $K3$). These zero-modes represent the bulk degrees of freedom transverse to the compact space X . When the brane configuration is stable there are no zero-modes for the adjoint scalars. The stable bundles hence represent configurations of branes which are completely decoupled from the bulk. The matrix string only propagates on the compact space X .

Consequently the infrared superconformal theory on the string world-sheet involving stable bundles describes the decoupled matrix string theory. The M(atr)ix conjecture as well as Maldacena's conjectures state that such a theory is dual to string/M theory in a non-trivial background given by the near horizon limit [38][37]. As far as the description in terms of matrix string theory is concerned the decoupling mechanism is exactly the same for both CY_3 and $K3$. Thus it seems to be natural to conjecture that the infrared conformal theory for the CY_3 case has an analogous dual description. The natural conjecture is duality with IIB string theory on $AdS_3 \times S^1 \times CY_3$. Here the AdS_3 space comes from the worldsheet and the norm of σ , while the S^1 is described by the phase of σ . There are several problems with such a relation. First of all, compactification on a Calabi-Yau 3-fold needs, in terms of type IIB, 7-branes wrapped around the Calabi-Yau. These are hard to describe, especially in the context of M-theory. Also, the near-horizon geometry for these 7-branes is not so well behaved. Secondly, the dilaton in this case is not constant, so that we can not tune it to a small value. This implies that we can not identify a region in moduli space where the string is weakly coupled. This makes it very hard indeed to make use of such a correspondence.

As a superconformal non-linear sigma-model the chiral rings can be described by a topological sigma-model [39][12][13]. These topological quantities will be important ingredients for checks of the M(atr)ix and Maldacena conjecture. Another interesting quantity is the elliptic genus (the half-twisted model). For the $K3$ case the elliptic genus of the world-sheet superconformal theory [40] is used to test the duality [41].

4. Applications: Twisted Models

Given a $\mathcal{N}_{ws} = (2, 2)$ GLSM natural objects to study are supersymmetric indices – the Euler characteristic, the elliptic genus, and the chiral rings. Those topological and pseudo

topological quantities contain interesting information both for physics and mathematics. The (pseudo) topological quantities are most naturally studied using topologically twisted versions of the supersymmetric theory we have been studying. These twisted versions are the subject of this section. It could also be a starting point for the study of the generalized mirror conjecture [26] from a sigma model viewpoint [13][42][14]. An obvious benefit of the GLSM is that those (pseudo) topological quantities attributed to the infrared superconformal non-linear model can be evaluated in a different regime of the theory.

The Euler characteristic of the moduli space of stable EH bundles corresponds to the holomorphic Casson invariant which was defined by Thomas [8][43]. The elliptic genus is the stringy generalization of this quantity. The elliptic genus is particularly relevant for the four-dimensional black-hole entropy. The correlation functions of the A -model correspond to the quantum intersection pairing of the moduli space of stable bundles. This gives a stringy generalization of Donaldson-Witten type invariants on a Calabi-Yau 3-fold. We also remark that the mathematical definition of these classical invariants involve hard technical obstacles. As a folk theorem one expects that string theory may soften many, if not all, of these problems.

We begin with the description of the A -model and then proceed with the B -model. The half-twisted model computing the elliptic genus can be treated along the lines of the A -model. We will not consider it here. As usual we perform a Wick-rotation on the worldsheet, and use holomorphic coordinates on Σ .

4.1. The A -Model

The A -model (and the half twisted model) can be defined following the standard recipe [13][14]. The observables of the theory are given by \mathcal{G} -equivariant differential forms on the target space \mathcal{A} . In the infrared limit these observables can be identified with differential forms on the moduli space \mathcal{M}_{EH} , and therefore flow to the usual observables in a topological non-linear sigma model [12].

In the A -model the twist on the worldsheet is performed such that $\epsilon_+ = \epsilon$ and $\bar{\epsilon}_- = \bar{\epsilon}$ become worldsheet scalars. They are then set equal to constants on Σ . The other generators $\bar{\epsilon}_+$ and ϵ_- are set to zero. Thus we are keeping the supercharges \mathbf{s}_+ and $\bar{\mathbf{s}}_-$, which now transform as world-sheet scalar under the two-dimensional rotation group. As there is no source for confusion, we leave out the subscript \pm in the rest of this subsection. The BRST operator of the model is then given by $\delta = \bar{\epsilon}\mathbf{s} + \epsilon\bar{\mathbf{s}}$. The resulting model computes the quantum cohomology ring of the moduli space of holomorphic vector bundles

over the Calabi-Yau 3-fold X . The twisting maps the $\mathcal{N}_{ws} = (2, 2)$ vector multiplet to a basic vector multiplet and an anti-ghost multiplet according to

$$\begin{aligned}(v_{++}, \bar{\eta}_+) &\rightarrow (v_z, \theta_z), \\ (v_{--}, \eta_-) &\rightarrow (v_{\bar{z}}, \theta_{\bar{z}}), \\ (\bar{\sigma}, \eta_+, \bar{\eta}_-, D) &\rightarrow (\bar{\sigma}, \eta, \bar{\eta}, D).\end{aligned}$$

The (anti) chiral multiplets containing the target space vector fields are twisted in the following way, giving rise to basic multiplets and anti-ghosts

$$\begin{aligned}(A^{1,0}, \bar{\psi}_-^{1,0}) &\rightarrow (A^{1,0}, \bar{\psi}^{1,0}), & (\bar{\psi}_+^{1,0}, H^{1,0}) &\rightarrow (\bar{\chi}_z^{1,0}, H_z^{1,0}), \\ (A^{0,1}, \psi_+^{0,1}) &\rightarrow (A^{0,1}, \psi^{0,1}), & (\psi_-^{0,1}, H^{0,1}) &\rightarrow (\chi_{\bar{z}}^{0,1}, H_{\bar{z}}^{0,1}).\end{aligned}$$

The BRST transformation laws for the basic fields are

$$\begin{aligned}\delta A^{1,0} &= i\epsilon \bar{\psi}^{1,0}, & \delta \bar{\psi}^{1,0} &= -\bar{\epsilon} \partial_A \sigma \\ \delta A^{0,1} &= i\bar{\epsilon} \psi^{0,1}, & \delta \psi^{0,1} &= -\epsilon \bar{\partial}_A \sigma, & \delta \sigma &= 0. \\ \delta v_z &= i\epsilon \theta_z, & \delta \theta_z &= -\bar{\epsilon} \nabla_z \sigma, \\ \delta v_{\bar{z}} &= i\bar{\epsilon} \theta_{\bar{z}}, & \delta \theta_{\bar{z}} &= -\epsilon \nabla_{\bar{z}} \sigma,\end{aligned}\tag{4.1}$$

For the anti-ghost multiplets we have

$$\begin{aligned}\delta \bar{\sigma} &= -i\bar{\epsilon} \eta - i\epsilon \bar{\eta}, \\ \delta \eta &= \epsilon \left(+iD - \frac{g_s}{2} f_{z\bar{z}} - \frac{1}{2} [\sigma, \bar{\sigma}] \right), & \delta \bar{\chi}_z^{1,0} &= -\epsilon H_z^{1,0} + \bar{\epsilon} (\partial_A v_z - \partial_z A^{1,0}), \\ \delta \bar{\eta} &= \bar{\epsilon} \left(-iD + \frac{g_s}{2} f_{z\bar{z}} - \frac{1}{2} [\sigma, \bar{\sigma}] \right), & \delta \chi_{\bar{z}}^{0,1} &= +\bar{\epsilon} H_{\bar{z}}^{0,1} + \epsilon (\bar{\partial}_A v_{\bar{z}} - \bar{\partial}_{\bar{z}} A^{0,1}).\end{aligned}\tag{4.2}$$

We omitted the transformation laws for the auxiliary fields D , $H_z^{1,0}$ and $H_{\bar{z}}^{0,1}$. They can easily be found from the general supersymmetry transformations. The worldsheet scaling dimensions for the fields are rearranged such that they correspond to their worldsheet form degree. Hence the fields $(v_z, \lambda_z, f_{z\bar{z}})$ have dimensions $(1, 1, 2)$, while all the other (worldsheet scalar) fields have zero dimension.

The two BRST supercharges are identified with the holomorphic and anti-holomorphic differentials of \mathcal{G} -equivariant Dolbeault cohomology satisfying the following commutation relations [44]

$$\mathbf{s}^2 = 0, \quad \{\mathbf{s}, \bar{\mathbf{s}}\} = -i\mathcal{L}(\sigma), \quad \bar{\mathbf{s}}^2 = 0.\tag{4.3}$$

They define the operators ∂ and $\bar{\partial}$ on the space $\mathcal{A}_{\mathcal{G}}$. The twisted theory is defined for arbitrary Riemann surfaces Σ . The $U(1)$ \mathcal{R} -charges (J_L, J_R) of the original fields before twisting are identified with the degrees of \mathcal{G} -equivariant differential forms on \mathcal{A} .

The localization equations are read off from the transformation rules,

$$\begin{aligned} F^{0,2} &= 0, & \partial_A v_z - \partial_z A^{1,0} &= 0, \\ \Lambda F - \zeta I - \frac{g_s^2}{2} f_{z\bar{z}} &= 0, & \bar{\partial}_A v_{\bar{z}} - \bar{\partial}_{\bar{z}} A^{1,0} &= 0, \end{aligned} \tag{4.4}$$

$$\nabla\sigma = 0, \quad d_A\sigma = 0, \quad [\sigma, \bar{\sigma}] = 0. \tag{4.5}$$

For $\Sigma = \mathbb{CP}^1$ one can show that, under certain condition, the path integral is localized to the moduli space of holomorphic maps $\Sigma \rightarrow \mathcal{M}_{EH}$. The EH condition is slightly changed from the condition for the IR target space \mathcal{M}_{EH} . In the IR however, the extra term proportional to the field strength on the worldsheet will become zero. As we are describing a topological theory, and the fixed points are not changed in the IR, taking the IR limit does not have any effect on the correlation functions of the theory. Therefore we can simply ignore this term.

Fermion Zero-Modes

As we have mentioned, the theory has two classically conserved ghost numbers. The ghost numbers are related to the \mathcal{R} -charges of the untwisted theory as $(-J_L, J_R)$. Note that the BRST operators \mathfrak{s} and $\bar{\mathfrak{s}}$ have ghost numbers $(1, 0)$ and $(0, 1)$ respectively. The basic bosonic fields v and A have vanishing ghost numbers. Furthermore, we find the following ghost numbers for the fermionic fields.

$$\begin{array}{cccc} \bar{\eta} & \eta & \psi^{0,1} & \bar{\psi}_+^{1,0} \\ \chi_z^{1,0} & \chi_{\bar{z}}^{0,1} & \theta_{\bar{z}} & \theta_z \\ (1, 0) & (0, 1) & (-1, 0) & (0, -1) \end{array}$$

On the first line are the worldsheet scalars, on the second line the worldsheet one-forms, and on the last line their ghost numbers.

As is well known at the quantum level these symmetries are broken due to the anomaly related to the index or the Riemann-Roch theorem. Basically, this is due to the fermionic zero-modes. So let us look in more detail to these zero-modes. We assume in the following that the gauge bundle on X is always stable, that is, semi-stability implies stability. In

this case, there are no covariantly constant adjoint scalar sections. Therefore, all the fields that are scalars on the X have no zero-modes. These include the fermionic fields η , $\bar{\eta}$ and θ . Therefore the remaining fermions that may have zero-modes are the one-forms on X . As we see above, there is a pair of worldsheet scalars, and a pair of worldsheet one-forms. The number of covariantly constant adjoint valued one-forms on X equals the complex dimension n of the moduli space \mathcal{M}_{EH} of bundles. Therefore, the total ghost number anomaly for Σ a genus g worldsheet is $n(1-g)$, for both the ghost numbers. This means that in order to have a nonvanishing correlation function $\langle \Pi \mathcal{O}_a \rangle$, the total ghost numbers of the observables \mathcal{O}_a should be equal to this number.

Observables and Correlation Functions

The observables of the A -model are easy to construct. We begin with observables to be inserted on a point in Σ . By definition those observables are \mathcal{G} -equivariant differential forms on the space \mathcal{A} of all connections. Those observables generate cohomology rings of the moduli space of EH connections via restriction and reduction. Equivalently those observables flow to the usual observables of the non-linear sigma model in the infrared limit.

From $\delta\sigma = 0$ we see that an arbitrary G -invariant polynomial $P(\sigma)$ of σ with degree r is an observables. It corresponds to an equivariant $2r$ -form, (more precisely an (r, r) -form). The other observables can be obtained by the usual descent procedure. Equivalently we may use the universal bundle to construct those observables. From the Bianchi identity $d_A F = 0$ and the transformation laws in (4.1), we have the following generalized Bianchi identity

$$\mathcal{D}\mathcal{F} = 0, \tag{4.6}$$

where

$$\begin{aligned} \mathcal{D} &= \mathbf{s} + \bar{\mathbf{s}} + \partial_A + \bar{\partial}_A, \\ \mathcal{F} &= \sigma + i\bar{\psi}^{1,0} + i\psi^{0,1} + F^{2,0} + F^{1,1} + F^{0,2} \end{aligned} \tag{4.7}$$

We define a generalized Chern class \mathbf{c}_n by

$$\mathbf{c}_n = \frac{(-1)^n}{(2\pi)^n n!} \text{Tr } \mathcal{F}^n. \tag{4.8}$$

We expand the generalized Chern class as

$$\mathbf{c}_n = \sum_{p+q+r+s=2n} \mathcal{V}_{p,q}^{r,s} \tag{4.9}$$

where the upper indices denote the form degree on X while the lower indices denote the degree of the ghost number. Now it follows from the Bianchi identity (4.6) that we have the following descent equations

$$(\mathbf{s} + \bar{\mathbf{s}} + \partial + \bar{\partial})\mathbf{c}_n = 0, \quad (4.10)$$

which can be written in terms of the observables as

$$\bar{\mathbf{s}}\mathcal{V}_{p,q}^{r,s} + \mathbf{s}\mathcal{V}_{p-1,q+1}^{r,s} + \bar{\partial}\mathcal{V}_{p,q+1}^{r,s-1} + \partial\mathcal{V}_{p,q+1}^{r-1,s} = 0. \quad (4.11)$$

We define

$$V_{p,q}(\alpha) = \int_X \alpha^{3-r,3-s} \wedge \mathcal{V}_{p,q}^{r,s} \quad (4.12)$$

where $\alpha^{3-r,3-s} \in H^{3-r,3-s}(X)$, $0 \leq r, s \leq 3$ and $0 \leq p, q$. Then we have equivalently

$$\mathbf{s}V_{p-1,q}^{r,s} + \bar{\mathbf{s}}V_{p,q-1}^{r,s} = 0. \quad (4.13)$$

The relation (4.11) implies that the $Q = \mathbf{s} + \bar{\mathbf{s}}$ cohomology depends on the d -cohomology on X . From the Hodge diamond for $h^{r,s}(X)$

$$\begin{array}{ccccc} & & & & 1 \\ & & & & 0 \\ & & 0 & & 0 \\ 0 & & h^{1,1} & & 0 \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & & 0 & & h^{1,1} & & 0 \\ & & 0 & & 0 & & 0 \\ & & & & & & 1 \end{array} \quad (4.14)$$

we see immediately that we can discard some of the $\mathcal{V}_{p,q}^{r,s}$ for defining non-trivial observables. In calculating correlation functions with these observables, the ghost numbers (r, s) should add up to the total ghost number anomaly, which is (d, d) , where d is the dimension of the moduli space. This is related to the fact that the calculation of the correlation function can be reduced to an integral over the moduli space of the corresponding form. As only the integral of a top form gives a non zero integral, we find only non zero correlation functions when the abovementioned condition is met.

Among other observables the equivariant Kähler form $\widetilde{\omega}$ (3.4) plays an important role (here ϕ should be replaced by σ). It can be identified with the first Chern class of a \mathcal{G} -equivariant determinant line bundle \mathcal{L} over $\mathcal{A}^{1,1}$. After reduction to \mathcal{M}_{EH} the line bundle becomes the determinant line bundle with the first Chern class given by the Kähler

form ϖ on \mathcal{M}_{EH} . The expectation value $\langle \exp \widetilde{\varpi} \rangle$ corresponds to the quantum volume form of \mathcal{M}_{EH} . It is also easy to introduce anti-symmetric tensor fields on \mathcal{M}_{EH} . If we pick a two-dimensional homology cycle γ_2 on X we may construct the following observable

$$\tilde{\alpha} = \frac{1}{4\pi^2} \int_{\gamma_2} \text{Tr} \left(i\sigma F + \psi^{0,1} \wedge \bar{\psi}^{1,0} \right), \quad (4.15)$$

The \mathfrak{s} and $\bar{\mathfrak{s}}$ cohomology class of $\tilde{\alpha}$ depends only on the homology class of γ_2 . On \mathcal{M}_{EH} $\tilde{\alpha}$ becomes an element of type $(1, 1)$ in the cohomology of \mathcal{M}_{EH} .

As a last remark, note that the EH condition depends on the class of the Kähler form ω on X . As we vary ω the target space \mathcal{M}_{EH} may undergo a sequence of birational transformations. However the quantum intersection form must depend smoothly on the Kähler form, as required by the supersymmetry. The difference in behaviour is due to sigma-model instanton corrections, which smooth out these transition [14].

4.2. The B-Model

We now turn to the B -twisting of the $\mathcal{N}_{ws} = (2, 2)$ model. In this B -model we set $\bar{\epsilon}_{\pm} = 0$, while the ϵ_{\pm} become constant functions on the worldsheet Σ . Therefore the operators $\bar{\mathfrak{s}}_{\pm}$ become the BRST charges for this topological model. We let the BRST generator be given by $\delta = \epsilon_+ \bar{\mathfrak{s}}_- + \epsilon_- \bar{\mathfrak{s}}_+$, satisfying $\delta^2 = 0$. After twisting some fields will transform differently under the two-dimensional Lorentz group. For example, the twisted $\mathcal{N}_{ws} = (2, 2)$ vector field contains several worldsheet one-forms. These are given by

$$\begin{aligned} (v_{++}, \bar{\eta}_+, \sigma) &\rightarrow (v_z, \bar{\eta}_z, \sigma_z), \\ (v_{--}, \bar{\eta}_-, \bar{\sigma}) &\rightarrow (v_{\bar{z}}, \bar{\eta}_{\bar{z}}, \sigma_{\bar{z}}), \end{aligned} \quad (4.16)$$

Hence these fields form a multiplet of worldsheet one-forms. Furthermore, the other fields in this multiplet become worldsheet scalars. They are anti-ghost in the twisted model. For the (anti) chiral multiplet containing the target space gauge fields, we find the following anti-ghosts

$$\psi_+^{0,1} \rightarrow \rho_z^{0,1}, \quad \psi_-^{0,1} \rightarrow \rho_{\bar{z}}^{0,1}.$$

All the other fields become worldsheet scalars after twisting.

The BRST charges $\bar{\mathfrak{s}}_{\pm}$ satisfy the following commutation relations

$$\bar{\mathfrak{s}}_{\pm}^2 = 0, \quad \{\bar{\mathfrak{s}}_+, \bar{\mathfrak{s}}_-\} = 0. \quad (4.17)$$

as remarked earlier, they are related to the (anti-)holomorphic derivatives on the space \mathcal{A} . We therefore introduce the following linear combinations of the BRST charges

$$\begin{aligned}\bar{\mathbf{s}} &= \bar{\mathbf{s}}_+ + \bar{\mathbf{s}}_-, \\ \mathbf{s}^\dagger &= \bar{\mathbf{s}}_+ - \bar{\mathbf{s}}_-.\end{aligned}\tag{4.18}$$

Then $\bar{\mathbf{s}}$ becomes the anti-holomorphic differential of the \mathcal{G} -equivariant cohomology on \mathcal{A} , while \mathbf{s}^\dagger is the adjoint of the holomorphic equivariant differential \mathbf{s} with respect to the inner product on \mathcal{A} . In the infrared limit $\bar{\mathbf{s}}$ and \mathbf{s} become the $\bar{\partial}$ and ∂^\dagger operators on the moduli space \mathcal{M}_{EH} of EH connections. In the following we will work exclusively with the operator $\bar{\mathbf{s}}$, as we are mainly interested in the $\bar{\partial}$ -cohomology on the moduli space.

It is also convenient to introduce the following combinations of the 'fermions' in the six dimensional gauge multiplet,

$$\begin{aligned}\bar{\psi}^{1,0} &= \bar{\psi}_+^{1,0} + \bar{\psi}_-^{1,0}, \\ \chi^{0,2} &= * \left((\bar{\psi}_+^{1,0} - \bar{\psi}_-^{1,0}) \wedge \omega^{0,3} \right)\end{aligned}\tag{4.19}$$

Note that $\chi^{0,2}$ could also be identified with a $(-1,0)$ form or vector, using the metric instead of $\omega^{0,3}$.

We have the following BRST transformation laws for the basic fields coming from the matter fields,

$$\begin{aligned}\bar{\mathbf{s}}A^{1,0} &= i\bar{\psi}^{1,0}, & \bar{\mathbf{s}}\bar{\psi}^{1,0} &= 0, \\ \bar{\mathbf{s}}A^{0,1} &= 0, & \bar{\mathbf{s}}\bar{\chi}^{0,2} &= F^{0,2},\end{aligned}\tag{4.20}$$

For the one-forms from the vector field we find

$$\begin{aligned}\bar{\mathbf{s}}v_z &= i\bar{\eta}_z, & \bar{\mathbf{s}}\sigma_z &= -i\bar{\eta}_z, & \bar{\mathbf{s}}\bar{\eta}_z &= 0, \\ \bar{\mathbf{s}}v_{\bar{z}} &= i\bar{\eta}_{\bar{z}}, & \bar{\mathbf{s}}\sigma_{\bar{z}} &= -i\bar{\eta}_{\bar{z}}, & \bar{\mathbf{s}}\bar{\eta}_{\bar{z}} &= 0.\end{aligned}\tag{4.21}$$

And finally for the anti-ghosts

$$\begin{aligned}\bar{\mathbf{s}}\rho_z^{0,1} &= -\bar{\partial}_A\sigma_z - \bar{\partial}_A v_z + \partial_z A^{0,1}, \\ \bar{\mathbf{s}}\rho_{\bar{z}}^{0,1} &= -\bar{\partial}_A\sigma_{\bar{z}} - \bar{\partial}_A v_{\bar{z}} + \partial_{\bar{z}} A^{0,1}, \\ \bar{\mathbf{s}}\eta_+ &= (\Lambda F - \zeta I) - \frac{1}{2}[\sigma_z, \sigma_{\bar{z}}] - \frac{1}{2}f_{z\bar{z}} - \nabla_z\sigma_{\bar{z}}, \\ \bar{\mathbf{s}}\eta_- &= (\Lambda F - \zeta I) + \frac{1}{2}[\sigma_z, \sigma_{\bar{z}}] + \frac{1}{2}f_{z\bar{z}} - \nabla_{\bar{z}}\sigma_z.\end{aligned}\tag{4.22}$$

From these transformations we read off the following fixed point equations (here we used both the $\bar{\mathbf{s}}_{\pm}$ transformation rules)

$$\begin{aligned}
F^{0,2} &= 0, \\
\Lambda F - \zeta I &= 0, \\
\nabla_{\bar{z}} \sigma_z &= 0, \\
f_{z\bar{z}} + [\sigma_z, \sigma_{\bar{z}}] &= 0,
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
\bar{\partial}_A \sigma_z &= \bar{\partial}_A \sigma_{\bar{z}} = 0, \\
\bar{\partial}_A v_z - \partial_z A^{0,1} &= 0, \\
\bar{\partial}_A v_{\bar{z}} - \partial_{\bar{z}} A^{0,1} &= 0,
\end{aligned} \tag{4.24}$$

Note that the last two equations in (4.23) are Hitchin's self-duality equations in two dimensions [45]. On a cylinder or \mathbb{CP}^1 these equations have no non-trivial solutions.¹¹ Thus $f_{z\bar{z}} = \sigma_z = 0$. Then the connection v_z is flat and can be gauge transformed away. What we are left with from the above equations are

$$\partial_z A^{0,1} = \partial_{\bar{z}} A^{0,1} = 0. \tag{4.25}$$

Thus the path integral is localized to a copy of the moduli space $\mathcal{M}_{EH}(X)$ of EH connections on X .

The action functional for the B -model can be written in the form

$$S(\epsilon^2) = \frac{1}{\epsilon^2} \bar{\mathbf{s}}V + \frac{1}{\epsilon^2} W, \tag{4.26}$$

modulo terms which vanish by the fermion equations of motion. The precise form for V and W is given in Appendix B. We introduced a coupling constant e for convenience later. The part W comes from the holomorphic potential; it is invariant under the BRST symmetry generated by $\bar{\mathbf{s}}$, although it is not exact. We may now follow the standard recipe for the B -model as put forward in [13]. The correlation functions of the theory are identified with periods of differential forms on \mathcal{M}_{EH} .

¹¹ If we consider a Riemann surface Σ with $genus(\Sigma) \geq 1$, the moduli space of the Hitchin equations may play an important role.

Fermion Zero-Modes

As for the A -model, also the B -model has two classically conserved ghost numbers, given by the \mathcal{R} -charges (J_L, J_R) . So the BRST operators have ghost number $(1, 0)$ and $(0, 1)$ respectively. We will consider here only the total ghost number $\frac{1}{2}(J_L + J_R)$. The bosonic fields v and A again have vanishing ghost number. For the fermions, the worldsheet scalars η_{\pm} and one-form $\rho^{0,1}$ have ghost number 1, while the worldsheet scalars $\bar{\psi}^{1,0}$, $\chi^{0,2}$ and the one-form $\bar{\eta}$ have ghost number -1 .

Again, there is an anomaly related to the index. As for the A -model we assume that the gauge bundle on X is always stable, so that the adjoint scalars on X η and $\bar{\eta}$ have no zero-modes. we remain with possible zero-modes for the adjoint forms. Note that the forms $\bar{\psi}^{1,0}$ and $\chi^{0,2}$ have the same number of zero-modes on X , as they can be related by using $\omega^{3,0}$. The number of covariantly constant adjoint-valued one-forms on X is the complex dimension n of the moduli space of bundles. Here it is essential that the condition $c_1(\mathcal{M}) = 0$ is met. Otherwise, the number of ρ zero-modes and $\bar{\psi}$ and χ zero-modes would be different. This would not even lead to an acceptable quantum theory, as this would mean that the fermion determinant is not real. The total ghost number anomaly for Σ a genus g worldsheet is $w = 2n(1 - g)$. This means that in order to have a nonvanishing correlation function, the total ghost number of the observables should be equal to this number.

With the assumption for the bundle on X that semi-stability implies stability, we find that there are no zero-modes on X for the adjoint scalars σ , η_{\pm} and $\bar{\eta}$. By going to the infrared theory, we may therefore disregard these fields completely in this case. We then only remain with the one-forms on X . In the more general case when there are strictly semi-stable bundles, the situation becomes much harder to analyze. We will not deal with this situation in this paper.

Some Observables

Now we consider the observables of the B -model. We will only be concerned with situations where semi-stability implies stability, that is there are no strictly semi-stable bundles. We also restrict to the case of genus zero. As remarked above, we can therefore disregard all the scalars of the theory, while also the worldsheet dependence is trivial. Therefore, we will look only at the sector that is left, and replace the worldsheet by a point.

The BRST transformation laws together with the Bianchi identity $d_A F = 0$ imply the following generalized Bianchi identity

$$(\bar{s} + \partial_A + \bar{\partial}_A) (i\bar{\psi}^{1,0} + F^{2,0} + F^{1,1} + F^{0,2}) = 0. \quad (4.27)$$

We remark that the above relation is part of the generalized Bianchi identity (4.6) of the A -model. Adopting the same procedure as for the A -model we have the following partial descent equations

$$\bar{s}\mathcal{V}_{0,q}^{r,s} + \bar{\partial}\mathcal{V}_{0,q+1}^{r,s-1} + \partial\mathcal{V}_{0,q+1}^{r-1,s} = 0. \quad (4.28)$$

Thus we can construct the following observables

$$V_{0,q} = \int_X \alpha^{3-r,3-s} \wedge \mathcal{V}_{0,q}^{r,s}, \quad (4.29)$$

satisfying $\bar{s}\mathcal{V}_{0,q} = 0$. Then $V_{0,q} \in H_{\bar{s}}^q(\mathcal{A}, \wedge^0 T^{1,0}\mathcal{A}) \equiv H_{\bar{s}}^{0,q}(\mathcal{A})$. Note that the \bar{s} -cohomology is the Dolbeault cohomology on \mathcal{A} . An interesting observable is

$$V_{0,3} = \frac{i}{48\pi^3} \int_X \omega^{0,3} \wedge \text{Tr}(\bar{\psi}^{1,0} \wedge \bar{\psi}^{1,0} \wedge \bar{\psi}^{1,0}). \quad (4.30)$$

It expresses the anti-holomorphic 3-form on the moduli space. Another interesting observable is

$$V_{0,1} = \frac{i}{8\pi^2} \int_X \alpha^{1,2} \wedge \text{Tr}(\bar{\psi}^{1,0} \wedge F^{1,1}). \quad (4.31)$$

The $(1,2)$ -form $\alpha^{1,2}$ parametrizes a deformation of the complex structure on the Calabi-Yau. It was proposed to define some special coordinates in the generalized mirror symmetry conjecture of [26] (or rather its complex conjugate).

To have a well-defined B -model we need to find observables corresponding to elements $V_{-p,q}$ of the Dolbeault cohomology $H_{\bar{s}}^q(\mathcal{A}, \wedge^p T^{1,0}\mathcal{A})$ with $p \neq 0$. The natural field to use to construct observables having $p \neq 0$ is $\chi^{2,0}$. However the transformation law $\bar{s}\chi^{0,2} = F^{0,2}$ in (4.20) implies that there are no such observables. We do however have $\bar{s}\chi^{0,2} = 0$ at the fixed point locus to which the path integral is localized. For example the candidate for the marginal operator $V_{-1,1}$ generating the complex structure deformation of the moduli space \mathcal{M}_{EM} is

$$V_{-1,1} = \frac{1}{8\pi^2} \int_X \alpha^{2,1} \wedge \text{Tr}(\bar{\psi}^{1,0} \wedge \chi^{0,2}), \quad (4.32)$$

where $\alpha^{2,1} \in H^{2,1}(X)$. We then have

$$\bar{s}V_{-1,1} = -\frac{1}{8\pi^2} \int_X \alpha^{2,1} \wedge \text{Tr}(\bar{\psi}^{1,0} \wedge F^{0,2}). \quad (4.33)$$

Thus $V_{-1,1}$ certainly reduces to an element of $H_{\bar{s}}^1(\mathcal{M}_{EH}, T^{1,0}\mathcal{M}_{EH})$. Following [13] one may try to add $V_{-1,1}$ to the action $S(e^2)$ and modify the \bar{s} transformation law in a suitable way, such that the total action deformed by this observable is invariant under \bar{s} . The problem with this approach however is that the condition $F^{0,2} = 0$ is not the equation of motion of any field. The resolution of this will involve a deformation to holomorphic Chern-Simons theory.

Mapping to Open String Field Theory of the B-Model on X

Our starting point is the observation that $\mathcal{W}(A^{0,1})$ is invariant under the BRST transformation of our B -model, since $\bar{s}A^{0,1} = 0$ and \mathcal{W} depends only on $A^{0,1}$. Thus we can regard \mathcal{W} as an "observable" in our B -model and consider the following generalized action functional¹²

$$S'(e^2) = -\frac{ik}{8\pi^2} \int_X \text{Tr} \omega^{3,0} \wedge \left(A^{0,1} \wedge \bar{\partial} A^{0,1} + \frac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right) + \frac{1}{e^2} \bar{s}V + \frac{1}{e^2} W. \quad (4.34)$$

Now the condition $F^{0,2} = 0$ may occur by the $A^{0,1}$ equation of motion. Such Chern-Simons like observables were also considered in [29], but in the theory at one dimension higher.

As noted above, we want to make sense out of the action functional deformed by the 'observable' (4.32). Thus we consider the following more general action functional, including both the deformation above and the deformation by (4.32),

$$S''(e^2, t^\alpha) = -\frac{ik}{8\pi^2} \int_X \text{Tr} \omega^{3,0} \wedge \left(A^{0,1} \wedge \bar{\partial} A^{0,1} + \frac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right) - \frac{k}{8\pi^2} t^\alpha \int_X \omega^{3,0} \wedge \mu_\alpha \wedge \text{Tr} (\bar{\psi}^{1,0} \wedge \chi^{0,2}) + \frac{1}{e^2} \bar{s}V + \frac{1}{e^2} W. \quad (4.35)$$

Here the $\mu_\alpha \in H^1(X, T^{1,0}X)$ ($\alpha = 1, \dots, h^{2,1}(X)$) form a basis. Note that the vector index of μ_α should be contracted in the action above. The above action functional is invariant under the following modified transformation laws (compare with (4.20))

$$\begin{aligned} \bar{s}A^{1,0} &= i\bar{\psi}^{1,0}, & \bar{s}\bar{\psi}^{1,0} &= 0, \\ \bar{s}A^{0,1} &= it^\alpha \mu_\alpha \bar{\psi}^{1,0}, & \bar{s}\chi^{0,2} &= F^{0,2}. \end{aligned} \quad (4.36)$$

¹² The holomorphic Chern-Simons form is not invariant under large gauge transformations, but transforms only by integral periods of the integral periods of the 3-form $\omega^{3,0}$ [8]. We will not concern ourselves here with this subtlety.

Note that we still have $\bar{s}^2 = 0$.¹³ We note that the above perturbation is the variation of complex structure on \mathcal{A} induced by the variation of complex structure on the Calabi-Yau X , i.e.,

$$\bar{s} \rightarrow \bar{s} + \hat{\mu}^i \mathbf{s}_i, \quad (4.37)$$

where $\hat{\mu}^i \in \Omega^1(\mathcal{A}, T^{1,0}\mathcal{A})$. This relates very elegantly to the fact that adding $t^\alpha \int_X \omega^{3,0} \mu_\alpha \wedge \text{Tr}(\bar{\psi}^{1,0} \wedge \chi^{2,0}) \in \Omega^1(\mathcal{A}, T^{1,0}\mathcal{A})$ to the action functional $S'(e^2)$ generates a marginal deformation corresponding to the variation of complex structure on \mathcal{A} !

Now we take $e^2 \rightarrow 0$ in $S''(e^2, t^\alpha)$ to see that the path integral (the partition function) is localized to the moduli space of stable bundles. The fermionic zero-modes $(\bar{\psi}^{\bar{i}}, \chi^i)$ of $(\bar{\psi}^{1,0}, \chi^{2,0})$ are identified¹⁴ as $\bar{\psi}^{\bar{i}} \in H^{1,0}(\text{End}(E), \partial_A)$ and $\chi_i \in H^{0,2}(\text{End}(E), \bar{\partial}_A) \simeq H^{0,1}(\text{End}(E), \bar{\partial}_A)$. Consequently the partition function for the action functional $S''(e^2, \tau^\alpha)$ is identified with the generating functional of the original B -model correlation functions of the marginal vertex operators

$$Z'' = \int \mathcal{D}(Bose) \mathcal{D}(Fermi) e^{-S''(e^2, t^\alpha)} = \langle \exp(k t^\alpha V_\alpha^{-1,1}) \rangle_B. \quad (4.38)$$

Now following the standard argument for the B -model [13] we should have

$$Z'' \sim \int_{\mathcal{M}_{EH}} \Omega \wedge \partial_{i_1} \dots \partial_{i_d} \Omega \quad (4.39)$$

where Ω is the holomorphic d -form on the moduli space \mathcal{M}_{EH} .

Finally we regard the action functional $S''(e^2, t^\alpha)$ in (4.35) as a BRST-exact deformation of a theory defined by the following action functional

$$I(t^\alpha) = -\frac{ik}{8\pi^2} \int_X \omega^{3,0} \wedge \text{Tr}\left(A \wedge \bar{\partial}A + \frac{2}{3} A \wedge A \wedge A\right) - \frac{k}{8\pi^2} t^\alpha \int_X \omega^{3,0} \mu_\alpha \wedge \text{Tr}(\bar{\psi}^{1,0} \wedge \chi^{0,2}). \quad (4.40)$$

Here we follow the recipe of [17][44]. Being a BRST exact deformation we expect that the theory is independent of e^2 since we have the same localization. Here we also assume that there are no zero-modes of $(\bar{\eta}_z, \bar{\eta}_{\bar{z}}, \eta_\pm, \rho_z^{0,1}, \rho_{\bar{z}}^{0,1})$.¹⁵ Then we may take an extreme limit

¹³ We obviously have $\bar{s}^2 A^{1,0} = \bar{s}^2 A^{0,1} = 0$, while $\bar{s}^2 \chi^{0,2} = it^\alpha \mu_\alpha \bar{\partial}_A \bar{\psi}^{1,0}$. However the latter is closed on shell, which is good enough. We can also make the algebra being closed off-shell by introducing an auxiliary field $H^{0,2}$, i.e., $\bar{s} \chi^{0,2} = F^{0,2} - H^{0,2}$ and $\bar{s} H^{0,2} = it^\alpha \mu_\alpha \bar{\partial}_A \bar{\psi}^{1,0}$.

¹⁴ This is modulo the gauge symmetry.

¹⁵ In such a situation our B -Model reduces to a cohomological field theory on the Calabi-Yau 3-fold X .

$e^2 \rightarrow \infty$ and simply drop the original $S(e^2)$ from the action $S''(e^2, t^\alpha)$ to arrive at the equivalent action functional $I(t^\alpha)$.

We remark that the fermionic term in $I(t^\alpha)$ is crucial for ensuring the global fermionic symmetry (4.36), relating the holomorphic Cherns-Simons theory with the variation of Hodge structure on the moduli space of stable bundles. The term also ensures a well-defined path integral measure similar to the situation in [17][44]. We view our model as a constructive definition of the path integral of holomorphic Chern-Simons theory.

So we argued that the B -model of our matrix string on a Calabi-Yau X is equivalent to Witten's open string field theory of the B -model [27]. Recently Vafa suggested such an extension of mirror symmetry involving stable bundles on Calabi-Yau [26]. It is based on the new understanding of mirror symmetry as T -duality of T^3 -fibered Calabi-Yau with D-branes [46]. The extended mirror conjecture involves stable bundles on one side and minimal Lagrangian submanifolds on the mirror side. For Calabi-Yau 3-folds Vafa conjectured mirror symmetry between Witten's open string field theories of the A - and B -models [27]. A closely related proposal was suggested by Kontsevich [47] and Tyurin [48]. It is not clear how our approach is related to Vafa's conjecture. We should mention that in fact Vafa gave a formula for the classical value of the holomorphic 3-form on the moduli space of bundles. This holomorphic 3-form basically is the observable (4.30). In our model it would not be very natural to calculate this observable, but rather the (quantum corrected) value of correlation functions involving this observable. This is closer to the integration of (powers of) this 3-form over 3-cycles in the moduli space.

Our B -model, equivalent to the model (4.40), computes the variation of Hodge structures on the moduli space of stable bundles. Our A -model computes the quantum cohomology ring of the moduli space of stable bundles. Following the well-known argument for conjectural mirror symmetry via $\mathcal{N}_{w,s} = (2, 2)$ superconformal theory, realized as a sigma model with the Calabi-Yau as a target space, we may conjecture that there are mirror pairs among our A - and B -models involving mirror Calabi-Yau's as well as mirror stable bundles (allow for torsion-free sheaves) along the lines of the mirror symmetry for higher dimensional Calabi-Yau [49].

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Appendix A. Supersymmetry Transformation Rules

In this appendix we write down the explicit $\mathcal{N}_{ws} = (2, 2)$ transformation rules. They are written in terms of the supersymmetry transformation $\delta = \bar{\epsilon}_- \mathbf{s}_+ + \bar{\epsilon}_+ \mathbf{s}_- + \epsilon_+ \bar{\mathbf{s}}_- + \epsilon_- \bar{\mathbf{s}}_+$.

For the *vector multiplet* $(v_{\pm\pm}, \eta_{\pm}, \bar{\eta}_{\pm}, \sigma, \bar{\sigma}, D)$ the supersymmetry transformations are given by

$$\begin{aligned}
\delta v_{++} &= i\bar{\epsilon}_+ \eta_+ + i\epsilon_+ \bar{\eta}_+, \\
\delta v_{--} &= i\bar{\epsilon}_- \eta_- + i\epsilon_- \bar{\eta}_-, \\
\delta \sigma &= -ig_s \bar{\epsilon}_+ \eta_- - ig_s \epsilon_- \bar{\eta}_+, \\
\delta \bar{\sigma} &= -ig_s \bar{\epsilon}_- \eta_+ - ig_s \epsilon_+ \bar{\eta}_-, \\
\delta \eta_+ &= +i\epsilon_+ D - \frac{1}{2g_s^2} \epsilon_+ [\sigma, \bar{\sigma}] - \frac{1}{2} \epsilon_+ f_{\Sigma} - \frac{1}{g_s} \epsilon_- \nabla_{++} \bar{\sigma}, \\
\delta \bar{\eta}_+ &= -i\bar{\epsilon}_+ D + \frac{1}{2g_s^2} \bar{\epsilon}_+ [\sigma, \bar{\sigma}] - \frac{1}{2} \bar{\epsilon}_+ f_{\Sigma} - \frac{1}{g_s} \bar{\epsilon}_- \nabla_{++} \sigma, \\
\delta \eta_- &= +i\epsilon_- D + \frac{1}{2g_s^2} \epsilon_- [\sigma, \bar{\sigma}] + \frac{1}{2} \epsilon_- f_{\Sigma} - \frac{1}{g_s} \epsilon_+ \nabla_{--} \sigma, \\
\delta \bar{\eta}_- &= -i\bar{\epsilon}_- D - \frac{1}{2g_s^2} \bar{\epsilon}_- [\sigma, \bar{\sigma}] + \frac{1}{2} \bar{\epsilon}_- f_{\Sigma} - \frac{1}{g_s} \bar{\epsilon}_+ \nabla_{--} \bar{\sigma}, \\
\delta D &= +\frac{1}{2} \bar{\epsilon}_- \nabla_{++} \eta_- + \frac{1}{2g_s} \bar{\epsilon}_- [\sigma, \eta_+] + \frac{1}{2} \bar{\epsilon}_+ \nabla_{--} \eta_+ + \frac{1}{2g_s} \bar{\epsilon}_+ [\bar{\sigma}, \eta_-] \\
&\quad - \frac{1}{2} \epsilon_- \nabla_{++} \bar{\eta}_- - \frac{1}{2g_s} \epsilon_- [\bar{\sigma}, \bar{\eta}_+] - \frac{1}{2} \epsilon_+ \nabla_{--} \bar{\eta}_+ - \frac{1}{2g_s} \epsilon_+ [\sigma, \bar{\eta}_-].
\end{aligned} \tag{A.1}$$

We have defined the covariant derivatives on the worldsheet as $\nabla_{\pm\pm} = \partial_{\pm\pm} + v_{\pm\pm}$, and its curvature is $f_{\Sigma} = [\nabla_{++}, \nabla_{--}]$.

The supersymmetry transformation rules for the adjoint *chiral multiplet* (A, ψ_{\pm}, H) are given by

$$\begin{aligned}
\delta A &= i\bar{\epsilon}_+ \psi_- + i\bar{\epsilon}_- \psi_+, \\
\delta \psi_+ &= +\bar{\epsilon}_+ H + \epsilon_- \nabla_{++} A + g_s^{-1} \epsilon_+ [\sigma, A], \\
\delta \psi_- &= -\bar{\epsilon}_- H + \epsilon_+ \nabla_{--} A + g_s^{-1} \epsilon_- [\bar{\sigma}, A], \\
\delta H &= -i\epsilon_- \nabla_{++} \psi_- - i\epsilon_- [\eta_+, A] + \frac{i}{g_s} \epsilon_- [\bar{\sigma}, \psi_+] \\
&\quad + i\epsilon_+ \nabla_{--} \psi_+ + i\epsilon_+ [\eta_-, A] - \frac{i}{g_s} \epsilon_+ [\sigma, \psi_-].
\end{aligned} \tag{A.2}$$

The transformation rules for the adjoint *anti-chiral multiplet* $(\bar{A}, \bar{\psi}_\pm, \bar{H})$ are given by

$$\begin{aligned}
\delta\bar{A} &= i\epsilon_+\bar{\psi}_- + i\epsilon_-\bar{\psi}_+, \\
\delta\bar{\psi}_+ &= +\epsilon_+\bar{H} + \bar{\epsilon}_-\nabla_{++}\bar{A} + g_s^{-1}\bar{\epsilon}_+[\bar{\sigma}, \bar{A}], \\
\delta\bar{\psi}_- &= -\epsilon_-\bar{H} + \bar{\epsilon}_+\nabla_{--}\bar{A} + g_s^{-1}\bar{\epsilon}_-[\sigma, \bar{A}], \\
\delta\bar{H} &= -i\bar{\epsilon}_-\nabla_{++}\bar{\psi}_- - i\bar{\epsilon}_-[\bar{\eta}_+, \bar{A}] + \frac{i}{g_s}\bar{\epsilon}_-[\sigma, \bar{\psi}_+] \\
&\quad + i\bar{\epsilon}_+\nabla_{--}\bar{\psi}_+ + i\bar{\epsilon}_+[\bar{\eta}_-, \bar{A}] - \frac{i}{g_s}\bar{\epsilon}_+[\bar{\sigma}, \bar{\psi}_-].
\end{aligned} \tag{A.3}$$

Appendix B. Action for the B -Model

In this appendix we give the explicit form of the action for the B model in terms of the action fermion V and the BRST invariant term W , appearing in (4.26).

The action fermion is given by

$$\begin{aligned}
V &= \int_X \text{Tr} \left(-\chi^{0,2} \wedge *F^{2,0} + \frac{i}{2}\sigma_z\partial_A\rho_{\bar{z}}^{0,1} + \frac{i}{2}\sigma_{\bar{z}}\partial_A\rho_z^{0,1} - \frac{i}{2}(\eta_+ + \eta_-)F^{1,1} \right. \\
&\quad - \frac{i}{2}[\nabla_z, \partial_A] \wedge \rho_{\bar{z}}^{0,1} - \frac{i}{2}[\nabla_{\bar{z}}, \partial_A] \wedge \rho_z^{0,1} \\
&\quad + \frac{1}{2} \left(\Lambda F - \zeta I + \frac{1}{2}[\sigma_z, \sigma_{\bar{z}}] + \frac{1}{2}f_{z\bar{z}} + \nabla_{\bar{z}}\sigma_z \right) \eta_+ \\
&\quad \left. + \frac{1}{2} \left(\Lambda F - \zeta I - \frac{1}{2}[\sigma_z, \sigma_{\bar{z}}] - \frac{1}{2}f_{z\bar{z}} + \nabla_z\sigma_{\bar{z}} \right) \eta_- \right).
\end{aligned} \tag{B.1}$$

The remaining terms in the action are given by

$$W = \int_X \left(F^{0,2} \wedge *F^{0,2} - \bar{\partial}_A\psi_+^{0,1} \wedge \psi_-^{0,1} \wedge \omega^{3,0} \right). \tag{B.2}$$

Note that it is invariant under the BRST symmetry \bar{s} of the B -model.

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