Random 3-D Young Diagrams and Representation Theory

Random 3-D Young-Diagrammen en Representatietheorie
(met een samenvatting in het Nederlands)

Proefschrift

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Chapter 1

Introduction and summary

1.1 Preface

The topic of the thesis is related to (mathematical) statistical mechanics and probability theory from one side, and to the representation theory of "big" groups on the other side. A typical example of a "big" group is the union of unitary groups $U(N)$ naturally embedded one into another; it is called the infinite–dimensional unitary group $U(\infty)$.

In statistical mechanics we deal with models of random discrete surfaces in $\mathbb{R}^3$ or, equivalently, random 3-D Young diagrams. The connection between random Young diagrams and representation theory has been known since 70s: in 1977 Logan, Shepp [56] and, independently, Vershik and Kerov [90] proved the law of large numbers for (2d) Young diagrams distributed according to the Plancherel measure, the latter being the object of representation-theoretic origin; from another direction, in papers [91],[92] Vershik and Kerov suggested an approach to representation theory of the infinite symmetric group $S(\infty)$ and infinite–dimensional unitary group $U(\infty)$ which reduces the classification of characters (i.e. positive definite central normalized continuous functions) of the group to the study of certain Markov chains on the set of Young diagrams.

The thesis starts from the study of the objects of purely combinatorial origin: we investigate various measures on 3-D Young diagrams confined to $a \times b \times c$ box and the limit behavior of these measures as $a, b, c \to \infty$. The most known example of a such measure, uniform distribution, has been studied by various authors for more than 10 years. In 1998 Cohn, Larsen and Propp [18] proved the law of large number for uniformly distributed 3-D Young diagrams in $a \times b \times c$ box, in other words, they showed that if $a, b, c \to \infty$ in such a way that their ratios remain unchanged, then properly rescaled stepped stepped surface (the boundary of a 3-D Young diagram) tends to a deterministic limit shape. Later Cohn, Kenyon and Propp [17] (see also paper by Destainville [19]) generalized these results to stepped surfaces with more general boundary conditions or, equivalently, to 3-D Young diagrams confined to more complicated volumes than

just a box. Kenyon, Okounkov and Scheffield [45], [46] gave a description of the arising limit shapes and found intriguing relations between the limit shapes and certain algebraic curves.

From other side, there are papers [89], [16], [68] which deal with free (i.e. not confined to any box) 3-D Young diagrams. Of course there is no uniform probability measure on the set of *all* 3-D Young diagram. Yet, one can replace the uniform measure with another natural probability distribution which assigns the weight q^{vol} to a Young diagram of volume vol.

In the first part of the thesis we study a more general 2–parametric family of measures deforming the uniform probability measure on 3-D Young diagrams in a box. A distribution of this family can be viewed as an interpolation between the measures q^{vol} and q^{-vol}. We prove a variety of the results about these measures: we construct a family of Markov chains which agree with the measures and, in particular, lead to an efficient random sampling algorithm; we find the corresponding generalization of the well-known MacMahon product formula for the number of 3-D Young diagrams in a given box; we compute the correlation functions which describe the local geometry of random 3-D Young diagrams and study their asymptotics as $a, b, c \to \infty$ and $q \to 1$.

The second part of the thesis deals with combinatorial structures related to the group $U(\infty)$. Vershik and Kerov [92] noted that characters of $U(\infty)$ are in bijection with probability measures on paths in the so-called Gelfand–Tsetlin graph possessing a certain Gibbs property. In this way the problem of the classification of the characters can be restated as the classification of Gibbs measures on paths. The former problem has a long history (see [93],[15],[92],[65]) and turns out to be related, in particular, to various classes of representations of the group $U(\infty)$ and to the study of totally positive Toeplitz matrices classified in [23].

Through combinatorial bijections paths in Gelfand–Tsetlin graph can be viewed as stepped surfaces or 3-D Young diagrams (this time, of the infinite volume). Then the Gibbs property turns into the uniformity of certain conditional measures on 3-D Young diagrams. In the thesis we study a 1-parametric deformation of this Gibbs property. Our deformation is closely related to the measure q^{vol} on 3-D Young diagrams and is, in some way, inspired by the study of that measure. Although the deformation is purely combinatorial, there are indications that these new q–Gibbs measures are still related to representation theory, but with unitary groups replaced by their q–deformations — quantum groups. We prove a classification theorem for the q–Gibbs measures and discuss their relations with certain classes of matrices with non-negative minors and with limits of symmetric polynomials as the number of variable tens to infinite.

Finally, in the last part of the thesis we study a problem of purely representation–theoretic origin. We deal with a two–parametric family of representations of $U(\infty)$ (to be more precise, we study a family of spherical representations of the Gelfand pair $(U(\infty)\times U(\infty), diag(U(\infty)))$) introduced by Olshanski [71] as a substitute of the non-existent regular representation of $U(\infty)$. We seek for an answer to the question whether these representations are disjoint (i.e. contain no isomorphic subrepresentations). In order to prove the disjointness

we reduce this problem to the analysis of certain measures on Gelfand–Tsetlin graph or, in other words, again to the study of random Young diagrams.

The thesis consists of the introductory chapter where we state and provide all necessary details on each of the problems solved in the paper and 3 chapters with main results. In principle, each of these chapters is a self–contained article and can be read separately.

Chapter 2 is based on the joint paper with A.Borodin and E.Rains published in Selecta Mathematica [6]. Chapter 3 is based on the paper [37]. Chapter 4 is based on the paper [34] published in Functional Analysis and its applications.

1.2 Informal summary

In this section we give a short informal summary of the results proved in the thesis. A more detailed and more exact exposition for each chapter is given in the next three sections which are also duplicated, for the convenience of the reader, at the beginnings of the corresponding chapters.

The thesis deals with probability and representation-theoretic problems which are related in various ways to the study of random *lozenge tilings* of planar domains.

A *lozenge* is a rhombus obtained by gluing two of the neighboring elementary triangles on the regular triangular lattice together. There are exactly 3 types of lozenges and they are shown in the left part of Figure 1.1.

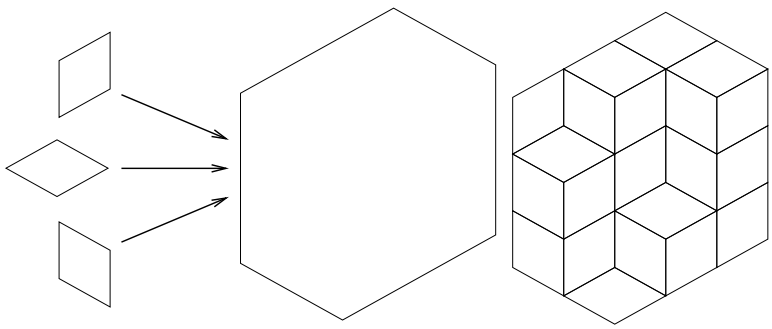


Figure 1.1: 3 types of lozenges, equiangular hexagon and a tiling.

Perhaps, the simplest tilable domain is an equiangular hexagon with integral side lengths a, b, c, a, b, c. An example of such hexagon and one of its tilings are shown in Figure 1.1.

Tilings are in bijection with various combinatorial objects. These bijections are best described pictorially. We show in Figure 1.2 a *plane partition* and a 3-D shape corresponding to the tiling of Figure 1.1. Recall that a *plane partition* is an $a \times b$ table filled with nonnegative integers not greater than c in such a way that the numbers are non-increasing along the rows and columns of the table.

A graph of the function encoded by a plane partition is a *stepped surface* in $\mathbb{Z}^3$. This stepped surface is a boundary of a certain 3-D body — 3-D Young diagram. Another equivalent way to construct this 3-D Young diagram is the following: place plane partition on the (x_1, x_2) coordinate plane in $\mathbb{Z}^3$ and then for each cell of the plane partition put k unit cubes on top of this cell, where k is the number written in the cell. Orthogonal projection of the stepped surface on the plane perpendicular to $(1, 1, 1)$ vector is precisely a lozenge tiling of a, b, c, a, b, c hexagon.

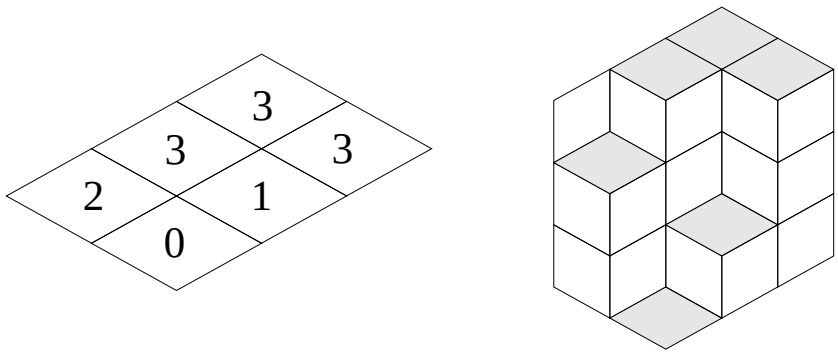


Figure 1.2: Plane partition and 3-D Young diagram/ stepped surface corresponding to the tiling of Figure 1.1

Similar bijections exist not only for a hexagon, but for arbitrary tilable planar domains. Figure 1.3 shows (left panel) a part of the tiling of the halfplane and corresponding 3-D shape/ stepped surface which is a graph of function shown at the right panel.

The first part of the thesis (Chapter 2) is devoted to the study of various probability measures on lozenge tilings of a, b, c, a, b, c hexagon. Our measures include, in particular: uniform measure, measure assigning the weight q^{vol} to the 3-D Young diagram with volume (i.e. sum of the numbers in the corresponding plane partition) vol, measures with weights interpolating between q^{vol} and q^{-vol}. Our aim is twofold. First, we introduce certain Markov chains on tilings, which, in particular, provide an efficient random sampling algorithm for tilings distributed according to our measures. Second, we are interested in the asymptotic behavior of random 3-D Young diagrams (random tilings) as proportions of the hexagon a, b, c tend to infinity. Nontriviality of this behavior can be understood by examining Figure 1.4, where a tiling of $70 \times 90 \times 70$ hexagon sampled from the measure with weights q^{vol} with $q = 0.96$ is shown. In particular, one can see formation of the so-called "frozen regions" which are the parts of the hexagon where the tiling is degenerate, i.e. it consists of only one type of lozenges.

Our main tool for studying random tilings is through their *correlation functions*. In order to define the notion of correlation functions let us fix one type

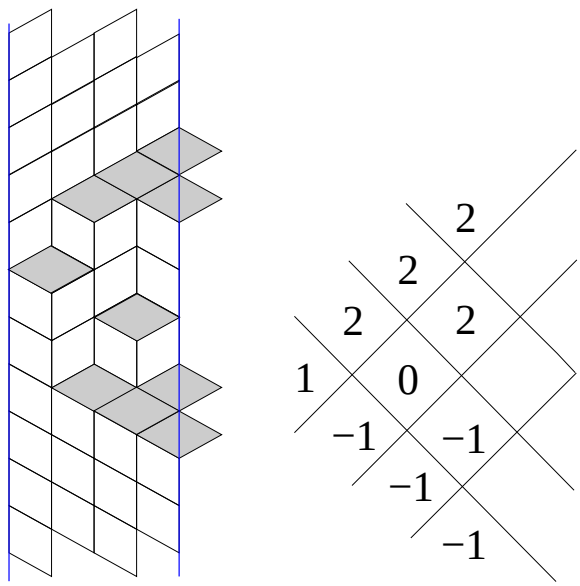


Figure 1.3: Part of the tiling of the halfplane/3-D shape/stepped surface.

of lozenges, say, "horizontal" ones which are grey in Figures 1.2, 1.3 and 1.4. The nth correlation function $\rho_n(Q_1, \ldots, Q_n)$ is a function of n distinct points on 2-dimensional lattice giving the probability that there are horizontal lozenges in the tiling in positions $Q_1, \ldots, Q_n$. A remarkable feature of the probability measures on tilings under consideration is that their correlation functions can be expressed through minors of a certain matrix K not depending on n. We call K the *correlation kernel*.

Investigating the limits of K we prove a theorem about asymptotical local behavior of the random tiling near a fixed point of the hexagon as $a, b, c \to \infty$ and $q \to 1$ (see the next section for the exact statement), in particular, we find frozen regions and give an explicit formula for their boundary.

Passing to the limit in the correlation kernel is complicated by the fact that there are no simple explicit formulas for K. We express K through the sums of certain *hypergeometric discrete orthogonal polynomials*. Uniform measure on tilings leads to *Hahn polynomials* and the most complicated measure we analyze is related to *q–Racah orthogonal polynomials*. We study the asymptotic of orthogonal polynomials by methods of spectral theory, i.e. analyzing certain difference operators which have these polynomials as eigenfunctions.

The second part of the thesis (Chapters 3 and 4) deals with tilings of the halfplane. We are interested in these tilings because of their relation to the representation theory of unitary groups.

A *Gelfand-Tsetlin pattern* of size N is a triangular array of integers $\{\lambda_i^j\}$, $j = 1, 2, \ldots, N$, $i = 1, 2, \ldots, j$ such that $\lambda_i^j \geq \lambda_i^{j+1} \geq \lambda_{i+1}^j$ for all meaningfull val-

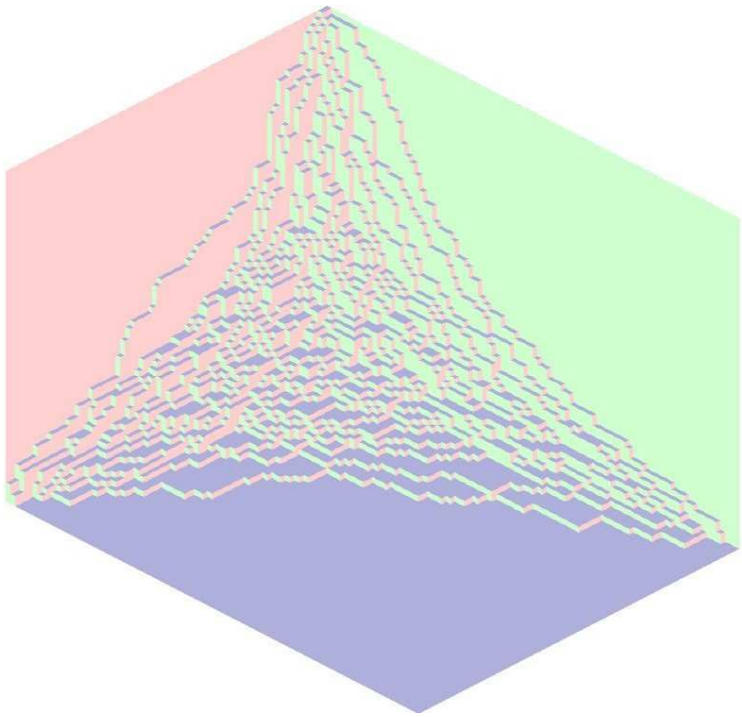


Figure 1.4: Sample from the measure q^{vol} with $q = 0.96$ on tilings of $70 \times 90 \times 70$ hexagon. Three types of lozenges are painted in 3 different colors.

ues of parameters. Similarly for $N = +\infty$ we get an *infinite Gelafand-Tsetlin pattern*.

Observe that at the right panel of Figure 1.3 a Gelfand-Tsetlin pattern of size 4 is shown. Indeed, here

$$\lambda_1^1 = 1,$$
$$\lambda_1^2 = 2, \quad \lambda_2^2 = -1,$$
$$\lambda_1^3 = 2, \quad \lambda_2^3 = 0, \quad \lambda_3^3 = -1,$$
$$\lambda_1^4 = 2, \quad \lambda_2^4 = 2, \quad \lambda_3^4 = -1, \quad \lambda_4^4 = -1.$$

Generalizing this observation we get a correspondence between infinite Gelfand-Tsetlin patterns and lozenge tilings of the halfplane. Also Gelfand-Tsetlin patterns can be interpreted as paths in a certain graded graph, and we intensively use this point of view in Chapters 3 and 4.

Historically, Gelfand-Tsetlin patterns first appeared in the representation theory. Consider N-tuple $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ of integers (such N-tuples are usually called signatures of size N) and let I_μ denote the set of all Gelfand-Tsetlin patterns $\{\lambda_i^j\}$ of size N such that $\lambda_i^N = \mu_i$ for $i = 1, \ldots, N$. Signatures

of size N parameterize *irreducible representations* of N–dimensional unitary group $U(N)$, and elements of I_μ parameterize the *Gelfand–Tsetlin* basis of the irreducible representation labeled by the signature μ.

Relationship between infinite Gelfand–Tsetlin patterns and representation theory is a little bit more complicated and we need some definitions to explain it.

We say that a probability measure P on infinite Gelfand–Tsetlin patterns $\{\lambda_i^j\}$ satisfies *the Gibbs property* if for every N and every signature $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ of size N the conditional distribution of $\{\lambda_i^j\}$, $j = 1, \ldots, N-1$ given $\lambda_1^N = \mu_1, \ldots, \lambda_N^N = \mu_N$ is uniform. In terms of lozenge tilings of the halfplane the Gibbs property can be restated as follows: given positions of lozenges on any vertical line, conditional distribution of tiling to the left of this line is uniform.

Consider the chain of the compact classical groups $U(N)$, $N = 1, 2, \ldots$, which are embedded one into another in a natural way. Let *infinite dimensional unitary group* $U(\infty)$ be their union.

$$U(\infty) = \bigcup_{N=1}^{\infty} U(N).$$

A *character* χ of a group is a positive-definite central (i.e. constant on conjugacy classes) continuous functions on the group normalized by the condition $\chi(e) = 1$, where e denotes the unit element of the group. The study of characters of the infinite-dimensional unitary group was initiated by Voiculescu in 1976 in the connection with *finite factor representations* of $U(\infty)$. Let us describe a remarkable relation between characters and tilings which was found later.

Let χ be a character of $U(\infty)$ and let χ^N denote its restriction to the subgroup $U(N) \subset U(\infty)$. Clearly χ^N is a character of $U(N)$. It is well known that the set of characters of a compact group is a simplex with extreme points being normalized conventional characters (i.e. matrix traces) of irreducible representations of the group. Therefore,

$$\chi^N(\cdot) = \sum_\lambda c_\lambda^N \frac{\chi^{\lambda(\cdot)}}{\chi^\lambda(e)},$$

where the sum is over all signatures of size N which are labels of irreducible representations of $U(N)$, coefficients c_λ^N are non-negative and sum up to 1, and χ^λ stays for the matrix trace in the irreducible representation of $U(N)$ indexed by λ. Note that the array of coefficients c_N^λ uniquely define χ.

From the other side, if P is a Gibbs probability measure of infinite Gelfand-Tsetlin patterns $\{\lambda_i^j\}$, then let P^N denote its projection on the coordinates $\{\lambda_i^N\}_{i=1,\ldots,N}$. We claim that the correspondence

$$c_\lambda^N \leftrightarrow P^N(\lambda)$$

yields an affine isomorphism between the set of characters of $U(\infty)$ and the set of Gibbs central probability measures on infinite Gelfand–Tsetlin patterns (equivalently, on lozenge tilings of the halfplane).

Chapter 4 uses the above correspondence to obtain certain representation-theoretic results through the study of probability measures on tilings. We deal with a family $T_{z,w}$ of representations related to the infinite-dimensional unitary group. More formally, these are $U(\infty) \times U(\infty)$ unitary representations possessing a distinguished vector invariant under the action of $U(\infty)$ diagonally embedded into $U(\infty) \times U(\infty)$. The representations $T_{z,w}$ were introduced by G. Olshanski as a natural substitute for the non-existent (bi-)regular representation of the group $U(\infty)$. The spaces of these representations are inductive limits of the spaces $L_2(U(N))$, but the embeddings depend on z and w. We study the question whether these representations are really different. The answer is affirmative and we prove that for general z, w the representations $T_{z,w}$ are pairwise disjoint, i.e. they contain no isomorphic subrepresentations.

Each representation $T_{z,w}$ corresponds to a certain character $\chi_{z,w}$ of $U(\infty)$ and, thus, there is a Gibbs probability measure $P_{z,w}$ on infinite Gelfand–Tsetlin patterns assigned to $T_{z,w}$. As it turns out, the disjointness of the representations $T_{z,w}$ is equivalent to the disjointness of the measures $P_{z,w}$ (i.e. disjointness of their supports). This is the latter disjointness that we prove in Chapter 4.

The aim of Chapter 3 is to introduce and study a q-deformation of the notion of a character of $U(\infty)$. We do this by deforming the notion of a Gibbs probability measure on infinite Gelfand–Tsetlin patterns. Our deformation is, in some sense, inspired by the study of probability measure on tilings in Chapter 2.

We say that a probability measure P on infinite Gelfand–Tsetlin patterns $\{\lambda_i^j\}$ satisfies *the q-Gibbs property* if for every N and every signature $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ of size N the conditional distribution of $\{\lambda_i^j\}$, $j = 1, \ldots, N-1$ given $\lambda_1^N = \mu_1, \ldots, \lambda_N^N = \mu_N$ is has weights $q^{|\lambda^1|+\cdots+|\lambda^{N-1}|}$, where $|\lambda^j| = \lambda_1^j + \cdots + \lambda_j^j$. In terms of lozenge tilings of the halfplane the q-Gibbs property can be restated as follows: given positions of lozenges on any vertical line, conditional distribution of tiling to the left of this line has weights q^{vol}, where vol is the volume enclosed under the stepped surface corresponding to the tiling.

We aim to describe all q-Gibbs probability measures on infinite Gelfand–Tsetlin patterns. We prove that these measures form a simplex and show that all extreme points of this simplex are parameterized by monotonous sequences of integers $\nu_1 \leq \nu_2 \leq \nu_3 \leq \ldots$. We also find a geometrical meaning of the parameters ν_j in terms of the probabilistic properties of the corresponding measures on infinite Gelfand–Tsetlin patterns.

As we explained above, the $q = 1$ case, i.e. the description of all Gibbs probability measures, is equivalent to the description of all characters of the infinite-dimensional unitary group. Conjecturally, q–Gibbs probability measures are similarly related to q-deformations of unitary groups, i.e. to the quantum groups.

Since the papers by Voiculescu it has been known that extreme points of the set of characters of $U(\infty)$ are in bijection with totally positive (i.e. with non-negative minors) infinite Toeplitz matrices. In Chapter 3 we extend this correspondence to the q–level: we introduce a notion of a q–Toeplitz matrix

and show that such matrices with certain non-negative minors are related to the extreme points of the set of q–Gibbs probability measures.

The main tools of Chapter 3 are various symmetric polynomials and relations between them. We intensively use a certain basis in the space of symmetric polynomials that we call the basis of *q–interpolation Schur polynomials*. These polynomials are particular cases of both factorial Schur polynomials studied by I. Macdonald and interpolation Macdonald polynomials studied, in particular, by F. Knop, S. Sahi and A. Okounkov. The binomial formula for Macdonald polynomials proved by A. Okounkov is a very important ingredient of our proofs.

1.3 Extended summary of Chapter 2

1.3.1 Preface

The uniform distribution on boxed plane partitions (equivalently, lozenge tilings of a hexagon) is one of the most studied models of random surfaces. There are four principal types of results regarding this model that have been proved.

(1) Law of large numbers: Under the global scaling (the bounding box/hexagon is fixed and the mesh is going to zero), the measure concentrates on surfaces that are close to a certain deterministic *limit shape*. The limit shape can be obtained as the unique solution to a suitable variational problem. The solution is encoded by a second degree polynomial in two variables, see [17], [18], [19], [20], [45].

(2) Locally near any point of the limit shape, the measure on tilings converges to a (uniquely defined, see [81]) translation-invariant ergodic Gibbs measure on lozenge tiling of the plane of a given slope, and the slope coincides with the slope of the tangent plane to the limit shape at the chosen point, see [35] and also [43], [44], [45], [46].

(3) The correlation kernel of the random point process of lozenges of one of the types is explicitly expressed in terms of classical Hahn orthogonal polynomials, see [39], [40], [41], [35].

(4) A few algorithms, both asymptotic and exact, have been proposed to generate the random tilings in question, see [5], [53], [75], [76], [96], [97].

These are complemented by the classical MacMahon product formula for the total number of plane partitions in a given box, see, e.g., Section 7.21 in [84].

In Chapter 2 we study measures on boxed plane partitions that generalize the uniform distribution. The weight of a tiling is defined as the product of certain simple factors over all lozenges of a fixed type, see Section 1.3.2 for definitions. One special case is the weight q^{volume}, where *volume* is the volume of the corresponding plane partition, and q is an arbitrary positive number.

In the most general case we consider, the weight of a lozenge is *elliptic*. Our initial motivation came from the fact that these weights lead to a nice generalization of the MacMahon formula mentioned above, see Theorem 2.10.5 in the Appendix of Chapter 2.

For the asymptotic analysis, we look at the top degeneration of the elliptic weight that is related to orthogonal polynomials.

Our asymptotic results amount to proving analogs of (2), (3), and (4) above. As for (1), we derive the corresponding variational problem (which differs from the one for the uniform case by the presence of an external potential), and show that the hypothetical limit shape (obtained from (2)) solves the corresponding Euler-Lagrange equation. However, we do not prove the concentration phenomenon rigorously.

One interesting feature of the limit shapes that arise is that the curve that bounds the frozen regions may have one or two nodes in vertices of the hexagon, see Section 2.9 for illustrations.

In terms of orthogonal polynomials, our models go all the way up to the top of the q-Askey scheme — the most general models we analyze asymptotically are related to the q-Racah classical orthogonal polynomials. We also show that the elliptic weights lead to the biorthogonal functions constructed in [83]. We hope to return to the asymptotic analysis of this case in a later publication.

Our proof of (2) follows the same steps as the argument for the uniform case in [35]. It is based on the general method of computing limits of correlation kernels suggested in [12] and [69]. The crucial property we need is the second order difference equation satisfied by the q-Racah orthogonal polynomials.

Our perfect sampling algorithm is obtained from a more general construction of relatively simple Markov chains that change the size of the box (one side increases by 1 and another side decreases by 1), and that map the measures from the class we consider to similar ones. The construction follows the approach of [4]; the key facts that make that approach possible reduce to certain recurrence relations for the q-Racah polynomials.

A computer simulation of the above-mentioned Markov chains can be found at [36].

1.3.2 Combinatorial interpretations

For any integers $a, b, c \geq 1$ consider a hexagon with sides a, b, c, a, b, c drawn on the regular triangular lattice. Denote by $\Omega_{a\times b\times c}$ the set of all tilings of this hexagon by rhombi obtained by gluing two of the neighboring elementary triangles together (such rhombi are called *lozenges*). An element of $\Omega_{3\times 3\times 3}$ is shown in Figure 1.5.

Lozenge tilings of a hexagon can be identified with 3-D Young diagrams (equivalently, boxed plane partitions) or with stepped surfaces. The bijection is best described pictorially. We show a 3-D shape corresponding to a tiling in Figure 1.5.

It is convenient for us to slightly modify both hexagon and lozenges by means of a simple affine transform of the plane.

We thus obtain a tiling of a slightly different hexagon.

In what follows we use different parameters instead of a, b, c. Set $N = a$, $T = b + c$, $S = c$. We will also denote the set $\Omega_{a\times b\times c}$ by $\Omega(N, T, S)$.

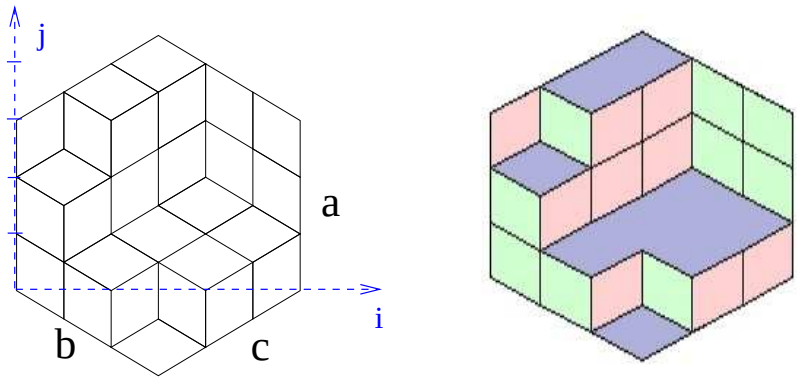


Figure 1.5: Tiling of a $3 \times 3 \times 3$ hexagon.

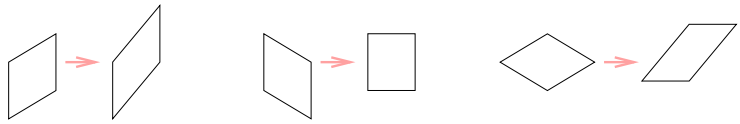


Figure 1.6: Affine modification of lozenges.

Each tiling corresponds to a family of nonintersecting paths as shown in Figure 1.7.

Consider a section of our family of paths by a vertical line $t = t_0$. Clearly, this gives an N-tuple of points in $\mathbb{Z}$. Thus, our tiling can be viewed as an N-point configuration varying in time $t = 0, 1, \ldots, T$. Note that when $t = 0$ the configuration consists of points $\{0, 1, \ldots, N-1\}$, while for $t = T$ the configuration consists of points $\{S, \ldots, S+N-1\}$.

1.3.3 Probability models

Let us introduce the probability measures on $\Omega(N, T, S)$ that are studied in this paper. For any $\mathcal{T} \in \Omega(N, T, S)$, we define its weight $w(\mathcal{T})$ and consider the probability distribution on $\Omega(N, T, S)$ given by the formula

$$\mathrm{Prob}\{\mathcal{T}\} = \frac{w(\mathcal{T})}{\sum\limits_{\mathcal{T}' \in \Omega(N,T,S)} w(\mathcal{T}')}$$

The weights we consider are such that the probability of a tiling is proportional to the product of certain weights corresponding to the horizontal lozenges in it, i.e.,

$$w(\mathcal{T}) = \prod_{\diamondsuit \in \mathcal{T}} w(\diamondsuit)$$

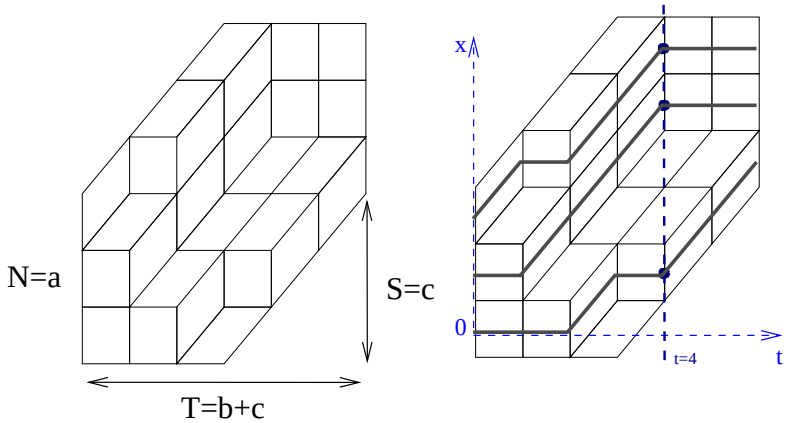


Figure 1.7: Modified tiling of a $3 \times 3 \times 3$ hexagon and the corresponding family of non-intersecting paths.

Note that the number of horizontal lozenges in a tiling of a given hexagon is fixed (i.e., it does not depend on the tiling). Hence, multiplying $w(\diamond)$ by a nonzero constant does not change the probability distribution.

In the most general case considered in Section 2.10, $w(\diamond)$ is an elliptic weight given by

$$w(\diamond) = \frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)}, \tag{1.1}$$

where the coordinates of the topmost point of $\diamond$ are (i, j) (the i and j axes are pictured in Figure 1.5), u_1, u_2, p, q are (generally speaking, complex) parameters,

$$\theta_p(x) = \prod_{i=0}^{\infty}(1 - p^i x)(1 - p^{i+1}/x)$$

and $\theta_p(a, b, c \dots) = \theta_p(a)\theta_p(b)\theta_p(c)\dots$

Mostly we will not be concerned with this most general case, nor with the most general trigonometric case obtained by taking $p \to 0$ (note $\theta_0(x) = 1 - x$), as in these cases the kernels involve biorthogonal functions. The most general orthogonal polynomial case is the limit

$$p \to 0, \quad u_1 = O(\sqrt{p}), \quad u_2 = O(\sqrt{p}), \quad u_1u_2 = p\kappa^2 q^{-S}, \tag{1.2}$$

in which case the weight function is

$$w(\diamond) = \kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}}; \tag{1.3}$$

this is also the most general case in which the weight is independent of i.

Clearly, the factor $q^{-(S+1)/2}$ can be removed if we replace κ by $\kappa' = \kappa \cdot q^{-(S+1)/2}$. However, this choice is more convenient for our further considerations.

We need to make sure that the probabilities of tilings are positive. This leads to certain restrictions on the parameters. There are three possible cases:

(i). *imaginary q-Racah case:* q is a positive real number, κ is an arbitrary pure imaginary complex number;

(ii). *real q-Racah case:* q is a positive real number, κ is a real number with additional restrictions depending on the size of a hexagon; κ cannot lie inside the interval $[q^{-N+1/2}, q^{(T-1)/2}]$ or $[q^{(T-1)/2}, q^{-N+1/2}]$, depending on whether $q > 1$ or $q < 1$;

(iii). *trigonometric q-Racah case:* q and κ are complex numbers of modulus 1, $q = e^{i\alpha}$, $\kappa = e^{i\beta}$, plus additional restrictions on κ depending on the size of a hexagon: both $-\alpha(T-1)/2+\beta$ and $\alpha(N-1/2)+\beta$ must lie in the same interval of the form $[\pi k, \pi(k+1)]$, $k \in \mathbb{Z}$. In this case

$$\kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}} = 2\sqrt{-1}\sin\left(\alpha(j-(S+1)/2)+\beta\right),$$

and the factor $2\sqrt{-1}$ here can be omitted.

The names of the cases are related to those of the classical orthogonal polynomials that appear in the analysis.

Denote the resulting measure on $\Omega(N,T,S)$ by $\mu(N,T,S,q,\kappa)$.

There are further limit transitions.

If we send $\kappa \to 0$ then we get the *q-Hahn case*

$$w(\diamond) = q^{-j}.$$

Thus, the probability of the plane partition of volume (=number of $1\times 1\times 1$ boxes) V is proportional to q^{-V}. On the other hand, if we send $\kappa \to \infty$ we will get the weights q^V. Therefore, the case of general κ can be viewed as an interpolation between the measures q^{volume} and $q^{-volume}$.

Another possibility is to set $\kappa = q^K$ and then send $q \to 1$. The weight of a horizontal lozenge tends to

$$w(\diamond) = K + j - (S+1)/2.$$

We call this case the *Racah case*. One has to impose restrictions to ensure positivity: K cannot lie inside the interval $[-N+1/2, (T-1)/2]$.

Finally, if we either send $\kappa \to 0$ and set $q = 1$, or send $q \to 1$ and then send $K \to \infty$, then we obtain the *Hahn case*

$$w(\diamond) = 1$$

In this case our probability distribution on $\Omega(N,T,S)$ is uniform.

Below we mostly work with the imaginary q-Racah case, but all the results can be carried over to all the other cases mentioned above by the appropriate substitutions of parameters and degenerations.

1.3.4 Representation of the weight

Let us view our tiling as a pile of $1 \times 1 \times 1$ cubes in the box located between the planes $x_1 = 0$, $x_1 = a$, $x_2 = 0$, $x_2 = b$, $x_3 = 0$, $x_3 = c$. Then Figure 1.5 represents a projection of the border of the 3-D-diagram to the plane $x_1 + x_2 + x_3 = 0$ parallel to the vector $(1, 1, 1)$.

For any $v \in \mathbb{R}^3$, denote by $h(v)$ the Euclidian distance from v to the plane $x_1 + x_2 + x_3 = 0$ divided by $\sqrt{3}$, and denote by $\hat{h}(v)$ the distance from v to the union of coordinate planes $x_1 x_2 x_3 = 0$ computed along the $(1, 1, 1)$-direction and divided by $\sqrt{3}$.

Recall that the weight of a tiling is given by the formula:

$$w(\mathcal{T}) = const \cdot \prod_{\diamond \in \mathcal{T}} w(j),$$

where j is as in Figure 1.5.

Grouping all the $1 \times 1 \times 1$ cubes of the plane partitions into columns with fixed coordinates (x_2, x_3), we can rewrite the weight in the form

$$w(\mathcal{T}) = const \cdot \prod_{\text{⬡}} \frac{w(j)}{w(j-1)},$$

where the product is taken over all cubes of the plane partition, and j stands for the j-coordinate (see Figure 1.5) of the top vertex of the cube.

Collecting factors with the same j, we obtain

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{w(j)}{w(j-1)} \right)^{h(v) - \hat{h}(v)},$$

where the product is taken over all points v on the border of the plane partition, whose three coordinates (x_1, x_2, x_3) are integers. Equivalently, one can think of the product being taken over all vertices of the triangular lattice inside the hexagon. Note that we may replace $h(v) - \hat{h}(v)$ by $h(v)$ since the remaining product depends only on (N, S, T). This gives

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{w(j)}{w(j-1)} \right)^{h(v)}.$$

For the q-Racah case we obtain

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{\kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}}{\kappa q^{j-(S+3)/2} - \dfrac{1}{\kappa q^{j-(S+3)/2}}} \right)^{h(v)},$$

while for the q-Hahn case

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{q^{-j}}{q^{-j+1}} \right)^{h(v)} = const \cdot \prod_v q^{-h(v)} = const \cdot q^{-volume}.$$

1.3.5 Results and variational interpretation

Our results are of two kinds.

First, for each of the probability distributions on tilings described above we construct explicit discrete time Markov chains that relate random tilings of hexagons of various sizes. The elementary steps of these chains change the size of the hexagon from $a \times b \times c$ to $a \times (b \mp 1) \times (c \pm 1)$.

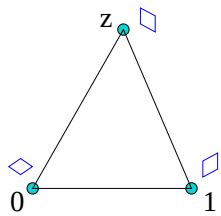
Randomness in each step consists in generating finitely many independent one-dimensional random variables. It takes $O(a(b+c))$ arithmetic operations to produce a tiling of the $a \times (b-1) \times (c+1)$ hexagon using a tiling of the $a \times b \times c$ hexagon.

Together with the trivial observation that there is exactly one tiling of an $a \times (b+c) \times 0$ hexagon, these chains provide, in particular, an efficient perfect sampling algorithm for random tilings distributed according to the real q-Racah, imaginary q-Racah, trigonometric q-Racah, q-Hahn, Racah and Hahn distributions.

A description of the algorithm can be found in Section 2.6, and in Section 2.9 we provide some pictures generated using this algorithm.

Second, we evaluate the asymptotics of the local behavior of our measures as all sides of the hexagon tend to infinity comparably.

It is known, see [43], [44], [46], [68], [81], [14], that for any three positive numbers (p_1, p_2, p_3) with $p_1 + p_2 + p_3 = 1$, there exists a unique translation-invariant Gibbs measure on lozenge tilings of the whole plane such that in a large box, the numbers $\{p_j\}_{j=1}^3$ provide asymptotic ratios of the number of lozenges of three types. It is convenient to encode the triple (p_1, p_2, p_3) by a complex number z in the upper-half plane so that the triangle $(0, 1, z)$ has angles $(\pi p_1, \pi p_2, \pi p_3)$. The correspondence between angles and lozenge types is indicated in the figure below.



If one of the p_j's tends to 1 (for example, z tends to a point in $\mathbb{R}$ away from $\{0, 1\}$), then the corresponding measure degenerates to the "frozen" (nonrandom) tiling of the plane by lozenges of the corresponding type.

Theorem (2.2.1)**.** *Introduce a small parameter $\varepsilon \ll 1$, and set*

$$S = \mathsf{S}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad T = \mathsf{T}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad N = \mathsf{N}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad q = \mathsf{q}^{\varepsilon + o(\varepsilon)}.$$

Then near a given point $(t, x) = (\mathsf{t}\varepsilon^{-1}, \mathsf{x}\varepsilon^{-1})$, the random tiling converges, as $\varepsilon \to 0$, to a certain ergodic translation-invariant Gibbs measure. The parameter

z of this measure (encoding the slope (p_1, p_2, p_3) via angles as described) is the unique solution in the upper half-plane to the (quadratic) equation

$$Q(u, v) = 0, \tag{1.4}$$

where

$$u = \frac{z\mathsf{q}^{\mathsf{t}} - \kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \qquad v = \frac{(1-z)\mathsf{q}^{\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \tag{1.5}$$

and Q is a degree 2 polynomial in (u, v):

$$\begin{aligned} Q(u,v) &= u^2 \\ &+ \left(\mathsf{q}^{\mathsf{T}-\mathsf{S}-\mathsf{N}} + \kappa^2(1 + \mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{T}} + \mathsf{q}^{-2\mathsf{S}+\mathsf{T}} + \mathsf{q}^{-\mathsf{S}-\mathsf{N}} - \mathsf{q}^{-\mathsf{S}} - \mathsf{q}^{-\mathsf{S}+\mathsf{T}}) + \kappa^4 \mathsf{q}^{-\mathsf{S}+\mathsf{N}}\right) v^2 \\ &+ \left(\mathsf{q}^{\mathsf{T}-\mathsf{S}} + \mathsf{q}^{-\mathsf{N}} + \kappa^2(\mathsf{q}^{\mathsf{N}} + \mathsf{q}^{-\mathsf{S}})\right) uv - \left(\mathsf{q}^{\mathsf{T}} + \mathsf{q}^{\mathsf{T}-\mathsf{S}-\mathsf{N}} + \kappa^2(1 + q^{\mathsf{N}-\mathsf{S}+\mathsf{T}})\right) v \\ &- (1 + \mathsf{q}^{\mathsf{T}})u + \mathsf{q}^{\mathsf{T}}. \end{aligned} \tag{1.6}$$

If the solutions to this equation in z are real, then the limit measure is frozen.

Let us now explain how one could guess these formulas. In Section 2.8 we present a rigorous proof of Theorem 2.2.1 which uses an argument of a different kind. The remainder of this section is purely empirical; we hope to address the same issues rigorously in a later publication.

Although Theorem 2.2.1 describes the microscopic behavior of our model, the parameters (p_1, p_2, p_3) of the limit measure are closely connected with macroscopic properties.

If we view tilings as stepped surfaces in a box and scale them in such a way that the bounding box remains fixed, then it is plausible that in the limit we will observe a deterministic limit shape. The normal vector to this limit shape at any point has to coincide with the vector (p_1, p_2, p_3) of the local limit measure at this point.

The concentration of the measure near the limit shape is known in the q-Hahn case, see [17], [45]. The limit shape is the unique solution of a certain variational problem. It is not hard to pose such a variational problem in the q-Racah case as well.

Recall that in Section 1.3.4 we found the following representation for the weight of a tiling:

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{\kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}}{\kappa q^{j-(S+3)/2} - \dfrac{1}{\kappa q^{j-(S+3)/2}}} \right)^{h(v)}.$$

Taking the logarithm of $w(\mathcal{T})$ and removing the constant we get

$$\sum_v h(v)\left[\ln\left(\kappa q^{j-\frac{S+1}{2}}-\frac{1}{\kappa q^{j-\frac{S+1}{2}}}\right)-\ln\left(\kappa q^{j-\frac{S+3}{2}}-\frac{1}{\kappa q^{j-\frac{S+3}{2}}}\right)\right].$$

This is a Riemannian sum for an integral, and as $\varepsilon\to 0$ this yields, up to second order terms,

$$\frac{1}{\varepsilon^2}\int\limits_{\text{hexagon}} \mathsf{h}(\mathsf{x},\mathsf{t})\frac{\partial\ln(\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}-\frac{1}{\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial\mathsf{j}},$$

where x, t are normalized coordinates inside the hexagon, h is the normalized height function, and $\mathsf{j}=\mathsf{x}-\mathsf{t}/2$.

Following [17] and [46] we know that the number of stepped surfaces in an ε-neighborhood of a given height function is asymptotically

$$\exp\left(\frac{1}{\varepsilon^2}\int\sigma(\nabla\mathsf{h})\right),$$

where σ is the surface tension that can be expressed through the Lobachevski function. (Note that the signs in [17] and [46] are different, we follow [17].)

Consequently, the probability of the stepped surfaces in the ε-neighborhood of a given height function is asymptotically proportional to

$$\exp\left[\frac{1}{\varepsilon^2}\left(\int\sigma(\nabla\mathsf{h})+\int\mathsf{h}(\mathsf{x},\mathsf{t})\frac{\partial\ln(\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}-\frac{1}{\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial\mathsf{j}}\right)\right],$$

and to find the limit shape one has to maximize this expression. We conclude that the limit shape can be found as a solution of the variational problem:

$$\int\sigma(\nabla\mathsf{h})+\int\mathsf{h}(\mathsf{x},\mathsf{t})\frac{\partial\ln(\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}-\frac{1}{\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial\mathsf{j}}\to\max$$

Let us write down the Euler-Lagrange equation for this variational problem. It is convenient to use our parameter z instead of partial derivatives of the limit height function h. We know (see [45]) that the Euler-Lagrange equation for the first term is the complex Burgers equation

$$\frac{z_t}{z}-\frac{z_x}{1-z}=0.$$

Adding it to the trivial Euler-Lagrange equation for the second term we obtain

$$\frac{z_t}{z}-\frac{z_x}{1-z}=\frac{\partial\ln(\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}-\frac{1}{\kappa\mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial\mathsf{j}},$$

or

$$\frac{z_t}{z}-\frac{z_x}{1-z}=\ln(\mathsf{q})\frac{\kappa^2\mathsf{q}^{-\mathsf{S}+2x-t}+1}{\kappa^2\mathsf{q}^{-\mathsf{S}+2x-t}-1}.$$

This is a quasilinear equation. The equations for characteristics have the form

$$\frac{dt}{ds} = \frac{1}{z}, \quad \frac{dx}{ds} = \frac{1}{z-1}, \quad \frac{dz}{ds} = \ln(\mathsf{q})\frac{\kappa^2\mathsf{q}^{-\mathsf{S}+2x-t}+1}{\kappa^2\mathsf{q}^{-\mathsf{S}+2x-t}-1}.$$

We find two first integrals

$$u = \frac{z\mathsf{q}^{\mathsf{t}} - \kappa^2\mathsf{q}^{-\mathsf{S}+2\mathsf{x}}}{1 - z\kappa^2\mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \qquad v = \frac{(1-z)\mathsf{q}^{\mathsf{x}}}{1 - z\kappa^2\mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}.$$

Any solution of the partial derivative equation has the form (cf. the proof of Corollary 1 in [45])

$$Q(u, v) = 0,$$

where Q is a suitable analytic function. This equation defines z as a function of t and x.

In the q-Hahn case, it is known that $Q(u, v)$ is a second degree polynomial. It is natural to assume that the same is true for the q-Racah case, and then the requirement that the (t, x)-curve where z degenerates to $\mathbb{R}$ is tangent to the six sides of the hexagon leads to the formula of Theorem 2.2.1. It remains a challenge to check if Q is still algebraic for more general polygonal domains, as was shown in [45] for the q-Hahn case.

1.4 Extended summary of Chapter 3

1.4.1 Preface

The infinite dimensional unitary group $U(\infty)$ is the union of the unitary groups $U(N)$ naturally embedded one into another. The study of *characters* of $U(\infty)$, i.e. normalized positive-definite central continuous functions on the group, was initiated by Voiculescu in 1976. He studied *finite factor representations* of $U(\infty)$. These representations are in bijection with *extreme characters* of $U(\infty)$, which are extreme points of the convex set of all characters of $U(\infty)$. In the paper [93] Voiculescu gave a list of extreme characters of $U(\infty)$. He also conjectured and partially proved that this list was complete.

There is a correspondence between extreme characters of $U(\infty)$ and *totally positive* Toeplitz matrices. Such matrices were studied much earlier by Shoenberg and his collaborators in the context of classical analysis. A few years after the paper [93], Boyer [15] and Vershik-Kerov [92] independently pointed out that the completeness of the Voiculescu's list of characters follows from the Edrei's result [23] on the classification of all totally positive Toeplitz matrices.

In the same paper [92] Vershik and Kerov suggested a new approach to the above problem. Their method is based on the approximation of the extreme characters of $U(\infty)$ by the normalized characters of the irreducible representations of the finite-dimensional unitary groups $U(N)$. As a result, classification of extreme characters of $U(\infty)$ is restated in purely combinatorial terms as the problem of the description of the *boundary of the Gelfand–Tsetlin graph.*

Okounkov and Olshanski in their paper [65] gave the detailed proof and further generalized the classification theorem for the characters of $U(\infty)$ using the Vershik-Kerov approach.

The aim of Chapter 3 is to introduce and study a q–deformation of the notion of a character of $U(\infty)$. We start from the Vershik-Kerov formulation and define a q–deformed version of the Gelfand–Tsetlin graph. Our main result is the complete description of the boundary of the q–Gelfand–Tsetlin graph. (See Section 1.4.2 for the details.)

There are several ways to interpret our results. We may go back to the characters and introduce their q–analogues which agree with our deformation of the Gelfand–Tsetlin graph. From this point of view, the problem that we solve in the present paper is the characterization of all possible limits of rational Schur functions normalized in a certain (depending on q) way as the number of variables grows to infinity. (See Section 1.4.4 for the details.)

Investigating a q–deformation of totally positive Toeplitz matrices, we arrive at a notion of a q–Toeplitz matrix. Every point of the boundary of the q–Gelfand–Tsetlin graph corresponds to a q–Toeplitz matrix with minors satisfying some non-negativity condition. (See Section 1.4.5 for the details.)

Paths in the Gelfand–Tsetlin graph are in bijection with tilings of the halfplane with rhombuses of 3 types. Through this correspondence our results turn into the classification theorem for certain Gibbs measures on the tilings of the halfplane. (See Section 3.2 for the details.)

There are strong reasons to believe that the q–Gelfand–Tsetlin graph is related to the representation theory of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. However we do not address this issue in the present paper.

1.4.2 Statement of the main result

The Gelfand-Tsetlin graph $\mathbb{GT}$ is a graded graph consisting of levels $\mathbb{GT}_N$, $N = 1, 2, \dots$. The vertices of $\mathbb{GT}_N$ are N–tuples $\lambda_1 \geq \cdots \geq \lambda_N$ of integers. Following the Weyl's book [95] we call these N–tuples *signatures*. We join two signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N+1}$ by an edge and write $\lambda \prec \mu$ if and only if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \lambda_N \geq \mu_{N+1}.$$

Let $\mathcal{T}$ denote the set of all infinite paths in $\mathbb{GT}$, i.e. $\tau = (\tau(1), \tau(2), \dots)$ belongs to $\mathcal{T}$ if and only if $\tau(N) \in \mathbb{GT}_N$ and $\tau(N)$ is connected with $\tau(N+1)$ by an edge in $\mathbb{GT}$ for every N.

For any probability measure P on $\mathcal{T}$ let $P_N(\cdot)$ denote its projection on $\mathbb{GT}_N$, in other words for any $\lambda \in \mathbb{GT}_N$

$$P_N(\lambda) = P(\tau \in \mathcal{T} \mid \tau(N) = \lambda).$$

For any finite path $\phi = (\phi(1) \prec \phi(2) \prec \cdots \prec \phi(N))$, $\phi(k) \in \mathbb{GT}_k$ let C_ϕ denote the corresponding cylinder set in $\mathcal{T}$, i.e.

$$C_\phi = \{\tau \in \mathcal{T} : \tau(1) = \phi(1), \dots, \tau(N) = \phi(N)\}.$$

Let us introduce the *weight* of ϕ:

$$w(\phi) = q^{|\phi(1)|+|\phi(2)|+\dots+|\phi(N-1)|},$$

where $|\lambda| = \lambda_1 + \dots + \lambda_k$ for $\lambda \in \mathbb{GT}_k$. In Section 3.2 we show that these weights have a simple combinatorial interpretation: given a path ϕ one constructs a 3D–body and $w(\phi)$ equals q^{vol}, where *vol* is the volume of this body. For any $\lambda \in \mathbb{GT}_N$ let

$$\mathrm{Dim}_q(\lambda) = \sum_{\phi(1)\prec\dots\prec\phi(N)|\phi(N)=\lambda} w(\phi).$$

A q–central measure on $\mathcal{T}$ is a probability measure P on $\mathcal{T}$ satisfying

$$P(C_\phi) = P_N(\phi(N))\frac{w(\phi)}{\mathrm{Dim}_q(\phi(N))}$$

for any finite path $\phi = (\phi(1) \prec \phi(2) \prec \dots \prec \phi(N))$, $\phi(k) \in \mathbb{GT}_k$. Put it otherwise, if we consider a family of finite paths ϕ ending at the same signature $\phi(N)$, then the probability of ϕ is proportional to the weight $w(\phi)$.

Let Ω_q denote the set of all q–central measures on $\mathcal{T}$. Clearly Ω_q is a convex set. *The minimal boundary of q–Gelfand–Tsetlin graph* is the set $\mathrm{Ex}(\Omega_q)$ of all extreme points of the set Ω_q. The main result of the present paper is the description of the minimal boundary of q–Gelfand–Tsetlin graph. We need to do some preparations before stating the theorem.

We say that signature $\lambda \in \mathbb{GT}_N$ is positive if all coordinates of λ are non-negative, i.e. $\lambda_N \geq 0$. Let $\mathbb{GT}_N^+ \subset \mathbb{GT}_N$ denote the subset of all positive signatures of size N.

Every extreme q–central measure P is uniquely defined by the array of numbers $P_N(\lambda)$. In the case studied by Voiculescu (i.e. $q = 1$) these numbers appear as the coefficients of the expansions of certain functions in the basis of rational Schur functions s_λ. It turns out that in the q–deformed case the formulas have the simplest form if instead of rational Schur functions we use other symmetric polynomials. It is convenient to use *q–interpolation Schur polynomials* $s^*_\lambda(x_1, \dots, x_N; q)$ defined for every $\lambda \in \mathbb{GT}_N^+$. These polynomials are a particular case of factorial Schur functions (see [59]). They are also a particular case ($q = t$) of interpolation Macdonald polynomials (see [51], [79], [63]). In one dimensional case we have

$$s^*_k(x; q) = (x-1)(x-q)\cdots(x-q^{k-1}).$$

For any probability measure P_N on $\mathbb{GT}_N^+$ we consider the following generating function:

$$\mathcal{S}^*(x_1, \dots, x_N; P_N) = \sum_{\lambda\in\mathbb{GT}_N^+} P_N(\lambda)\frac{s^*_\lambda(q^{N-1}x_1, \dots, q^{N-1}x_N; q^{-1})}{s^*_\lambda(0, \dots, 0;, q^{-1})} \tag{1.7}$$

Define $\mathcal{N}$ to be the set of all non-decreasing sequences of integers:

$$\mathcal{N} = \{\nu_1 \leq \nu_2 \leq \nu_3 \leq \dots\} \subset \mathbb{Z}^\infty.$$

Finally, let A_ℓ be the isomorphism of the Gelfand–Tsetlin graph taking a signature $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{GT}_N$ to $A_\ell(\lambda) = (\lambda_1 + \ell, \dots, \lambda_N + \ell)$.

Now we are ready to state the main theorem of Chapter 3.

Theorem (3.1.1)**.** *Let $0 < q < 1$. We have:*

(i). $\mathrm{Ex}(\Omega_q)$ *is parameterized by points of $\mathcal{N}$ with $\nu \in \mathcal{N}$ corresponding to the extreme q–central measure $\mathcal{E}^\nu$.*

(ii). If $\nu_1 \geq 0$ then

$$\mathcal{S}^*(x_1, \dots, x_N; \mathcal{E}^\nu_N) = H^\nu(x_1) \cdots H^\nu(x_N),$$

where

$$H^\nu(t) = \frac{\prod_{i=0}^\infty (1 - q^i t)}{\prod_{j=1}^\infty (1 - q^{\nu_j + j - 1} t)}$$

and $\mathcal{E}^\nu_N$ is the projection of $\mathcal{E}^\nu$ on $\mathbb{GT}_N$.

(iii). For general ν the measure $\mathcal{E}^\nu$ is the image of the measure $\mathcal{E}^{\nu'}$ with $\nu' = (0, \nu_2 - \nu_1, \nu_3 - \nu_1, \dots)$ under isomorphism A_{ν_1} of the graph $\mathbb{GT}$.

(iv). Ω_q is a simplex meaning that every $P \in \Omega_q$ is a unique average of measures $\mathcal{E}^\nu$. Put it otherwise, for every $P \in \Omega_q$ there exist a unique probability measure Q on $\mathcal{N}$ such that

$$P = \int_{\mathcal{N}} \mathcal{E}^\nu dQ.$$

There is a natural involution of the graph $\mathbb{GT}$ mapping signature $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ to $-\lambda_N \geq -\lambda_{N-1} \geq \dots \geq -\lambda_1$. Is is easy to see that this map provides a bijection between q–central measures and q^{-1}–central measures. Thus, Theorem 3.1.1 also provides a description of q–central measures with $q > 1$.

The following proposition describes the support of measure $\mathcal{E}^\nu$ and explains the geometric meaning of the parameter ν.

Proposition (3.1.2)**.** *Let τ be a random element of $\mathcal{T}$ distributed according to $\mathcal{E}^\nu$, then almost surely the last coordinates of $\tau(N)$ tend to $\{\nu_j\}$, i.e. for every j*

$$\lim_{N \to \infty} \tau(N)_{N+1-j} = \nu_j.$$

The proof of Theorem 3.1.1 relies on the so-called ergodic method, first proposed by Vershik in [88]. This method was used for the graph $\mathbb{GT}$ in [92] and for the generalizations of $\mathbb{GT}$ in [65]. Essentially the same method of the identification of the boundary of a graph was proposed by Diaconis and Freedman [22] in the context of *partial exchangeability*.

For the q–Gelfand–Tsetlin graph the ergodic method shows that every extreme q–central measure on $\mathcal{T}$ is, in some precise sense, a limit of extreme

measures on $\mathbb{GT}_N$. Passing from the measures to their generating functions we arrive at the problem of finding all possible limits of rational Schur functions normalized in a certain way as the number of variables grows to infinity. (In Section 1.4.4 we discuss this point in more details.) It seems natural to describe extreme q–central measures in terms of the normalized rational Schur functions too. However, if we try to do that, then the formulas turn out to be quite ugly. In the contrast, if we use the polynomials s^* instead of rational Schur functions, then we arrive at simple multiplicative formulas presented in Theorem 3.1.1.

q–Interpolation Schur polynomials and their properties play a crucial rule in our proofs. One of the key ingredients is the *binomial formula*, which relates q–interpolation Schur polynomials with ordinary Schur polynomials and is a particular case of the binomial formula for interpolation Macdonald polynomials proved by Okounkov in [62].

1.4.3 Representations of $U(\infty)$ and the origin of the Gelfand–Tsetlin graph

In this section we explain how the study of characters of $U(\infty)$ leads to the Gelfand–Tsetlin graph and compare Theorem 3.1.1 with the solution of the problem studied by Voiculescu.

There are two different approaches to the representation theory of $U(\infty)$ which lead to a reasonable class of representations (i.e. finite factor representations, see e.g. [86], or spherical representations of Gelfand pair $(U(\infty) \times U(\infty), U(\infty))$, see [72], [73]). In both approaches representations are in correspondence with *characters*, which are central (i.e. constant on conjugacy classes) positive definite continuous functions on $U(\infty)$ taking value 1 on the unit element of the group.

The Gelfand-Tsetlin graph $\mathbb{GT}$ is a convenient tool for studying characters of $U(\infty)$. Recall that irreducible representations of $U(N)$ are parameterized by their dominant weights, which are N–tuples $\lambda_1 \geq \cdots \geq \lambda_N$ of integers. (See e.g. [87].) Thus, the vertices of $\mathbb{GT}_N$ symbolize irreducible representations of $U(N)$. Furthermore, the edges of $\mathbb{GT}$ encode inclusion relations between irreducible representations of $U(N+1)$ and $U(N)$.

Given any character χ of the group $U(\infty)$ we can expand its restriction to the subgroup $U(N)$ into a convex combination of normalized characters (in the conventional sense) of irreducible representations of $U(N)$:

$$\chi\big|_{U(N)} = \sum_{\lambda \in \mathbb{GT}_N} P_N(\lambda) \frac{\chi^\lambda(\cdot)}{\chi^\lambda(e)}. \tag{1.8}$$

It is known (see e.g. [87]) that the characters of irreducible representations of unitary groups are rational Schur functions, i.e.

$$\frac{\chi^\lambda(U)}{\chi^\lambda(e)} = \frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(1, \ldots, 1)},$$

where $u_1, \ldots, u_N$ are eigenvalues of U.

The coefficients $P_N(\lambda)$ determine a probability distribution on the set $\mathbb{GT}_N$. In this way we get a bijection between characters χ and sequences $\{P_N\}_{N=1}^{\infty}$ of probability distributions. These sequences are called *coherent systems* because P_N and P_{N+1} satisfy a certain relation.

Any coherent system $\{P_N\}$ defines a measure P on $\mathcal{T}$ in the following way. For a finite path $\phi = (\phi(1) \prec \cdots \prec \phi(N))$ and corresponding cylinder set C_ϕ set

$$P(C_\phi) = \frac{P_N(\phi(N))}{\mathrm{Dim}(\phi(N))},$$

where $\mathrm{Dim}(\lambda)$ is the number of paths $\tau(1) \prec \cdots \prec \tau(N)$ such that $\tau(N) = \lambda$. The coherency relations between P_N and P_{N+1} imply that P is well-defined. Measure P is *a central measure* in the sense that $P(C_\phi)$ depends only on $\phi(N)$. In other words, projection of P on the set of all finite paths $\phi(1), \dots \phi(N)$ ending at a fixed $\phi(N) = \lambda \in \mathbb{GT}_N$ is uniform. Note that P_N is a projection of measure P on $\mathbb{GT}_N$. Also note that a q–central measure becomes a central measure if we set $q = 1$.

Extreme points of the convex set of characters of $U(\infty)$ (in other words, extreme characters) correspond to *irreducible spherical representations* of pair $(U(\infty) \times U(\infty), U(\infty))$ (or, again, finite factor representations). Extreme points of the convex set of all central measures on $\mathcal{T}$ correspond to extreme characters via the above bijections.

Theorem ([93],[15],[92],[65])**.** *Extreme characters of $U(\infty)$ are parameterized by the points ω of the infinite-dimensional domain*

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

The corresponding extreme character is given by the formula

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spectrum}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)}. \tag{1.9}$$

The remarkable feature shared by the above theorem and Theorem 3.1.1 is the appearance of the multiplicative functions. There is an independent representation-theoretic argument proving that any extreme character of $U(\infty)$

is multiplicative (i.e. that it should be a product of the values of some function over the eigenvalues of the element of $U(\infty)$). However, the author knows no conceptual reason for the appearance of the multiplicativity in Theorem 3.1.1 (independent of the classification theorem itself), and it would be interesting to find one.

It is very natural to ask whether extreme central measures on $\mathcal{T}$ can be obtained as $q \to 1$ limits of extreme q–central measures on $\mathcal{T}$. The answer is "Yes" for extreme central measures such that $\alpha_i^+ = \alpha_i^- = 0$. For example, if we chose $\nu(q)$ such that

$$H^{\nu(q)}(t) = (1 - q^{x_1(q)}t)(1 - q^{x_2(q)}t) \cdots (1 - q^{x_k(q)}t)$$

and $q^{x_i(q)} \to \beta_i^+$ as $q \to 1$, then the measures $\mathcal{E}^{\nu(q)}$ tend as $q \to 1$ to the extreme central measure on $\mathcal{T}$ parameterized by $\beta_1^+, \dots, \beta_k^+$. Using similar simple arguments one can obtain extreme central measures on $\mathcal{T}$ with arbitrary parameters $\gamma^\pm$, $\{\beta_i^\pm\}$. As for the general case, the author was not able to point out a sequence of extreme q–central measures (with q tending to 1) that converges to an extreme central measure with at least one nonzero coordinate $\alpha_i^\pm$.

1.4.4 Approximation of characters and limits of symmetric polynomials

Vershik and Kerov proved in [92] that every extreme character of $U(\infty)$ is a limit of normalized characters of irreducible representations of $U(N)$, i.e. of the functions

$$\frac{s_\lambda(u_1, \dots, u_N)}{s_\lambda(1, \dots, 1)}. \tag{1.10}$$

In other words, they reduced the classification problem to the following question: What are the possible limits of symmetric polynomials (1.10) as the number of variables grows to infinity? The answers for the similar questions for more general polynomials were obtained by Okounkov and Olshanski. See [65, Theorem 1.1] and [66, Theorem 1.4].

The results of the present paper can be also interpreted as an answer to a certain asymptotic problem for symmetric polynomials as the number of variables grows to infinity.

Let $N(i)$ be an increasing sequence of positive integers and $\lambda(i) \in \mathbb{GT}_{N(i)}$. We call the sequence of signatures $\lambda(i)$ *regular* if for any k the sequence of functions

$$\frac{s_{\lambda(i)}(x_1, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-N(i)})} \tag{1.11}$$

converges uniformly on the set $\{(x_1, \dots, x_k) \in \mathbb{C}^k \mid |x_i| = q^{1-i}\}$.

The normalization $s_\lambda(1, q^{-1}, \dots, q^{1-k})$, that we use, is well-known. See e.g. [58, Examples in Section 3, Chapter 1].

Let Λ be an algebra of symmetric functions, i.e. a projective limit of the graded algebras $Sym(N)$ of symmetric polynomials in N variables (see e.g. [58]).

Theorem (3.1.3)**.** *Let $0 < q < 1$. We have:*

(i). Sequence of signatures $\lambda(i)$ is regular if and only if the last coordinates of $\lambda(i)$ stabilize, i.e. if there is a nondecreasing sequence of integers $\nu = \{\nu_j\}$ such that for any $j > 0$

$$\lim_{i\to\infty} \lambda(i)_{N(i)+1-j} = \nu_j.$$

(ii). The limit function $Q_k^\nu(x_1, \dots, x_k)$ is an entire function in $\mathbb{C}^k$.

(iii). There is a bijection between limit functions Q_k^ν and measures $\mathcal{E}_k^\nu$. More precisely

$$Q_k^\nu = \sum_{\mu\in\mathbb{GT}_k} \mathcal{E}_k^\nu(\mu) \frac{s_{\lambda(i)}(x_1, \dots, x_k)}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-k})}$$

(iv). If ν and ν' are such that $\nu_i = \nu'_i + \ell$ for every i, then

$$Q_k^\nu = \frac{x_1^\ell \dots, x_k^\ell}{(1 \cdot q^{-1} \cdots q^{1-k})^\ell} Q_k^{\nu'}.$$

(v). If $\nu_1 \geq 0$, then

$$Q_k^\nu = \sum_{\mu\in\mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k; q),$$

where the series converges everywhere in $\mathbb{C}^k$ and Spec_ν is a specialization of algebra of symmetric function Λ (put it otherwise, homomorphism from Λ to $\mathbb{C}$) with H–generating function

$$\sum_{j=0}^{\infty} \mathrm{Spec}_\nu(h_j) t^j = \frac{\prod\limits_{i\geq 0}(1 - q^i t)}{\prod\limits_{j=1}^{\infty}(1 - q^{\nu_j+j-1} t)}.$$

(Here h_j stays for the complete symmetric function of degree j.)

Remark 1. The author knows no general explicit formulas for the functions Q_k^ν. This is the main reason why we do not use these functions as a description of measures $\mathcal{E}^\nu$. Instead we use $\mathcal{S}^*(x_1, \dots, x_k; \mathcal{E}_k^\nu)$, for which a simple multiplicative formula is provided in Theorem 3.1.1

Remark 2. In a special case $\nu_n = \nu_{n+1} = \cdots = a$ the functions Q_k^ν admit a simple interpretation. Let us consider only the case $a = 0$. Note that if

$0 < q < 1$, then for any $f \in \Lambda$ sequence $f_n = f(1, q, \dots, q^k)$ converges as $n \to \infty$, and a specialization map

$$f(y_1, y_2, \dots) \mapsto f(1, q, q^2, q^3, \dots)$$

is a well-defined homomorphism from Λ to $\mathbb{R}$. In the same way the map

$$f(y_1, y_2, \dots) \mapsto f(x_1^{-1}, \dots, x_k^{-1}, q^k, q^{k+1}, \dots)$$

is a homomorphism from Λ to the ring of Laurent polynomials in $x_1, \dots, x_k$.

Now let λ be a Young diagram with $n-1$ rows such that $\lambda_1 = -\nu_1, \dots, \lambda_{n-1} = -\nu_{n-1}$ and consider Schur function s_λ — element of Λ. The image of s_λ under the map

$$f(y_1, y_2, \dots) \mapsto \frac{f(x_1^{-1}, \dots, x_k^{-1}, q^k, q^{k+1}, \dots)}{f(1, q, q^2, \dots)}$$

coincides with Q_k^ν.

Remark 3. The functions

$$\frac{s_\lambda(x_1, \dots, x_N)}{s_{\lambda(i)}(1, q, \dots, q^{1-N})}$$

can be viewed as quantum traces of irreducible representations of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_N)$. Furthermore, one can show that the definition of a q–central measure is related to the branching rules for quantum characters of irreducible representations of $U_\epsilon(\mathfrak{gl}_N)$. (This is parallel to the fact that the Gelfand-Tsetlin graph itself is related to the branching rules of irreducible representations of the unitary group.) Thus, it is natural to expect that the functions Q_k^ν and q–central measures are related to certain representations of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. The author hopes to address this issue in a later publication.

Remark 4. It is natural to ask what happens if one replaces geometric series q^{1-i} in (1.11) by an arbitrary sequence ξ_i. The answer to this question remains unknown.

1.4.5 Toeplitz and q–Toeplitz matrices

Let us explain the connection between extreme characters of the group $U(\infty)$ and total positivity of Toeplitz matrices. Recall that any extreme character of $U(\infty)$ is a multiplicative function:

$$\chi(U) = \prod_{u_i} \widehat{\chi}(u_i), \quad u_i \in \text{Spectrum}(U).$$

Thus, χ is uniquely defined by $\widehat{\chi}(u)$ which is a continuous function on S^1. $\widehat{\chi}(u)$ can be represented as

$$\widehat{\chi}(u) = \sum_{l \in \mathbb{Z}} c_l u^l.$$

Introduce an infinite Toeplitz matrix

$$c[i,j] \stackrel{def}{=} c_{i-j}.$$

As a corollary of the fact that χ is a positive–definite function, one proves that the matrix $c[i,j]$ is *totally positive*, i.e. all minors of $c[i,j]$ are non-negative. Furthermore, this correspondence is a bijection between extreme characters of $U(\infty)$ and totally positive infinite Toeplitz matrices such that the sum of the matrix elements along the row equals 1. (See [93], [15], [92].)

In order to extend the correspondence between extreme measures and certain matrices to the case of general q we deform the notion of a Toeplitz matrix. We call a semi-infinite matrix $d[i,j]$, $i>0$, $j>0$ a *semi-infinite q–Toeplitz matrix* if

$$d[i,j+1] = d[i-1,j] + (q^{1-j} - q^{1-i})d[i,j] \tag{1.12}$$

for all $i>0$, $j>0$. Here we agree that $d[i,j]=0$ is either $i<1$ or $j<1$. Note that when $q=1$, the relation (1.12) turns into

$$d[i,j+1] = d[i-1,j].$$

Hence, a q–Toeplitz matrix becomes a Toeplitz matrix.

Recall that according to Theorem 3.1.1 extreme q–central measures correspond to multiplicative functions $H(x_1)\cdots H(x_N)$. Given a function $H(t)$ we construct a lower-triangular semi-infinite q–Toeplitz matrix $d[i,j]$ in the following way: expand H in series

$$H(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1}(1-tq^i)$$

and let $d[i,j]$, $i>0$, $j>0$ be a unique semi-infinite q–Toeplitz matrix such that

$$d[i,1] = c_{i-1}, \quad i=1,2,\ldots.$$

Initial minor of size N of matrix $d[i,j]$ is a minor corresponding to either the first N columns and arbitrary N rows of matrix $d[i,j]$ or the first N rows and arbitrary N columns of $d[i,j]$.

Proposition (3.1.4)**.** *Let ν be a non-decreasing sequence of non-negative integers $0\le\nu_1\le\nu_2\le\ldots$, let $\mathcal{E}^\nu$ be the extreme q–central measure parameterized by ν and corresponding to the function H^ν. If $d^\nu[i,j]$ is a semi-infinite q–Toeplitz matrix constructed by H^ν, then all initial minors of $d^\nu[i,j]$ are non-negative.*

A general theorem (see [27]) says that if A is a finite non-degenerate matrix with non-negative initial minors, then A is totally positive, in other words *all* minors of A are non-negative. But, alas, the q–Toeplitz matrices corresponding to q–central measures are usually degenerate. These matrices are triangular and some elements on the main diagonal vanish. Thus, we cannot guarantee that q–Toeplitz matrices corresponding to q–central measures are totally positive, we

can only claim that their certain top-left corners are. And, indeed, straightforward computations show that even some matrix elements of these matrices are negative.

However, it might be still interesting to classify all q–Toeplitz matrices with non-negative initial minors. We have the following conjecture here, which is a straightforward analogue of $q = 1$ case.

Conjecture (3.1.5)**.** *If $d[i, j]$ is a lower-triangular semi-infinite q–Toeplitz matrix such that*

$$\sum_{i=1}^{\infty} d[i, 1] = 1,$$

then $d[i, j]$ coincides with one of the matrices from Proposition 3.1.4, i.e. $d[i, j] = d^{\nu}[i, j]$ for some ν.

1.4.6 Other branching graphs and general formalism

The Gelfand–Tsetlin graph and the q–Gelfand–Tsetlin graph are two examples of *branching graphs*. (This term was introduced by Vershik and Kerov.) There is a wide class of problems that can be stated as a problem of identification the boundary of a branching graph.

A bunch of examples comes from the representation theory of "big" groups. Representations of the infinite symmetric group $S(\infty)$ are related to the boundary of the Young graph (see [85], [91], [64]), projective representations of $S(\infty)$ are related to the Schur graph (see [60], [38]), "block characters" of $S(\infty)$ are related to the Euler graph (see [29]), and (as we already mentioned) representation of the infinite dimensional unitary group $U(\infty)$ are related to the Gelfand–Tsetlin graph.

However, there are other examples of purely probabilistic and combinatorial nature. Perhaps, the most known example is De Finetti's theorem (see. e.g [26, Chapter VII, §4] or [1]) which states that every probability measure on $\{0, 1\}^{\infty}$ invariant with respect to finite permutations of coordinates is a mixture of Bernoulli measures. Here the underlying branching graph is the Pascal graph.

Motivated by a problem of population genetics Kingman introduced in [50] the notion of a partition structure. Kingman's classification of partition structures is equivalent to the description of the boundary of a certain graph, that is now called the Kingman graph, see also [48].

Other examples can be found in [67], [30], [31], [32], [33].

Let us consider a subgraph of $\mathbb{GT}$ consisting of zero–one signatures, i.e. $\lambda_1 \geq \cdots \geq \lambda_N$ such that $1 \geq \lambda_1 \geq \lambda_N \geq 0$. q–central measures on paths in this subgraph can be identified with central measures on paths in the q–Pascal graph studied by Gnedin and Olshanski [30]. These measures are related to q–analogues of De Finetti's theorem. (See also earlier paper by Kerov [47, Chapter 1.4]) The authors of [30] proved that for $0 < q < 1$ the boundary of the q–Pascal graph is parameterized by points of the set $\{1, q, q^2, \dots\} \cup \{0\}$. This result agrees with our description of the boundary of the q–Gelfand–Tsetlin

graph. Indeed, the only extreme q–central measures concentrated on zero–one signatures are those parameterized by ν with $0 \leq \nu_i \leq 1$ for every i. The extreme central measure on q–Pascal graph parameterized by q^k corresponds to q–central measure on $\mathcal{T}$ parameterized by $\nu = 0 \leq 0 \leq \cdots \leq 0 \leq 1 \leq 1 \ldots$ with exactly k zeros.

In another article [31] Gnedin and Olshanski studied a multidimensional generalization of the model of [30]. The common feature of the results of both these two papers and the present paper is that in the contrast to $q = 1$ case, parameters of the extreme q–central measures (and of the measures studied in [30], [31]) are discrete.

As a final remark of this section we want to mention the paper [22] where Diaconis and Freedman introduced a notion of partial exchangeability. Both central and q–central measures of $\mathcal{T}$ are particular cases of *partially exchangeable probabilities*.

1.4.7 Further connections and developments

Representation theory of $U(\infty)$ shows numerous connections with $S(\infty)$, i.e. inductive limit of symmetric groups S_n. (See [70] and [49] for reviews of the representation theory of $S(\infty)$) For example, while extreme characters of $U(\infty)$ are related to infinite totally positive Toeplitz matrices, extreme character of $S(\infty)$ are similarly related to semi-infinite totally positive Toeplitz matrices. The description of the simplex of q–central measures (in particular, their "right" definition) on the Young graph (substitute of the graph $\mathbb{GT}$ for the group $S(\infty)$) is yet to be done.

The are two important problems in the representation theory of the infinite-dimensional unitary group: identification of all irreducible representations and decomposition of natural representations into irreducible ones (see [71]). Irreducible representations (again, we should speak either about finite factor representations or irreducible spherical representations here) lead to extreme central measures on $\mathcal{T}$ and in the present paper we study a q–analogue of these measures. The natural representations associated with $U(\infty)$, in turn, lead to a remarkable family of central measures on $\mathcal{T}$, i.e. so called (z, w)–measures. At the moment it is unclear whether one can introduce some natural q–central measures that will serve as a q–analogue of (z, w)–measures.

Although, the study of central measures on $\mathcal{T}$ has a representation-theoretic origin, later these measures led to a number of very interesting probability models related to random Young diagrams and random stepped surfaces, see [7], [4], [3]. The author hopes that q–central measures introduced in the present paper will also provide a source of new interesting probability models.

1.5 Extended summary of Chapter 4

Harmonic analysis on a group studies the regular representation and another natural representations connected with this group. Chapter 4 deals with the

infinite-dimensional unitary group $U(\infty)$, that is, the union of the classical groups $U(N)$ naturally embedded in one another. The conventional definition of the regular representation does not apply to $U(\infty)$ because this group is not locally compact. In his pioneering paper [74] Pickrell proposed a construction for circumventing this difficulty in a similar situation. This paper served as a starting point for [71] where Olshanski introduced a natural generalization of the regular representation for the the "big" group $U(\infty)$.

The space of the representations introduced by Olshanski is an inductive limit of the spaces $L_2(U(N))$. For this limit to be well defined one should choose embeddings $L_2(U(N)) \hookrightarrow L_2(U(N+1))$. It turns out that this choice is far from being unique. Different possible choices lead to a family of representations $T_{z,w}$ which depend on two complex parameters z and w. The representation $T_{z,w}$ does not change if z or w is replaced by $\overline{z}$ or $\overline{w}$, respectively. The structure of this representation strongly depends on whether the parameters are integer or not. In this paper we handle the latter case.

Since all representations $T_{z,w}$ are inductive limits of the same sequence of spaces, these arises a very natural question: Are these representations really different? We answer this question by proving that the representations $T_{z,w}$ are pairwise disjoint. Recall that two representations T and T' are said to be disjoint if they have no equivalent nonzero subrepresentations. In Chapter 4 we prove the following theorem:

Theorem (4.1.1)**.** *Suppose that the parameters z, w, z' and w' are noninteger, $\{z, \overline{z}\} \neq \{z', \overline{z'}\}$, $\{w, \overline{w}\} \neq \{w', \overline{w'}\}$ (all pairs are unordered), and* $\mathrm{Re}(z+w) > -1/2$. *Then representations $T_{z,w}$ and $T_{z',w'}$ are disjoint.*

Decompositions of representations $T_{z,w}$ into irreducible representations are determined by certain measures, which we call spectral. To prove the disjointness of representations $T_{z,w}$, it suffices to prove that of the corresponding spectral measures. These measures were fully described in [10]. There exist explicit formulas for the correlation functions of spectral measures. But, as far as the author knows, there is no no direct method for proving the disjointness of measures by looking at their correlation functions. Thus, we have to use a different approach. We reduce the disjointness of spectral measures to that of certain probability measures $\rho_{z,w}$ on the space of paths in the Gelfand-Tsetlin graph; it is these measures $\rho_{z,w}$ which are the main object of study in this chapter.

The disjointness of representations and measures associated with "big" groups has been discussed in numerous papers, and still, many related problems remain unsolved.

The authors of [49] studied a family of representations T_z of the infinite symmetric group and obtained a result similar to Theorem 4.1.1. We use some ideas from that paper, although the case of the unitary group turned out to be much more complicated and requires new arguments.

Another closely related example was given in [11], where Borodin and Olshanski studied the so-called Hua-Pickrell s–measures on the space of infinite Hermitian matrices and, in particular proved the pairwise disjointness of these measures.

In [61] Neretin generalized the original construction of Pickrell to the ten classical series of compact Riemannian symmetric spaces. The disjointness problem can be posed in each of these ten cases. Perhaps, disjointness can be proved by methods similar to those used in this paper.

Note that Theorem 4.1.1 imposes a constraint on the parameters z and w, namely: $\mathrm{Re}(z+w) > -1/2$. Actually, the construction of the representations $T_{z,w}$ makes sense for all $z, w \in \mathbb{C}$. But for $\mathrm{Re}(z+w) \leq -1/2$, we obtain representations without so-called distinguished K–invariant vector. It would be interesting to study these representations too, but our methods do not apply to them.

In Section 4.7 we present a result of a different kind. A path in the Gelfand-Tsetlin graph is a sequence $\{\lambda(N)\}$ of dominant weights of irreducible representations of the unitary groups $U(N)$, and each $\lambda(N)$ can be regarded as the pair $(\lambda^+(N), \lambda^-(N))$ of Young diagrams. Thus, each measure $\rho_{z,w}$ generates two random sequences, $\{\lambda^+(N)\}$ and $\{\lambda^-(N)\}$, of Young diagrams which form a Markov growth process. As yet another application of the method described in Section 4.4, we prove that the length of the diagonal in $\{\lambda^\pm(N)\}$ grows at most logarithmically in N.

Chapter 2

q-Distributions on boxed plane partitions

2.1 Introduction

The uniform distribution on boxed plane partitions (equivalently, lozenge tilings of a hexagon) is one of the most studied models of random surfaces. There are four principal types of results regarding this model that have been proved.

(1) Law of large numbers: Under the global scaling (the bounding box/hexagon is fixed and the mesh is going to zero), the measure concentrates on surfaces that are close to a certain deterministic *limit shape*. The limit shape can be obtained as the unique solution to a suitable variational problem. The solution is encoded by a second degree polynomial in two variables, see [17], [18], [19], [20], [45].

(2) Locally near any point of the limit shape, the measure on tilings converges to a (uniquely defined, see [81]) translation-invariant ergodic Gibbs measure on lozenge tiling of the plane of a given slope, and the slope coincides with the slope of the tangent plane to the limit shape at the chosen point, see [35] and also [43], [44], [45], [46].

(3) The correlation kernel of the random point process of lozenges of one of the types is explicitly expressed in terms of classical Hahn orthogonal polynomials, see [39], [40], [41], [35].

(4) A few algorithms, both asymptotic and exact, have been proposed to generate the random tilings in question, see [5], [53], [75], [76], [96], [97].

These are complemented by the classical MacMahon product formula for the total number of plane partitions in a given box, see, e.g., Section 7.21 in [84].

In this paper we study measures on boxed plane partitions that generalize the uniform distribution. The weight of a tiling is defined as the product of certain simple factors over all lozenges of a fixed type, see Section 2.2.2 for definitions. One special case is the weight q^{volume}, where *volume* is the volume of the corresponding plane partition, and q is an arbitrary positive number.

In the most general case we consider, the weight of a lozenge is *elliptic*. Our initial motivation came from the fact that these weights lead to a nice generalization of the MacMahon formula mentioned above, see Theorem 2.10.5 in the Appendix.

For the asymptotic analysis, we look at the top degeneration of the elliptic weight that is related to orthogonal polynomials.

Our asymptotic results amount to proving analogs of (2), (3), and (4) above. As for (1), we derive the corresponding variational problem (which differs from the one for the uniform case by the presence of an external potential), and show that the hypothetical limit shape (obtained from (2)) solves the corresponding Euler-Lagrange equation. However, we do not prove the concentration phenomenon rigorously.

One interesting feature of the limit shapes that arise is that the curve that bounds the frozen regions may have one or two nodes in vertices of the hexagon, see Section 2.9 for illustrations.

In terms of orthogonal polynomials, our models go all the way up to the top of the q-Askey scheme — the most general models we analyze asymptotically are related to the q-Racah classical orthogonal polynomials. We also show that the elliptic weights lead to the biorthogonal functions constructed in [83]. We hope to return to the asymptotic analysis of this case in a later publication.

Our proof of (2) follows the same steps as the argument for the uniform case in [35]. It is based on the general method of computing limits of correlation kernels suggested in [12] and [69]. The crucial property we need is the second order difference equation satisfied by the q-Racah orthogonal polynomials.

Our perfect sampling algorithm is obtained from a more general construction of relatively simple Markov chains that change the size of the box (one side increases by 1 and another side decreases by 1), and that map the measures from the class we consider to similar ones. The construction follows the approach of [4]; the key facts that make that approach possible reduce to certain recurrence relations for the q-Racah polynomials.

A computer simulation of the above-mentioned Markov chains can be found at [36].

2.2 Model and results

2.2.1 Combinatorial interpretations

For any integers $a, b, c \geq 1$ consider a hexagon with sides a, b, c, a, b, c drawn on the regular triangular lattice. Denote by $\Omega_{a\times b\times c}$ the set of all tilings of this hexagon by rhombi obtained by gluing two of the neighboring elementary triangles together (such rhombi are called *lozenges*). An element of $\Omega_{3\times 3\times 3}$ is shown in Figure 2.1.

Lozenge tilings of a hexagon can be identified with 3-D Young diagrams (equivalently, boxed plane partitions) or with stepped surfaces. The bijection is best described pictorially. We show a 3-D shape corresponding to a tiling in

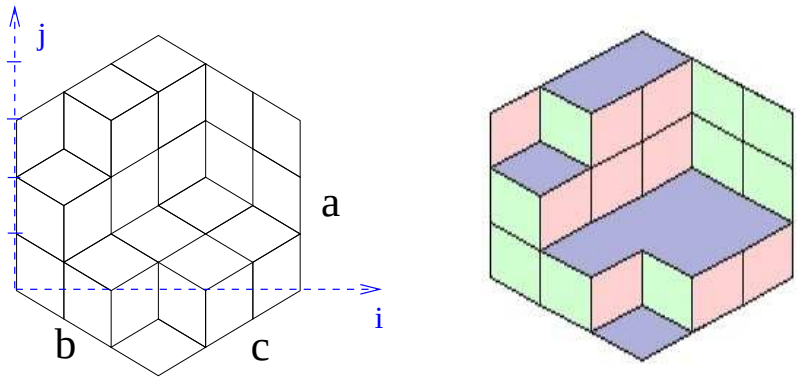


Figure 2.1: Tiling of a $3 \times 3 \times 3$ hexagon.

Figure 2.1.

It is convenient for us to slightly modify both hexagon and lozenges by means of a simple affine transform of the plane.

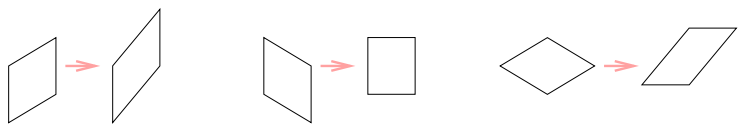


Figure 2.2: Affine modification of lozenges.

We thus obtain a tiling of a slightly different hexagon.

In what follows we use different parameters instead of a, b, c. Set $N = a$, $T = b + c$, $S = c$. We will also denote the set $\Omega_{a\times b\times c}$ by $\Omega(N, T, S)$.

Each tiling corresponds to a family of nonintersecting paths as shown in Figure 2.3.

Consider a section of our family of paths by a vertical line $t = t_0$. Clearly, this gives an N-tuple of points in $\mathbb{Z}$. Thus, our tiling can be viewed as an N-point configuration varying in time $t = 0, 1, \ldots, T$. Note that when $t = 0$ the configuration consists of points $\{0, 1, \ldots, N - 1\}$, while for $t = T$ the configuration consists of points $\{S, \ldots, S + N - 1\}$.

2.2.2 Probability models

Let us introduce the probability measures on $\Omega(N, T, S)$ that are studied in this paper. For any $\mathcal{T} \in \Omega(N, T, S)$, we define its weight $w(\mathcal{T})$ and consider the

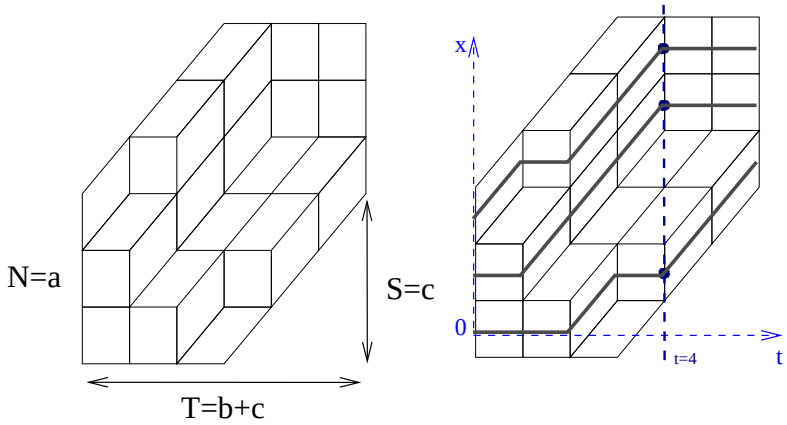


Figure 2.3: Modified tiling of a $3 \times 3 \times 3$ hexagon and the corresponding family of non-intersecting paths.

probability distribution on $\Omega(N, T, S)$ given by the formula

$$\mathrm{Prob}\{\mathcal{T}\} = \frac{w(\mathcal{T})}{\sum\limits_{\mathcal{T}' \in \Omega(N,T,S)} w(\mathcal{T}')}$$

The weights we consider are such that the probability of a tiling is proportional to the product of certain weights corresponding to the horizontal lozenges in it, i.e.,

$$w(\mathcal{T}) = \prod_{\diamondsuit \in \mathcal{T}} w(\diamondsuit)$$

Note that the number of horizontal lozenges in a tiling of a given hexagon is fixed (i.e., it does not depend on the tiling). Hence, multiplying $w(\diamondsuit)$ by a nonzero constant does not change the probability distribution.

In the most general case considered in Section 2.10, $w(\diamondsuit)$ is an elliptic weight given by

$$w(\diamondsuit) = \frac{(u_1 u_2)^{1/2} q^{j-1/2} \theta_p(q^{2j-1} u_1 u_2)}{\theta_p(q^{j-3i/2-1} u_1, q^{j-3i/2} u_1, q^{j+3i/2-1} u_2, q^{j+3i/2} u_2)}, \tag{2.1}$$

where the coordinates of the topmost point of $\diamondsuit$ are (i, j) (the i and j axes are pictured in Figure 2.1), u_1, u_2, p, q are (generally speaking, complex) parameters,

$$\theta_p(x) = \prod_{i=0}^{\infty} (1 - p^i x)(1 - p^{i+1}/x)$$

and $\theta_p(a, b, c \dots) = \theta_p(a)\theta_p(b)\theta_p(c) \dots$

Mostly we will not be concerned with this most general case, nor with the most general trigonometric case obtained by taking $p \to 0$ (note $\theta_0(x) = 1 - x$), as in these cases the kernels involve biorthogonal functions. The most general orthogonal polynomial case is the limit

$$p \to 0, \quad u_1 = O(\sqrt{p}), \quad u_2 = O(\sqrt{p}), \quad u_1 u_2 = p\kappa^2 q^{-S}, \tag{2.2}$$

in which case the weight function is

$$w(\diamondsuit) = \kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}}; \tag{2.3}$$

this is also the most general case in which the weight is independent of i.

Clearly, the factor $q^{-(S+1)/2}$ can be removed if we replace κ by $\kappa' = \kappa \cdot q^{-(S+1)/2}$. However, this choice is more convenient for our further considerations.

We need to make sure that the probabilities of tilings are positive. This leads to certain restrictions on the parameters. There are three possible cases:

(i). *imaginary q-Racah case:* q is a positive real number, κ is an arbitrary pure imaginary complex number;

(ii). *real q-Racah case:* q is a positive real number, κ is a real number with additional restrictions depending on the size of a hexagon; κ cannot lie inside the interval $[q^{-N+1/2}, q^{(T-1)/2}]$ or $[q^{(T-1)/2}, q^{-N+1/2}]$, depending on whether $q > 1$ or $q < 1$;

(iii). *trigonometric q-Racah case:* q and κ are complex numbers of modulus 1, $q = e^{i\alpha}$, $\kappa = e^{i\beta}$, plus additional restrictions on κ depending on the size of a hexagon: both $-\alpha(T-1)/2 + \beta$ and $\alpha(N - 1/2) + \beta$ must lie in the same interval of the form $[\pi k, \pi(k+1)]$, $k \in \mathbb{Z}$. In this case

$$\kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}} = 2\sqrt{-1} \sin\left(\alpha(j - (S+1)/2) + \beta\right),$$

and the factor $2\sqrt{-1}$ here can be omitted.

The names of the cases are related to those of the classical orthogonal polynomials that appear in the analysis.

Denote the resulting measure on $\Omega(N, T, S)$ by $\mu(N, T, S, q, \kappa)$.

There are further limit transitions.

If we send $\kappa \to 0$ then we get the *q-Hahn case*

$$w(\diamondsuit) = q^{-j}.$$

Thus, the probability of the plane partition of volume (=number of $1 \times 1 \times 1$ boxes) V is proportional to q^{-V}. On the other hand, if we send $\kappa \to \infty$ we will get the weights q^V. Therefore, the case of general κ can be viewed as an interpolation between the measures q^{volume} and $q^{-volume}$.

Another possibility is to set $\kappa = q^K$ and then send $q \to 1$. The weight of a horizontal lozenge tends to

$$w(\diamond) = K + j - (S+1)/2.$$

We call this case the *Racah case*. One has to impose restrictions to ensure positivity: K cannot lie inside the interval $[-N + 1/2, (T-1)/2]$.

Finally, if we either send $\kappa \to 0$ and set $q = 1$, or send $q \to 1$ and then send $K \to \infty$, then we obtain the *Hahn case*

$$w(\diamond) = 1$$

In this case our probability distribution on $\Omega(N, T, S)$ is uniform.

Below we mostly work with the imaginary q-Racah case, but all the results can be carried over to all the other cases mentioned above by the appropriate substitutions of parameters and degenerations.

2.2.3 Representation of the weight

Let us view our tiling as a pile of $1 \times 1 \times 1$ cubes in the box located between the planes $x_1 = 0$, $x_1 = a$, $x_2 = 0$, $x_2 = b$, $x_3 = 0$, $x_3 = c$. Then Figure 2.1 represents a projection of the border of the 3-D-diagram to the plane $x_1 + x_2 + x_3 = 0$ parallel to the vector $(1, 1, 1)$.

For any $v \in \mathbb{R}^3$, denote by $h(v)$ the Euclidian distance from v to the plane $x_1 + x_2 + x_3 = 0$ divided by $\sqrt{3}$, and denote by $\hat{h}(v)$ the distance from v to the union of coordinate planes $x_1 x_2 x_3 = 0$ computed along the $(1, 1, 1)$-direction and divided by $\sqrt{3}$.

Recall that the weight of a tiling is given by the formula:

$$w(\mathcal{T}) = const \cdot \prod_{\diamond \in \mathcal{T}} w(j),$$

where j is as in Figure 2.1.

Grouping all the $1 \times 1 \times 1$ cubes of the plane partitions into columns with fixed coordinates (x_2, x_3), we can rewrite the weight in the form

$$w(\mathcal{T}) = const \cdot \prod_{\square} \frac{w(j)}{w(j-1)},$$

where the product is taken over all cubes of the plane partition, and j stands for the j-coordinate (see Figure 2.1) of the top vertex of the cube.

Collecting factors with the same j, we obtain

$$w(\mathcal{T}) = const \cdot \prod_{v} \left(\frac{w(j)}{w(j-1)} \right)^{h(v) - \hat{h}(v)},$$

where the product is taken over all points v on the border of the plane partition, whose three coordinates (x_1, x_2, x_3) are integers. Equivalently, one can think of the product being taken over all vertices of the triangular lattice inside the hexagon. Note that we may replace $h(v) - \hat{h}(v)$ by $h(v)$ since the remaining product depends only on (N, S, T). This gives

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{w(j)}{w(j-1)} \right)^{h(v)} .$$

For the q-Racah case we obtain

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{\kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}}{\kappa q^{j-(S+3)/2} - \dfrac{1}{\kappa q^{j-(S+3)/2}}} \right)^{h(v)} ,$$

while for the q-Hahn case

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{q^{-j}}{q^{-j+1}} \right)^{h(v)} = const \cdot \prod_v q^{-h(v)} = const \cdot q^{-volume}.$$

2.2.4 Results and variational interpretation

Our results are of two kinds.

First, for each of the probability distributions on tilings described above we construct explicit discrete time Markov chains that relate random tilings of hexagons of various sizes. The elementary steps of these chains change the size of the hexagon from $a \times b \times c$ to $a \times (b \mp 1) \times (c \pm 1)$.

Randomness in each step consists in generating finitely many independent one-dimensional random variables. It takes $O(a(b+c))$ arithmetic operations to produce a tiling of the $a \times (b-1) \times (c+1)$ hexagon using a tiling of the $a \times b \times c$ hexagon.

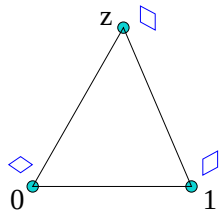
Together with the trivial observation that there is exactly one tiling of an $a \times (b + c) \times 0$ hexagon, these chains provide, in particular, an efficient perfect sampling algorithm for random tilings distributed according to the real q-Racah, imaginary q-Racah, trigonometric q-Racah, q-Hahn, Racah and Hahn distributions.

A description of the algorithm can be found in Section 2.6, and in Section 2.9 we provide some pictures generated using this algorithm.

Second, we evaluate the asymptotics of the local behavior of our measures as all sides of the hexagon tend to infinity comparably.

It is known, see [43], [44], [46], [68], [81], [14], that for any three positive numbers (p_1, p_2, p_3) with $p_1 + p_2 + p_3 = 1$, there exists a unique translation-invariant Gibbs measure on lozenge tilings of the whole plane such that in a large box, the numbers $\{p_j\}_{j=1}^3$ provide asymptotic ratios of the number of

lozenges of three types. It is convenient to encode the triple (p_1, p_2, p_3) by a complex number z in the upper-half plane so that the triangle $(0, 1, z)$ has angles $(\pi p_1, \pi p_2, \pi p_3)$. The correspondence between angles and lozenge types is indicated in the figure below.



If one of the p_j's tends to 1 (for example, z tends to a point in $\mathbb{R}$ away from $\{0, 1\}$), then the corresponding measure degenerates to the "frozen" (nonrandom) tiling of the plane by lozenges of the corresponding type.

Theorem 2.2.1. *Introduce a small parameter $\varepsilon \ll 1$, and set*

$$S = \mathsf{S}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad T = \mathsf{T}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad N = \mathsf{N}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad q = \mathsf{q}^{\varepsilon + o(\varepsilon)}.$$

Then near a given point $(t, x) = (\mathsf{t}\varepsilon^{-1}, \mathsf{x}\varepsilon^{-1})$, the random tiling converges, as $\varepsilon \to 0$, to a certain ergodic translation-invariant Gibbs measure. The parameter z of this measure (encoding the slope (p_1, p_2, p_3) via angles as described) is the unique solution in the upper half-plane to the (quadratic) equation

$$Q(u, v) = 0, \tag{2.4}$$

where

$$u = \frac{z\mathsf{q}^{\mathsf{t}} - \kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \qquad v = \frac{(1-z)\mathsf{q}^{\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \tag{2.5}$$

and Q is a degree 2 polynomial in (u, v):

$$\begin{aligned} Q(u,v) &= u^2 \\ &+ \left(\mathsf{q}^{\mathsf{T}-\mathsf{S}-\mathsf{N}} + \kappa^2(1 + \mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{T}} + \mathsf{q}^{-2\mathsf{S}+\mathsf{T}} + \mathsf{q}^{-\mathsf{S}-\mathsf{N}} - \mathsf{q}^{-\mathsf{S}} - \mathsf{q}^{-\mathsf{S}+\mathsf{T}}) + \kappa^4 \mathsf{q}^{-\mathsf{S}+\mathsf{N}}\right) v^2 \\ &+ \left(\mathsf{q}^{\mathsf{T}-\mathsf{S}} + \mathsf{q}^{-\mathsf{N}} + \kappa^2(\mathsf{q}^{\mathsf{N}} + \mathsf{q}^{-\mathsf{S}})\right) uv - \left(\mathsf{q}^{\mathsf{T}} + \mathsf{q}^{\mathsf{T}-\mathsf{S}-\mathsf{N}} + \kappa^2(1 + q^{\mathsf{N}-\mathsf{S}+\mathsf{T}})\right) v \\ &- (1 + \mathsf{q}^{\mathsf{T}})u + \mathsf{q}^{\mathsf{T}}. \end{aligned} \tag{2.6}$$

If the solutions to this equation in z are real, then the limit measure is frozen.

Let us now explain how one could guess these formulas. In Section 2.8 we present a rigorous proof of Theorem 2.2.1 which uses an argument of a different kind. The remainder of this section is purely empirical; we hope to address the same issues rigorously in a later publication.

Although Theorem 2.2.1 describes the microscopic behavior of our model, the parameters (p_1, p_2, p_3) of the limit measure are closely connected with macroscopic properties.

If we view tilings as stepped surfaces in a box and scale them in such a way that the bounding box remains fixed, then it is plausible that in the limit we will observe a deterministic limit shape. The normal vector to this limit shape at any point has to coincide with the vector (p_1, p_2, p_3) of the local limit measure at this point.

The concentration of the measure near the limit shape is known in the q-Hahn case, see [17], [45]. The limit shape is the unique solution of a certain variational problem. It is not hard to pose such a variational problem in the q-Racah case as well.

Recall that in Section 2.2.3 we found the following representation for the weight of a tiling:

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{\kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}}{\kappa q^{j-(S+3)/2} - \dfrac{1}{\kappa q^{j-(S+3)/2}}} \right)^{h(v)} .$$

Taking the logarithm of $w(\mathcal{T})$ and removing the constant we get

$$\sum_v h(v) \left[\ln \left(\kappa q^{j-\frac{S+1}{2}} - \frac{1}{\kappa q^{j-\frac{S+1}{2}}} \right) - \ln \left(\kappa q^{j-\frac{S+3}{2}} - \frac{1}{\kappa q^{j-\frac{S+3}{2}}} \right) \right] .$$

This is a Riemannian sum for an integral, and as $\varepsilon \to 0$ this yields, up to second order terms,

$$\frac{1}{\varepsilon^2} \int\limits_{\text{hexagon}} \mathsf{h}(\mathsf{x}, \mathsf{t}) \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}},$$

where x, t are normalized coordinates inside the hexagon, h is the normalized height function, and $\mathsf{j} = \mathsf{x} - \mathsf{t}/2$.

Following [17] and [46] we know that the number of stepped surfaces in an ε-neighborhood of a given height function is asymptotically

$$\exp \left(\frac{1}{\varepsilon^2} \int \sigma(\nabla \mathsf{h}) \right),$$

where σ is the surface tension that can be expressed through the Lobachevski function. (Note that the signs in [17] and [46] are different, we follow [17].)

Consequently, the probability of the stepped surfaces in the ε-neighborhood of a given height function is asymptotically proportional to

$$\exp \left[\frac{1}{\varepsilon^2} \left(\int \sigma(\nabla \mathsf{h}) + \int \mathsf{h}(\mathsf{x}, \mathsf{t}) \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}} \right) \right],$$

and to find the limit shape one has to maximize this expression. We conclude that the limit shape can be found as a solution of the variational problem:

$$\int \sigma(\nabla \mathsf{h}) + \int \mathsf{h}(\mathsf{x},\mathsf{t}) \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}} \to \max$$

Let us write down the Euler-Lagrange equation for this variational problem. It is convenient to use our parameter z instead of partial derivatives of the limit height function h. We know (see [45]) that the Euler-Lagrange equation for the first term is the complex Burgers equation

$$\frac{z_t}{z} - \frac{z_x}{1-z} = 0.$$

Adding it to the trivial Euler-Lagrange equation for the second term we obtain

$$\frac{z_t}{z} - \frac{z_x}{1-z} = \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}},$$

or

$$\frac{z_t}{z} - \frac{z_x}{1-z} = \ln(\mathsf{q}) \frac{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t} + 1}{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t} - 1}.$$

This is a quasilinear equation. The equations for characteristics have the form

$$\frac{dt}{ds} = \frac{1}{z}, \quad \frac{dx}{ds} = \frac{1}{z-1}, \quad \frac{dz}{ds} = \ln(\mathsf{q}) \frac{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t} + 1}{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t} - 1}.$$

We find two first integrals

$$u = \frac{z\mathsf{q}^{\mathsf{t}} - \kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \qquad v = \frac{(1-z)\mathsf{q}^{\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}.$$

Any solution of the partial derivative equation has the form (cf. the proof of Corollary 1 in [45])

$$Q(u,v) = 0,$$

where Q is a suitable analytic function. This equation defines z as a function of t and x.

In the q-Hahn case, it is known that $Q(u,v)$ is a second degree polynomial. It is natural to assume that the same is true for the q-Racah case, and then the requirement that the (t,x)-curve where z degenerates to $\mathbb{R}$ is tangent to the six sides of the hexagon leads to the formula of Theorem 2.2.1. It remains a challenge to check if Q is still algebraic for more general polygonal domains, as was shown in [45] for the q-Hahn case.

2.3 Weight sums

A horizontal lozenge in the tiling interpretation corresponds to a hole (=absence of a particle) in the nonintersecting paths or N-point configuration interpretation. The coordinates (i, j) of the horizontal lozenge in the formulas (2.1) and (2.3) correspond to the coordinates (t, x) of a hole in the following way:

$$\begin{cases} i = t, \\ j = x - t/2 + 1. \end{cases}$$

Now consider any vertical section $t = t_0$ of the family of nonintersecting paths (see Figure 2.3) and fix the corresponding N-point configuration $x_1 < x_2 < \cdots < x_N$.

Denote by $C(t; x_1, \ldots, x_N)$ the product of weights corresponding to the holes on this vertical line.

Denote by $L(t; x_1, \ldots, x_N)$ the sum of the products of weights corresponding to holes situated to the left of the vertical line. The sum is taken over all families of paths connecting the points $\{(0, 0), (0, 1), \ldots, (0, N-1)\}$ to the points $\{(t_0, x_1), (t_0, x_2), \ldots, (t_0, x_N)\}$.

Denote by $R(t; x_1, \ldots, x_N)$ the sum of the products of weights corresponding to holes situated to the right of the vertical line. The sum is taken over all families of paths connecting the points $\{(t_0, x_1), (t_0, x_2), \ldots, (t_0, x_N)\}$ to the points $\{(T, S), (T, S + 1), \ldots, (T, S + N - 1)\}$.

The following three propositions correspond to the case of the q-Racah weight (2.3).

Set

$$\mu_{t,S}(x) = q^{-x} + \kappa^2 q^{x-S-t+1}$$

Here and below we use the q-Pochhammer symbol:

$$(a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1}).$$

Proposition 2.3.1. *We have*

$$L_t(x_1, \ldots, x_N) = const_t \cdot \prod_{1 \le i < j \le N} (\mu_{t,S}(x_i) - \mu_{t,S}(x_j)) \\ \times \prod_{i=1}^{N} \frac{q^{x_i(t+N-1)}(1 - \kappa^2 q^{2x_i - S - t + 1})}{(q^{-1}; q^{-1})_{t+N-1-x_i} \cdot (q; q)_{x_i} \cdot (\kappa^2 q^{x_i - t - S + 1}; q)_{t+N}}.$$

Proof. Here and in the proof of the next lemma we should consider four cases depending on the value of t (see formulas (2.8)-(2.11)). The proofs are similar in all four cases and we consider only the one that corresponds to the pictures in Lemma 2.10.2 and Lemma 2.10.3 (i.e., $S < t < T - S$)

Let us use Lemma 2.10.2 that expresses the required weight sum in terms of the point configuration complementary to $\{x_i\}$, i.e., in terms of the positions of the horizontal lozenges that we call holes. Denote the positions of holes by $\{y_i\}$. Performing the limit transition (2.2) we get.

$$L_t(x_1,\dots,x_N) = const \cdot \prod_{1\le i<j\le S} (\mu_{t,S}(y_i)-\mu_{t,S}(y_j)) \times$$
$$\prod_{i=1}^{S} \frac{q^{(S-t)(S+N-1-y_i)}(q^{t-S+1};q)_{S+N-1-y_i}(q^{-2N-S+t+1}/\kappa^2;q)_{S+N-1-y_i}}{(q;q)_{S+N-1-y_i}(q^{-2N+1}/\kappa^2;q)_{S+N-1-y_i}}.$$

To finish the proof we rewrite the last formula in terms of particles $\{x_i\}$ instead of holes $\{y_i\}$. For the second factor this procedure is simple, while for the first one we use the following observation:

$$\prod_{1\le i<j\le N} (\mu_{t,S}(y_i)-\mu_{t,S}(y_j)) = \prod_{1\le i<j\le N} (\mu_{t,S}(x_i)-\mu_{t,S}(x_j))$$
$$\times \prod_{0\le u<v\le S+N-1} (\mu_{t,S}(u)-\mu_{t,S}(v))$$
$$\times \prod_{1\le i\le N} \frac{1}{\prod_{0\le u<x_i}(\mu_{t,S}(x_i)-\mu_{t,S}(u)) \prod_{x_i<u\le S+N-1}(\mu_{t,S}(u)-\mu_{t,S}(x_i))}.$$

The product over $u<v$ depends only on t, while the last two products over u are easily expressible in terms of q-Pochhammer symbols.

□

Proposition 2.3.2. *We have*

$$R_t(x_1,\dots,x_N) = const_t \cdot \prod_{1\le i<j\le N} (\mu_{t,S}(x_i)-\mu_{t,S}(x_j))$$
$$\times \prod_{i=1}^{N} \frac{(1-\kappa^2 q^{2x_i-S-t+1})q^{x_i(T-t+N-1)}}{(q^{-1};q^{-1})_{S+N-1-x_i}\cdot(q;q)_{x_i+T-t-S}\cdot(\kappa^2 q^{x_i-T+1};q)_{N+T-t}}.$$

Proof. Performing the limit transition (2.2) in Lemma 2.10.3 we get

$$R_t(x_1,\dots,x_N) = const \cdot \prod_{1\le i<j\le S} (\mu_{t,S}(y_i)-\mu_{t,S}(y_j))$$
$$\times \prod_{i=1}^{S} \frac{q^{(S-T+t)(S+N-1-y_i)}(q^{-N-S+1};q)_{S+N-1-y_i}(q^{T-S-N+1}/\kappa^2;q)_{S+N-1-y_i}}{(q^{-N+1-T+t};q)_{S+N-1-y_i}(q^{t-N+1}/\kappa^2;q)_{S+N-1-y_i}}$$

Again expressing all the factors in terms of $\{x_i\}$ we obtain the desired result.

□

Proposition 2.3.3. *We have*

$$C_t(x_1,\dots,x_N) = const_t \cdot \prod_{i=1}^{N} \frac{q^{x_i}}{1-\kappa^2 q^{2x_i-S-t+1}}.$$

Proof. Clearly,

$$C_t(x_1,\dots,x_N)=\prod_{i=1}^{S}\left(\kappa q^{y_i-S/2-t/2+1/2}-\frac{1}{\kappa q^{y_i-S/2-t/2+1/2}}\right),$$

where $\{y_i\}$ is the point configuration complementary to $\{x_i\}$. Expressing the product in terms of $\{x_i\}$ we get the result. □

2.4 Distributions and transition probabilities

We consider our probability measure on the set of tilings of a given hexagon as a Markov chain in the space of N-tuples of integers. The Markov property can be easily seen in the following form: The past and the future are independent given the present. In this way the Markov property reduces to the fact that a lozenge tiling of the hexagon is a union of a tiling to the left of a vertical line $t = const$ and a tiling to the right of this vertical line. Denote this Markov chain by $X(t)$, $t = 0, 1, \dots T$. Set

$$\mathfrak{X}_{N,T}^{S,t}=\{x\in\mathbb{Z}:\max(0,t+S-T)\le x\le\min(t+N-1,S+N-1)\}$$

and

$$\mathcal{X}_{N,T}^{S,t}=\{(x_1,x_2,\dots,x_N)\in(\mathfrak{X}_{N,T}^{S,t})^N:x_1<x_2<\dots<x_N\}.$$

$\mathfrak{X}_{N,T}^{S,t}$ is the section of our hexagon by the vertical line with coordinate t, and $\mathcal{X}_{N,T}^{S,t}$ is the set of all N-tuples in this section. Clearly, $X(t)$ takes values in $\mathcal{X}_{N,T}^{S,t}$.

The following theorem gives the one-dimensional distributions of our chain at the time t, which are probability distributions on N-tuples of integers. Denote by $\rho_{S,t}$ the one-dimensional distribution of the process $X(t)$ (below we explain why we keep S in the notation but omit all the other parameters).

Theorem 2.4.1. *We have*

$$\mathrm{Prob}\{X(t)=(x_1,x_2,\dots,x_N)\}=const\cdot\prod_{i<j}(\mu_{t,S}(x_i)-\mu_{t,S}(x_j))^2\prod_{i=1}^{N}w_{t,S}(x_i),$$

where

$$\mu_{t,S}(x)=q^{-x}+\kappa^2q^{x-S-t+1}$$

and

$$w_{t,S}(x)=\frac{(-1)^{t+S}q^{x(2N+T-1)}(1-\kappa^2q^{2x-t-S+1})}{(q;q)_x(q;q)_{T-S-t+x}(q^{-1};q^{-1})_{t+N-x-1}(q^{-1};q^{-1})_{S+N-x-1}}\times\frac{1}{(\kappa^2q^{x-T+1};q)_{T+N-t}(\kappa^2q^{x-t-S+1};q)_{N+t}}.$$

Proof. Clearly,

$$\begin{aligned}&\mathrm{Prob}\{X(t)=(x_1,x_2,\dots,x_N)\}\\&\qquad\qquad\propto L(t;x_1,\dots,x_N)C(t;x_1,\dots,x_N)R(t;x_1,\dots,x_N)\end{aligned}$$

Propositions 2.3.1, 2.3.2, 2.3.3 imply the result. □

Observe that $w_{t,S}(x)$ is (up to the factor not depending on x) the weight function of the q-Racah orthogonal polynomials, see e. g. [52, Section 3.2]. These polynomials are given by the formula

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta\mid q)={}_4\phi_3\left(\begin{matrix}q^{-n},\alpha\beta q^{n+1},q^{-x},\gamma\delta q^{x+1}\\ \alpha q,\beta\delta q,\gamma q\end{matrix}\,\Big|\,q;q\right),\tag{2.7}$$

where

$$\mu(x)=q^{-x}+\gamma\delta q^{x+1},$$

and $\alpha q=q^{-M}$ or $\beta\delta q=q^{-M}$ or $\gamma q=q^{-M}$ for a nonnegative integer M. They are orthogonal on $\{0,1,\dots,M\}$ with respect to the weight function

$$w(x)=\frac{(\alpha q,\beta\delta q,\gamma q,\gamma\delta q;q)_x}{(q,\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_x}\frac{1-\gamma\delta q^{2x+1}}{(\alpha\beta q)^x(1-\gamma\delta q)}.$$

The correspondence between the parameters of polynomials and parameters of our model is established in the following way.

(i). $t<S,\quad T-t-S>0,\quad 0\le x\le t+N-1,$

$$\begin{aligned}q_{(qRacah)}&=q\\ \alpha_{(qRacah)}&=q^{-S-N}\\ \beta_{(qRacah)}&=q^{S-T-N}\\ \gamma_{(qRacah)}&=q^{-t-N}\\ \delta_{(qRacah)}&=\kappa^2q^{-S+N}\end{aligned}\tag{2.8}$$

(ii). $S-1<t<T-S+1,\quad 0\le x\le S+N-1,$

$$\begin{aligned}q_{(qRacah)}&=q\\ \alpha_{(qRacah)}&=q^{-t-N}\\ \beta_{(qRacah)}&=q^{t-T-N}\\ \gamma_{(qRacah)}&=q^{-S-N}\\ \delta_{(qRacah)}&=\kappa^2q^{-t+N}\end{aligned}\tag{2.9}$$

(iii). $T-S-1<t<S,\quad 0\le x-(t+S-T)\le T-S+N-1,$

$$\begin{aligned}q_{(qRacah)}&=q\\ \alpha_{(qRacah)}&=q^{-T+t-N}\\ \beta_{(qRacah)}&=q^{-t-N}\\ \gamma_{(qRacah)}&=q^{-T-N+S}\\ \delta_{(qRacah)}&=\kappa^2q^{-T+t+N}\\ x_{(qRacah)}&=T-t-S+x\end{aligned}\tag{2.10}$$

(iv). $t > T - S - 1, \quad t > S - 1, \quad 0 \le x - (t + S - T) \le T - t + N - 1,$

$$\begin{array}{rcl} q_{(qRacah)} & = & q \\ \alpha_{(qRacah)} & = & q^{-T-N+S} \\ \beta_{(qRacah)} & = & q^{-S-N} \\ \gamma_{(qRacah)} & = & q^{-T+t-N} \\ \delta_{(qRacah)} & = & \kappa^2 q^{-T+N+S} \\ x_{(qRacah)} & = & T - t - S + x \end{array} \tag{2.11}$$

Let us also describe what happens if one performs the limit transitions described in Section 2.2.2.

If we send $\kappa \to 0$ (the weight of a plane partition becomes proportional to $q^{-volume}$), then we obtain the weight function

$$w(x) = \frac{q^{x(2N+T-1)}}{(q;q)_x (q;q)_{T-S-t+x} (q^{-1};q^{-1})_{t+N-x-1} (q^{-1};q^{-1})_{S+N-x-1}}$$

This is exactly (up to the factor not depending on x) the weight function of the q-Hahn polynomials.

If we set $\kappa = q^K$ and send $q \to 1$, then the weight function becomes

$$w(x) = \frac{1}{x!(T-S-t+x)!(t+N-x-1)!(S+N-x-1)!} \times \frac{K+2x-t-S+1}{(K+x-T+1)_{T+N-t}(K+x-t-S+1)_{N+t}}.$$

This is the weight function of the Racah orthogonal polynomials.

Finally, if we send $\kappa \to 0$ and set $q = 1$ (this case corresponds to the uniform measure on the set of lozenge tilings of the hexagon) then we get the weight function

$$w(x) = \frac{1}{x!(T-S-t+x)!(t+N-x-1)!(S+N-x-1)!}.$$

This is the weight function of the Hahn polynomials. This case was previously studied in [35], [39], [40], see also references therein.

We also need the transition probabilities of the Markov chain $X(t)$.

Proposition 2.4.2.

$$\begin{aligned} &\mathrm{Prob}\{X(t+1) = Y | X(t) = X\} \\ &\qquad = const \cdot \prod_{i<j} \frac{\mu_{t+1,S}(y_i) - \mu_{t+1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \prod_{y_i = x_i + 1} w_1(x_i) \prod_{y_i = x_i} w_0(x_i), \end{aligned}$$

where

$$w_0(x) = -(1 - q^{x+T-t-S}) \frac{1 - \kappa^2 q^{x+N-t}}{1 - \kappa^2 q^{2x-t-S+1}}$$

and

$$w_1(x) = q^{T+N-1-t}(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}.$$

Proof. We use

$$\begin{aligned}\mathrm{Prob}\{X(t+1)=Y|X(t)=X\} &= \frac{L_t(X)C_t(X)C_{t+1}(Y)R_{t+1}(Y)}{L_t(X)C_t(X)R_t(X)} \\ &= \frac{R_{t+1}(Y)C_{t+1}(Y)}{R_t(X)}\end{aligned}$$

and Propositions 2.3.3, 2.3.2. □

Next let us compute the *cotransition* probabilities ($t \to t-1$).

Proposition 2.4.3.

$$\begin{aligned}&\mathrm{Prob}\{X(t-1)=Y|X(t)=X\} = \\ &\qquad const \cdot \prod_{i<j} \frac{\mu_{t-1,S}(y_i)-\mu_{t-1,S}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)} \tilde{w}_1(x_i) \prod_{y_i=x_i} \tilde{w}_0(x_i),\end{aligned}$$

where

$$\tilde{w}_0(x) = -(1-q^{x-t-N+1})\frac{1-\kappa^2 q^{x-S-t+1}}{1-\kappa^2 q^{2x-t-S+1}}$$

and

$$\tilde{w}_1(x) = q^{-(t+N-1)}(1-q^x)\frac{1-\kappa^2 q^{x+N-S}}{1-\kappa^2 q^{2x-t-S+1}}$$

Proof. We use

$$\begin{aligned}\mathrm{Prob}\{X(t-1)=Y|X(t)=X\} &= \frac{L_{t-1}(Y)C_{t-1}(Y)C_t(X)R_t(X)}{L_t(X)C_t(X)R_t(X)} \\ &= \frac{L_{t-1}(Y)C_{t-1}(Y)}{L_t(X)}\end{aligned}$$

and Propositions 2.3.1, 2.3.3. □

2.5 Families of stochastic matrices

This section and the next one are similar to [5], where the Hahn case was treated, and we have tried to keep the notations and statements of theorems unchanged where possible.

2.5.1 Definition of matrices

We want to introduce four families of stochastic matrices $P^{S,t}_{t+}$, $P^{S,t}_{t-}$, $P^{S,t}_{S+}$, $P^{S,t}_{S-}$.

$P^{S,t}_{t+}(X,Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S,t+1}|$ matrix, $X = (x_1 < \cdots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \cdots < y_N) \in \mathcal{X}^{S,t+1}$;

if $y_i - x_i \in \{0,1\}$ for every i, then

$$P^{S,t}_{t+}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t+1,S}(y_i) - \mu_{t+1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \prod_{y_i = x_i+1} w_1(x_i) \prod_{y_i = x_i} w_0(x_i),$$

where

$$w_0(x) = -(1 - q^{x+T-t-S}) \frac{1 - \kappa^2 q^{x+N-t}}{1 - \kappa^2 q^{2x-t-S+1}},$$

$$w_1(x) = q^{T+N-1-t}(1 - q^{x-S-N+1}) \frac{1 - \kappa^2 q^{x-T+1}}{1 - \kappa^2 q^{2x-t-S+1}},$$

and $P^{S,t}_{t+}(X,Y) = 0$ otherwise.

$P^{S,t}_{S+}(X,Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S+1,t}|$ matrix, $X = (x_1 < \cdots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \cdots < y_n) \in \mathcal{X}^{S+1,t}$;

If $y_i - x_i \in \{0,1\}$ for every i, then

$$P^{S,t}_{S+}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t,S+1}(y_i) - \mu_{t,S+1}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \prod_{y_i = x_i+1} w_1(x_i) \prod_{y_i = x_i} w_0(x_i),$$

where

$$w_0(x) = -(1 - q^{x+T-t-S}) \frac{1 - \kappa^2 q^{x+N-S}}{1 - \kappa^2 q^{2x-t-S+1}},$$

$$w_1(x) = q^{T+N-1-S}(1 - q^{x-t-N+1}) \frac{1 - \kappa^2 q^{x-T+1}}{1 - \kappa^2 q^{2x-t-S+1}},$$

and $P^{S,t}_{S+}(X,Y) = 0$ otherwise.

$P^{S,t}_{t-}(X,Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S,t-1}|$ matrix, $X = (x_1 < \cdots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \cdots < y_n) \in \mathcal{X}^{S,t-1}$;

If $y_i - x_i \in \{-1,0\}$ for every i, then

$$P^{S,t}_{t-}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t-1,S}(y_i) - \mu_{t-1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \prod_{y_i = x_i-1} \tilde{w}_1(x_i) \prod_{y_i = x_i} \tilde{w}_0(x_i),$$

where

$$\tilde{w}_0(x) = -(1 - q^{x-t-N+1}) \frac{1 - \kappa^2 q^{x-S-t+1}}{1 - \kappa^2 q^{2x-t-S+1}},$$

$$\tilde{w}_1(x) = q^{-(t+N-1)}(1 - q^{x}) \frac{1 - \kappa^2 q^{x+N-S}}{1 - \kappa^2 q^{2x-t-S+1}},$$

and $P^{S,t}_{t-}(X,Y) = 0$ otherwise.

$P^{S,t}_{S-}(X,Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S-1,t}|$ matrix, $X = (x_1 < \cdots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \cdots < y_n) \in \mathcal{X}^{S-1,t}$;

If $y_i - x_i \in \{-1, 0\}$ for every i, then

$$P^{S,t}_{S-}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t,S-1}(y_i) - \mu_{t,S-1}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \prod_{y_i = x_i - 1} \tilde{w}_1(x_i) \prod_{y_i = x_i} \tilde{w}_0(x_i),$$

where

$$\tilde{w}_0(x) = -(1 - q^{x-S-N+1}) \frac{1 - \kappa^2 q^{x-S-t+1}}{1 - \kappa^2 q^{2x-t-S+1}},$$

$$\tilde{w}_1(x) = q^{-(S+N-1)}(1 - q^x) \frac{1 - \kappa^2 q^{x+N-t}}{1 - \kappa^2 q^{2x-t-S+1}},$$

and $P^{S,t}_{S-}(X,Y) = 0$ otherwise.

Looking at the sets that parameterize rows and columns of these matrices one can say that $P^{S,t}_{t+}$ increases t, $P^{S,t}_{t-}$ decreases t, while $P^{S,t}_{S+}$ increases S and $P^{S,t}_{S-}$ decreases S. This explains our notation.

Theorem 2.5.1. *With appropriate choices of normalizing constants, all four types of matrices defined above are stochastic. They preserve the family of measures $\rho_{S,t}$. In other words*

$$\sum_{Y \in \mathcal{X}^{S,t\pm1}} P^{S,t}_{t\pm}(X,Y) = 1, \qquad \sum_{Y \in \mathcal{X}^{S,S\pm1}} P^{S,t}_{t\pm}(X,Y) = 1, \tag{2.12}$$

$$\rho_{S,t\pm1}(Y) = \sum_{X \in \mathcal{X}^{S,t}} P^{S,t}_{t\pm}(X,Y) \cdot \rho_{S,t}(X),$$

$$\rho_{S\pm1,t}(Y) = \sum_{X \in \mathcal{X}^{S,t}} P^{S,t}_{S\pm}(X,Y) \cdot \rho_{S,t}(X).$$

Proof. Propositions 2.4.2 and 2.4.3 imply the claim for $P^{S,t}_{t+}(X,Y)$ and $P^{S,t}_{t-}(X,Y)$.

Now observe that the space $\mathcal{X}^{S,t}$ is unaffected when we interchange parameters t and S, i.e.,

$$\mathcal{X}^{S,t} = \mathcal{X}^{t,S}.$$

Moreover, the measures $\rho_{S,t}$ are also invariant under $S \leftrightarrow t$, i.e.,

$$\rho_{S,t} = \rho_{t,S}.$$

(This is a consequence of our special choice of the parameter κ which included additional factor $q^{-S/2}$.)

Finally, note that $P^{S,t}_{t+}(X,Y)$ becomes $P^{S,t}_{S+}(X,Y)$ under $S \leftrightarrow t$ and $P^{S,t}_{t-}(X,Y)$ becomes $P^{S,t}_{S-}(X,Y)$.

Therefore, applying $S \leftrightarrow t$ to the relations for $P_{t\pm}$ we obtain the needed relations for $P_{S\pm}$. □

2.5.2 Determinantal representation

In this section we write our stochastic matrices in a determinantal form. This representation is very convenient for various computations.

First, we introduce 4 new two-diagonal matrices.

For $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S,t+1}$,

$$U^{S,t}_{t+}(x,y) = \begin{cases} -q^{T+N-1-t}(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x+1, \\ (1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise;} \end{cases}$$

for $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S+1,t}$,

$$U^{S,t}_{S+}(x,y) = \begin{cases} -q^{T+N-1-S}(1-q^{x-t-N+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x+1, \\ (1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-S}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise;} \end{cases}$$

for $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S,t-1}$,

$$U^{S,t}_{t-}(x,y) = \begin{cases} q^{-(t+N-1)}(1-q^{x})\frac{1-\kappa^2 q^{x+N-S}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x-1, \\ -(1-q^{x-t-N+1})\frac{1-\kappa^2 q^{x-S-t+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise;} \end{cases}$$

and for $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S-1,t}$,

$$U^{S,t}_{S-}(x,y) = \begin{cases} q^{-(S+N-1)}(1-q^{x})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x-1, \\ -(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x-S-t+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

It is possible to express the stochastic matrices $P^{S,t}_{t\pm}$, $P^{S,t}_{S\pm}$ as certain minors of the matrices defined above.

Proposition 2.5.2. *We have*

$$P^{S,t}_{t+}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t+1,S}(y_i) - \mu_{t+1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U^{S,t}_{t+}(x_i,y_j)]_{i,j=1,\dots,N}$$

$$P^{S,t}_{S+}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t,S+1}(y_i) - \mu_{t,S+1}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U^{S,t}_{S+}(x_i,y_j)]_{i,j=1,\dots,N}$$

$$P^{S,t}_{t-}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t-1,S}(y_i) - \mu_{t-1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U^{S,t}_{t-}(x_i,y_j)]_{i,j=1,\dots,N}$$

$$P^{S,t}_{S-}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t,S-1}(y_i) - \mu_{t,S-1}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U^{S,t}_{S-}(x_i,y_j)]_{i,j=1,\dots,N}$$

Proof. Straightforward computation using the definitions of the stochastic matrices $P^{S,t}_{t\pm}$, $P^{S,t}_{S\pm}$ and the matrices $U^{S,t}_{t\pm}$, $U^{S,t}_{S\pm}$.

Any submatrix of a two-diagonal matrix, which has a nonzero determinant, is block-diagonal, where each block is either an upper or a lower triangular matrix. Thus, any nonzero minor is a product of suitable matrix elements. □

2.5.3 Commutativity

Theorem 2.5.3. *The families of stochastic matrices $P^{S,t}_{t\pm}$ and $P^{S,t}_{S\pm}$ commute, that is*

$$\begin{aligned}
P^{S,t}_{t+} \cdot P^{S,t+1}_{S-} &= P^{S,t}_{S-} \cdot P^{S-1,t}_{t+},\\
P^{S,t}_{t-} \cdot P^{S,t-1}_{S-} &= P^{S,t}_{S-} \cdot P^{S-1,t}_{t-},\\
P^{S,t}_{t+} \cdot P^{S,t+1}_{S+} &= P^{S,t}_{S+} \cdot P^{S+1,t}_{t+},\\
P^{S,t}_{t-} \cdot P^{S,t-1}_{S+} &= P^{S,t}_{S+} \cdot P^{S+1,t}_{t-},
\end{aligned}$$

for any meaningful values of S and t.

Proof. Proofs of all four cases are very similar and we consider only the first one.

$$\begin{aligned}
(P^{S,t}_{t+} \cdot P^{S,t+1}_{S-})(X,Y) &= \sum_{Z\in\mathcal{X}^{S,t+1}} P^{S,t}_{t+}(X,Z)\cdot P^{S,t+1}_{S-}(Z,Y)\\
&= const \cdot \prod_{i<j} \frac{\mu_{t+1,S-1}(y_i)-\mu_{t+1,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\\
&\quad\times \sum_{Z\in\mathcal{X}^{S,t+1}} \det[U^{S,t}_{t+}(x_i,z_j)]_{i,j=1,\dots,N}\det[U^{S,t+1}_{S-}(z_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}$$

Applying the Cauchy-Binet identity we obtain

$$\begin{aligned}
\sum_{Z\in\mathcal{X}^{S,t+1}} \det[U^{S,t}_{t+}(x_i,z_j)]_{i,j=1,\dots,N}\det[U^{S,t+1}_{S-}(z_i,y_j)]_{i,j=1,\dots,N}&\\
= \det[(U^{S,t}_{t+}\cdot U^{S,t+1}_{S-})(x_i,y_j)]_{i,j=1,\dots,N}&.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(P^{S,t}_{t+} \cdot P^{S,t+1}_{S-})(X,Y)\\
&= const \cdot \prod_{i<j} \frac{\mu_{t+1,S-1}(y_i)-\mu_{t+1,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\det[(U^{S,t}_{t+}\cdot U^{S,t+1}_{S-})(x_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(P^{S,t}_{S-} \cdot P^{S-1,t}_{t+})(X,Y)\\
&= const \cdot \prod_{i<j} \frac{\mu_{t+1,S-1}(y_i)-\mu_{t+1,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\det[(U^{S,t}_{S-}\cdot U^{S-1,t}_{t+})(x_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}$$

Our claim reduces to verifying the equality

$$U_{t+}^{S,t} \cdot U_{S-}^{S,t+1} = U_{S-}^{S,t} \cdot U_{t+}^{S-1,t}.$$

Note that this will also imply the coincidence of normalization constants, since all matrices under consideration are stochastic.

A straightforward computation yields

$$U_{t+S-}^{S,t} = U_{t+}^{S,t} \cdot U_{S-}^{S,t+1} = U_{S-}^{S,t} \cdot U_{t+}^{S-1,t},$$

where

$$U_{t+S-}^{S,t}(x,y) = \begin{cases} u_1, & \text{if } y = x+1, \\ u_0, & \text{if } y = x, \\ u_{-1}, & \text{if } y = x-1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_1 = q^{T+N-1-t}(1-q^{x-S-N+1})(1-q^{x-S-N+2}) \\ \times \frac{(1-\kappa^2 q^{x-S-t+1})(1-\kappa^2 q^{x-T+1})}{(1-\kappa^2 q^{2x-t-S+2}))1-\kappa^2 q^{2x-t-S+1})},$$

$$u_0 = -(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}} \\ \times \left(q^{T-S-t}(1-q^{x+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+2}} + (1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x-S-t}}{1-\kappa^2 q^{2x-t-S}}\right),$$

$$u_{-1} = q^{-(S+N-1)}(1-q^{x+T-t-S})(1-q^x)\frac{(1-\kappa^2 q^{x+N-t})(1-\kappa^2 q^{x+N-t-1})}{(1-\kappa^2 q^{2x-t-S+1})(1-\kappa^2 q^{2x-t-S})}.$$

□

2.6 Perfect sampling algorithm

2.6.1 Definition of transition matrices

In this section we aim to define two new stochastic matrices

$$P_{S\mapsto S+1}^{S}(X,Y), \quad X \in \Omega(N,T,S), \quad Y \in \Omega(N,T,S+1)$$

and

$$P_{S\mapsto S-1}^{S}(X,Y), \quad X \in \Omega(N,T,S), \quad Y \in \Omega(N,T,S-1)$$

that preserve the measures $\mu(N,T,S,q,\kappa)$. Both $P_{S\mapsto S+1}^{S}$ and $P_{S\mapsto S-1}^{S}$ depend on parameters N, T, q, κ but we omit these parameters from the notation.

Suppose we are given a sequence $X = (X(0), X(1), \ldots, X(T)) \in \Omega(N,T,S)$ (recall that $X(t) \in \mathcal{X}^{S,t}$). Below we construct a random sequence $Y =$

$(Y(0),\dots,Y(T)) \in \Omega(N,T,S+1)$ and therefore define the transition probability (or, equivalently, stochastic matrix) $P^S_{S\mapsto S+1}(X,Y)$.

First note that $Y(0) \in \mathcal{X}^{S+1,0}$ and $|\mathcal{X}^{S+1,0}| = 1$. Thus, $Y(0)$ is uniquely defined. We will perform a *sequential update*. Suppose $Y(0), Y(1), \dots, Y(t)$ have been already defined. Define the conditional distribution of $Y(t+1)$ given X, $Y(0), Y(1), \dots, Y(t)$ by

$$\begin{aligned}\mathrm{Prob}\{Y(t+1)=Z\} &= \frac{P^{S+1,t}_{t+}(Y(t),Z)\cdot P^{S+1,t+1}_{S-}(Z,X(t+1))}{(P^{S+1,t}_{t+}P^{S+1,t+1}_{S-})(Y(t),X(t+1))}\\ &= \frac{P^{S,t+1}_{S+}(X(t+1),Z)\cdot P^{S+1,t+1}_{t-}(Z,Y(t))}{(P^{S,t+1}_{S+}P^{S+1,t+1}_{t-})(X(t+1),Y(t))}. \qquad (2.13)\end{aligned}$$

(The second equality follows from $\rho_{S+1,t+1}(X)P^{S+1,t+1}_{t-}(X,Y) = \rho_{S+1,t}(Y)P^{S+1,t}_{t+}(Y,X)$.)

This definition follows the idea of [21, Section 2.3], see also [4].

Observe that $(P^{S+1,t}_{t+}P^{S+1,t+1}_{S-})(Y(t),X(t+1)) > 0$ (here and below see [5] for more details).

One could say that we choose $Y(t+1)$ using conditional distribution of the middle point in the successive application of $P^{S+1,t}_{t+}$ and $P^{S+1,t+1}_{S-}$ (or $P^{S,t+1}_{S+}$ and $P^{S+1,t+1}_{t-}$), provided that we start at $Y(t)$ and finish at $X(t+1)$ (or start at $X(t+1)$ and finish at $Y(t)$).

After performing T updates we obtain the sequence Y.

Equivalently, define $P^S_{S\mapsto S+1}$ by

$$P^S_{S\mapsto S+1}(X,Y) = \begin{cases} \prod\limits_{t=0}^{T-1} \dfrac{P^{S+1,t}_{t+}(Y(t),Y(t+1))P^{S+1,t+1}_{S-}(Y(t+1),X(t+1))}{(P^{S+1,t}_{t+}P^{S+1,t+1}_{S-})(Y(t),X(t+1))}, \\ \qquad \text{if } \prod\limits_{t=0}^{T-1}(P^{S+1,t}_{t+}P^{S+1,t+1}_{S-})(Y(t),X(t+1)) > 0, \\ 0, \text{ otherwise.}\end{cases}$$

Theorem 2.6.1. *The matrix $P^S_{S\mapsto S+1}$ on $\Omega(N,T,S)\times\Omega(N,T,S+1)$ is stochastic. The transition probabilities $P^S_{S\mapsto S+1}(X,Y)$ preserve the measures $\mu(N,T,S,q,\kappa)$:*

$$\mu(N,T,S+1,q,\kappa)(Y) = \sum_{X\in\Omega(N,T,S)} P^S_{S\mapsto S+1}(X,Y)\mu(N,T,S,q,\kappa)(X).$$

Proof. See [5]. □

Similarly to $P_{S\mapsto S+1}$, one defines a transition matrix

$$P^S_{S\mapsto S-1}(X,Y), \quad X\in\Omega(N,T,S), \quad Y\in\Omega(N,T,S-1),$$

by

$$P^S_{S\mapsto S-1}(X,Y)=\begin{cases}\prod\limits_{t=0}^{T-1}\dfrac{P^{S-1,t}_{t+}(Y(t),Y(t+1))P^{S-1,t+1}_{S+}(Y(t+1),X(t+1))}{(P^{S-1,t}_{t+}P^{S-1,t+1}_{S+})(Y(t),X(t+1))},\\ \qquad\text{if } \prod\limits_{t=0}^{T-1}(P^{S-1,t}_{t+}P^{S-1,t+1}_{S+})(Y(t),X(t+1))>0,\\ 0,\ \text{otherwise.}\end{cases}$$

Similarly to (2.13) there is another way to write $P^S_{S\mapsto S-1}$ because of the equality

$$\frac{P^{S-1,t}_{t+}(Y(t),Y(t+1))\cdot P^{S-1,t+1}_{S+}(Y(t+1),X(t+1))}{(P^{S-1,t}_{t+}P^{S-1,t+1}_{S+})(Y(t),X(t+1))}$$
$$=\frac{P^{S,t+1}_{S-}(X(t+1),Y(t+1))\cdot P^{S-1,t+1}_{t-}(Y(t+1),Y(t))}{(P^{S,t+1}_{S-}P^{S-1,t+1}_{t-})(X(t+1),Y(t))}$$

Similarly to Theorem 2.6.1 one proves the following claim.

Theorem 2.6.2. *The matrix $P^S_{S\mapsto S-1}$ on $\Omega(N,T,S)\times\Omega(N,T,S-1)$ is stochastic. The transition probabilities $P^S_{S\mapsto S-1}(X,Y)$ preserve the measures $\mu(N,T,S,q,\kappa)$:*

$$\mu(N,T,S-1,q,\kappa)(Y)=\sum_{X\in\Omega(N,T,S)}P^S_{S\mapsto S-1}(X,Y)\mu(N,T,S,q,\kappa)(X).$$

Remark. The above construction performs a sequential update from $t=0$ to $t=T$. One can equally well update from $t=T$ to $t=0$ by suitably modifying the definitions. The resulting Markov chains also preserve the measures $\mu(N,T,S,q,\kappa)$, and they are different from the Markov chains defined above.

2.6.2 Algorithm for the $S\mapsto S+1$ step.

Now we suggest an algorithmic description of the Markov chain from the previous section.

Denote

$$p(x,t,q,\kappa,S,T)=\frac{1-q^{x+T-t-S-1}}{q^{T-t-S-1}(1-q^{x+1})}\frac{1-\kappa^2q^{x-S-t-1}}{1-\kappa^2q^{x-T+1}}\frac{1-\kappa^2q^{2x-t-S+1}}{1-\kappa^2q^{2x-t-S-1}}$$

and

$$P(x,t,q,\kappa,S,T;k)=\prod_{i=1}^{k}p(x+i-1,t,q,\kappa,S,T)$$
$$=\frac{(q^{x+T-t-S-1};q)_k}{q^{k(T-t-S-1)}(q^{x+1};q)_k}\frac{(\kappa^2q^{x-S-t-1};q)_k}{(\kappa^2q^{x-T+1};q)_k}\frac{(\kappa^2q^{2x-t-S+1};q^2)_k}{(\kappa^2q^{2x-t-S-1};q^2)_k}.$$

Denote by $D(x,t,S;n)$ (it also depends on q,κ,T, but we omit these parameters) the probability distribution on $\{0,1,\dots,n\}$ given by

$$\mathrm{Prob}(\{k\}) = D(x,t,S;n)\{k\} = \frac{P(x,t,q,\kappa,S,T;k)}{\sum_{j=0}^{n} P(x,t,q,\kappa,S,T;j)}\,. \tag{2.14}$$

Suppose we are given $X=(X(0),X(1),\dots,X(T))\in\Omega(N,T,S)$. We want to construct $Y=(Y(0),Y(1),\dots,Y(T))\in\Omega(N,T,S+1)$.

In the first place we note that $Y(0)$ is uniquely defined,

$$Y(0)=(0,1,\dots,N-1).$$

Then we perform T sequential updates, i.e., for $t=0,1,\dots T-1$ we construct $Y(t+1)$ using $Y(t)$ and $X(t+1)$. Let us describe each step.

Let $Y(t)=(y_1<y_2<\dots<y_N)$ and $X(t+1)=(x_1<x_2<\dots<x_N)$. We are going to construct $Y(t+1)=(z_1<z_2<\dots<z_N)$.

Recall that

$$z_i\in\mathfrak{X}^{S+1,t+1}=\{x\in\mathbb{Z}\mid \max(0,t+S-T+2)\le x\le \min(t+N,S+N)\}.$$

Observe that $Y(t)$ and $X(t+1)$ satisfy $(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))>0$. This implies that x_i-y_i is equal to either -1, 0 or 1 for every i.

• First, consider all indices i such that $x_i-y_i=1$. For every such i we set $z_i=x_i$.

• Second, consider all indices i such that $x_i-y_i=-1$ and set $z_i=y_i$.

• Finally, consider all remaining indices, i.e., all i such that $x_i=y_i$. Divide the corresponding x_i's into blocks of neighboring integers of distance at least one from each other. Call such a block a (k,l)-block, where k is the smallest number in the block and l is its size. Thus, we have

$$x_i=y_i=k,\quad x_{i+1}=y_{i+1}=k+1,\quad \dots,\quad x_{i+l-1}=y_{i+l-1}=k+l-1$$

and

$$y_{i-1}<k-1,\quad y_{i+l}>k+l.$$

For each (k,l)-block we perform the following procedure: consider a random variable ξ distributed according to $D(k,t,S;l)$ (ξ's corresponding to different (k,l)-blocks are independent). Set $z_i=x_i$ for the first ξ integers of the block (their coordinates are $k,k+1,\dots,k+\xi-1$) and set $z_i=x_i+1$ for the rest of the block.

At Figure 2.4 we provide an example of constructing $Y(t+1)$ using $X(t+1)$ and $Y(t)$: there is only one (k,l)-block and it splits into two groups, here $\xi=2$.

Theorem 2.6.3. *The algorithm described above is precisely the $S\mapsto S+1$ Markov step given by $P^{S}_{S\mapsto S+1}$.*

Proof. Straightforward computations. See [5] for some details. □

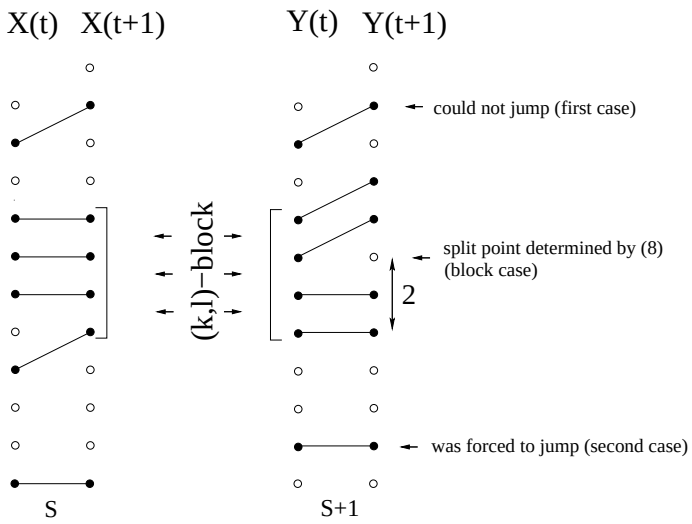


Figure 2.4: Example of (k, l)-block split, $l = 4$, $\xi = 2$.

Remarks. Setting $\kappa = 0$ in the formulas for the distribution $D(x, t, S; n)$ we obtain the perfect sampling algorithm for boxed plane partitions distributed as $q^{-volume}$.

Sending $q \to 1$ in the formulas for the distribution $D(x, t, S; n)$ as described in Section 2.2.2, we get a perfect sampling algorithm for the Racah case (recall that in this case the weight of a horizontal lozenge is proportional to a linear function of its vertical coordinate).

2.6.3 Algorithm for $S \mapsto S - 1$ step

Using similar methods we can also obtain $S \mapsto S - 1$ Markov step which gives alternative way to sample a random tiling: We start from the case $T = S$ and then perform some amount of $S \mapsto S - 1$ steps.

The $S \mapsto S - 1$ step algorithm is very similar to the $S \mapsto S + 1$ one.

Denote

$$\hat{p}(x, t, q, \kappa, S, T, N) = \frac{q^{t+1-S}(1 - q^{x-t-N-1})}{(1 - q^{x-S-N+1})} \times \frac{1 - \kappa^2 q^{x+N-t-1}}{1 - \kappa^2 q^{x+N-S+1}} \frac{1 - \kappa^2 q^{2x-t-S+1}}{1 - \kappa^2 q^{2x-t-S-1}}$$

and

$$\hat{P}(x,t,q,\kappa,S,T,N;k) = \prod_{i=1}^{k} \hat{p}(x+i-1,t,q,\kappa,S,T,N)$$
$$= \frac{q^{k(t+1-S)}(q^{x-t-N-1};q)_k}{(q^{x-S-N+1};q)_k} \frac{(\kappa^2 q^{x+N-t-1};q)_k}{(\kappa^2 q^{x+N-S+1};q)_k} \frac{(\kappa^2 q^{2x-t-S+1};q^2)_k}{(\kappa^2 q^{2x-t-S-1};q^2)_k}.$$

Denote by $\hat{D}(x,t,S;n)$ the probability distribution on $\{0,1,\dots,n\}$ given by

$$\mathrm{Prob}(\{k\}) = \hat{D}(x,t,S;n)\{k\} = \frac{\hat{P}(x,t,q,\kappa,S,T,N;k)}{\sum_{j=0}^{n} \hat{P}(x,t,q,\kappa,S,T,N;j)}. \tag{2.15}$$

Suppose we are given $X = (X(0), X(1), \dots, X(T)) \in \Omega(N,T,S)$. We want to construct $Y = (Y(0), Y(1), \dots, Y(T)) \in \Omega(N,T,S-1)$.

As above, note that $Y(0)$ is uniquely defined,

$$Y(0) = (0, 1, \dots, N-1).$$

Then we again perform T sequential updates, i.e., for $t = 0, 1, \dots T-1$ we construct $Y(t+1)$ using $Y(t)$ and $X(t+1)$. Let us describe each step.

Let $Y(t) = (y_1 < y_2 < \dots < y_N)$ and $X(t+1) = (x_1 < x_2 < \dots < x_N)$. We are going to construct $Y(t+1) = (z_1 < z_2 < \dots < z_N)$.

Recall that

$$z_i \in \mathfrak{X}^{S-1,t+1} = \{x \in \mathbb{Z} \mid \max(0, t+S-T) \le x \le \min(t+N, S+N-2)\}.$$

$Y(t)$ and $X(t+1)$ satisfy $(P_{t+}^{S-1,t} P_{S+}^{S-1,t+1})(Y(t), X(t+1)) > 0$. This implies that $x_i - y_i$ is equal to either 0, 1 or 2 for every i.

• First, consider all indices i such that $x_i - y_i = 0$. For every such i we set $z_i = x_i$.

• Second, consider all indices i such that $x_i - y_i = 2$ and set $z_i = y_i + 1$.

• Finally, consider all remaining indices, i.e., all i such that $x_i = y_i + 1$. Divide the corresponding x_i's into blocks of neighboring integers of distance at least one from each other. Call such a block a $(k,l)'$-block, where k is the smallest number in the block and l is its size. Thus, we have

$$x_i = y_i + 1 = k, \quad x_{i+1} = y_{i+1} + 1 = k+1, \quad \dots, \quad x_{i+l-1} = y_{i+l-1} = k+l-1.$$

For each $(k,l)'$-block we perform the following procedure: consider random variable ξ distributed according to $\hat{D}(k,t,S;l)$ (ξ's corresponding to different $(k,l)'$-blocks are independent). Set $z_i = y_i$ for the first ξ integers of the block (their coordinates are $k-1, k, \dots, k+\xi-2$) and set $z_i = y_i + 1$ for the rest of the block.

Theorem 2.6.4. *The algorithm described above is precisely $S \mapsto S-1$ Markov step defined by $P^S_{S\mapsto S-1}$.*

The proof is similar to Theorem 2.6.3.

2.6.4 Markov evolution of the top path

The $S \mapsto S+1$ Markov step described in the previous section has the following property: Its projection to the set of topmost horizontal lozenges (or the topmost holes in terms of nonintersecting paths and point configurations) is also a Markov chain. This Markov chain is an exclusion type process. Let us describe it.

The general setting is as follows. The state space of our discrete time Markov chain consists of semi-infinite particle configurations $\{e_1 < e_2 < e_3 < \dots\}$ in $\mathbb{Z}$. At each time moment every particle either stays or jumps to the left (any distance) avoiding collisions and jumps over neighbors. Jumps are performed sequentially. First, the leftmost particle (e_1) jumps, then the second one and so on. The distribution D of the length of the jump of a particle depends on the number of the particle, moment of time, current position of the particle (e_i) and the distance between the current position of the particle and the position of the previous particle (e_{i-1}) in the next moment of time. At time 0 we have the step initial condition, i.e., $e_i = i + const$.

Now let us turn back to our situation. All particles are enumerated by the parameter t and our time parameter is S that changes from 0 to T. Consider a sequence $\{u_t^S\}_{t=1,\dots}$, where u_t^S is the vertical coordinate (in our notation - x) corresponding to the topmost hole inside the hexagon for $t \le S$ and $u_t^S = N+t-1$ for $t > S$. (We can also view u_t^S as the vertical coordinate corresponding to the tth hole, if we count all holes, not just the ones inside a hexagon, starting from the line $x = 0$.)

The evolution of $\{u_t^S\}$ is precisely our Markov process. When $S = 0$ the configuration consists of points $N, N+1, N+2, \dots$. The distribution of the length of jump of the particle with coordinate u_t^S at the time moment S is given by the distribution $D(u_{t-1}^{S+1} + 1, t, S; u_t^S - u_{t-1}^{S+1} - 1)$ (see (2.14) for the definition).

Note that when $S = T$ the configuration consists of points $0, 1, 2, \dots$.

We can also obtain in a similar way a Markov chain for the bottommost holes.

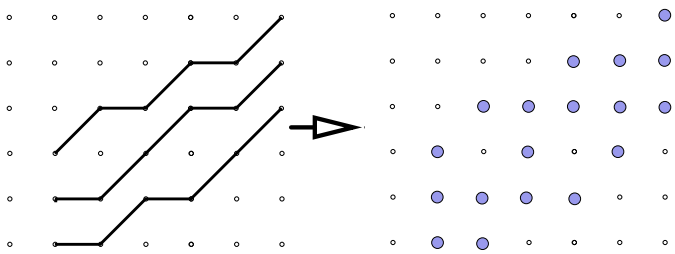
Finally, we may construct two more similar processes using the $S \mapsto S - 1$ Markov chain (for this chains the direction of particle jumps changes and distributions D are replaced by distributions $\hat{D}$).

2.7 Correlation kernel

The aim of this section is to obtain the formulas for the correlation functions of random point configurations in $\mathbb{Z}^2$ obtained from the random tilings we are interested in.

2.7.1 Expression via orthogonal polynomials

Recall that a tiling of a hexagon corresponds to some family of nonintersecting paths that can be viewed as a point configuration in $\mathbb{Z}^2$. Let us denote this configuration by $\mathbb{M}$.



As above, we denote the horizontal coordinate by t and the vertical coordinate by x.

We want to compute the correlation functions of this random point configuration.

Recall that the nth correlation function is defined by

$$\rho_n(t_1, x_1; \ldots; t_n, x_n) = \mathrm{Prob}\{(t_1, x_1) \in \mathbb{M}, \ldots, (t_n, x_n) \in \mathbb{M}\}$$

for any collection $\{(t_i, x_i)\}_{i=1,\ldots,n}$ of distinct points in $\mathbb{Z}^2$.

To compute the correlation functions ρ_n we are going to use a variant of the Eynard-Mehta theorem (see [25] and [12, Section 7.4]). Let us state it first.

Proposition 2.7.1. *Assume that for every time moment t we are given an orthonormal system $\{f_n^t\}_{n\geq 0}$ in $l_2(\{0, 1, \ldots, L\})$ and a set of numbers $c_0^t, c_1^t, \ldots$. Denote*

$$v_{t,t+1}(x, y) = \sum_{n\geq 0} c_n^t f_n^t(x) f_n^{t+1}(y).$$

Assume also that we are given a discrete time Markov process $\mathcal{P}_t$ taking values in N-tuples of elements of the set $\{0, 1, \ldots, L\}$, with one-dimensional distributions

$$\left(\det\left[f_{i-1}^t(x_j)\right]_{i,j=1,\ldots,N}\right)^2$$

and transition probabilities

$$\frac{\det\left[v_{t,t+1}(x_i, y_j)\right]_{i,j=1,\ldots,N} \det\left[f_{i-1}^{t+1}(y_j)\right]_{i,j=1,\ldots,N}}{\det\left[f_{i-1}^t(x_j)\right]_{i,j=1,\ldots,N} \prod_{n=0}^{N-1} c_n^t}.$$

Then

$$\mathrm{Prob}\{x_1 \in \mathcal{P}_{k_1}, \ldots, x_n \in \mathcal{P}_{k_n}\} = \det\left[K(k_i, x_i; k_j, x_j)\right]_{i,j=1,\ldots,n},$$

where

$$K(k,x;l,y) = \sum_{i=0}^{N-1} \frac{1}{c_i^{l,k}} f_i^k(x) f_i^l(y),\ k \geq l;$$
$$K(k,x;l,y) = -\sum_{i \geq N} c_i^{k,l} f_i^k(x) f_i^l(y),\ k < l;$$
$$c_i^{k,k} = 1,\ c_i^{k,l} = c_i^k \cdot c_i^{k+1} \cdot \dots \cdot c_i^{l-1}.$$

Theorem 2.7.2. *The Markov process $X(t)$ meets the assumptions of Proposition 2.7.1.*

The Markov process $\mathcal{P}_t$ is precisely our Markov process $X(t)$. The orthonormal functions $f_n^t(x)$ are the normalized q-Racah polynomials multiplied by the square root of their weight function (see Section 2.4 for the definition of q-Racah polynomials, their weight function, and the correspondence between parameters of these polynomials and our parameters t, q, κ, N, T, S):

$$f_n^t(x) = \sqrt{w_{t,S}(x)} \frac{R_n^t(x)}{\sqrt{(R_n^t, R_n^t)}}, \tag{2.16}$$

where (R_n^t, R_n^t) is the squared norm of the q-Racah polynomials with respect to the weight function $w_{t,S}(x)$. This norm can be obtained from the norm of the q-Racah polynomials provided in [52] ($w_{t,S}(x)$ differs from the weight function of [52] by a factor not depending on x). The explicit formula is a little bit different in the four cases of correspondence between parameters of polynomials and t, q, κ, N, T, S. For instance, in the case given by formula (2.8):

$$\begin{aligned}(R_n^t, R_n^t) = {} & \frac{(-1)^{t+S}(q^{-2N-T+2}, k^{-2}q^{S-N}; q)_{t+N-1}}{(\kappa^{-2}q^{-2N+1}, q^{S-T-N+1}, \kappa^2 q^{-t-S+2}, q^{-t-N+1}; q)_{t+N-1}} \\ & \times \frac{(1-q^{-T-2N})(q, q^{-T-N+t+1}, \kappa^{-2}q^{-2N+1}, q^{S-T-N}; q)_n}{(1-q^{-2N-T+2n+1})(q^{-S-N}, q^{-2N-T+1}, \kappa^2 q^{-T}, q^{-t-N+1}; q)_n} \\ & \times \frac{\kappa^{2n} q^{-n(S+t+1)}}{(q;q)_{T-S-t}(q^{-S-N+1}; q)_{S+N-1}(\kappa^2 q^{-T+1}; q)_{T+N-t}}.\end{aligned}$$

However, this long formula is not important for us, since factors involving it always cancel out. In particular, in the case given by (2.8) the quotient $(R_n^{t+1}, R_n^{t+1})/(R_n^t, R_n^t)$, that is crucial for us, is simply

$$\frac{(R_n^{t+1}, R_n^{t+1})}{(R_n^t, R_n^t)} = -\frac{(1-q^{T+N-t-n-1})(1-q^{-t-N+n})}{(1-q^{-t-N})^2}.$$

The constants c_n^t are given by

$$c_n^t = \sqrt{(1-q^{-N-t+n})(1-q^{T+N-t-n-1})}. \tag{2.17}$$

Proof. Theorem 2.4.1 yields

$$\mathrm{Prob}\{X(t) = (x_1, x_2, \dots, x_N)\} = \frac{1}{Z} \prod_{i<j} (\mu_{t,S}(x_i) - \mu_{t,S}(x_j))^2 \prod_{i=1}^{N} w_{t,S}(x_i).$$

On the other hand

$$\det \left[f_{i-1}^t(x_j) \right]_{i,j=1,\dots,N} = const \cdot \prod_{i=1}^{N} \sqrt{w_{t,S}(x_i)} \det \left[R_{i-1}(x_j) \right]_{i,j=1,\dots,N}.$$

The last determinant is a Vandermonde determinant in variables $\mu_{t,S}(x_j)$, hence

$$\left(\det \left[f_{i-1}^t(x_j) \right]_{i,j=1,\dots,N} \right)^2 = \frac{1}{Z'} \prod_{i<j} (\mu_{t,S}(x_i) - \mu_{t,S}(x_j))^2 \prod_{i=1}^{N} w_{t,S}(x_i).$$

Coincidence of the constants ($Z = Z'$) follows from the fact that the left-hand side in the last equality defines a probability distribution.

Thus, the one-dimensional distributions of our process have the required form.

Next, we need the following standard facts.

Lemma 2.7.3. *The following relation for the basic hypergeometric function holds:*

$$(c-w)(1-d)\,{}_4\phi_3 \left(\begin{matrix} a, b, c, qd \\ u, v, qw \end{matrix} \middle| q; q \right) + (w-d)(1-c)\,{}_4\phi_3 \left(\begin{matrix} a, b, qc, d \\ u, v, qw \end{matrix} \middle| q; q \right)$$
$$= (c-d)(1-w)\,{}_4\phi_3 \left(\begin{matrix} a, b, c, d \\ u, v, w \end{matrix} \middle| q; q \right). \quad (2.18)$$

In terms of q-Racah polynomials, this relation can be rewritten as

$$(q^{-x} - q\gamma)(1 - \gamma\delta q^{x+1}) R_n \Big(\mu(x); \alpha, \beta, q\gamma, \delta \mid q \Big)$$
$$+ (q\gamma - \gamma\delta q^{x+1})(1 - q^{-x}) R_n \Big(\mu(x-1); \alpha, \beta, q\gamma, \delta \mid q \Big)$$
$$= (q^{-x} - \gamma\delta q^{x+1})(1 - q\gamma) R_n \Big(\mu(x); \alpha, \beta, \gamma, \delta \mid q \Big) \quad (2.19)$$

or as

$$(q^{-x} - q\alpha)(1 - \gamma\delta q^{x+1}) R_n \Big(\mu(x); q\alpha, q^{-1}\beta, \gamma, q\delta \mid q \Big)$$
$$+ (q\alpha - \gamma\delta q^{x+1})(1 - q^{-x}) R_n \Big(\mu(x-1); q\alpha, q^{-1}\beta, \gamma, q\delta \mid q \Big)$$
$$= (q^{-x} - \gamma\delta q^{x+1})(1 - q\alpha) R_n \Big(\mu(x); \alpha, \beta, \gamma, \delta \mid q \Big), \quad (2.20)$$

where R_n is given by (2.7).

For the balanced terminating ${}_4\phi_3\left(\begin{smallmatrix} a,b,c,d \\ u,v,w \end{smallmatrix}\middle| q;q\right)$ (i.e., one of a, b, c or d equals q^{-n} and $a\cdot b\cdot c\cdot d = u\cdot v\cdot w$) we also have the following relation:

$$\begin{aligned}(c-u)(1-vc^{-1})(wq^{-1}-1){}_4\phi_3\left(\begin{matrix} a,b,q^{-1}c,d \\ u,v,q^{-1}w \end{matrix}\middle| q;q\right)\\ +(u-d)(1-vd^{-1})(wq^{-1}-1){}_4\phi_3\left(\begin{matrix} a,b,c,q^{-1}d \\ u,v,q^{-1}w \end{matrix}\middle| q;q\right)\\ =(c-d)(wq^{-1}-b)(1-aqw^{-1}){}_4\phi_3\left(\begin{matrix} a,b,c,d \\ u,v,w \end{matrix}\middle| q;q\right). \end{aligned} \tag{2.21}$$

In terms of q-Racah polynomials, the last relation can be rewritten as

$$\begin{aligned}&(q^{-x}-q\gamma)(1-\beta\delta q^{x+1})(\alpha-1)R_n\Big(\mu(x+1);q^{-1}\alpha,q\beta,\gamma,q^{-1}\delta\mid q\Big)\\ &\quad+(q\gamma-\gamma\delta q^{x+1})(1-q^{-x}\beta\gamma^{-1})(\alpha-1)R_n\Big(\mu(x);q^{-1}\alpha,q\beta,\gamma,q^{-1}\delta\mid q\Big)\\ &=(q^{-x}-\gamma\delta q^{x+1})(\alpha-\alpha\beta q^{n+1})(1-\alpha^{-1}q^{-n})R_n\Big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\Big)\end{aligned} \tag{2.22}$$

or as

$$\begin{aligned}&(q^{-x}-q\alpha)(1-\beta\delta q^{x+1})(\gamma-1)R_n\Big(\mu(x+1);\alpha,\beta,q^{-1}\gamma,\delta\mid q\Big)\\ &\quad+(q\alpha-\gamma\delta q^{x+1})(1-q^{-x}\beta\gamma^{-1})(\gamma-1)R_n\Big(\mu(x);\alpha,\beta,q^{-1}\gamma,\delta\mid q\Big)\\ &=(q^{-x}-\gamma\delta q^{x+1})(\gamma-\alpha\beta q^{n+1})(1-\gamma^{-1}q^{-n})R_n\Big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\Big),\end{aligned} \tag{2.23}$$

Proof. To prove the first relation for the basic hypergeometric function we expand ${}_4\phi_3$ into series in q and perform straightforward computations in every term.

To obtain (2.19) and (2.20) we simply rewrite the relation (2.18) in terms of q-Racah polynomials using their definition (2.7).

Next, we observe that q-Racah polynomials form an orthogonal basis in the corresponding l_2 space. Consequently, we can write dual relations for (2.19) and (2.20) and these are precisely (2.22) and (2.23). It is easily seen that two last relations are equivalent to just one relation for basic hypergeometric function (2.21). □

Using the last lemma we obtain the following one:

Lemma 2.7.4.

$$v_{t,t+1}(x,y) \;=\; \sum_{n\geq 0} c_n^t f_n^t(x) f_n^{t+1}(y) \;=\; \sqrt{\frac{w_{t,S}(x)}{w_{t+1,S}(y)}}\left(\delta_{x+1}^y w_1(x) + \delta_x^y w_0(x)\right),$$

where

$$w_0(x) = -(1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}},$$

$$w_1(x) = q^{T+N-1-t}(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}},$$

while $w_{t,S}(x)$ stands for the weight function corresponding to the parameters t, q, κ, N, T, S (see Section 2.4 and Theorem 2.4.1 for details).

Proof. First, we substitute the parameters of q-Racah polynomials given by formulas (2.8)-(2.11) into the statement of Lemma 2.7.3.

We use (2.19), (2.20) in cases (2.8), (2.9), and we use (2.22), (2.23) in cases (2.10), (2.11).

In all 4 cases we rewrite the corresponding relation in terms of orthogonal functions $f_n^t(x)$ and get the following

$$c_n^t f_n^{t+1}(y) = \sqrt{\frac{w_{t,S}(y-1)}{w_{t+1,S}(y)}} f_n^t(y-1) w_1(y-1) + \sqrt{\frac{w_{t,S}(y)}{w_{t+1,S}(y)}} f_n^t(y) w_0(y) \quad (2.24)$$

Multiply the last relation by $f_n^t(x)$ and sum over all meaningful n.

Since functions $f_n^t(y)$ form an orthonormal basis in the corresponding l_2 space,

$$\sum_n f_n^t(x) f_n^t(y) = \delta_x^y,$$

and the needed relation follows. □

Proposition 2.5.2 implies that the transition probabilities $P_{t+}^{S,t}(X,Y)$ have a determinantal form. The last lemma yields that this form is exactly the one required for the application of Proposition 2.7.1. Thus, the theorem is proved. □

Applying Proposition 2.7.1 for the process $X(t)$ we obtain the following statement.

Theorem 2.7.5.

$$\rho_n(t_1, x_1; \ldots; t_n, x_n) = \det\left[K(k_i, x_i; k_j, x_j)\right]_{i,j=1,\ldots,n},$$

where

$$K(k,x;l,y)=\sum_{i=0}^{N-1}\frac{1}{c_i^{l,k}}f_i^k(x)f_i^l(y),\ k\geq l;$$
$$K(k,x;l,y)=-\sum_{i\geq N}c_i^{k,l}f_i^k(x)f_i^l(y),\ k<l;$$
$$c_i^{k,k}=1,\ c_i^{k,l}=c_i^k\cdot c_i^{k+1}\cdot\dots\cdot c_i^{l-1}$$

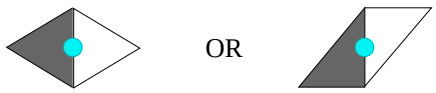
and functions $f_i^k(x)$ *and numbers* c_i^t *are given by the formulas* (2.16) *and* (2.17).

2.7.2 Inverse Kasteleyn matrix

Let us present another way to view the correlation kernel derived in the previous section.

Recall that we deal with lozenge tilings of a hexagon. Divide every lozenge into two unit triangles and color the resulting triangles into black and white (west triangle is black). In this way a tiling turns into a perfect matching of the part of the dual hexagonal lattice that fits in our hexagon. Correlation functions of the perfect matchings can be computed using Kasteleyn's theorem (see [42]). Let us describe it.

Associate to every triangle the midpoint of its vertical side. Note that in this way both black and white triangles can be parameterized by the points of the two-dimensional lattice. Thus, we can use our usual coordinates (t,x) for the triangles.



The Kasteleyn matrix $\mathrm{Kast}(t,x;r,y)$ is a weighted adjacency matrix. Here (t,x) stand for the coordinates of a white triangle and (r,y) stand for the coordinates of a black triangle. In our case,

$$\mathrm{Kast}(t,x;r,y)=\begin{cases}\kappa q^{-\frac{S}{2}+x-\frac{t}{2}+1/2}-\dfrac{1}{\kappa q^{-\frac{S}{2}+x-\frac{t}{2}+\frac{1}{2}}}, & (t,x)=(r,y),\\ 1, & (t,x)=(r-1,y-1),\\ 1, & (t,x)=(r-1,y),\\ 0, & \text{otherwise.}\end{cases}$$

Set

$$g(t,x)=\frac{1}{\sqrt{w_{t,S}(x)}}$$
$$\times\frac{(-1)^{t+x}\kappa^{-t}q^{x(T+N-t-1)+t(S/2-1/2)+t(t+1)/4}(1-\kappa^2q^{2x-t-S+1})}{(q^{-1};q^{-1})_{S+N-1-x}(q;q)_{T-S+x-t}(\kappa^2q^{x-T+1};q)_{T+N-t}}.\quad(2.25)$$

Theorem 2.7.6. *Consider n lozenges enumerated by pairs of triangles $((t_i, x_i), (r_i, y_i))$. The probability that a random tiling contains these lozenges equals*

$$\prod_{i=1}^{n} \mathrm{Kast}(t_i, x_i; r_i, y_i) \cdot \det \left[K^{ext}(r_i, y_i; t_j, x_j) \right]_{i,j=1,\dots,n},$$

where

$$K^{ext}(r, y; t, x) = \frac{g(r, y)}{g(t, x)} \left(\delta_{(r,y)}^{(t,x)} - K(r, y; t, x) \right),$$

the function g is given by (2.25)*, and $K(r, y; t, x)$ is given in Theorem 2.7.5.*

Proof. Kasteleyn's theorem states that the probability to find lozenges

$$((t_1, x_1); (r_1, y_1)), \dots, ((t_n, x_n); (r_n, y_n))$$

can be expressed via the inverse of the Kasteleyn matrix:

$$\prod_{i=1}^{n} \mathrm{Kast}(t_i, x_i; r_i, y_i) \cdot \det \left[\mathrm{Kast}^{-1}(r_i, y_i; t_j, x_j) \right]_{i,j=1,\dots,n}$$

We can compare this statement with Theorem 2.7.5. Note that Theorem 2.7.5 describes the correlation functions of the particles. Consequently the correlation kernel

$$\hat{K}(t, x; r, y) = \delta_{(t,x)}^{(r,y)} - K(t, x; r, y)$$

(where $K(t, x; r, y)$ is the correlation kernel of Theorem 2.7.5) is the correlation kernel of holes or, equivalently of horizontal lozenges. (See, e.g., [8, Appendix A.3] for some details on the particle-hole involution.)

Since the correlation kernel appears only in a determinant, it is only determined up to conjugation: the transformation

$$\hat{K}(t, x; r, y) \mapsto \frac{g(t, x)}{g(r, y)} \hat{K}(t, x; r, y)$$

does not change correlation functions. We conclude that

$$\frac{\hat{K}(t, x; r, y)}{\kappa q^{-S/2+x-t/2+1/2} - (\kappa q^{-S/2+x-t/2+1/2})^{-1}}$$

should be (perhaps, after some conjugation) the inverse Kasteleyn matrix.

Let us verify this fact and find the appropriate conjugation factor.

We have

$$\sum_{(h,z)} \frac{g(t, x)}{g(h, z)} \frac{\hat{K}(t, x; h, z)}{\kappa q^{-S/2+x-t/2+1/2} - (\kappa q^{-S/2+x-t/2+1/2})^{-1}} \mathrm{Kast}(h, z; r, y) = \delta_{(t,x)}^{(r,y)}. \quad (2.26)$$

First, suppose that $t < r-1$. In this case the $((t,x);(r,y))$ matrix element of the right-hand side is zero while the one of the left-hand side of (2.26) is

$$\frac{g(t,x)}{\kappa q^{-S/2+t-x/2+1/2} - (\kappa q^{-S/2+t-x/2+1/2})^{-1}} \sum_{i \geq N} f_i^t(x) c_i^{t,r-1}$$
$$\times \left[\frac{\left(\kappa q^{-S/2+y-r/2+1/2} - \frac{1}{\kappa q^{-S/2+y-r/2+1/2}}\right) c_i^{r-1} f_i^r(y)}{g(r,y)} \right.$$
$$\left. + \frac{f_i^{r-1}(y)}{g(r-1,y)} + \frac{f_i^{r-1}(y-1)}{g(r-1,y-1)} \right].$$

Let us find such function g that for every i

$$\frac{\left(\kappa q^{-S/2+y-r/2+1/2} - \frac{1}{\kappa q^{-S/2+y-r/2+1/2}}\right) c_i^{r-1} f_i^r(y)}{g(r,y)}$$
$$+ \frac{f_i^{r-1}(y)}{g(r-1,y)} + \frac{f_i^{r-1}(y-1)}{g(r-1,y-1)} = 0. \quad (2.27)$$

We know that (see Lemma 2.7.4)

$$c_i^t f_i^{t+1}(y) - \sqrt{\frac{w_{t,S}(y-1)}{w_{t+1,S}(y)}} w_1^t(y-1) f_i^t(y-1) - \sqrt{\frac{w_{t,S}(y)}{w_{t+1,S}(y)}} w_0^t(y) f_i^t(y) = 0, \quad (2.28)$$

where

$$w_0^t(x) = -(1 - q^{x+T-t-S}) \frac{1 - \kappa^2 q^{x+N-t}}{1 - \kappa^2 q^{2x-t-S+1}},$$

$$w_1^t(x) = q^{T+N-1-t}(1 - q^{-(S+N-1-x)}) \frac{1 - \kappa^2 q^{x-T+1}}{1 - \kappa^2 q^{2x-t-S+1}},$$

while $w_{t,S}(x)$ stands for the weight function corresponding to the parameters t, q, κ, N, T, S (see Section 2.4 and Theorem 2.4.1 for details).

For every pair (r,y) we get the following three equations defining g

$$g(r,y) \propto \frac{\kappa q^{-S/2+y-r/2+1/2} - \frac{1}{\kappa q^{-S/2+z-r/2+1/2}}}{\sqrt{w_{r,S}(y)}},$$

$$g(r-1,y-1) \propto -\frac{1}{\sqrt{w_{r-1,S}(y-1)} w_1^{r-1}(y-1)},$$

$$g(r-1,y) \propto -\frac{1}{\sqrt{w_{r-1,S}(y)} w_0^{r-1}(y)},$$

where the proportionality coefficient is the same for all three equations (but it may depend on the pair (r,y)). One checks that g given by (2.25) satisfies these

relations and after the conjugation with g the $((t,x);(r,y))$ matrix element of the left-hand side of (2.26) is zero.

Next, suppose that $t > r$. In this case the $((t,x);(r,y))$ matrix element of the left-hand side of (2.26) is zero by the similar reasoning.

If either $t = r$ or $t = r-1$ the argument becomes a little more involved, but the computation requires no new ideas.

□

2.8 Bulk limits. Limit shapes.

2.8.1 Bulk limit theorem

In this section we compute so-called "bulk limits" of the correlation functions introduced in the previous section.

We are interested in the following limit regime. Fix positive numbers S, T, N, t, x, q. Introduce a small parameter $\varepsilon \ll 1$, and set

$$S = \mathsf{S}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad T = \mathsf{T}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad N = \mathsf{N}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad q = \mathsf{q}^{\varepsilon + o(\varepsilon)}.$$

Consider also integer valued functions $t_i = t_i(\varepsilon)$ and $x_i = x_i(\varepsilon)$, $i = 1, \dots, n$, such that

$$\lim_{\varepsilon \to 0} \varepsilon t_i(\varepsilon) = \mathsf{t}, \quad \lim_{\varepsilon \to 0} \varepsilon x_i(\varepsilon) = \mathsf{x}, \quad i = 1, \dots, n,$$

and pairwise differences $t_i - t_j$, and $x_i - x_j$ do not depend on ε.

Then the correlation functions ρ_n computed in Theorem 2.7.5 tend to a limit $\hat{\rho}_n$ which depends on the parameters of the limit regime q, S, T, N, t, x and the original parameter κ.

We consider the region where the limit correlation functions are nontrivial. This region is commonly referred to as the "bulk", sometimes also called the "liquid region". This is a simply connected domain inside the hexagon.

The main result of this section is Theorem 2.8.1.

Note that the first limit correlation function allows us to predict *the limit shape* which appears in our model.

Theorem 2.8.1. *We have*

$$\lim_{\varepsilon \to 0} \rho_n(t_1, x_1; \dots; t_n, x_n) = \det \left[\hat{K}(t_i, x_i; t_j, x_j) \right]_{i,j=1,\dots,n},$$

where

$$\hat{K}(x, s; y, t) = \frac{1}{2\pi i} \oint_{e^{-i\phi}}^{e^{i\phi}} (1 + cw)^{t-s} \, w^{x-y-1} dw.$$

Here the integration is to be performed over the right side of the unit circle when $s \geq t$ and over the left side otherwise,

$$c = \left(\frac{\mathsf{q}^{\mathsf{T}-2\mathsf{t}}(1 - \mathsf{q}^{-(\mathsf{S}+\mathsf{N}-\mathsf{x})})(1 - \mathsf{q}^{\mathsf{x}})}{(1 - \mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1 - \mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})} \frac{(1 - \kappa^2 \mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{S}})(1 - \kappa^2 \mathsf{q}^{\mathsf{x}-\mathsf{T}})}{(1 - \kappa^2 \mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{t}})(1 - \kappa^2 \mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})} \right)^{\frac{1}{2}},$$

and ϕ is given by the formula:

$$\phi = \arccos \frac{\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2 + A + B}{2\sqrt{AB}},$$

where

$$A = (1-\mathsf{q}^{-\mathsf{S}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{T}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{x}}),$$

$$B = \mathsf{q}^{-2\mathsf{N}-\mathsf{T}}(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{T}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{x}}).$$

If the expression under arccos *is greater than 1, then we set $\phi = 0$. If the expression is less than -1, then $\phi = \pi$.*

Setting $v = -cw$ (and omitting some "conjugation factors" again) we get the incomplete beta-kernel form of the integral, cf. [68],

$$K(x,s;y,t) = \frac{1}{2\pi i}\oint_{-c\cdot e^{-i\phi}}^{-c\cdot e^{i\phi}} (1-v)^{t-s}\, v^{x-y-1} dv.$$

Here the contour of integration intersects $(-\infty, 0)$, if $s \geq t$, and intersects $(0,1)$ otherwise. For an explanation of the relation of the incomplete beta-kernel and Gibbs measures see [46], [14].

It is not hard to compute that $z = -ce^{i\phi}$ has the form

$$\begin{aligned} z = \frac{1}{2}\, & \frac{\mathsf{q}^{\mathsf{T}+\mathsf{N}-\mathsf{t}}}{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})} \\ & \cdot \Bigg[\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2 + A + B \\ & + i\sqrt{4AB - \left(\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2 + A + B\right)^2}\Bigg], \end{aligned}$$

with

$$A = (1-\mathsf{q}^{-\mathsf{S}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{T}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{x}})$$

and

$$B = \mathsf{q}^{-2\mathsf{N}-\mathsf{T}}(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{T}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{x}}).$$

Proposition 2.8.2. *The parameter z defined above coincides with the one defined in Theorem 2.2.1.*

Proof. The quadratic equation satisfied by z is

$$Z^2 - (z+\bar{z})Z + z\bar{z} = 0.$$

Substituting the expression for z given above, one obtains a relation equivalent to the one in Theorem 2.2.1. □

2.8.2 Proof of the bulk limit theorem

In this section we prove Theorem 2.8.1.

Recall that the correlation kernel before the limit is given by

$$K(x,k;y,l) = \sum_{i=0}^{N-1} \frac{1}{c_i^{l,k}} f_i^k(x) f_i^l(y),\ k \geq l;$$
$$K(x,k;y,l) = -\sum_{i \geq N} c_i^{k,l} f_i^k(x) f_i^l(y),\ k < l;$$
$$c_i^{k,k} = 1,\ c_i^{k,l} = c_i^k \cdot c_i^{k+1} \cdot \dots \cdot c_i^{l-1}.$$

The functions $f_i^k(x)$ and the coefficients c_i^k were defined in Section 2.7.

First, let us consider the case $k = l$. We want to find a limit of the projection kernel

$$\mathcal{P}_t(x,y) = \sum_{n=0}^{N-1} f_n^t(x) f_n^t(y).$$

In order to find a limit of $\mathcal{P}_t(x,y)$ we use the spectral projection method proposed by G. Olshanski, see [9] and [69].

We want to consider $\mathcal{P}_t(x,y)$ as a matrix element of the operator $\mathcal{P}_t$. It turns out that finding the limit of the operator is easier than computing the limit of the matrix elements. Note that functions $f_n^t(x)$ are eigenvectors of some difference operator (it will be explicitly given below). The projection operator can be regarded as the spectral projection on the segment containing the first N eigenvalues of this difference operator. Now, to find the limit of the spectral projection operators we will take the limit of the difference operators. Note that both the difference operator and the spectral segment are varying simultaneously.

To justify the limit transition we use some facts from functional analysis.

Consider the set $l_2^0(\mathbb{Z})$ of the finite vectors from $l_2(\mathbb{Z})$ (i.e., the algebraic span of the basis elements δ_x) as a common essential domain of all considered difference operators. It will be clear from the following that the difference operators strongly converge on this domain. It follows that the operators converge in the strong resolvent sense (see [77], Theorem VIII.25). The last fact, continuity of the spectrum of the limit operator, and Theorem VIII.24 from [77] imply that the spectral projections associated with the difference operators strongly converge on the set of finite vectors to the limit spectral projection associated with the limit difference operator.

Now we present some details and computations.

Note that since q-Racah polynomials are eigenfunctions of a certain difference operator (see [52]), the same is true for the functions $f_n^t(x)$. The difference operator is

$$q^{-n}(1-q^n)(1-\alpha\beta q^{n+1})f_n^t(x) = B(x)f_n^t(x+1)\sqrt{\frac{w_{t,S}(x)}{w_{t,S}(x+1)}}$$
$$-[B(x)+D(x)]f_n^t(x) + D(x)f_n^t(x-1)\sqrt{\frac{w_{t,S}(x)}{w_{t,S}(x-1)}}, \quad (2.29)$$

where $w_{t,S}(x)$ is the weight function corresponding to the parameters t, q, κ, N, T, S (see Theorem 2.4.1), and

$$B(x) = \frac{(1-\alpha q^{x+1})(1-\beta\delta q^{x+1})(1-\gamma q^{x+1})(1-\gamma\delta q^{x+1})}{(1-\gamma\delta q^{2x+1})(1-\gamma\delta q^{2x+2})},$$

$$D(x) = \frac{q(1-q^x)(\alpha-\gamma\delta q^x)(\beta-\gamma q^x)(1-\delta q^x)}{(1-\gamma\delta q^{2x})(1-\gamma\delta q^{2x+1})}.$$

(Here α, β, γ, δ are the corresponding parameters of q-Racah polynomials.)

We find

$$w_{t,S}(x+1)/w_{t,S}(x) = \frac{q^{2N+T-1}(1-\kappa^2 q^{2x-t-S+3})}{1-\kappa^2 q^{2x-t-S+1}}$$
$$\times \frac{(1-q^{x-t-N+1})(1-q^{x-S-N+1})(1-\kappa^2 q^{x-T+1})(1-\kappa^2 q^{x-t-S+1})}{(1-q^{x+1})(1-q^{T-S-t+x+1})(1-\kappa^2 q^{x+N-t+1})(1-\kappa^2 q^{x+N-S+1})},$$

$$w_{t,S}(x-1)/w_{t,S}(x) = \frac{q^{-2N-T+1}(1-\kappa^2 q^{2x-t-S-1})}{1-\kappa^2 q^{2x-t-S+1}}$$
$$\times \frac{(1-q^x)(1-q^{T-S-t+x})(1-\kappa^2 q^{x+N-t})(1-\kappa^2 q^{x+N-S})}{(1-q^{x-t-N})(1-q^{x-S-N})(1-\kappa^2 q^{x-T})(1-\kappa^2 q^{x-t-S})}.$$

Substituting the q-Racah parameters of our model (see Section 2.4) into $B(x)$ and $D(x)$ one computes

$$B(x)$$
$$= \frac{(1-q^{-S-N+x+1})(1-\kappa^2 q^{-T+x+1})(1-q^{-t-N+x+1})(1-\kappa^2 q^{-t-S+x+1})}{(1-\kappa^2 q^{-t-S+2x+1})(1-\kappa^2 q^{-t-S+2x+2})},$$

$$D(x)$$
$$= \frac{q(1-q^x)(q^{-S-N}-\kappa^2 q^{-t-S+x})(q^{S-T-N}-q^{-t-N+x})(1-\kappa^2 q^{-S+N+x})}{(1-\kappa^2 q^{-t-S+2x})(1-\kappa^2 q^{-t-S+2x+1})}$$
$$= q^{1-2N-T}\frac{(1-q^x)(1-\kappa^2 q^{-t+N+x})(1-q^{-t-S+T+x})(1-\kappa^2 q^{-S+N+x})}{(1-\kappa^2 q^{-t-S+2x})(1-\kappa^2 q^{-t-S+2x+1})}.$$

It follows that

$$\frac{B(x)}{\sqrt{w_{t,S}(x+1)/w_{t,S}(x)}} = \Big(q^{1-2N-T}$$
$$\times (1-q^{x+1})(1-q^{T-S-t+x+1})(1-q^{-S-N+x+1})(1-q^{-t-N+x+1})\Big)^{\frac{1}{2}}$$
$$\times \left(\frac{(1-\kappa^2 q^{-T+x+1})(1-\kappa^2 q^{-t-S+x+1})}{(1-\kappa^2 q^{-t-S+2x+1})(1-\kappa^2 q^{2x-t-S+3})(1-\kappa^2 q^{-t-S+2x+2})^2}\right.$$
$$\left.\times (1-\kappa^2 q^{x+N-t+1})(1-\kappa^2 q^{x+N-S+1})\right)^{\frac{1}{2}},$$

$$\frac{D(x)}{\sqrt{w_{t,S}(x-1)/w_{t,S}(x)}}$$
$$= \left(q^{1-2N-T}(1-q^x)(1-q^{x-t-N})(1-q^{x-S-N})(1-q^{x-t-S+T})\right)^{\frac{1}{2}}$$
$$\times \left(\frac{(1-\kappa^2 q^{-t+N+x})(1-\kappa^2 q^{-S+N+x})(1-\kappa^2 q^{x-T})(1-\kappa^2 q^{x-t-S})}{(1-\kappa^2 q^{-t-S+2x+1})(1-\kappa^2 q^{2x-t-S-1})(1-\kappa^2 q^{-t-S+2x})^2}\right)^{\frac{1}{2}}.$$

The eigenvalues in the left-hand side of (2.29) become

$$q^{-n}(1-q^n)(1-q^{-T-2N+n+1}), \quad n = 0, \ldots, N-1$$

Note that in the "bulk limit" regime, $q \to 1$, $N, T, S \to \infty$ as $\varepsilon \to 0$ in such a way that q^N, q^T, q^S have finite limits.

In the limit $\frac{B(x)}{\sqrt{w(x+1)/w(x)}}$ tends to some constant and $\frac{D(x)}{\sqrt{w(x-1)/w(x)}}$ tends to the very same constant. After dividing by twice this constant the limit operator becomes

$$f(x) \mapsto \frac{f(x+1)+f(x-1)}{2} - f(x)\frac{A+B}{2\sqrt{AB}},$$

where

$$A = (1-\mathsf{q}^{-\mathsf{S}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{T}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{x}})$$

and

$$B = \mathsf{q}^{-2\mathsf{N}-\mathsf{T}}(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{T}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{x}}),$$

while the spectral interval becomes

$$\left[\frac{\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2}{2\sqrt{AB}}, 0\right]$$

Since the spectrum of the operator $f \to \frac{f(x+1)+f(x-1)}{2}$ is $[-1,1]$, while $\frac{\sqrt{A+B}}{2\sqrt{AB}} > 1$, one can equivalently write the operator in the form

$$f \to \frac{f(x+1)+f(x-1)}{2}$$

with spectral interval

$$\left[\frac{\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2+A+B}{2\sqrt{AB}},1\right]. \tag{2.30}$$

If we perform the Fourier transform $l_2(\mathbb{Z}) \to L_2(S^1)$, where S^1 is the unit circle in $\mathbb{C}$, we get an operator of the projection on the part of the unit circle with x-coordinate varying over precisely the spectral interval (2.30).

If we do the inverse Fourier transform we will get the discrete sine kernel, cf. the end of Section 3.2 in [35].

Next, let us consider the case $k < l$. The prelimit correlation kernel is given by

$$K(x,k;y,l) = -\sum_{i\geq N} c_i^{k,l} f_i^k(x) f_i^l(y), \qquad k<l;$$
$$c_i^{k,k} = 1, \qquad c_i^{k,l} = c_i^k \cdot c_i^{k+1} \cdot \dots \cdot c_i^{l-1}.$$

Let us decompose the correlation kernel (i.e., the operator given by it) into the product of the static projection kernel:

$$\mathcal{P}'_t(x,y) = -\sum_{i\geq N}^{N-1} f_i^t(x) f_i^t(y),$$

and a collection of transition operators (or their inverses) $U_h(x,y)$ with

$$U_h(x,y) = \begin{cases} \sum\limits_{i\geq 0} c_i^h f_i^h(x) f_i^{h+1}(y), & x\in\mathfrak{X}_h, y\in\mathfrak{X}_{h+1}, \\ 0 & \text{for other } x,y. \end{cases}$$

For the operators $\mathcal{P}'_t(x,y)$ we can use the same methods as for $\mathcal{P}_t(x,y)$. We get minus the operator of projection on the part of the unit circle complementary to the spectral interval (2.30).

Let us turn to the transition operators U_t.

By virtue of already proved facts (see Section 2.7), we obtain

$$U_t(x,y) = const \cdot \sqrt{\frac{w_{t,S}(x)}{w_{t+1,S}(y)}}\left[w_1(x)\delta_{x+1}^y + w_0(x)\delta_x^y\right],$$

where

$$w_{t,S}(x) = \frac{q^{x(2N+T-1)}(1-\kappa^2 q^{2x-t-S+1})}{(q;q)_x (q;q)_{T-S-t+x}(q^{-1};q^{-1})_{t+N-x-1}(q^{-1};q^{-1})_{S+N-x-1}} \cdot \frac{1}{(\kappa^2 q^{x-T+1};q)_{T+N-t}(\kappa^2 q^{x-t-S+1};q)_{N+t}},$$

$$w_0(x) = -(1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}\,,$$

and

$$w_1(x) = q^{T+N-1-t}(1-q^{-(S+N-1-x)})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}.$$

Thus,

$$U_t(x,y) = const\cdot\left[\tilde{w}_1(x)\delta^y_{x+1} + \tilde{w}_0(x)\delta^y_x\right],$$

where

$$\tilde{w}_0(x) = \left(\frac{(1-q^{x+T-t-S})(1-q^{-t-N+x})(1-\kappa^2 q^{x-t-S})(1-\kappa^2 q^{x+N-t})}{(1-\kappa^2 q^{2x-t-S+1})(1-\kappa^2 q^{2x-t-S})}\right)^{\frac{1}{2}},$$

and

$$\tilde{w}_1(x) = \left(\frac{q^{T-t-1}(1-q^{x-S-N+1})(1-q^{x+1})(1-\kappa^2 q^{x+N-S+1})(1-\kappa^2 q^{x-T+1})}{(1-\kappa^2 q^{2x-t-S+1})(1-\kappa^2 q^{2x-t-S+2})}\right)^{\frac{1}{2}}.$$

Passing to the limit we get the operator

$$\mathsf{U}(x,y) = const\cdot\left[\mathsf{U}_1\delta^y_{x+1} + \mathsf{U}_0\delta^y_x\right],$$

where

$$\mathsf{U}_0 = \left(\frac{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{t}})}{(1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})\ (1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})}\right)^{\frac{1}{2}}$$

and

$$\mathsf{U}_1 = \left(\frac{\mathsf{q}^{\mathsf{T}-2t}(1-\mathsf{q}^{\mathsf{x}-\mathsf{S}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{T}})(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{S}})}{(1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})}\right)^{\frac{1}{2}}.$$

Equivalently,

$$\mathsf{U}(x,y) = const\cdot\left[\mathsf{u}\delta^y_{x+1} + \delta^y_x\right],$$

where

$$\mathsf{u} = \frac{\mathsf{U}_1}{\mathsf{U}_0} = \left(\frac{\mathsf{q}^{\mathsf{T}-2\mathsf{t}}(1-\mathsf{q}^{\mathsf{x}-\mathsf{S}-\mathsf{N}})(1-\mathsf{q}^{\mathsf{x}})}{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})}\frac{(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{T}})}{(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{t}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})}\right)^{\frac{1}{2}}.$$

Fourier transform gives us the operator of multiplication by $const(1+\mathsf{u}/w)$, where w is the coordinate on the circle $|w|=1$.

If we now multiply all necessary operators, perform inverse Fourier transform and substitute $w\to 1/w$ in the resulting integral, we get the desired limit kernel. Note that the constant prefactor in U can be omitted since it corresponds to the conjugation of the kernel that does not affect the correlation functions.

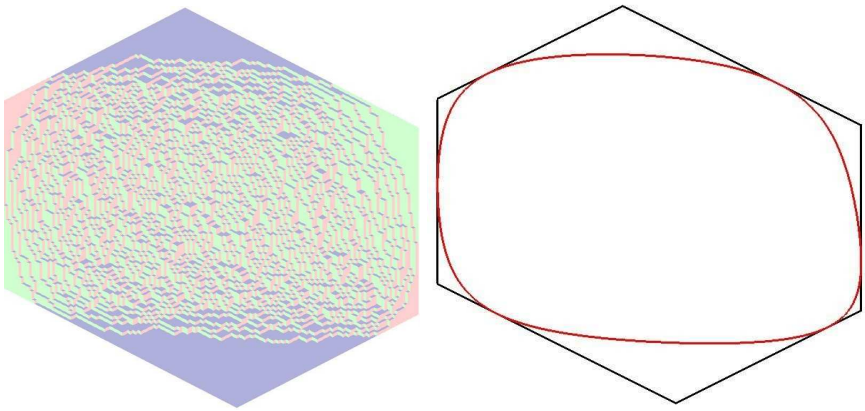
The final case $k>l$ is similar. The interested reader can find some details in [35] where similar computations (with Hahn polynomials instead of q-Racah polynomials) were performed.

2.9 Computer simulations. Different limit regimes.

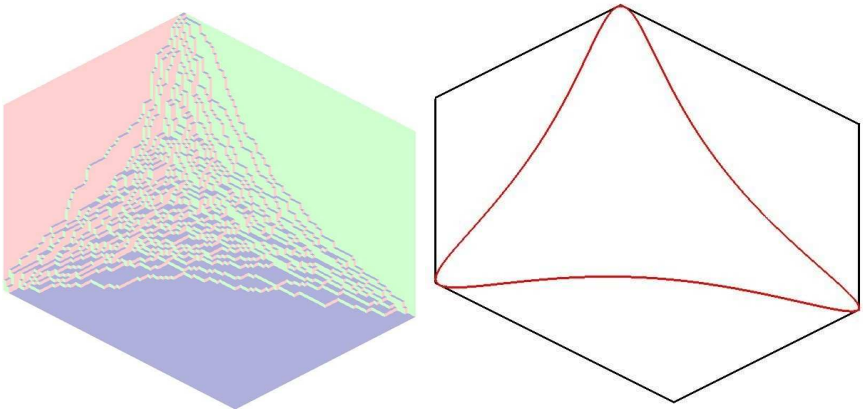
Using the perfect sampling algorithm described in Section 2.6, we performed some computer simulations; the program that we used can be found at [36]. We are mostly interested in the case when the hexagon is large, since in this case we can see some limit shapes appearing. In all pictures we color the three types of lozenges in three different colors (as in Figure 2.1). When we draw big pictures, we erase the borders between lozenges and get some coloring of a hexagon in three colors which can also be viewed as a stepped surface in $\mathbb{R}^3$.

Although we mostly show not very big pictures, our algorithm can generate random tilings of a $1000 \times 1000 \times 1000$ hexagon in a reasonable amount of time.

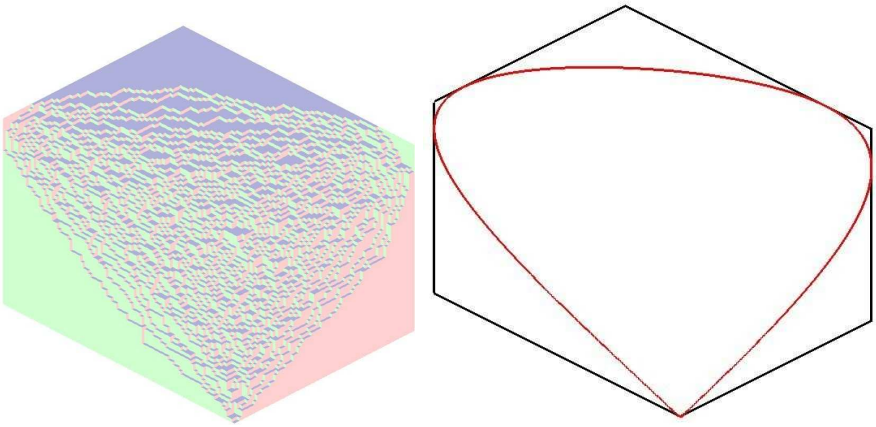
The following picture shows a plane partition in a $70 \times 90 \times 70$ box sampled from the distributions with parameters $q = 0.97$, $\kappa = i$. The formation of a limit shape with frozen regions is clearly visible on the picture, and the next picture shows the border of the frozen region as predicted by Theorem 2.2.1.



By the appropriate limit transition, we get plane partitions distributed as $q^{-volume}$. The following picture shows a random plane partitions in a $70 \times 90 \times 70$ box and the corresponding theoretical frozen boundary with $q = 1.04$.

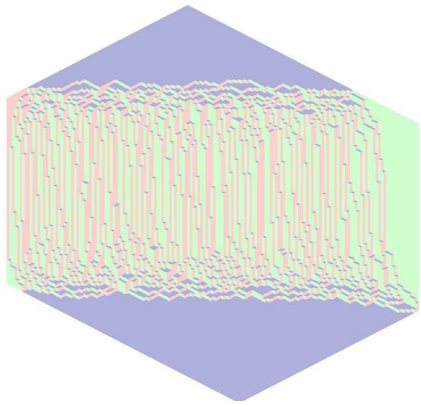


As was explained in Section 2.4, if we send $q \to 1$ in the original model, then the weight of a horizontal lozenge becomes a linear function in the vertical coordinate. If we tune the parameters in such a way that our linear function has a zero at the bottommost point of a hexagon, then we get the following random plane partition in a $70 \times 90 \times 70$ box, cf. the corresponding theoretical frozen boundary.

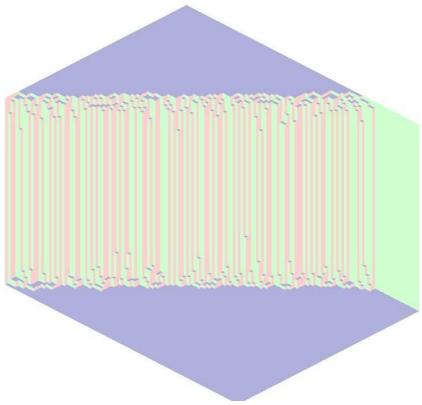


We see that the border of frozen region has a node near the bottommost point of a hexagon.

Another interesting case to consider is when q does not tend to 1 as the size of the hexagon tends to infinity. Look at the following picture where the random plane partition with parameters $q = 0.9$, $\kappa = i$ in a $70 \times 90 \times 70$ box is shown. We see that the surface becomes different from the ones shown above. We call this new class of surfaces *waterfalls*.

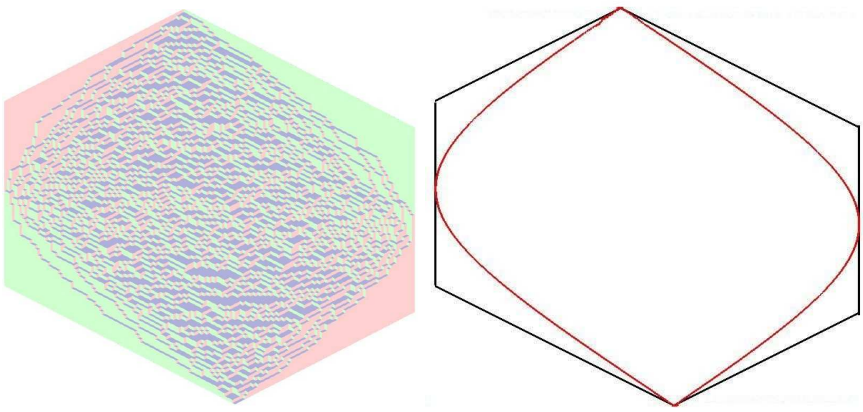


The next picture shows an even more degenerate case $q = 0.7$.



We hope to study the asymptotic behavior of these cases in a later publication.

Finally, let us turn to the trigonometric q-Racah case. In this case the weight of a horizontal lozenge is $\sin(\alpha(j-(S+1)/2)+\beta)$. One can tune the parameters α and β in such a way that this weight becomes zero at both the topmost and the bottommost points of the hexagon. The boundary of the frozen region has two nodes in this case.



2.10 Appendix. Lozenge tilings and biorthogonal functions

We were led to consider the q-Racah and Racah cases via the realization that, in much the same way that uniform lozenge tilings are related to Hahn polynomials, so were q-weighted lozenge tilings related to q-Hahn polynomials. This naturally led to the question of whether more general (discrete) hypergeometric orthogonal polynomials could arise in this way. In fact, as we will consider in this section, one can generalize even further, to the elliptic analogue, certain biorthogonal elliptic functions due to Spridonov and Zhedanov [83].

One way to derive the required lozenge weights (essentially equivalent weights were considered by Schlosser in [80], although the connection to plane partitions or lozenge tilings was not made explicit there) is via the following desiderata:

First, we want the total weight of all tilings of a hexagon to be "nice", in the sense that it should be expressible as a product of simple theta functions

$$\theta_p(x) := \prod_{0\leq k}(1-p^k x)(1-p^{k+1}/x);$$

the same should apply to the individual weights as well. Note that, as is traditional in much recent work on elliptic special functions, we use the multiplicative form for our elliptic curves and theta functions. This can be translated to the usual doubly-periodic form by composing with the singly-periodic function $x \mapsto \exp(2\pi i x)$, but the multiplicative form makes certain degenerations simpler to obtain.

Next, the sums that arise should be hypergeometric, in the sense that any parameters should vary along geometric progressions as one moves around in the hexagon.

Finally, the form of the weight of a given cube (i.e., the ratio of the weights of the two ways to tile any given unit hexagon) should be invariant under the

symmetry group of the triangular lattice, except that certain reflections should invert the weight. Note that the weights of cubes are gauge-invariant, and any set of choices of cube weights corresponds to a unique gauge-equivalence class.

We may rewrite these criteria in terms of plane partitions, noting that the requirement that the cube weights come from lozenge weights places the additional restriction that if $w(x,y,z)$ is the weight of a cube with centroid (x,y,z), then

$$w(x,y,z) = w(x+1,y+1,z+1).$$

If we consider plane partitions inside a $1 \times 1 \times n$ box, we thus require that

$$\sum_{0 \le l \le n} \prod_{0 \le m < l} w(1/2, 1/2, m+1/2)$$

should be nice, and should indeed correspond to a hypergeometric sum. The isotropy condition then implies more generally that

$$\sum_{0 \le l \le n} \prod_{0 \le m < l} w(x,y,z+m)$$

is a hypergeometric sum for any half-integer vector (x,y,z). In particular, this sum is doubly-telescoping, in the sense that the sum over any subinterval is nice. In particular, if we assume that the sum should be a special case of the Frenkel-Turaev sum [28], we find that it should have the form (up to some convenient changes of parameters)

$$\sum_{0 \le l \le n} \frac{q^l \theta_p(abq^{2l-1})}{\theta_p(q^{l-1}a, q^l a, q^{l-1}b, q^l b)} \frac{\theta_p(a/q, a, b/q, b)}{\theta_p(ab/q)} = \frac{\theta_p(q^{n-1}ab, q^{n+1}, a, b)}{\theta_p(ab/q, q, q^n a, q^n b)},$$

where a and b depend on (x,y,z). In particular, we find

$$\begin{aligned} &w(x,y,z+m) \\ &\qquad = \frac{q\theta_p(q^{m-1}a, q^{m-1}b, abq^{2m+1})}{\theta_p(q^{m+1}a, q^{m+1}b, abq^{2m-1})} = \frac{q^3\theta_p(q^{m-1}a, q^{m-1}b, q^{-2m-1}/ab)}{\theta_p(q^{m+1}a, q^{m+1}b, q^{1-2m}/ab)}, \end{aligned}$$

where a,b depend on (x,y,z); consistency then implies that $q^{-z}a$ and $q^{-z}b$ are independent of z.

Rotating the picture by 120 degrees gives a similar expression for $w(i,j,k)$ with the dependence on i or j factored out; comparing the results leads us to an expression of the form

$$w(x,y,z) = \frac{q^3\theta_p(q^{y+z-2x-1}u_1, q^{x+z-2y-1}u_2, q^{x+y-2z-1}u_3)}{\theta_p(q^{y+z-2x+1}u_1, q^{x+z-2y+1}u_2, q^{x+y-2z+1}u_3)},$$

where $u_1 = a$, $u_2 = b$, $u_3 = 1/ab$; i.e., u_1, u_2, u_3 are generic such that $u_1u_2u_3 = 1$. We will see that this indeed gives rise to a factored sum over plane partitions

in a cube. Note that if we rewrite this expression in terms of new variables $\tilde{u}_1 = q^{y+z-2x}u_1$, $\tilde{u}_2 = q^{x+z-2y}u_2$, $\tilde{u}_3 = q^{x+y-2z}u_3$, then

$$w(x,y,z) = \frac{q^3\theta_p(\tilde{u}_1/q, \tilde{u}_2/q, \tilde{u}_3/q)}{\theta_p(q\tilde{u}_1, q\tilde{u}_2, q\tilde{u}_3)},$$

which can be described in a canonical way: w is the value at q of the unique elliptic function with simple zeros at $\tilde{u}_i$, simple poles at $\tilde{u}_i^{-1}$, and the value 1 at 1. It follows, in particular, that w is invariant under shifting any parameter by p, as well as under all modular transformations.

To write this in terms of lozenge weights, there are, of course, a number of choices one could make. One convenient choice is to allow only horizontal lozenges to have nonzero weights. We must thus have

$$\frac{w(i,j+1)}{w(i,j)} = \frac{q\theta_p(q^{j-3i/2-1}u_1, q^{j+3i/2-1}u_2, q^{2j+1}u_1u_2)}{\theta_p(q^{j-3i/2+1}u_1, q^{j+3i/2+1}u_2, q^{2j-1}u_1u_2)}$$

where $w(i,j)$ denotes the weight of a horizontal lozenge with upper corner at (i,j) (recall that in terms of 3-D coordinates, $i = x-y$, $j = z-(x+y)/2$). This recurrence is straightforward to solve, and one obtains

$$w(i,j) = C(i)\frac{q^{j-1/2}(u_1u_2)^{1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)},$$

where $C(i)$ is an arbitrary (non-vanishing) function of i.

If we view (with a mind to applying Kasteleyn's theorem) the lozenge weight as a matrix indexed by a right-pointing and a left-pointing triangle, we find that, coordinatizing triangles by their upper corners,

$$w((i,j),(i,j)) = C(i)\frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)}$$
$$w((i,j),(i+1,j+1/2)) = 1, \qquad w((i,j),(i+1,j-1/2)) = 1,$$

and all other values are 0.

Let $\Pi_{x_0,x_1}^{y_0,y_1}$ represent the parallelogram

$$x_0 \le i \le x_1, y_0 \le j + i/2 \le y_1,$$

and observe that the restriction of w to triangles in $\Pi_{x_0,x_1}^{y_0,y_1}$ is a square matrix, and we can thus attempt to invert w in such a region. In fact, not only can we explicitly invert w in such parallelograms, but the result is independent of the choice of parallelogram.

Theorem 2.10.1. *The inverse transpose of w in $\Pi_{x_0,x_1}^{y_0,y_1}$ has the form*

$$W((i_0,j_0),(i_1,j_1)) = \delta_{i_0<i_1}\prod_{i_0\le k<i_1} C(k)\cdot(u_1u_2)^{(i_1-i_0-1)/2}$$
$$\times\,\delta_{j_1+i_1/2\le j_0+i_0/2}(-1)^{j_0-i_0/2-j_1+i_1/2-1}q^{(i_1-i_0-1)(i_1-i_0+4j_1-2)/4}$$
$$\times\frac{\theta_p(q^{j_0+i_0/2-j_1-i_1/2+1}, q^{j_0+i_0/2+j_1-i_1/2}u_1u_2;q)_{i_1-i_0-1}}{\theta_p(q, q^{j_0-i_0/2-i_1}u_1, q^{j_1-3i_1/2+1}u_1, q^{j_1+i_0+i_1/2}u_2, q^{j_0+3i_0/2+1}u_2;q)_{i_1-i_0-1}},$$

where

$$\theta_p(x;q)_k := \prod_{0\le i<k} \theta_p(q^i x).$$

Proof. Note that by Kasteleyn's theorem, W is the total weight (up to sign) of all lozenge tilings of the parallelogram that omit the two given triangles. In particular, $W((i_0,j_0),(i_1,j_1))$ must vanish if $i_0 \ge i_1$ or $j_0 + i_0/2 < j_1 + i_1/2$ simply because then no such tiling exists.

Now, the claim that W is the inverse transpose of w reduces to the statement

$$\begin{aligned} W((i_0,j_0),(i_1,j_1))w(i_1,j_1) + W((i_0,j_0),(i_1+1,j_1-1/2)) \\ + W((i_0,j_0),(i_1+1,j_1+1/2)) = \delta_{(i_0,j_0),(i_1,j_1)}. \end{aligned}$$

This holds trivially for $i_0 > i_1$, since all three terms vanish, and similarly for $i_0 = i_1$, $j_1 > j_0$. When $i_0 = i_1$, $j_0 = j_1$, the claim reduces to

$$W((i_0,j_0),(i_0+1,j_0-1/2)) = 1,$$

while for $i_0 = i_1$, $j_0 \ge j_1$, we have

$$\begin{aligned} W((i_0,j_0),(i_0+1,j_1-1/2)) + W((i_0,j_0),(i_0+1,j_1+1/2)) \\ = (-1)^{j_0-j_1-1} + (-1)^{j_0-j_1} = 0, \end{aligned}$$

as required.

It remains to consider the case $i_0 < i_1$. If $j_1 + i_1/2 > j_0 + i_0/2$, then all three terms again vanish, while if $j_1 + i_1/2 = j_0 + i_0/2$, the third term vanishes, and

$$W((i_0,j_0),(i_1,j_1))w(i_1,j_1) + W((i_0,j_0),(i_1+1,j_1-1/2)) = 0$$

as required. Finally, when $j_1 + i_1/2 < j_0 + i_0/2$, so that all three terms survive, if we divide by

$$\frac{W((i_0,j_0),(i_1,j_1))w(i_1,j_1)}{\theta_p(q^{i_1-i_0}, q^{j_0-i_0/2-i_1-1}u_1, q^{j_0+i_0/2+i_1}u_2, q^{2j_1-1}u_1u_2)},$$

we simply obtain a special case of the addition law for θ_p, in the form

$$\begin{aligned} \theta_p(a_0z, a_1z, a_2z, a_0a_1a_2/z) - \theta_p(a_0a_1, a_0a_2, a_1a_2, z^2) \\ + \theta_p(z/a_0, a_1/z, a_2/z, a_0a_1a_2z)za_0 = 0 \end{aligned}$$

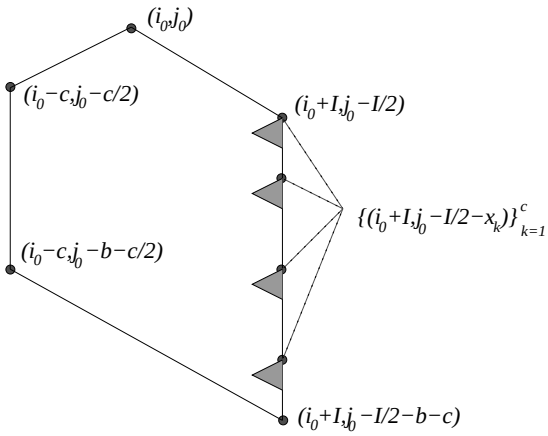
with

$$\begin{aligned} a_0 &= q^{-j_0/2-3i_0/4+j_1/2+3i_1/4}, \quad a_1 = q^{j_0/2-i_0/4+j_1/2-5i_1/4-1}u_1, \\ a_2 &= q^{j_0/2+3i_0/4+j_1/2+3i_1/4}u_2, \quad z = q^{j_0/2-i_0/4-j_1/2+i_1/4}. \end{aligned}$$

□

Remark. One major source of guidance regarding the form of W is that it corresponds to an enumeration of plane partitions in a rectangular parallelepiped with dimensions $m \times n \times 1$, say; i.e., a sum over ordinary partitions. If we first sum over the first part of the partition, we find that W should look like the term of a (singly) telescoping hypergeometric sum. One can also, of course, compute "small" values of W in the case $p = 0$, and look for patterns in the resulting factorizations.

Lemma 2.10.2. *Consider the domain*



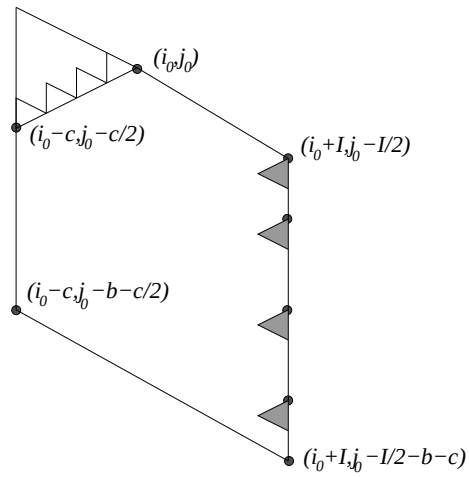
The total weight of lozenge tilings of this domain is equal to a constant independent of $\{x_k\}$ *times*

$$\prod_{1\le l\le c} \frac{(-1)^{x_l}\theta_p(q^{I+1}, q^{I-j_0+3i_0/2}/u_1, q^{-I+2-j_0-3i_0/2}/u_2, q^{I+1-2j_0}/u_1u_2; q)_{x_l}}{\theta_p(q, q^{2I-j_0+3i_0/2}/u_1, q^{2-j_0-3i_0/2}/u_2, q^{1-2j_0}/u_1u_2; q)_{x_l}}$$

times

$$\frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le l\le c}\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1, q^{x_l-j_0-3i_0/2+2}/u_2; q)_{c-1}}.$$

Proof. The problem is equivalent to computing the weight of tilings of the domain



According to Kasteleyn's theorem, cf. [42], we obtain

$$\det\left[W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))\right]_{k,l=1}^{c}.$$

Now, we can write

$$\begin{aligned}\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}\\ =\frac{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}\\ \times\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))},\end{aligned}$$

noting that (for generic values of the parameters) $W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))\neq 0$. Since

$$\begin{aligned}&\frac{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l-1))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}\\ &=-\frac{\theta_p(q^{x_l}q^{I+1},q^{x_l}q^{I-j_0+3i_0/2}/u_1,q^{x_l}q^{-I+2-j_0-3i_0/2}/u_2,q^{x_l}q^{I+1-2j_0}/u_1u_2)}{\theta_p(q^{x_l}q,q^{x_l}q^{2I-j_0+3i_0/2}/u_1,q^{x_l}q^{2-j_0-3i_0/2}/u_2,q^{x_l}q^{1-2j_0}/u_1u_2)},\end{aligned}$$

we find

$$\begin{aligned}&\frac{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}\\ &=(-1)^{x_l}\frac{\theta_p(q^{I+1},q^{I-j_0+3i_0/2}/u_1,q^{-I+2-j_0-3i_0/2}/u_2,q^{I+1-2j_0}/u_1u_2;q)_{x_l}}{\theta_p(q,q^{2I-j_0+3i_0/2}/u_1,q^{2-j_0-3i_0/2}/u_2,q^{1-2j_0}/u_1u_2;q)_{x_l}}.\end{aligned}$$

For the other factor, we similarly have

$$\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))} \propto \frac{\theta_p(q^{-x_l},q^{1+x_l-2j_0+I}/u_1u_2;q)_{k-1}}{\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1,q^{x_l-j_0-3i_0/2+2}/u_2;q)_{k-1}},$$

where we have removed factors independent of x_l, and observed that the right-hand side vanishes when $0 \le x_l < k$ as required. As a function of

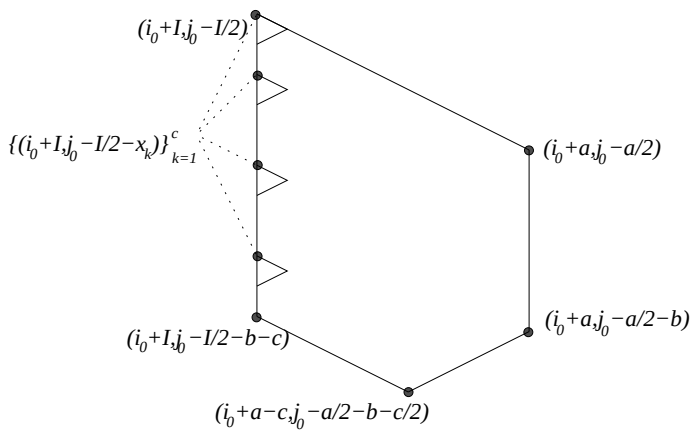
$$z_l = q^{x_l+I/2-j_0+1/2}(u_1u_2)^{-1/2},$$

this is invariant under $z_l \mapsto 1/z_l$, and we may thus apply Corollary 5.4 of [94] to conclude that

$$\det\left[\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}\right]_{k,l=1}^{c} \propto \frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l},q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le l\le c}\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1,q^{x_l-j_0-3i_0/2+2}/u_2;q)_{c-1}}.$$

□

Lemma 2.10.3. *Similarly, the total weight of lozenge tilings of the domain*



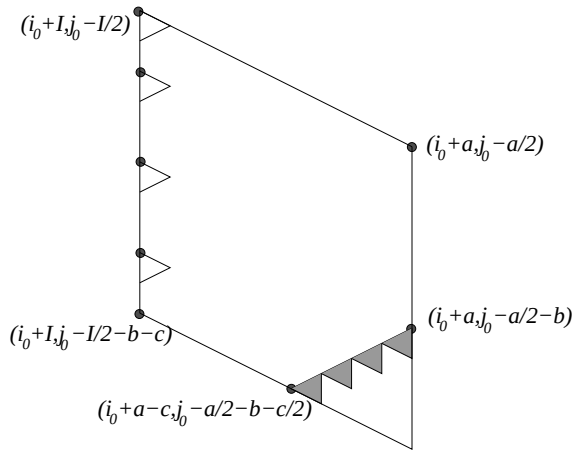
is equal to a constant independent of $\{x_k\}$ *times*

$$\prod_{k=1}^{c}\frac{(-1)^{x_k}\theta_p\left(q^{1-b-c},\frac{q^{2I-j_0+2+3i_0/2}}{u_1},\frac{q^{-a+c-j_0-3i_0/2}}{u_2},\frac{q^{-2j_0+a+b+1}}{u_1u_2};q\right)_{x_k}}{\theta_p\left(q^{I-a-b+1},\frac{q^{I+2-j_0+3i_0/2+a-c}}{u_1},\frac{q^{-I-j_0-3i_0/2}}{u_2},\frac{q^{I-2j_0+b+c+1}}{u_1u_2};q\right)_{x_k}}$$

times

$$\frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l},q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le k\le c}\theta_p(q^{x_k+I-j_0+3i_0/2+2+a-c}/u_1,q^{j_0+3i_0/2+a+1-c-x_k}u_2;q)_{c-1}}$$

Proof. The problem is equivalent to computing the weight of tiling of the domain



Again, by the Kasteleyn theorem, we need to compute

$$\det\left[W((i_0+I, j_0-I/2-x_k),(i_0+a-c+l, j_0-a/2-b-c/2+l/2))\right].$$

Again,

$$W((i_0+I, j_0-I/2-x_k),(i_0+a-c+1, j_0-a/2-b-c/2+1/2)) \neq 0,$$

allowing us to factor the weights accordingly:

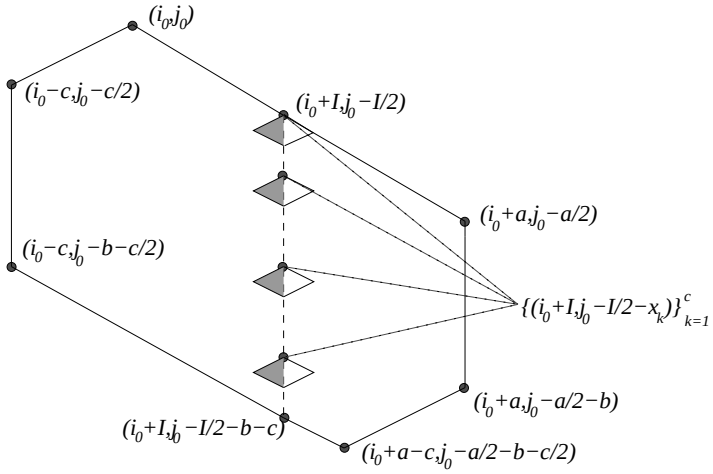
$$\frac{W((i_0+I, j_0-I/2-x_k),(i_0+a-c+1, j_0-a/2-b-c/2+1/2))}{W((i_0+I, j_0-I/2),(i_0+a-c+1, j_0-a/2-b-c/2+1/2))}$$
$$= (-1)^{x_k}\frac{\theta_p\left(q^{1-b-c}, \frac{q^{2I-j_0+2+3i_0/2}}{u_1}, \frac{q^{-a+c-j_0-3i_0/2}}{u_2}, \frac{q^{-2j_0+a+b+1}}{u_1u_2}; q\right)_{x_k}}{\theta_p\left(q^{I-a-b+1}, \frac{q^{I+2-j_0+3i_0/2+a-c}}{u_1}, \frac{q^{-I-j_0-3i_0/2}}{u_2}, \frac{q^{I-2j_0+b+c+1}}{u_1u_2}; q\right)_{x_k}}$$

and

$$\frac{W((i_0+I, j_0-I/2-x_k),(i_0+a-c+l, j_0-a/2-b-c/2+l/2))}{W((i_0+I, j_0-I/2-x_k),(i_0+a-c+1, j_0-a/2-b-c/2+1/2))}$$
$$\propto \frac{\theta_p(q^{x_k-b-c+1}, q^{2j_0-I-b-c-x_k}u_1u_2; q)_{l-1}}{\theta_p(q^{I-j_0+3i_0/2+2+a-c}/u_1, q^{j_0+3i_0/2+a+1-c-x_k}u_2; q)_{l-1}}.$$

□

Lemma 2.10.4. *The weight of all tilings of the hexagon*



such that the horizonal tiles along the line $i = i_0+I$ are as specified, is a constant independent of $\{x_k\}$ times

$$\prod_{1\le k\le c} \frac{\theta_p(q^{2x_k+I+1-2j_0}/u_1u_2)}{\theta_p(q^{I+1-2j_0}/u_1u_2)} \cdot \frac{q^{x_k}\theta_p(q^{I+1},q^{1-b-c})}{\theta_p(q,q^{I-a-b+1})}$$
$$\times \frac{\theta_p\left(\frac{q^{I-j_0+3i_0/2}}{u_1}, \frac{q^{-a+c-j_0-3i_0/2}}{u_2}, \frac{q^{I+1-2j_0}}{u_1u_2}, \frac{q^{-2j_0+a+b+1}}{u_1u_2}; q\right)_{x_k}}{\theta_p\left(\frac{q^{I+2-j_0+3i_0/2+a-c}}{u_1}, \frac{q^{2-j_0-3i_0/2}}{u_2}, \frac{q^{1-2j_0}}{u_1u_2}, \frac{q^{I-2j_0+b+c+1}}{u_1u_2}; q\right)_{x_k}},$$

times

$$\prod_{1\le k<l\le c} q^{-2x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)^2$$
$$\times \prod_{1\le k\le c} \frac{1}{\theta_p(q^{x_k+I-j_0+3i_0/2+2+a-c}/u_1, q^{I-j_0+3i_0/2+2+a-c}/u_1; q)_{c-1}}$$
$$\times \prod_{1\le k\le c} \frac{1}{\theta_p(q^{j_0+3i_0/2+a+1-c-x_k}u_2, q^{j_0+3i_0/2+a+1-c-x_k}u_2; q)_{c-1}}.$$

Remark. The first "univariate" factor is just the product over the variables of a special case of the weight function considered by Spiridonov and Zhedanov in [83], and the two sets of elliptic hypergeometric biorthogonal functions constructed there are triangular with respect to the functions appearing in the two determinants. We could thus replace the second "cross-term" factor by a product of Vandermonde-style determinants in which the k-th row consists of the $(k-1)$st biorthogonal function evaluated at x_1 through x_c, analogously to our calculations for the q-Racah case above.

Using either lemma, we can in particular obtain a formula for the total weight of all tilings of a hexagon. Written in terms of plane partitions, we obtain the following elliptic analogue of MacMahon's identity.

Theorem 2.10.5. *Let p, q, u_1, u_2, u_3 be generic parameters such that $|p| < 1$, $u_1u_2u_3 = 1$. Then*

$$\sum_{\Pi\subset a\times b\times c}\prod_{(i,j,k)\in\Pi}\frac{q^3\theta_p(q^{j+k-2i-1}u_1,q^{i+k-2j-1}u_2,q^{i+j-2k-1}u_3)}{\theta_p(q^{j+k-2i+1}u_1,q^{i+k-2j+1}u_2,q^{i+j-2k+1}u_3)}$$
$$= q^{abc}\prod_{1\le i\le a,1\le j\le b,1\le k\le c}\frac{\theta_p(q^{i+j+k-1},q^{j+k-i-1}u_1,q^{i+k-j-1}u_2,q^{i+j-k-1}u_3)}{\theta_p(q^{i+j+k-2},q^{j+k-i}u_1,q^{i+k-j}u_2,q^{i+j-k}u_3)}.$$

Proof. The term corresponding to a specific plane partition Π is easily seen by induction to be the ratio of the weight of the corresponding tiling to the weight of the tiling associated to the empty plane partition. The claim follows by simplifying the corresponding determinant of W using Corollary 5.4 of [94]. □

Although it is possible to arrange for the elliptic weights to be positive, there are difficulties in the analysis. For one thing, the algebra required to replace orthogonal polynomials by biorthogonal functions in constructing the kernel has not been fully developed. A further complication in computing the limit kernel is that, although the biorthogonal functions do satisfy reasonably simple difference equations, they are not eigenfunctions of any difference operators. In addition, the corresponding variational problem is more difficult; while we can indeed solve the associated PDE, we have so far been unable to *derive* the solution. This is why we focus on a limiting case in the present paper.

Fix $u_1u_2 = pq\zeta^2$, let $u_1, u_2 = \Omega(\sqrt{p})$ as $p \to 0$, and similarly let $C(i) \sim p^{1/2}$. In this limit, one has

$$\lim_{p\to 0} w(i,j) = C'(i)\left(\zeta q^j - \frac{1}{\zeta q^j}\right),$$

or in other words the q-Racah weight(s) discussed above. In the notations of Sections 2.2-2.9, $\zeta = \kappa q^{-(S+1)/2}$. The corresponding limit of the elliptic MacMahon identity is

$$\sum_{\Pi\subset a\times b\times c}\prod_{(i,j,k)\in\Pi}\frac{q^{2k+1}\zeta^2 - q^{i+j-1}}{q^{2k}\zeta^2 - q^{i+j}}$$
$$= \prod_{1\le i\le a,1\le j\le b,1\le k\le c}\frac{(1-q^{i+j+k-1})(\zeta^2 - q^{i+j-k-2})}{(1-q^{i+j+k-2})(\zeta^2 - q^{i+j-k-1})},$$

or equivalently (performing the products over k and simplifying)

$$\sum_{\Pi\subset a\times b\times c} q^{|\Pi|}\prod_{1\le i\le a,1\le j\le b}\frac{\zeta^2 - q^{i+j-2\Pi_{ij}-2}}{\zeta^2 - q^{i+j-c-2}} = \prod_{1\le i\le a,1\le j\le b}\frac{1-q^{i+j+c-1}}{1-q^{i+j-1}}.$$

This, of course, becomes the usual MacMahon identity upon taking the limit $\zeta \to \infty$, cf. Section 7.21 in [84].

Chapter 3

The q–Gelfand–Tsetlin graph, Gibbs measures and q–Toeplitz matrices

3.1 Introduction

3.1.1 Preface

The infinite dimensional unitary group $U(\infty)$ is the union of the unitary groups $U(N)$ naturally embedded one into another. The study of *characters* of $U(\infty)$, i.e. normalized positive-definite central continuous functions on the group, was initiated by Voiculescu in 1976. He studied *finite factor representations* of $U(\infty)$. These representations are in bijection with *extreme characters* of $U(\infty)$, which are extreme points of the convex set of all characters of $U(\infty)$. In the paper [93] Voiculescu gave a list of extreme characters of $U(\infty)$. He also conjectured and partially proved that this list was complete.

There is a correspondence between extreme characters of $U(\infty)$ and *totally positive* Toeplitz matrices. Such matrices were studied much earlier by Shoenberg and his collaborators in the context of classical analysis. A few years after the paper [93], Boyer [15] and Vershik-Kerov [92] independently pointed out that the completeness of the Voiculescu's list of characters follows from the Edrei's result [23] on the classification of all totally positive Toeplitz matrices.

In the same paper [92] Vershik and Kerov suggested a new approach to the above problem. Their method is based on the approximation of the extreme characters of $U(\infty)$ by the normalized characters of the irreducible representations of the finite-dimensional unitary groups $U(N)$. As a result, classification of extreme characters of $U(\infty)$ is restated in purely combinatorial terms as the problem of the description of the *boundary of the Gelfand–Tsetlin graph*. Okounkov and Olshanski in their paper [65] gave the detailed proof and further generalized the classification theorem for the characters of $U(\infty)$ using the

Vershik-Kerov approach.

The aim of the present paper is to introduce and study a q–deformation of the notion of a character of $U(\infty)$. We start from the Vershik-Kerov formulation and define a q–deformed version of the Gelfand–Tsetlin graph. Our main result is the complete description of the boundary of the q–Gelfand–Tsetlin graph. (See Section 3.1.2 for the details.)

There are several ways to interpret our results. We may go back to the characters and introduce their q–analogues which agree with our deformation of the Gelfand–Tsetlin graph. From this point of view, the problem that we solve in the present paper is the characterization of all possible limits of rational Schur functions normalized in a certain (depending on q) way as the number of variables grows to infinity. (See Section 3.1.4 for the details.)

Investigating a q–deformation of totally positive Toeplitz matrices, we arrive at a notion of a q–Toeplitz matrix. Every point of the boundary of the q–Gelfand–Tsetlin graph corresponds to a q–Toeplitz matrix with minors satisfying some non-negativity condition. (See Section 3.1.5 for the details.)

Paths in the Gelfand–Tsetlin graph are in bijection with tilings of the halfplane with rhombuses of 3 types. Through this correspondence our results turn into the classification theorem for certain Gibbs measures on the tilings of the halfplane. (See Section 3.2 for the details.)

There are strong reasons to believe that the q–Gelfand–Tsetlin graph is related to the representation theory of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. However we do not address this issue in the present paper.

3.1.2 Statement of the main result

The Gelfand-Tsetlin graph $\mathbb{GT}$ is a graded graph consisting of levels $\mathbb{GT}_N$, $N = 1, 2, \dots$. The vertices of $\mathbb{GT}_N$ are N–tuples $\lambda_1 \geq \dots \geq \lambda_N$ of integers. Following the Weyl's book [95] we call these N–tuples *signatures*. We join two signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N+1}$ by an edge and write $\lambda \prec \mu$ if and only if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \dots \geq \lambda_N \geq \mu_{N+1}.$$

Let $\mathcal{T}$ denote the set of all infinite paths in $\mathbb{GT}$, i.e. $\tau = (\tau(1), \tau(2), \dots)$ belongs to $\mathcal{T}$ if and only if $\tau(N) \in \mathbb{GT}_N$ and $\tau(N)$ is connected with $\tau(N+1)$ by an edge in $\mathbb{GT}$ for every N.

For any probability measure P on $\mathcal{T}$ let $P_N(\cdot)$ denote its projection on $\mathbb{GT}_N$, in other words for any $\lambda \in \mathbb{GT}_N$

$$P_N(\lambda) = P(\tau \in \mathcal{T} \mid \tau(N) = \lambda).$$

For any finite path $\phi = (\phi(1) \prec \phi(2) \prec \dots \prec \phi(N))$, $\phi(k) \in \mathbb{GT}_k$ let C_ϕ denote the corresponding cylinder set in $\mathcal{T}$, i.e.

$$C_\phi = \{\tau \in \mathcal{T} : \tau(1) = \phi(1), \dots, \tau(N) = \phi(N)\}.$$

Let us introduce the *weight* of ϕ:

$$w(\phi) = q^{|\phi(1)| + |\phi(2)| + \dots + |\phi(N-1)|},$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_k$ for $\lambda \in \mathbb{GT}_k$. In Section 3.2 we show that these weights have a simple combinatorial interpretation: given a path ϕ one constructs a 3D–body and $w(\phi)$ equals q^{vol}, where *vol* is the volume of this body. For any $\lambda \in \mathbb{GT}_N$ let

$$\mathrm{Dim}_q(\lambda) = \sum_{\phi(1) \prec \cdots \prec \phi(N) | \phi(N) = \lambda} w(\phi).$$

A q–central measure on $\mathcal{T}$ is a probability measure P on $\mathcal{T}$ satisfying

$$P(C_\phi) = P_N(\phi(N)) \frac{w(\phi)}{\mathrm{Dim}_q(\phi(N))}$$

for any finite path $\phi = (\phi(1) \prec \phi(2) \prec \cdots \prec \phi(N))$, $\phi(k) \in \mathbb{GT}_k$. Put it otherwise, if we consider a family of finite paths ϕ ending at the same signature $\phi(N)$, then the probability of ϕ is proportional to the weight $w(\phi)$.

Let Ω_q denote the set of all q–central measures on $\mathcal{T}$. Clearly Ω_q is a convex set. *The minimal boundary of q–Gelfand–Tsetlin graph* is the set $\mathrm{Ex}(\Omega_q)$ of all extreme points of the set Ω_q. The main result of the present paper is the description of the minimal boundary of q–Gelfand–Tsetlin graph. We need to do some preparations before stating the theorem.

We say that signature $\lambda \in \mathbb{GT}_N$ is positive if all coordinates of λ are non-negative, i.e. $\lambda_N \geq 0$. Let $\mathbb{GT}_N^+ \subset \mathbb{GT}_N$ denote the subset of all positive signatures of size N.

Every extreme q–central measure P is uniquely defined by the array of numbers $P_N(\lambda)$. In the case studied by Voiculescu (i.e. $q = 1$) these numbers appear as the coefficients of the expansions of certain functions in the basis of rational Schur functions s_λ. It turns out that in the q–deformed case the formulas have the simplest form if instead of rational Schur functions we use other symmetric polynomials. It is convenient to use *q–interpolation Schur polynomials* $s^*_\lambda(x_1, \ldots, x_N; q)$ defined for every $\lambda \in \mathbb{GT}_N^+$. These polynomials are a particular case of factorial Schur functions (see [59]). They are also a particular case ($q = t$) of interpolation Macdonald polynomials (see [51], [79], [63]). In one dimensional case we have

$$s^*_k(x; q) = (x-1)(x-q)\cdots(x-q^{k-1}).$$

For any probability measure P_N on $\mathbb{GT}_N^+$ we consider the following generating function:

$$\mathcal{S}^*(x_1, \ldots, x_N; P_N) = \sum_{\lambda \in \mathbb{GT}_N^+} P_N(\lambda) \frac{s^*_\lambda(q^{N-1}x_1, \ldots, q^{N-1}u_N; q^{-1})}{s^*_\lambda(0, \ldots, 0; , q^{-1})} \tag{3.1}$$

Define $\mathcal{N}$ to be the set of all non-decreasing sequences of integers:

$$\mathcal{N} = \{\nu_1 \leq \nu_2 \leq \nu_3 \leq \ldots\} \subset \mathbb{Z}^\infty.$$

Finally, let A_ℓ be the isomorphism of the Gelfand–Tsetlin graph taking a signature $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{GT}_N$ to $A_\ell(\lambda) = (\lambda_1 + \ell, \ldots, \lambda_N + \ell)$.

Now we are ready to state the main theorem of the present paper.

Theorem 3.1.1. *Let $0 < q < 1$. We have:*

(i). $\mathrm{Ex}(\Omega_q)$ *is parameterized by points of $\mathcal{N}$ with $\nu \in \mathcal{N}$ corresponding to the extreme q–central measure $\mathcal{E}^\nu$.*

(ii). If $\nu_1 \geq 0$ then

$$\mathcal{S}^*(x_1, \dots, x_N; \mathcal{E}_N^\nu) = H^\nu(x_1) \cdots H^\nu(x_N),$$

where

$$H^\nu(t) = \frac{\prod_{i=0}^{\infty}(1 - q^i t)}{\prod_{j=1}^{\infty}(1 - q^{\nu_j + j - 1} t)}$$

and $\mathcal{E}_N^\nu$ is the projection of $\mathcal{E}^\nu$ on $\mathbb{GT}_N$.

(iii). For general ν the measure $\mathcal{E}^\nu$ is the image of the measure $\mathcal{E}^{\nu'}$ with $\nu' = (0, \nu_2 - \nu_1, \nu_3 - \nu_1, \dots)$ under isomorphism A_{ν_1} of the graph $\mathbb{GT}$.

(iv). Ω_q is a simplex meaning that every $P \in \Omega_q$ is a unique average of measures $\mathcal{E}^\nu$. Put it otherwise, for every $P \in \Omega_q$ there exist a unique probability measure Q on $\mathcal{N}$ such that

$$P = \int_{\mathcal{N}} \mathcal{E}^\nu dQ.$$

There is a natural involution of the graph $\mathbb{GT}$ mapping signature $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ to $-\lambda_N \geq -\lambda_{N-1} \geq \cdots \geq -\lambda_1$. Is is easy to see that this map provides a bijection between q–central measures and q^{-1}–central measures. Thus, Theorem 3.1.1 also provides a description of q–central measures with $q > 1$.

The following proposition describes the support of measure $\mathcal{E}^\nu$ and explains the geometric meaning of the parameter ν.

Proposition 3.1.2. *Let τ be a random element of $\mathcal{T}$ distributed according to $\mathcal{E}^\nu$, then almost surely the last coordinates of $\tau(N)$ tend to $\{\nu_j\}$, i.e. for every j*

$$\lim_{N \to \infty} \tau(N)_{N+1-j} = \nu_j.$$

The proof of Theorem 3.1.1 relies on the so-called ergodic method, first proposed by Vershik in [88]. This method was used for the graph $\mathbb{GT}$ in [92] and for the generalizations of $\mathbb{GT}$ in [65]. Essentially the same method of the identification of the boundary of a graph was proposed by Diaconis and Freedman [22] in the context of *partial exchangeability*.

For the q–Gelfand–Tsetlin graph the ergodic method shows that every extreme q–central measure on $\mathcal{T}$ is, in some precise sense, a limit of extreme measures on $\mathbb{GT}_N$. Passing from the measures to their generating functions we arrive at the problem of finding all possible limits of rational Schur functions normalized in a certain way as the number of variables grows to infinity. (In Section 3.1.4 we discuss this point in more details.) It seems natural to describe

extreme q–central measures in terms of the normalized rational Schur functions too. However, if we try to do that, then the formulas turn out to be quite ugly. In the contrast, if we use the polynomials s^* instead of rational Schur functions, then we arrive at simple multiplicative formulas presented in Theorem 3.1.1.

q–Interpolation Schur polynomials and their properties play a crucial rule in our proofs. One of the key ingredients is the *binomial formula*, which relates q–interpolation Schur polynomials with ordinary Schur polynomials and is a particular case of the binomial formula for interpolation Macdonald polynomials proved by Okounkov in [62].

3.1.3 Representations of $U(\infty)$ and the origin of the Gelfand–Tsetlin graph

In this section we explain how the study of characters of $U(\infty)$ leads to the Gelfand–Tsetlin graph and compare Theorem 3.1.1 with the solution of the problem studied by Voiculescu.

There are two different approaches to the representation theory of $U(\infty)$ which lead to a reasonable class of representations (i.e. finite factor representations, see e.g. [86], or spherical representations of Gelfand pair $(U(\infty) \times U(\infty), U(\infty))$, see [72], [73]). In both approaches representations are in correspondence with *characters*, which are central (i.e. constant on conjugacy classes) positive definite continuous functions on $U(\infty)$ taking value 1 on the unit element of the group.

The Gelfand-Tsetlin graph $\mathbb{GT}$ is a convenient tool for studying characters of $U(\infty)$. Recall that irreducible representations of $U(N)$ are parameterized by their dominant weights, which are N–tuples $\lambda_1 \geq \cdots \geq \lambda_N$ of integers. (See e.g. [87].) Thus, the vertices of $\mathbb{GT}_N$ symbolize irreducible representations of $U(N)$. Furthermore, the edges of $\mathbb{GT}$ encode inclusion relations between irreducible representations of $U(N+1)$ and $U(N)$.

Given any character χ of the group $U(\infty)$ we can expand its restriction to the subgroup $U(N)$ into a convex combination of normalized characters (in the conventional sense) of irreducible representations of $U(N)$:

$$\chi\big|_{U(N)} = \sum_{\lambda \in \mathbb{GT}_N} P_N(\lambda) \frac{\chi^\lambda(\cdot)}{\chi^\lambda(e)}. \tag{3.2}$$

It is known (see e.g. [87]) that the characters of irreducible representations of unitary groups are rational Schur functions, i.e.

$$\frac{\chi^\lambda(U)}{\chi^\lambda(e)} = \frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(1, \ldots, 1)},$$

where $u_1, \ldots, u_N$ are eigenvalues of U.

The coefficients $P_N(\lambda)$ determine a probability distribution on the set $\mathbb{GT}_N$. In this way we get a bijection between characters χ and sequences $\{P_N\}_{N=1}^\infty$ of probability distributions. These sequences are called *coherent systems* because P_N and P_{N+1} satisfy a certain relation.

Any coherent system $\{P_N\}$ defines a measure P on $\mathcal{T}$ in the following way. For a finite path $\phi = (\phi(1) \prec \cdots \prec \phi(N))$ and corresponding cylinder set C_ϕ set

$$P(C_\phi) = \frac{P_N(\phi(N))}{\mathrm{Dim}(\phi(N))},$$

where $\mathrm{Dim}(\lambda)$ is the number of paths $\tau(1) \prec \cdots \prec \tau(N)$ such that $\tau(N) = \lambda$. The coherency relations between P_N and P_{N+1} imply that P is well-defined. Measure P is *a central measure* in the sense that $P(C_\phi)$ depends only on $\phi(N)$. In other words, projection of P on the set of all finite paths $\phi(1), \dots \phi(N)$ ending at a fixed $\phi(N) = \lambda \in \mathbb{GT}_N$ is uniform. Note that P_N is a projection of measure P on $\mathbb{GT}_N$. Also note that a q–central measure becomes a central measure if we set $q = 1$.

Extreme points of the convex set of characters of $U(\infty)$ (in other words, extreme characters) correspond to *irreducible spherical representations* of pair $(U(\infty) \times U(\infty), U(\infty))$ (or, again, finite factor representations). Extreme points of the convex set of all central measures on $\mathcal{T}$ correspond to extreme characters via the above bijections.

Theorem ([93],[15],[92],[65])**.** *Extreme characters of $U(\infty)$ are parameterized by the points ω of the infinite-dimensional domain*

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

The corresponding extreme character is given by the formula

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spectrum}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)}. \tag{3.3}$$

The remarkable feature shared by the above theorem and Theorem 3.1.1 is the appearance of the multiplicative functions. There is an independent representation-theoretic argument proving that any extreme character of $U(\infty)$ is multiplicative (i.e. that it should be a product of the values of some function over the eigenvalues of the element of $U(\infty)$). However, the author knows no conceptual reason for the appearance of the multiplicativity in Theorem 3.1.1 (independent of the classification theorem itself), and it would be interesting to find one.

It is very natural to ask whether extreme central measures on $\mathcal{T}$ can be obtained as $q \to 1$ limits of extreme q–central measures on $\mathcal{T}$. The answer is "Yes" for extreme central measures such that $\alpha_i^+ = \alpha_i^- = 0$. For example, if we chose $\nu(q)$ such that

$$H^{\nu(q)}(t) = (1 - q^{x_1(q)}t)(1 - q^{x_2(q)}t) \cdots (1 - q^{x_k(q)}t)$$

and $q^{x_i(q)} \to \beta_i^+$ as $q \to 1$, then the measures $\mathcal{E}^{\nu(q)}$ tend as $q \to 1$ to the extreme central measure on $\mathcal{T}$ parameterized by $\beta_1^+, \ldots, \beta_k^+$. Using similar simple arguments one can obtain extreme central measures on $\mathcal{T}$ with arbitrary parameters $\gamma^\pm$, $\{\beta_i^\pm\}$. As for the general case, the author was not able to point out a sequence of extreme q–central measures (with q tending to 1) that converges to an extreme central measure with at least one nonzero coordinate $\alpha_i^\pm$.

3.1.4 Approximation of characters and limits of symmetric polynomials

Vershik and Kerov proved in [92] that every extreme character of $U(\infty)$ is a limit of normalized characters of irreducible representations of $U(N)$, i.e. of the functions

$$\frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(1, \ldots, 1)}. \tag{3.4}$$

In other words, they reduced the classification problem to the following question: What are the possible limits of symmetric polynomials (3.4) as the number of variables grows to infinity? The answers for the similar questions for more general polynomials were obtained by Okounkov and Olshanski. See [65, Theorem 1.1] and [66, Theorem 1.4].

The results of the present paper can be also interpreted as an answer to a certain asymptotic problem for symmetric polynomials as the number of variables grows to infinity.

Let $N(i)$ be an increasing sequence of positive integers and $\lambda(i) \in \mathbb{GT}_{N(i)}$. We call the sequence of signatures $\lambda(i)$ *regular* if for any k the sequence of functions

$$\frac{s_{\lambda(i)}(x_1, \ldots, x_k, q^{-k}, q^{-k-1}, \ldots, q^{1-N(i)})}{s_{\lambda(i)}(1, q^{-1}, \ldots, q^{1-N(i)})} \tag{3.5}$$

converges uniformly on the set $\{(x_1, \ldots, x_k) \in \mathbb{C}^k \mid |x_i| = q^{1-i}\}$.

The normalization $s_\lambda(1, q^{-1}, \ldots, q^{1-k})$, that we use, is well-known. See e.g. [58, Examples in Section 3, Chapter 1].

Let Λ be an algebra of symmetric functions, i.e. a projective limit of the graded algebras $Sym(N)$ of symmetric polynomials in N variables (see e.g. [58]).

Theorem 3.1.3. *Let $0 < q < 1$. We have:*

(i). Sequence of signatures $\lambda(i)$ is regular if and only if the last coordinates of $\lambda(i)$ stabilize, i.e. if there is a nondecreasing sequence of integers $\nu = \{\nu_j\}$ such that for any $j > 0$

$$\lim_{i\to\infty} \lambda(i)_{N(i)+1-j} = \nu_j.$$

(ii). The limit function $Q_k^\nu(x_1, \dots, x_k)$ is an entire function in $\mathbb{C}^k$.

(iii). There is a bijection between limit functions Q_k^ν and measures $\mathcal{E}_k^\nu$. More precisely

$$Q_k^\nu = \sum_{\mu\in\mathbb{GT}_k} \mathcal{E}_k^\nu(\mu) \frac{s_{\lambda(i)}(x_1, \dots, x_k)}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-k})}$$

(iv). If ν and ν' are such that $\nu_i = \nu'_i + \ell$ for every i, then

$$Q_k^\nu = \frac{x_1^\ell \dots, x_k^\ell}{(1 \cdot q^{-1} \cdots q^{1-k})^\ell} Q_k^{\nu'}.$$

(v). If $\nu_1 \geq 0$, then

$$Q_k^\nu = \sum_{\mu\in\mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k; q),$$

where the series converges everywhere in $\mathbb{C}^k$ and Spec_ν is a specialization of algebra of symmetric function Λ (put it otherwise, homomorphism from Λ to $\mathbb{C}$) with H–generating function

$$\sum_{j=0}^{\infty} \mathrm{Spec}_\nu(h_j) t^j = \frac{\prod\limits_{i\geq 0}(1 - q^i t)}{\prod\limits_{j=1}^{\infty}(1 - q^{\nu_j+j-1} t)}.$$

(Here h_j stays for the complete symmetric function of degree j.)

Remark 1. The author knows no general explicit formulas for the functions Q_k^ν. This is the main reason why we do not use these functions as a description of measures $\mathcal{E}^\nu$. Instead we use $\mathcal{S}^*(x_1, \dots, x_k; \mathcal{E}_k^\nu)$, for which a simple multiplicative formula is provided in Theorem 3.1.1

Remark 2. In a special case $\nu_n = \nu_{n+1} = \dots = a$ the functions Q_k^ν admit a simple interpretation. Let us consider only the case $a = 0$. Note that if $0 < q < 1$, then for any $f \in \Lambda$ sequence $f_n = f(1, q, \dots, q^k)$ converges as $n \to \infty$, and a specialization map

$$f(y_1, y_2, \dots) \mapsto f(1, q, q^2, q^3, \dots)$$

is a well-defined homomorphism from Λ to $\mathbb{R}$. In the same way the map

$$f(y_1, y_2, \dots) \mapsto f(x_1^{-1}, \dots, x_k^{-1}, q^k, q^{k+1}, \dots)$$

is a homomorphism from Λ to the ring of Laurent polynomials in $x_1, \dots, x_k$.

Now let λ be a Young diagram with $n-1$ rows such that $\lambda_1 = -\nu_1, \dots, \lambda_{n-1} = -\nu_{n-1}$ and consider Schur function s_λ — element of Λ. The image of s_λ under the map

$$f(y_1, y_2, \dots) \mapsto \frac{f(x_1^{-1}, \dots, x_k^{-1}, q^k, q^{k+1}, \dots)}{f(1, q, q^2, \dots)}$$

coincides with Q_k^ν.

Remark 3. The functions

$$\frac{s_\lambda(x_1, \dots, x_N)}{s_{\lambda(i)}(1, q, \dots, q^{1-N})}$$

can be viewed as quantum traces of irreducible representations of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_N)$. Furthermore, one can show that the definition of a q–central measure is related to the branching rules for quantum characters of irreducible representations of $U_\epsilon(\mathfrak{gl}_N)$. (This is parallel to the fact that the Gelfand-Tsetlin graph itself is related to the branching rules of irreducible representations of the unitary group.) Thus, it is natural to expect that the functions Q_k^ν and q–central measures are related to certain representations of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. The author hopes to address this issue in a later publication.

Remark 4. It is natural to ask what happens if one replaces geometric series q^{1-i} in (3.5) by an arbitrary sequence ξ_i. The answer to this question remains unknown.

3.1.5 Toeplitz and q–Toeplitz matrices

Let us explain the connection between extreme characters of the group $U(\infty)$ and total positivity of Toeplitz matrices. Recall that any extreme character of $U(\infty)$ is a multiplicative function:

$$\chi(U) = \prod_{u_i} \widehat{\chi}(u_i), \quad u_i \in \text{Spectrum}(U).$$

Thus, χ is uniquely defined by $\widehat{\chi}(u)$ which is a continuous function on S^1. $\widehat{\chi}(u)$ can be represented as

$$\widehat{\chi}(u) = \sum_{l \in \mathbb{Z}} c_l u^l.$$

Introduce an infinite Toeplitz matrix

$$c[i,j] \stackrel{def}{=} c_{i-j}.$$

As a corollary of the fact that χ is a positive–definite function, one proves that the matrix $c[i,j]$ is *totally positive*, i.e. all minors of $c[i,j]$ are non-negative. Furthermore, this correspondence is a bijection between extreme characters of

$U(\infty)$ and totally positive infinite Toeplitz matrices such that the sum of the matrix elements along the row equals 1. (See [93], [15], [92].)

In order to extend the correspondence between extreme measures and certain matrices to the case of general q we deform the notion of a Toeplitz matrix. We call a semi-infinite matrix $d[i,j]$, $i>0$, $j>0$ a *semi-infinite q–Toeplitz matrix* if

$$d[i,j+1] = d[i-1,j] + (q^{1-j} - q^{1-i})d[i,j] \tag{3.6}$$

for all $i>0$, $j>0$. Here we agree that $d[i,j]=0$ is either $i<1$ or $j<1$. Note that when $q=1$, the relation (3.6) turns into

$$d[i,j+1] = d[i-1,j].$$

Hence, a q–Toeplitz matrix becomes a Toeplitz matrix.

Recall that according to Theorem 3.1.1 extreme q–central measures correspond to multiplicative functions $H(x_1)\cdots H(x_N)$. Given a function $H(t)$ we construct a lower-triangular semi-infinite q–Toeplitz matrix $d[i,j]$ in the following way: expand H in series

$$H(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1}(1-tq^i)$$

and let $d[i,j]$, $i>0$, $j>0$ be a unique semi-infinite q–Toeplitz matrix such that

$$d[i,1] = c_{i-1}, \quad i=1,2,\ldots.$$

Initial minor of size N of matrix $d[i,j]$ is a minor corresponding to either the first N columns and arbitrary N rows of matrix $d[i,j]$ or the first N rows and arbitrary N columns of $d[i,j]$.

Proposition 3.1.4. *Let ν be a non-decreasing sequence of non-negative integers $0\le\nu_1\le\nu_2\le\ldots$, let $\mathcal{E}^\nu$ be the extreme q–central measure parameterized by ν and corresponding to the function H^ν. If $d^\nu[i,j]$ is a semi-infinite q–Toeplitz matrix constructed by H^ν, then all initial minors of $d^\nu[i,j]$ are non-negative.*

A general theorem (see [27]) says that if A is a finite non-degenerate matrix with non-negative initial minors, then A is totally positive, in other words *all* minors of A are non-negative. But, alas, the q–Toeplitz matrices corresponding to q–central measures are usually degenerate. These matrices are triangular and some elements on the main diagonal vanish. Thus, we cannot guarantee that q–Toeplitz matrices corresponding to q–central measures are totally positive, we can only claim that their certain top-left corners are. And, indeed, straightforward computations show that even some matrix elements of these matrices are negative.

However, it might be still interesting to classify all q–Toeplitz matrices with non-negative initial minors. We have the following conjecture here, which is a straightforward analogue of $q=1$ case.

Conjecture 3.1.5. *If $d[i,j]$ is a lower-triangular semi-infinite q–Toeplitz matrix such that*

$$\sum_{i=1}^{\infty} d[i,1] = 1,$$

then $d[i,j]$ coincides with one of the matrices from Proposition 3.1.4, i.e. $d[i,j] = d^{\nu}[i,j]$ for some ν.

3.1.6 Other branching graphs and general formalism

The Gelfand–Tsetlin graph and the q–Gelfand–Tsetlin graph are two examples of *branching graphs*. (This term was introduced by Vershik and Kerov.) There is a wide class of problems that can be stated as a problem of identification the boundary of a branching graph.

A bunch of examples comes from the representation theory of "big" groups. Representations of the infinite symmetric group $S(\infty)$ are related to the boundary of the Young graph (see [85], [91], [64]), projective representations of $S(\infty)$ are related to the Schur graph (see [60], [38]), 'block characters" of $S(\infty)$ are related to the Euler graph (see [29]), and (as we already mentioned) representation of the infinite dimensional unitary group $U(\infty)$ are related to the Gelfand–Tsetlin graph.

However, there are other examples of purely probabilistic and combinatorial nature. Perhaps, the most known example is De Finetti's theorem (see. e.g [26, Chapter VII, §4] or [1]) which states that every probability measure on $\{0,1\}^{\infty}$ invariant with respect to finite permutations of coordinates is a mixture of Bernoulli measures. Here the underlying branching graph is the Pascal graph.

Motivated by a problem of population genetics Kingman introduced in [50] the notion of a partition structure. Kingman's classification of partition structures is equivalent to the description of the boundary of a certain graph, that is now called the Kingman graph, see also [48].

Other examples can be found in [67], [30], [31], [32], [33].

Let us consider a subgraph of $\mathbb{GT}$ consisting of zero–one signatures, i.e. $\lambda_1 \geq \cdots \geq \lambda_N$ such that $1 \geq \lambda_1 \geq \lambda_N \geq 0$. q–central measures on paths in this subgraph can be identified with central measures on paths in the q–Pascal graph studied by Gnedin and Olshanski [30]. These measures are related to q–analogues of De Finetti's theorem. (See also earlier paper by Kerov [47, Chapter 1.4]) The authors of [30] proved that for $0 < q < 1$ the boundary of the q–Pascal graph is parameterized by points of the set $\{1, q, q^2, \dots\} \cup \{0\}$. This result agrees with our description of the boundary of the q–Gelfand–Tsetlin graph. Indeed, the only extreme q–central measures concentrated on zero–one signatures are those parameterized by ν with $0 \leq \nu_i \leq 1$ for every i. The extreme central measure on q–Pascal graph parameterized by q^k corresponds to q–central measure on $\mathcal{T}$ parameterized by $\nu = 0 \leq 0 \leq \cdots \leq 0 \leq 1 \leq 1 \dots$ with exactly k zeros.

In another article [31] Gnedin and Olshanski studied a multidimensional generalization of the model of [30]. The common feature of the results of both

these two papers and the present paper is that in the contrast to $q = 1$ case, parameters of the extreme q–central measures (and of the measures studied in [30], [31]) are discrete.

As a final remark of this section we want to mention the paper [22] where Diaconis and Freedman introduced a notion of partial exchangeability. Both central and q–central measures of $\mathcal{T}$ are particular cases of *partially exchangeable probabilities*.

3.1.7 Further connections and developments

Representation theory of $U(\infty)$ shows numerous connections with $S(\infty)$, i.e. inductive limit of symmetric groups S_n. (See [70] and [49] for reviews of the representation theory of $S(\infty)$) For example, while extreme characters of $U(\infty)$ are related to infinite totally positive Toeplitz matrices, extreme character of $S(\infty)$ are similarly related to semi-infinite totally positive Toeplitz matrices. The description of the simplex of q–central measures (in particular, their "right" definition) on the Young graph (substitute of the graph $\mathbb{GT}$ for the group $S(\infty)$) is yet to be done.

The are two important problems in the representation theory of the infinite-dimensional unitary group: identification of all irreducible representations and decomposition of natural representations into irreducible ones (see [71]). Irreducible representations (again, we should speak either about finite factor representations or irreducible spherical representations here) lead to extreme central measures on $\mathcal{T}$ and in the present paper we study a q–analogue of these measures. The natural representations associated with $U(\infty)$, in turn, lead to a remarkable family of central measures on $\mathcal{T}$, i.e. so called (z, w)–measures. At the moment it is unclear whether one can introduce some natural q–central measures that will serve as a q–analogue of (z, w)–measures.

Although, the study of central measures on $\mathcal{T}$ has a representation-theoretic origin, later these measures led to a number of very interesting probability models related to random Young diagrams and random stepped surfaces, see [7], [4], [3]. The author hopes that q–central measures introduced in the present paper will also provide a source of new interesting probability models.

Organization of the chapter

In Section 2 we introduce all the basic objects of the study and give a purely combinatorial definition of q–central measures. Sections 3 and 4 contain the information about the technical tools that we use for analyzing q–central measures. In Section 3 we introduce various symmetric polynomials and Section 4 is devoted to the definitions of certain special probability generating functions for q–central measures. In Section 5 we prove the main theorems of the present paper. Some of the proofs are quite involved and contain lots of technicalities, and we moved these proofs to Section 6. Finally, in Section 7 we discuss the relations with q–Toeplitz matrices.

3.2 Combinatorial setup

$\star$ **Throughout the paper we assume** $0 < q < 1$**.** $\star$

In this section we introduce the basic definitions and give a combinatorial interpretation for the notion of q–centrality. We are going to follow the notations of [58] when it is possible.

A *Young diagram* μ is a finite collection of boxes arranged in rows with nonincreasing row lengths μ_i. The total number of boxes in μ is denoted by $|\mu|$. Every box of μ has two coordinates (i, j), the first one is increasing from top to bottom and the second one is increasing from left to right. The top-left corner of a diagram has coordinates $(1, 1)$. If we reflect Young diagram μ with respect to the diagonal $i = j$, then we obtain a *transposed diagram* μ'. Row lengths μ'_i of μ' are column lengths of μ.

The set of all Young diagrams has a natural partial order by inclusion relation, i.e. we write $\lambda \subset \mu$ if for every i we have $\lambda_i \leq \mu_i$.

A *signature* λ of size N is an ordered collection of integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N,$$

we call λ_i the ith coordinate of λ.

Given a signature λ we construct two Young diagrams λ^+ and λ^- which are called a *positive diagram* and a *negative diagram*, respectively. The row lengths of the former one are all the positive λ_is, while the row lengths of the latter are absolute values of negative λ_is. An example is shown in Figure 3.1. Let $|\lambda|$ denote the sum of coordinates of λ. Clearly, $|\lambda| = |\lambda^+| - |\lambda^-|$.

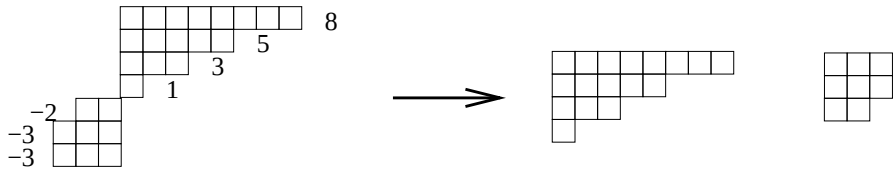


Figure 3.1: Signature $8 \geq 5 \geq 3 \geq 1 \geq 0 \geq -2 \geq -3 \geq -3$ and two corresponding Young diagrams.

Let us write $\lambda \leq \mu$ for two signatures of the same size if $\lambda_i \leq \mu_i$ for every i. Clearly, $\lambda \leq \mu$ is equivalent to $\lambda^+ \subset \mu^+$ and $\mu^- \subset \lambda^-$.

Let $\mathbb{GT}_N$ denote the set of all signatures of size N and let $\mathbb{GT}_N^+$ denote the set of al signatures of size N with nonnegative coordinates. Every element of $\mathbb{GT}_N^+$ can be identified with a Young diagram, however note, that a signature has an additional information, its size N.

A path $\tau \in \mathcal{T}$ in $\mathbb{GT}$ is a sequence of signatures $\{\lambda(n) \in \mathbb{GT}_n\}_{1,\ldots}$ such that for every n, $\lambda(n) \prec \lambda(n+1)$. In other words,

$$\lambda(n+1)_1 \geq \lambda(n)_1 \geq \lambda(n+1)_2 \geq \cdots \geq \lambda(n)_n \geq \lambda(n+1)_{n+1}.$$

Paths in $\mathbb{GT}$ are usually called *Gelfand-Tsetlin schemes* in representation-theoretic literature.

Let $\mathcal{T}_N$ denote the set of all paths $\tau(1) \prec \cdots \prec \tau(N)$ of length N with $\tau(i) \in \mathbb{GT}_i$. The set $\mathcal{T}$ of all infinite paths is a projective limit of sets $\mathcal{T}_N$. (Here projection is just a removal of the last step of a path.)

We define $\mathbb{GT}^+$ to be the part of $\mathbb{GT}$ consisting of signatures with nonnegative coordinates. Let $\mathcal{T}^+$ and $\mathcal{T}_N^+$ denote the corresponding sets of paths.

A (semistandard Young) *tableau* T of shape $\lambda \in \mathbb{GT}_N^+$ is an assignment of numbers $1, \ldots, N$ to boxes of the Young diagram λ in such a way that the numbers are increasing along the columns and non-decreasing along the rows. Given a path $\tau \in \mathcal{T}_N$ we construct two Young tableaux T_τ^+ and T_τ^- as follows: Shape of T_τ^+ is $\tau(N)^+$, $T_\tau^+(i,j) = k$ if and only if $(i,j) \in \tau(k)^+ \setminus \tau(k-1)^+$, where we agree that $\tau(0)^+ = \emptyset$. Similarly, shape of T_τ^- is $\tau(N)^-$, $T_\tau^-(i,j) = k$ if and only if $(i,j) \in \tau(k)^- \setminus \tau(k-1)^-$.

Every tableau T of shape $\lambda \in \mathbb{GT}_N$ corresponds to a *3-D Young diagram* in the following way: For every k put $N-k$ unit cubes on all the boxes of T with number k. The union of all these unit cubes is the desired 3-D Young diagram. An example of the above procedure is shown in Figure 3.2.

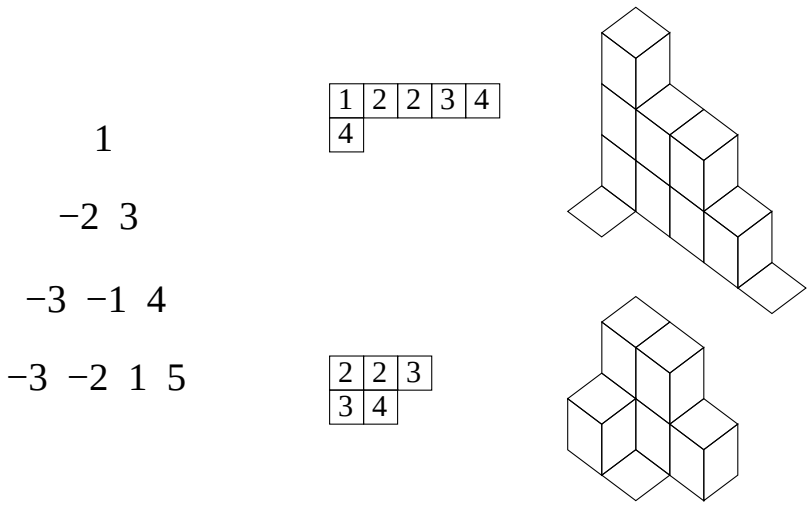


Figure 3.2: Finite path from $\mathcal{T}_4$, corresponding semistandard Young tableaux and 3-D Young diagrams.

Let $V(T)$ denote the volume of 3-D Young diagram corresponding to T.

Lemma 3.2.1. *The following formulas hold for an arbitrary $\tau \in \mathcal{T}_N$:*

$$V(T_\tau^+) - V(T_\tau^-) = \sum_{i=1}^{N-1} |\tau(i)| = \sum_{i=1}^{N} (N-i)(|\tau(i)| - |\tau(i-1)|),$$

where we agree that $|\tau(0)| = 0$

We leave the proof to the reader.

For a finite path $\tau \in \mathcal{T}_N$ let C_τ be a corresponding cylinder set in $\mathcal{T}$, i.e.

$$C_\tau = \{\lambda(1) \prec \lambda(2) \prec \cdots \in \mathcal{T} \mid \lambda(1) = \tau(1), \ldots, \lambda(N) = \tau(N)\}.$$

We equip $\mathcal{T}$ with a σ-algebra spanned by all cylinder sets. We are going to consider various probability measures on $\mathcal{T}$. For any finite path τ we usually write $P(\tau)$ instead of $P(C_\tau)$ where it leads to no confusion.

For any probability measure P on $\mathcal{T}$ let P_N be its projection on $\mathbb{GT}_N$, i.e.

$$P_N(\{\lambda\}) = P(\{\tau \in \mathcal{T} : \quad \tau(N) = \lambda\}).$$

To simplify the notation we usually write $P_N(\lambda)$ instead of $P_N(\{\lambda\}$

Recall that a probability measure P on $\mathcal{T}$ is called *q–central* if probabilities of paths $\tau \in \mathcal{T}_N$ ending at the same signature λ are proportional to $q^{V(T_\tau^+)-V(T_\tau^-)}$, i.e. for any $\tau \in \mathcal{T}_N$ such that $\tau(N) = \lambda$ we have:

$$P(\tau) = P_N(\lambda) \frac{q^{V(T_\tau^+)-V(T_\tau^-)}}{\sum\limits_{\theta \in \mathcal{T}_N : \theta(N)=\lambda} q^{V(T_\theta^+)-V(T_\theta^-)}}.$$

The main goal of the present paper is to describe the convex set of all q–central probability measures on $\mathcal{T}$.

Statistical mechanical interpretation

Before starting the proofs of the main theorems let us show a way to interpret q–central measures in the spirit of statistical mechanics. (The material of this subsection is not further used throughout the paper.)

Consider a tiling of the halfplane by rhombuses of 3 types (such rhombuses are usually called *lozenges*) as shown in Figure 3.3.

A tiling is uniquely defined by the positions of *horizontal* lozenges (i.e. lozenges with no sides parallel to the vertical line; such lozenge is the middle one in the left part of Figure 3.3). Given a path $\lambda(1) \prec \lambda(2) \prec \ldots$ in $\mathbb{GT}$ consider a tiling with coordinates of horizontal lozenges $(N, \lambda(N)_i + N - i - 1)$ as shown in Figure 3.4, left panel.

There is a well–known bijection between tilings and discrete stepped surfaces. To obtain such surface we arrange coordinates of the signatures of a path as in Figure 3.4, right panel; we get some function, which is usually called *the height function*. Graph of this function is a stepped surface and projection of the surface in the direction $(1, 1, 1)$ is precisely a lozenge tiling.

q–Centrality of probability measures on paths in $\mathbb{GT}$ turns into the following Gibbs property of measures on tilings: given positions of horizontal lozenges on a vertical line, tilings of the strip to the left from this line are distributed with weight q^{vol}, where *vol* is the volume enclosed underneath the corresponding stepped surface. Note that the differences between the volumes underneath

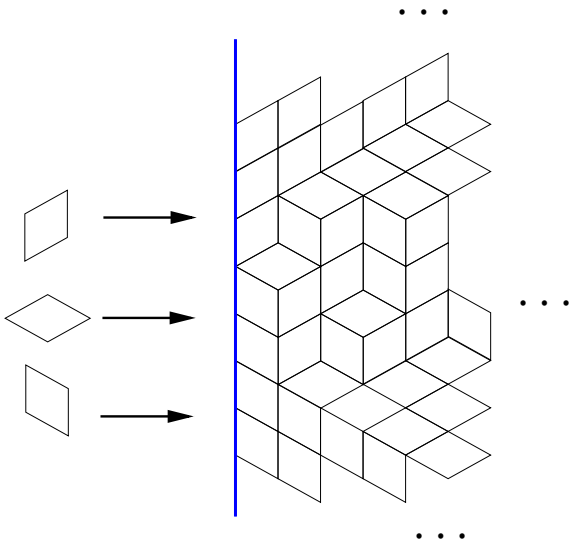


Figure 3.3: Lozenge tiling of the halfplane

surfaces corresponding to the distinct tilings of the strip are finite. Thus, the weight q^{vol} is well defined.

Therefore, the problem solved in the present paper can be restated as the classification of certain Gibbs measures on the lozenge tilings of the halfplane.

3.3 Symmetric polynomials

In this section we introduce various symmetric functions and the algebras they belong to. These functions play an important role in our study of q–central measures.

3.3.1 Schur functions and factorial Schur functions

Recall that a (rational) Schur function $s_\lambda(x_1, \dots, x_N)$ parameterized by an arbitrary signature $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N$, is a symmetric Laurent polynomial given by

$$s_\lambda = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{i,j=1,\dots,N}}{\prod\limits_{i<j} (x_i - x_j)}.$$

If $\lambda \in \mathbb{GT}^+$, then s_λ is an ordinary symmetric polynomial. It is known that rational Schur functions form a linear basis in the space of all symmetric Laurent

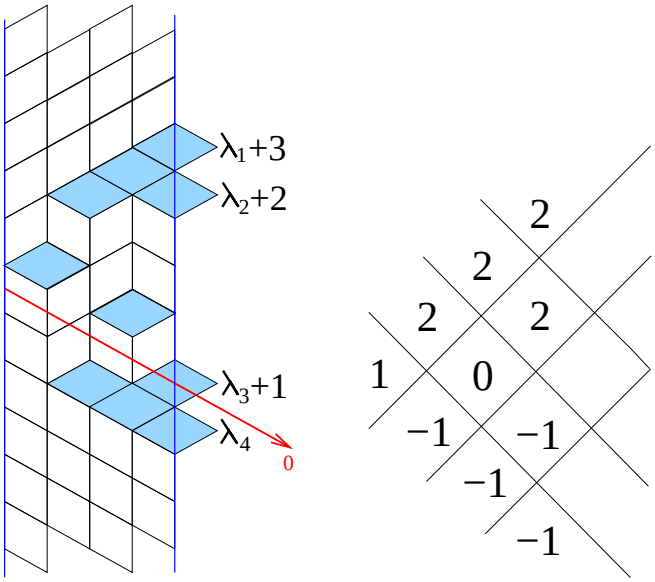


Figure 3.4: Part of the tiling corresponding to the path in $\mathbb{GT}$ $(1) \prec (2 \geq -1) \prec (2 \geq 0 \geq -1) \prec (2 \geq 2 \geq -1 \geq -1)$

polynomials. We recommend [58] as a general source of information about symmetric polynomials and Schur functions.

Proposition 3.3.1 (The branching rule for Schur functions)**.** *For any $\lambda \in \mathbb{GT}_N$ we have:*

$$s_\lambda(x_1, \dots, x_N) = \sum_{\mu \prec \lambda} s_\mu(x_1, \dots, x_{N-1}) x_N^{|\lambda|-|\mu|},$$

where $|\lambda|$ and $|\mu|$ stand for the sum of the coordinates of signatures λ and μ correspondingly.

See e.g. [58, Chapter 1, Section 5] for the proof. Iterating the branching rule for Schur functions we get the following:

Proposition 3.3.2 (The combinatorial formula)**.** *For any $\lambda \in \mathbb{GT}_k$ we have:*

$$s_\lambda(x_1, \dots, x_N) = \sum_{\tau(0) \prec \dots \prec \tau(N)} x_1^{|\tau(1)|-|\tau(0)|} \dots x_N^{|\tau(N)|-|\tau(N-1)|},$$

where the sum is taken over all paths in $\mathbb{GT}$ $\tau(0) \prec \dots \prec \tau(N)$ such that $\tau(N) = \lambda$ and $|\tau(i)|$ stands for the sum of the coordinates of signature $\tau(i)$.

From now on assume that $\lambda \in \mathbb{GT}_N^+$, i.e. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$.
Let $\{a_n\}_{n \in \mathbb{Z}}$ be any sequence of numbers. For any $r \geq 0$ let

$$(x \mid a)^r = (x + a_1) \dots (x + a_r).$$

Factorial Schur function $s_\lambda(x \mid a)(x_1, \dots, x_N)$ is a symmetric polynomial in variables $x_1, \dots x_N$ defined through

$$s_\lambda(x \mid a) = \frac{\det\left[(x_i \mid a)^{\lambda_j + N - j}\right]_{i,j=1,\dots,N}}{\prod\limits_{i<j}(x_i - x_j)}.$$

See [59, 6th Variation] for the properties of these polynomials.

Proposition 3.3.3 (Combinatorial formula for factorial Schur polynomials)**.** *For any $\lambda \in \mathbb{GT}_N^+$ we have:*

$$s_\lambda(x \mid a) = \sum_T \prod_{(i,j)\in\lambda} (x_{T(i,j)} + a_{T(i,j)+j-i}),$$

where the sum is taken over all semistandard Young tableau $T(i,j)$ of shape λ filled with numbers $1, \dots, N$.

3.3.2 q–interpolation Schur polynomials

q–Interpolation Schur polynomials $s^*_\lambda(x; q)$ are factorial Schur polynomials with $a_i = -q^{i-N}$:

$$s^*_\lambda(x_1, \dots, x_N; q) = s_\lambda(x \mid a), \quad a = \{-q^{i-N}\}.$$

This polynomials are a particular case of Macdonald interpolation polynomials for the case $q = t$. (See [51], [79], [63].)

Let us introduce some notations. Suppose $\mu \in \mathbb{GT}_N^+$, and recall that μ can be identified with a Young diagram with not more than N rows. Let $\mu_1 \geq \mu_2 \geq \dots$ be the row lengths of μ and let $\mu'_1 \geq \mu'_2 \geq \dots$ be the row lengths of the transposed diagram μ', or, equivalently, the column lengths of μ. For any box $(i,j) \in \mu$ set $c(i,j) = j - i$ and $h(i,j) = \mu_i - i + \mu'_j - j + 1$. Denote $n(\mu) = \sum(i-1)\mu_i$.

Proposition 3.3.4 (Interpolation property, [63])**.** *The q–interpolation Schur polynomial $s^*_\mu(x_1, \dots, x_N; q)$ is the unique symmetric polynomial in $x_1, \dots, x_N$ such that:*

(i). $\deg(s^*_\mu(x_1, \dots, x_N; q)) = |\mu|$

(ii). $s^*_\mu(q^{\mu-\delta}; q) = q^{n(\mu') - 2n(\mu)} \prod\limits_{(i,j)\in\mu} (q^{h(i,j)} - 1)$

(iii). $s^*_\mu(q^{\lambda-\delta}; q) = 0$ *for all positive signatures (Young diagrams)* $\lambda \in \mathbb{GT}_N^+$ *such that* $\mu \nsubseteq \lambda$,

where $q^{\lambda-\delta}$ *is* $(q^{\lambda_1}, q^{\lambda_2 - 1}, \dots, q^{\lambda_N - N + 1})$

Rewriting the combinatorial formula for the factorial Schur polynomials we get:

Proposition 3.3.5. *We have*

$$s^*_\mu(x_1,\dots,x_N;q)=\sum_T\prod_{(i,j)\in\mu}\left(x_{T(i,j)}-q^{j-i+T(i,j)-N}\right),$$

where the sum is taken over all semistandard Young tableau $T(i,j)$ of shape λ filled with numbers $1,\dots,N$.

In one-dimensional case interpolation Schur polynomials are enumerated by nonnegative integers and

$$s^*_k(x;q)=(x-1)\dots(x-q^{k-1})$$

Polynomials $s^*_\mu(x_1,x_2\cdot,\dots,x_N;q)$ form a linear basis in the space of symmetric polynomials in N variables. The leading term of $s^*_\mu(x_1,x_2\cdot,\dots,x_N\cdot;q)$ coincides with ordinary Schur function $s_\mu(x_1,x_2,\dots,x_N)$.

The basis s^*_μ is connected with the basis s_μ via the following formulas due to Okounkov [62]

$$s_\lambda(x_1,x_2,\dots,x_k)=\sum_{\mu\in\mathbb{GT}^+_k}\frac{s^*_\mu(q^{\lambda-\delta};q)}{s^*_\mu(q^{\mu-\delta};q)}\frac{s_\lambda(1,q^{-1},\dots,q^{1-k})}{s_\mu(1,q^{-1},\dots,q^{1-k})}s^*_\mu(x_1,\dots,x_k;q), \tag{3.7}$$

$$s^*_\lambda(x_1,x_2,\dots,x_k;q)=\sum_{\mu\in\mathbb{GT}^+_k}\frac{s^*_\mu(q^{-(\lambda-\delta)};q^{-1})}{s^*_\mu(q^{-(\mu-\delta)});q^{-1})}\frac{s^*_\lambda(0,\dots,0;q)}{s^*_\mu(0,\dots,0;q)}s_\mu(x_1,\dots,x_k). \tag{3.8}$$

Observe that in both formulas only diagrams μ such that $\mu\subset\lambda$ give a nonzero contribution. Thus, both sums are actually finite. Also note that the formulas (3.7) and (3.8) look similar, we are going to further use this symmetry.

3.3.3 Algebras of symmetric functions

Let Λ_N be the graded algebra of symmetric polynomials in $x_1,\dots,x_N$. Let Λ be a projective limit of graded algebras Λ_N with respect to projections ρ_N:

$$\rho_N:\Lambda_{N+1}\to\Lambda_N,\quad \rho_N(f(x_1,\dots,x_{N+1}))=f(x_1,\dots,x_N,0).$$

Λ is usually called *the algebra of symmetric functions.* There are 3 well-known systems of algebraic generators of Λ. They are *Newton power sums* p_k

$$p_k=\sum_{i=1}^\infty x_i^k,$$

elementary symmetric functions e_k

$$e_k=\sum_{\ell_1<\ell_2<\dots<\ell_k}x_{\ell_1}\cdots x_{\ell_k}$$

and *complete symmetric functions* h_k

$$h_k = \sum_{\ell_1 \le \ell_2 \le \cdots \le \ell_k} x_{\ell_1} \cdots x_{\ell_k}.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{GT}_k^+$ and let $\widehat{\lambda} \in \mathbb{GT}_{k+1}^+$ be a signature obtained by adding one zero to λ, i.e. $\widehat{\lambda} = (\lambda_1, \ldots, \lambda_k, 0)$. Schur polynomials corresponding to non-negative signatures are stable in the sense that

$$s_{\widehat{\lambda}}(x_1, \ldots, x_k, 0) = s_\lambda(x_1, \ldots, x_k)$$

for any $\lambda \in \mathbb{GT}_k^+$. Thus, the Schur polynomial corresponding to a non-negative signature λ defines an element of Λ that we denote s_λ. Functions s_λ form a linear basis in algebra Λ.

Let $\widehat{\Lambda}$ be a projective limit of filtered algebras $\widehat{\Lambda}_N$, where $\widehat{\Lambda}_N$ is the ordinary algebra of symmetric polynomials in N variables filtered by the degree, and the projections $\widehat{\rho}_N$ are given by the following formula:

$$\widehat{\rho}_N : \widehat{\Lambda}_{N+1} \to \widehat{\Lambda}_N, \quad \widehat{\rho}_N(f(x_1, \ldots, x_{N+1})) = f(x_1, \ldots, x_N, q^{-N})$$

Polynomials s_μ^* are stable in the sense that

$$s_{\widehat{\mu}}^*(x_1, \ldots, x_N, q^{-N}; q) = s_\mu^*(x_1, \ldots, x_N; q)$$

for any $\mu \in \mathbb{GT}_N^+$. This fact follows from Proposition 3.3.4. Thus, these polynomials can be viewed as elements of a filtered algebra $\widehat{\Lambda}$. Furthermore, s_μ^* form a linear basis of $\widehat{\Lambda}$.

3.4 q–Central measures and probability generating functions

In this section we state a number of propositions about q–central measures and probability generating functions related to them. Some of the proofs are quite technical and we omit them for the convenience of the reader. All the missing proofs are given in Section 3.6.2.

For any $\mu \in \mathbb{GT}_N$ denote

$$\mathrm{Dim}_q(\mu) = \sum_{\tau \in \mathcal{T}_N,\, \tau(N) = \mu} q^{|\tau(1)| + \cdots + |\tau(N-1)|}.$$

Lemma 3.4.1. *We have*

$$\mathrm{Dim}_q(\lambda) = s_\lambda(1, q, \ldots, q^{N-1}).$$

Proof. Applying the combinatorial formula for Schur functions we obtain

$$\begin{aligned} s_\lambda(1,\dots,q^{N-1}) &= s_\lambda(q^{N-1},\dots,1) \\ &= \sum_{\tau(0)\prec\dots\prec\tau(N)} q^{(N-1)\cdot(|\tau(1)|-|\tau(0)|)+\dots+0\cdot(|\tau(N)|-|\tau(N-1)|)} \\ &= \sum_{\tau(0)\prec\dots\prec\tau(N)} q^{|\tau(1)|+|\tau(2)|+\dots+|\tau(N-1)|} = \mathrm{Dim}_q(\lambda) \end{aligned}$$

□

Let $\lambda \in \mathbb{GT}_{N+1}$, $\mu \in \mathbb{GT}_N$. We define *cotransitional probability* $P(\lambda \to \mu)$:

$$P(\lambda \to \mu) = \begin{cases} q^{|\mu|}\frac{\mathrm{Dim}_q(\mu)}{\mathrm{Dim}_q(\lambda)}, & \mu \prec \lambda, \\ 0, & \textit{otherwise}. \end{cases}$$

Proposition 3.4.2. *For any $\lambda \in \mathbb{GT}_{N+1}$ we have*

$$\sum_{\mu\in\mathbb{GT}_N} P(\lambda \to \mu) = 1.$$

Proof. Proposition 3.3.1 implies that

$$s_\lambda(1,\dots,q^N) = \sum_{\mu\prec\lambda} s_\mu(q,q^2,\dots,q^N) = \sum_{\mu\prec\lambda} q^{|\mu|} s_\mu(1,q,\dots,q^{N-1}).$$

Dividing by $s_\lambda(1,\dots,q^N)$ we get the desired equality. □

Suppose that P_N and P_{N+1} are probability distributions on $\mathbb{GT}_N$ and $\mathbb{GT}_{N+1}$, respectively. We call P_N and P_{N+1} *q–coherent* if for any $\mu \in \mathbb{GT}_N$:

$$P_N(\mu) = \sum_{\lambda\in\mathbb{GT}_{N+1}} P_{N+1}(\lambda)P(\lambda \to \mu).$$

We call probability distributions $P_1, P_2, \dots$ on $\mathbb{GT}_1, \mathbb{GT}_2, \dots$ respectively *a q–coherent system*, if P_i and P_{i+1} are q–coherent for every $i = 1, 2, \dots$.

Note that for any probability distribution P_N on $\mathbb{GT}_N$, there exist unique distributions $P_1, \dots, P_{N-1}$ on $\mathbb{GT}_1, \dots, \mathbb{GT}_{N-1}$, respectively, such that $P_1, \dots, P_N$ is a q–coherent system.

Let P be an arbitrary probability measure on $\mathcal{T}_N$. Let P_k denote a projection of P on $\mathbb{GT}_k$. The following two propositions are proved in Section 3.6.2:

Proposition 3.4.3. *If measure P is such that*

$$P(\tau(1) \prec \dots \tau(N)) = \frac{q^{|\tau(1)|+\dots+|\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))} P_N(\tau(N)), \tag{3.9}$$

for any path $\tau \in \mathcal{T}_N$, then $P_1, P_2, \dots, P_N$ is a q–coherent system. In particular, if P is a q–central measure on $\mathcal{T}$, then $P_1, P_2, \dots$ is a q–coherent system.

Proposition 3.4.4. *Let $P_1, P_2, \dots$ be a q–coherent system. There exists a unique q–central measure P such that P_k is a projection of P on $\mathbb{GT}_k$ for every $k = 1, 2, \dots$.*

Next, we want to introduce a convenient tool for studying q–central measures and q–coherent systems.

Suppose that P is a probability measure on $\mathbb{GT}_N$. *q–Schur generating function* of P is a symmetric function in $x_1, \dots, x_N$ given by:

$$\mathcal{S}(x_1, \dots, x_N; P) = \sum_{\mu \in \mathbb{GT}_N} P(\mu) \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})}. \tag{3.10}$$

Note that when $N = 1$, this definition turns into the usual definition of probability generating function:

$$F(t) = \sum_\ell c_\ell t^\ell.$$

For every N we define

$$T_N = \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid |x_i| = q^{1-i}\}$$

and

$$D_N = \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid |x_i| \le q^{1-i}\}.$$

Proposition 3.4.5. *The series* (3.10) *converges uniformly on T_N. If* $\operatorname{supp}(P) \subset \mathbb{GT}_N^+$, *then the series* (3.10) *converges uniformly on D_N.*

Proof. The proposition follows from the fact that

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})} \right| \le 1,$$

for all $(x_1, \dots, x_N) \in T_N$, and if $\mu \in \mathbb{GT}_N^+$, then

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})} \right| \le 1,$$

for all $(x_1, \dots, x_N) \in D_N$.

These inequalities, in turn, easily follow from the combinatorial formula for rational Schur functions. □

The following proposition is proved in Section 3.6.2:

Proposition 3.4.6. *Suppose that P_N and P_{N+1} are two probability measures on $\mathbb{GT}_N$ and $\mathbb{GT}_{N+1}$ respectively. P_N and P_{N+1} are q–coherent if and only if*

$$\mathcal{S}(x_1, \dots, x_N; P_N) = \mathcal{S}(x_1, \dots, x_N, q^{-N}; P_{N+1}).$$

If $\mathrm{supp}(P) \subset \mathbb{GT}_N^+$ (i.e. P is a probability measure on $\mathbb{GT}_N^+$) then we define *q–interpolation Schur generating function* of P through

$$\mathcal{S}^*(x_1, \dots, x_N; P) = \sum_{\mu \in \mathbb{GT}_N^+} P(\mu) \frac{s_\mu^*(q^{N-1}x_1, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})}. \tag{3.11}$$

Proposition 3.4.7. *Series* (3.11) *converges uniformly on compact subsets of* $\mathbb{C}^N$.

Proof. Using the combinatorial formula for q–interpolation polynomials we get:

$$\begin{aligned}
&s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1}) \\
&\qquad = q^{(N-1)|\mu|} \sum_T \prod_{(i,j)\in\mu} \left(x_{T(i,j)} - q^{i-j-1+T(i,j)} \right) \\
&\qquad = q^{(N-1)|\mu|} (-1)^{|\mu|} \left(\prod_{(i,j)\in\mu} q^{i-j-1} \right) \\
&\qquad \times \sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right) \prod_{(i,j)\in\mu} \left(1 - x_{T(i,j)} q^{j-i-T(i,j)+1} \right).
\end{aligned}$$

Let M be a constant such that $|x_i| < M$ for all i. We have

$$\begin{aligned}
&\left| \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})} \right| \\
&\qquad = \frac{\left| \sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right) \prod_{(i,j)\in\mu} \left(1 - x_{T(i,j)} q^{j-i-T(i,j)+1} \right) \right|}{\sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right)} \\
&\qquad \le \frac{\sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right) \prod_{(i,j)\in\mu} \left(1 + M q^{j-i-T(i,j)+1} \right)}{\sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right)} \\
&\qquad \le \max_T \prod_{(i,j)\in\mu} \left(1 + M q^{j-i-T(i,j)+1} \right) \\
&\qquad \le \max_T \exp \left(\sum_{(i,j)\in\mu} M q^{j-i-T(i,j)+1} \right) \\
&\qquad \le \exp \left(M \sum_{(i,j)\in\mu} q^{j-i-N} \right).
\end{aligned}$$

Since μ has at most N rows,

$$\sum_{(i,j)\in\mu} q^{j-i-N} < N(q^{1-2N} + q^{2-2N} + q^{3-2N} + \dots) = \frac{N q^{1-2N}}{1-q}.$$

We conclude that if $(x_1, \dots, x_N)$ is such that $|x_i| < M$ for every i, then

$$\left| \frac{s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})} \right| < A(M).$$

Consequently,

$$\sum_{\mu \in \mathbb{GT}^+_N} c_\mu \left| \frac{s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})} \right| < \infty,$$

(3.11) absolutely converges and this convergence is uniform on compact subsets of $\mathbb{C}^N$. □

The following proposition is proved in Section 3.6.2:

Proposition 3.4.8. *Suppose that P_N and P_{N+1} are two probability measures on $\mathbb{GT}^+_N$ and $\mathbb{GT}^+_{N+1}$ respectively. P_N and P_{N+1} are q–coherent if and only if*

$$\mathcal{S}^*(x_1, \dots, x_N; P_N) = \mathcal{S}^*(x_1, \dots, x_N, 0; P_{N+1}).$$

Next, we want to show that convergence of q–central probability measures is equivalent to uniform convergence of their q–Schur generating functions.

Let P_N and P^i_N, $i = 1, 2, \dots$ be probability measures on $\mathbb{GT}_N$. We say that P^i_N *weakly converges* to P^N as $i \to \infty$ if $\lim_{i\to\infty} P^i_N(\mu) = P_N(\mu)$ for every $\mu \in \mathbb{GT}_N$.

Let P and P^i be probability measures on $\mathcal{T}$. We say that P^i *weakly converges* to P as $i \to \infty$ if $\lim_{i\to\infty} P^i(C_\tau) = P(C_\tau)$ for any cylinder set C_τ.

Suppose that P and P^i are q–central probability measures on $\mathcal{T}$ and let P_N, P^i_N be the corresponding q–coherent systems (i.e. projections of the measures on $\mathbb{GT}_N$).

Proposition 3.4.9. *Measures P^i weakly converge to P if and only if P^i_N weakly converge to P_N for every N.*

Proof. This proposition follows from the correspondence between q–central measures and q–coherent systems. (See Proposition 3.4.3 and Proposition 3.4.4). □

Proposition 3.4.10. *Let P_N, P^i_N be probability measures on $\mathbb{GT}_N$. If measures P^i_N weakly converge to P_N, then*

$$\mathcal{S}(x_1, \dots, x_N; P^i_N) \rightrightarrows \mathcal{S}(x_1, \dots, x_N; P_N)$$

uniformly on T_N. If P_N and P^i_N are supported on $\mathbb{GT}^+_N$, then the convergence is uniform on D_N.

Proposition 3.4.11. *Let P^i_N be probability measures on $\mathbb{GT}_N$. If*

$$\mathcal{S}(x_1, \dots, x_N; P^i_N)$$

converge uniformly on T_N to a function $S(x_1, \dots, x_N)$, then there exists a probability measure P_N such that

$$S(x_1, \dots, x_N) = \mathcal{S}(x_1, \dots, x_N; P_N)$$

and P_N^i weakly converge to P_N.

Proposition 3.4.12. *Let P_N, P_N^i be probability measures on $\mathbb{GT}_N^+$. P_N^i weakly converge to P_N if and only if*

$$\mathcal{S}^*(x_1, \dots, x_N; P_N^i) \rightrightarrows \mathcal{S}^*(x_1, \dots, x_N; P_N)$$

uniformly on compact subsets of $\mathbb{C}^N$.

The proofs are quite involved and we give them in Section 3.6.2.

3.5 The boundary of the q–Gelfand–Tsetlin graph

Suppose that $\lambda \in \mathbb{GT}_N$. Let P_i^λ, $i = 1, \dots, N$ be probability measures on $\mathbb{GT}_i$ such that $P_1^\lambda, \dots, P_N^\lambda$ is a q–coherent system and P_N^λ is delta-measure on λ (i.e. $P_N^\lambda(\lambda) = 1$). Clearly, such q–coherent system exists and is unique. We call $\{P_i^\lambda\}$ *a primitive q–coherent system* corresponding to λ.

Let $N(i)$ be an increasing sequence of positive integers and $\lambda(i) \in \mathbb{GT}_{N(i)}$. We call $\lambda(i)$ a *regular* sequence if for every k probability measures $P_k^{\lambda(i)}$ weakly converge to a certain probability measure P_k. When we vary k, we see that the limit measures P_k form a q–coherent system. The set of all possible q–coherent system that can be obtained in such a way is called *the Martin boundary* of the graph. In Sections 3.5.1-3.5.4 we describe the Martin boundary of the q–Gelfand–Tsetlin graph. And in Section 3.5.5 we prove that the minimal boundary of the q–Gelfand–Tsetlin graph coincides with the Martin boundary of the q–Gelfand–Tsetlin graph.

In Section 3.3.3 we defined the algebra of symmetric function Λ. *A specialization* Spec of Λ is an arbitrary homomorphism from algebra Λ to $\mathbb{C}$. H–generating function of the specialization is the following power series

$$H(t) = \sum_{i=0}^{\infty} \mathrm{Spec}(h_k) t_k.$$

Since complete symmetric functions h_k generate Λ, H–generating function uniquely defines the specialization.

Recall that $\mathcal{N}$ is the set of all non-decreasing sequences of integers $\nu = (\nu_1 \le \nu_2 \le \dots)$. We prove the following theorem in Sections 3.5.1-3.5.2.

Theorem 3.5.1. *The Martin boundary of the q–Gelfand–Tsetlin graph $\{\mathcal{E}_\nu^k\}$ is parameterized by points of $\mathcal{N}$.*

If a sequence of signatures $\lambda(i)$ is regular, then there exists $\nu \in \mathcal{N}$ such that the last coordinates of $\lambda(i)$ stabilize to ν, i.e. for any j

$$\lim_{i\to\infty} \lambda(i)_{N(i)+1-j} = \nu_j.$$

If the last coordinates of $\lambda(i)$ stabilize to ν, then the measures $P_k^{\lambda(i)}$ weakly tend to $\mathcal{E}_k^\nu$.

The q–Schur generating function of the measure $\mathcal{E}_\nu^k$ can be uniquely extended to a function analytic everywhere in $\mathbb{C}^k$ except, perhaps, at the origin.

If $\nu' = \nu + \ell$, i.e. for every j we have $\nu'_j = \nu_j + \ell$, then

$$\mathcal{S}(x_1, \dots, x_k; \mathcal{E}_{\nu'}^k) = \frac{x_1^\ell \cdots x_k^\ell}{1^\ell q^{-\ell} \cdots q^{(1-k)\ell}} \mathcal{S}(x_1, \dots, x_k; \mathcal{E}_{\nu'}^k)$$

If $\nu_1 \geq 0$, then

$$\mathcal{S}(x_1, \dots, x_k; \mathcal{E}_\nu^k) = \sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k; q),$$

the series converges everywhere in $\mathbb{C}^k$ and Spec_ν is a specialization of algebra Λ with H–generating function

$$H^\nu(t) = \frac{\prod\limits_{i\geq 0}(1 - q^i t)}{\prod\limits_{j=1}^{\infty}(1 - q^{\nu_j + j - 1} t)}$$

It is worth stressing that Propositions 3.4.10 and 3.4.11 imply that $\lambda(i)$ is regular if and only if for any k functions

$$\frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_\lambda(i)(1, q, \dots, q^{1-N(i)})}$$

converge uniformly on T_k. Thus, Theorems 3.1.3 and 3.5.1 are equivalent.

3.5.1 Simple necessary conditions for convergence

Recall that A_ℓ is an automorphisms of graph $\mathbb{GT}$ acting on an arbitrary signature λ by

$$A_\ell(\lambda_1 \geq \cdots \geq \lambda_N) = (\lambda_1 + \ell) \geq \cdots \geq (\lambda_N + \ell).$$

We call A_ℓ an *ℓ–shift*.

Lemma 3.5.2. *If $P_1, P_2, \dots$ is a q–coherent system, then $A_\ell(P_1), A_\ell(P_2), \dots$ is a q–coherent system too.*

Proof. This follows form the fact that $P(\lambda \to \mu) = P(A_\ell(\lambda) \to A_\ell(\mu))$, which is straightforward. □

Proposition 3.5.3. *For any $N > 0$ and any $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{GT}_N$ we have*

$$P_1^\lambda(\lambda_N) \geq \prod_{i=1}^{\infty}(1-q^i).$$

Here λ_N stands for a signature of $\mathbb{GT}_1$ with coordinate λ_N.

Proof. Note that

$$P_1^\lambda(\lambda_N) = P_1^{A_\ell(\lambda)}(\lambda_N + \ell).$$

Thus, without loss of generality we may assume that $\lambda_N = 0$.

Since P_N^λ is concentrated on the signature λ, we have

$$\mathcal{S}(x_1, \ldots, x_N; P_N^\lambda) = \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1, \ldots, q^{1-N})}.$$

Using Proposition 3.4.6 we conclude that

$$\mathcal{S}(x; P_1^\lambda) = \frac{s_\lambda(x, q^{-1}, \ldots, q^{1-N})}{s_\lambda(1, \ldots, q^{1-N})}.$$

Recall that

$$\mathcal{S}(x; P_1^\lambda) = \sum_\ell P_1^\lambda(\ell) x^\ell.$$

Since $\mathcal{S}(x_1, \ldots, x_N; P_N^\lambda)$ is a polynomial, so is $\mathcal{S}(x; P_1^\lambda)$. It follows that

$$P_1^\lambda(\lambda_N) = P_1^\lambda(0) = \mathcal{S}(0; P_1^\lambda) = \frac{s_\lambda(0, q^{-1}, \ldots, q^{1-N})}{s_\lambda(1, \ldots, q^{1-N})} = \frac{s_\lambda(q^{-1}, \ldots, q^{1-N})}{s_\lambda(1, \ldots, q^{1-N})}$$

The value $s_\lambda(1, q^{-1}, \ldots, q^{1-N})$ can be computed using the definition of the rational Schur function (see e.g. [58, Example 3.1]). Recall that $n(\lambda) = \sum(i-1)\lambda_i$. We have

$$s_\lambda(1, q, \ldots, q^{1-N}) = q^{-n(\lambda)} \prod_{1 \leq i < j \leq N} \frac{1 - q^{\lambda_j - \lambda_i + i - j}}{1 - q^{i-j}} \tag{3.12}$$

It follows that

$$\frac{s_\lambda(q^{-1}, \ldots, q^{1-N})}{s_\lambda(1, \ldots, q^{1-N})} = \frac{q^{-|\lambda|} \prod\limits_{1 \leq i < j \leq N-1} \dfrac{1 - q^{-\lambda_i + \lambda_j + i - j}}{1 - q^{i-j}}}{\prod\limits_{1 \leq i < j \leq N} \dfrac{1 - q^{-\lambda_i + \lambda_j + i - j}}{1 - q^{i-j}}} = \frac{q^{-|\lambda|}}{\prod\limits_{i=1}^{N-1} \dfrac{1 - q^{-\lambda_i + i - N}}{1 - q^{i-N}}}$$

$$= \prod_{i=1}^{N-1} \frac{q^{-\lambda_i} - q^{-\lambda_i + i - N}}{1 - q^{-\lambda_i + i - N}} = \prod_{i=1}^{N-1} \frac{1 - q^{N-i}}{1 - q^{\lambda_i - i + N}} \geq \prod_{i=1}^{N-1}(1 - q^{N-i}) \geq \prod_{i=1}^{\infty}(1 - q^i).$$

Thus,

$$P_1^\lambda(\lambda_N) \geq \prod_{i=1}^{\infty}(1 - q^i).$$

□

Corollary 3.5.4. *If a sequence of measures $P_1^{\lambda(i)}$ with $\lambda(i) \in \mathbb{GT}_{N_i}$, $N_i \to \infty$, weakly converges to a certain probability measure P_1, then $\lambda(i)_{N_i}$ is bounded from below.*

Proof. Let k be an integer such that

$$P_1(\{\lambda \in \mathbb{GT}_1 : -k < \lambda < k\}) > 1 - \frac{1}{2}\prod_{j=1}^{\infty}(1-q^j).$$

Since $P_1^{\lambda(i)}$ weakly converges to P_1 and the set $\{\lambda \in \mathbb{GT}_1 : -k < \lambda < k\}$ is finite, we conclude that

$$P_1^{\lambda(i)}(\{\lambda \in \mathbb{GT}_1 : -k < \lambda < k\}) > 1 - \prod_{j=1}^{\infty}(1-q^j)$$

for $i > i_0$. Thus, if $\lambda(i)_{N(i)} < -k$, then $P_1^{\lambda(i)}(\lambda_{N(i)}) < \prod_{j=1}^{\infty}(1-q^j)$. This is a contradiction with Proposition 3.5.3, consequently, $\lambda(i)_{N(i)} \geq (-k)$ for $i \geq i_0$. □

Corollary 3.5.4 implies that for the full description of the Martin boundary of the q–Gelfand–Tsetlin graph it is enough to study only measures concentrated on positive signatures and their ℓ–shifts.

Proposition 3.5.5. *Let $k < N$, $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_k$. If $P_k^{\lambda}(\mu) > 0$, then $\mu \geq (\lambda_{N-k+1}, \lambda_{N-k+2}, \ldots, \lambda_N)$.*

Proof. From the definitions of measures P_k^{λ} and q–coherent systems it follows that if $P_k^{\lambda}(\mu) > 0$, then there exists a sequence $\tau(k) \prec \tau(k+1) \prec \cdots \prec \tau(N)$ such that $\tau(k) = \mu$ and $\tau(N) = \lambda$. Consequently, for $i = 0, 1, \ldots, k-1$ we have

$$\mu_{k-i} = \tau(k)_{k-i} \leq \tau(k+1)_{k+1-i} \leq \cdots \leq \tau(N)_{N-i} = \lambda_{N-i}.$$

□

Corollary 3.5.6. *If $\lambda(i) \in \mathbb{GT}_{N(i)}$ is a sequence of signatures such that the measures $P_k^{\lambda(i)}$ weakly converges, then the sequence of integers $\lambda(i)_{N(i)-m}$ is bounded from above for any $0 \leq m \leq k-1$.*

3.5.2 Proof of Theorems 3.1.3 and 3.5.1

We start with the following compactness result

Proposition 3.5.7. *Let $\lambda(i)$ be a sequence of signatures stabilizing to ν. The family of functions*

$$g_i(x_1, \ldots, x_k) = \frac{s_{\lambda(i)}(x_1, x_2, \ldots, x_k, q^{-k}, q^{-k-1}, \ldots, q^{1-N(i)})}{s_{\lambda}(i)(1, q, \ldots, q^{1-N(i)})}$$

is a relatively compact subset of the set of continuous functions on k–dimensional torus T_k with uniform convergence topology.

The proof is quite technical and we present it in Section 3.6.3. The main idea is to find a uniform estimate for the derivatives of functions $g_i(x_1, \dots, x_k)$.

In order to identify all the possible limits of the functions g_i, we want to decompose the functions in q–interpolation polynomials series. We need the following proposition that is proved in Section 3.6.1. Recall that $q^{\lambda-\delta} = (q^{\lambda_1}, q^{\lambda_2-1}, \dots, q^{\lambda_N-N+1})$.

Proposition 3.5.8. *Let P_N be a probability measure on $\mathbb{GT}_N$ such that $\operatorname{supp}(P_N) \subset \mathbb{GT}_N^+$, in other words the probability of any signature with at least one negative coordinate is zero. The q–Schur generating function of P_N is well-defined for all $(x_1, \dots, x_N) \in D_N$ and it can be uniquely represented as*

$$\mathcal{S}(x_1, \dots, x_N; P_N) = \sum_{\mu \in \mathbb{GT}_N^+} c_\mu s_\mu^*(x_1, \dots, x_N; q). \tag{3.13}$$

The series converges uniformly on any ball $B(0, r)$ with radius $0 < r < 1$ and in every point $q^{\lambda-\delta}$.

Furthermore, suppose that P_N^i and P_N are probability measures on $\mathbb{GT}_N$ such that $\operatorname{supp}(P_N^i) \subset \mathbb{GT}_N^+$ and $\operatorname{supp}(P_N) \subset \mathbb{GT}_N^+$. Let c_μ^i and c_μ be the coefficients of the decomposition (3.13) for the q–Schur generating functions of P_N^i and P_N, respectively. If P_N^i weakly converge to P_N as $i \to \infty$ (in other words, if the q–Schur generating functions of the measures uniformly converge), then for every μ

$$\lim_{i \to \infty} c_\mu^i = c_\mu.$$

Remark. Note that here we use functions $s_\mu^*(\cdot; q)$ while in the definition of a q–interpolation Schur generating function we use $s_\mu^*(\cdot; q^{-1})$.

Let $\nu = (0 \le \nu_1 \le \nu_2 \le \dots)$ be an arbitrary nondecreasing sequence of non-negative integers.

Recall that $H^\nu(t)$ is the following function:

$$H^\nu(t) = \frac{\prod_{i \ge 0} (1 - q^i t)}{\prod_{j=1}^{\infty} (1 - q^{\nu_j + j - 1} t)},$$

$H^\nu(t)$ is an analytic function in $\mathbb{C}$.

There is a one-to-one correspondence X between νs and subsets of $\mathbb{Z}_{\ge 0}$ with infinite complement given by

$$X(\nu) = \mathbb{Z}_{\ge 0} \setminus \{\nu_i + i - 1\}_{i=1,2,\dots}.$$

Observe that

$$H^\nu(t) = \prod_{x \in X(\nu)} (1 - q^x t).$$

The main part of the proof of Theorem 3.1.3 is contained in the following proposition.

Proposition 3.5.9. *Let $N(i)$ be an increasing sequence of positive integers and $\lambda(i) \in \mathbb{GT}^+_{N(i)}$. Suppose that $P_k^{\lambda(i)}$ weakly converges as $i \to \infty$ to a certain probability measure P_k for $k = 1, 2, \dots$. Then the last coordinates of $\lambda(i)$ stabilize, i.e. there exists a nondecreasing sequence of integers ν_j such that for every $j = 1, 2, \dots$*

$$\lambda(i)_{N(i)+1-j} \to \nu_j.$$

The q–Schur generating function of the limit measure $P_k = \mathcal{E}_k^\nu$ has the following decomposition:

$$\mathcal{S}(x_1, \dots, x_k; \mathcal{E}_k^\nu) = \sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s^*_\mu(x_1, \dots, x_k; q),$$

where Spec_ν *is a specialization of algebra Λ with H–generating function $H^\nu(t)$*

Proof. The probability measure $P_{N(i)}^{\lambda(i)}$ is concentrated on a single signature $\lambda(i)$, thus

$$\mathcal{S}\left(x_1, \dots, x_{N(i)}; P_{N(i)}^{\lambda(i)}\right) = \frac{s_{\lambda(i)}(x_1, \dots, x_{N(i)})}{s_{\lambda(i)}(1, \dots, q^{-N(i)})}.$$

Measures $P_1^{\lambda(i)}, \dots, P_{N(i)}^{\lambda(i)}$ form a q–coherent system. Consequently, Proposition 3.4.6 yields

$$\mathcal{S}\left(x_1, \dots, x_k; P_k^{\lambda(i)}\right) = \frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_\lambda(i)(1, q, \dots, q^{1-N(i)})},$$

Let us expand the right side of the last formula into the sum of q-interpolation Schur polynomials. Coefficients of this expansion are given by the formula (3.7):

$$\begin{aligned} &\frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, \dots, q^{1-N(i)})}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-N(i)})} \\ &= \sum_{\mu \in \mathbb{GT}^+_{N(i)}} \frac{q^{(N(i)-1)|\mu|} s^*_\mu(q^{\lambda(i)-\delta}; q)}{s^*_\mu(q^{\mu-\delta}; q)} \frac{s^*_\mu(x_1, \dots, x_k, q^{-k}, \dots, q^{1-N(i)})}{s_\mu(1, q, \dots, q^{N(i)-1})} \end{aligned} \tag{3.14}$$

The combinatorial formula for interpolation Schur polynomials (see Proposition 3.3.5) implies that if $\mu_{k+1} \neq 0$, then

$$s^*_\mu(x_1, \dots, x_k, q^{-k}, \dots, q^{1-N(i)}) = 0.$$

Furthermore, polynomials s^*_μ are stable, i.e. if $\mu_{k+1} = \mu_{k+2} = \dots = 0$, then

$$s^*_\mu(x_1, \dots, x_k, q^{-k}, \dots, q^{1-N(i)}) = s^*_{(\mu_1, \dots, \mu_k)}(x_1, \dots, x_k).$$

It follows that

$$\frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, \dots, q^{1-N(i)})}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-N(i)})}$$
$$= \sum_{\mu \in \mathbb{GT}_k^+} \frac{q^{(N(i)-1)|\mu|} s_\mu^*(q^{\lambda(i)-\delta}; q)}{s_\mu^*(q^{\mu-\delta}; q)} \frac{s_\mu^*(x_1, \dots, x_k)}{s_\mu(1, q, \dots, q^{N(i)-1})}, \quad (3.15)$$

Using Proposition 3.4.10 and Proposition 3.5.8 we conclude that weak convergence of measures $P_k^{\lambda(i)}$ implies that for any μ

$$\frac{q^{(N(i)-1)|\mu|} s_\mu^*(q^{\lambda(i)-\delta}; q)}{s_\mu^*(q^{\mu-\delta}; q) s_\mu(1, q, \dots, q^{N(i)-1})}$$

converges as $i \to \infty$.

Using again [58, Example 3.1] we obtain

$$\lim_{N\to\infty} s_\mu(1, q, \dots, q^N) = \lim_{N\to\infty} \frac{q^{n(\mu)} \prod\limits_{(i,j)\in\mu} (1 - q^{N+j-i})}{\prod\limits_{(i,j)\in\mu} (1 - q^{h(i,j)})} = \frac{q^{n(\mu)}}{\prod\limits_{(i,j)\in\mu} (1 - q^{h(i,j)})},$$

Proposition 3.3.4 yields

$$s_\mu^*(q^\mu; q) = q^{n(\mu')-2n(\mu)} \prod_{(i,j)\in\mu} \left(q^{h(i,j)} - 1\right).$$

Thus,

$$\frac{1}{s_\mu^*(q^\mu; q) s_\mu(1, q, \dots)} \to (-1)^{|\mu|} q^{n(\mu)-n(\mu')}$$

Consequently,

$$q^{(N(i)-1)|\mu|} s_\mu^*(q^{\lambda(i)-\delta}; q) \quad (3.16)$$

should tend to a limit as $i \to \infty$.

Recall that functions s_μ^* form a linear basis of filtered algebra $\widehat{\Lambda}$. Therefore, (3.16) has a limit for any μ if and only if

$$\lim_{i\to\infty} q^{(N(i)-1)k} f(q^{\lambda(i)-\delta})$$

for any $f \in \widehat{\Lambda}$ of the degree k.

Let us introduce an analogue of Newton power sums in algebra $\widehat{\Lambda}$:

$$p_k^* = \sum_{j\geq 1} x_j^k - (q^{1-j})^k$$

(with $p_0^* = 1$). Note that functions

$$p_{k_1}^* \cdots p_{k_l}^*$$

also form a linear basis of $\widehat{\Lambda}$,

We conclude that (3.16) has a limit for every μ if and only if

$$q^{(N(i)-1)(k_1+\dots+k_l)}p^*_{k_1}(q^{\lambda(i)-\delta})\cdots p^*_{k_1}(q^{\lambda(i)-\delta})$$

converges for any positive integers $k_1,\dots,k_l$. The last limits exist if and only if

$$q^{(N(i)-1)k}p^*_k(q^{\lambda(i)-\delta})$$

tends to a finite limit as $i\to\infty$ (for any k). We claim that then the last coordinates of $\lambda(i)$ should stabilize.

Indeed, we have

$$q^{(N(i)-1)k}p^*_k(q^{\lambda(i)-\delta};q)=\sum_{j=1}^{N(i)}(q^{k\lambda(i)_j}-1)q^{k(N(i)-i)}=\sum_{j=0}^{N(i)}q^{kj}(q^{k\lambda(i)_{N(i)-j}}-1).$$

Denote $f(i,j)=\lambda(i)_{N(i)-j}+j$. Note that $\sum_{j=0}^{N(i)}q^{kj}(-1)$ converges for any k. Thus,

$$\sum_{j=0}^{N(i)}q^{kj}q^{k\lambda(i)_{N(i)-j}}=\sum_{j=0}^{N(i)}q^{kf_{i,j}}$$

should also converge. But since $f(i,j)$ is increasing in j, we have

$$\left|\sum_{j=0}^{N(i)}q^{kf(i,j)}-q^{kf(i,0)}\right|<q^{k(f(i,0)+1)}(1+q+q^2+\dots)=q^{kf(i,0)}\frac{q^k}{1-q}.$$

Informally speaking, if k is large enough, then

$$\sum_{j=0}^{N(i)}q^{kf(i,j)}\approx q^{kf(i,0)}.$$

Recall that $f(i,0)$ is an integer. Therefore, if $\sum_{j=0}^{N(i)}q^{kf(i,j)}$ converges, then either $f(i,0)\to+\infty$ or $f(i,0)$ stabilize as $i\to\infty$. Repeating the same argument for $\sum_{j=w}^{N(i)}q^{kf_{i,j}}$, $w=1,2,\dots$ we conclude that for every w, either $f(i,w)\to+\infty$ or $f(i,w)$ stabilize as $i\to\infty$. In the former case $\lambda(i)_{N(i)-w}$ is unbounded which is a contradiction with Corollary 3.5.6. Thus, $f(i,w)$ stabilize, in other words there exists a sequence $0\le\nu_1\le\nu_2\le\dots$ such that

$$\lambda(i)_{N(i)+1-w}\to\nu_w.$$

In this case

$$\lim_{i\to\infty}q^{(N(i)-1)k}p^*_k(q^{\lambda(i)-\delta};q)=\sum_{i\ge1}(q^{k\nu_i}-1)q^{k(i-1)}.$$

For any Young diagram (i.e. positive signature) let s_λ denote the element of Λ corresponding to Schur function $s_\lambda(x_1,\dots,x_N)$. Newton power sums p_k are

algebraically independent generators of Λ. Thus, for any λ there exists a unique polynomial R_λ, such that

$$s_\lambda = R_\lambda(p_1, \dots, p_m).$$

(Here m also depends on λ, but we omit this dependence to simplify the notations.)

Now consider the following element of $\widehat{\Lambda}$:

$$r^*_\lambda = s^*_\lambda - R_\lambda(p^*_1, \dots, p^*_m).$$

Note that, if we work with finite sets of variables (i.e. in algebras Λ_N and $\widehat{\Lambda}_N$) then the highest homogenous component of s^*_λ is exactly s_λ, and the same is true for p^*_k and p_k. It follows that the degree of r^*_λ is less than $|\lambda|$. Consequently,

$$\lim_{i\to\infty} q^{(N(i)-1)|\lambda|} r^*_\lambda(q^{\lambda(i)-\delta}) = 0.$$

Let Spec_ν be a specialization of Λ defined on generators p_k through

$$\mathrm{Spec}_\nu(p_k) = \sum_{i\geq 1} (q^{k\nu_i} - 1) q^{k(i-1)}.$$

The above arguments show that

$$\begin{aligned}\lim_{i\to\infty} q^{(N(i)-1)|\lambda|} s^*_\lambda &= \lim_{i\to\infty} q^{(N(i)-1)|\lambda|} R_\lambda(p^*_1, \dots, p^*_m) \\ &= R_\lambda(\mathrm{Spec}_\nu(p_1), \dots, \mathrm{Spec}_\nu(p_m)) = \mathrm{Spec}_\nu(s_\lambda).\end{aligned}$$

Thus, we have proved that if $P_k^{\lambda(i)}$ weakly converges as $i \to \infty$ to a certain probability measure P_k for any $k = 1, 2, \dots$, then $\lambda(i)_{N(i)+1-j} \to \nu_j$ and

$$\begin{aligned}\mathcal{S}(x_1, \dots, x_k; P_k) &= \lim_{i\to\infty} \mathcal{S}\left(x_1, \dots, x_k; P_k^{\lambda(i)}\right) \\ &= \sum_{\mu\in\mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\mu) s^*_\mu(x_1, \dots, x_k) \qquad (3.17)\end{aligned}$$

It remains to prove that H–generating function of Spec_ν is $H^\nu(t)$.

P–generating function of Spec_ν is the following power series

$$P(t) = \sum_{k=1}^{\infty} \mathrm{Spec}_\nu(p_k) t^{k-1}$$

The following equality relates P–generating function to H–generating function: $P(t) = H'(t)/H(t)$. See e.g. [58, 2.10] for the proof. The following computation completes the proof:

$$P(t) = \sum_{k=1}^{\infty} \mathrm{Spec}_\nu(p_k) t^{k-1} = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} (q^{k\nu_i} - 1) q^{k(i-1)} t^{k-1}$$
$$= \sum_{i=1}^{\infty} \frac{q^{\nu_i+i-1}}{1-q^{\nu_i+i-1}t} - \sum_{i=1}^{\infty} \frac{q^{i-1}}{1-q^{i-1}t},$$

$$H(t) = \sum_{k=0}^{\infty} \mathrm{Spec}_\nu(h_k) t^k = \exp\left(\int \mathrm{Spec}_\nu(P(t))\right)$$
$$= \exp\left(\sum_{i\geq 1} -\ln(1-q^{\nu_i+i-1}t) + \sum_{i\geq 1} \ln(1-q^{i-1}t)\right)$$
$$= \frac{\prod_{i\geq 0}(1-q^i t)}{\prod_{j=1}^{\infty}(1-q^{\nu_j+j-1}t)} = H^\nu(t). \quad (3.18)$$

□

We also need yet another technical proposition that will be proved in Section 3.6.4:

Proposition 3.5.10. *For every ν the series*

$$\sum_{\mu\in\mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\mu) s_\mu^*(x_1,\dots,x_k)$$

converges for all $x_1,\dots x_k$ and defines an entire function in $\mathbb{C}^k$.

Now we are ready to finish the proof of Theorems 3.1.3 and 3.5.1.

Proof of Theorem 3.1.3. Let $\lambda(i) \in \mathbb{GT}_{N(i)}$ be a regular sequence of signatures. It means that for any k probability measures $P_k^{\lambda(i)}$ weakly converge to a certain probability measure P_k. Corollary 3.5.4 implies that there exists ℓ such that $\lambda(i)_j \geq \ell$ for every i, j. Let $\mu(i) = A_{-\ell}(\lambda(i))$, then $\mu(i) \in \mathbb{GT}^+_{N(i)}$ and for any k probability measures $P_k^{\mu(i)}$ weakly converge to a measure $\widetilde{P}_k$. Applying Proposition 3.5.9 we conclude that the last coordinates of $\mu(i)$ stabilize to some ν_j. Thus, the last coordinates of $\lambda(i)$ also stabilize

$$\lim_{i\to\infty} \lambda(i)_{N(i)+1-j} \to \nu_j + \ell.$$

If all the coordinates of $\lambda(i)$ are non-negative starting from some i (equivalently, if the last coordinates stabilize to non-negative numbers), then Proposition 3.5.9 and Proposition 3.5.10 imply that q–Schur generating function of P_k

is an analytic function with desired interpolation polynomials series decomposition. Again using the correspondence between weak convergence of probability measures and convergence of their q–Schur generating functions (Proposition 3.4.11 and Proposition 3.4.10) we conclude that polynomials of Theorem 3.1.3 converge to the desired analytic function. For general $\lambda(i)$ the limit function is a product of the analytic q–Schur generating function corresponding to measure $\mathcal{E}_k^\nu$ and polynomial

$$\frac{x_1^\ell \cdots x_k^\ell}{1^\ell q^{-\ell} \cdots q^{-(k-1)\ell}},$$

consequently the limit function is analytic.

Now suppose that a sequence of signatures $\lambda(i) \in \mathbb{GT}_{N(i)}$, $N(i) \to \infty$ is such that

$$\lim_{i\to\infty} \lambda(i)_{N(i)+1-j} \to \nu_j$$

for some sequence $\nu_1 \leq \nu_2, \ldots$. Proposition 3.5.7 yields that the sequence of functions

$$g_i(x_1, \ldots, x_k) = \frac{s_{\lambda(i)}(x_1, x_2, \ldots, x_k, q^{-k}, q^{-k-1}, \ldots, q^{1-N(i)})}{s_\lambda(i)(1, q, \ldots, q^{1-N(i)})}$$

has converging subsequences. But in the first part of the theorem we have identified all the possible subsequential limits and their are the same. Thus, $g_i(x_1, \ldots, x_k)$ converges uniformly on T_k. □

3.5.3 Limit measures

The only formula we have for q–Schur generating functions of $\mathcal{E}_k^\nu$ is an infinite series expansion. The situation is different if we turn to q–interpolation Schur generating functions.

Proposition 3.5.11. *We have*

$$\mathcal{S}^*(x_1, \ldots, x_k; \mathcal{E}_k^\nu) = H^\nu(x_1) \cdots H^\nu(x_k).$$

Recall that $\mathcal{S}^*(x_1, \ldots, x_k; P_k)$ is an entire function for any measure P_k with $\mathrm{supp}(P_k) \subset \mathbb{GT}_k^+$. We need the following lemma:

Lemma 3.5.12. *Let P_k be a probability measure of $\mathbb{GT}_k^+$ Suppose that $\mathcal{S}^*(x_1, \ldots, x_k; P_k)$ has the following Taylor series expansion*

$$\mathcal{S}^*(x_1, \ldots, x_k; P_k) = \sum_{\mu \in \mathbb{GT}_k^+} a_\mu \frac{s_\mu(x_1, \ldots, x_k)}{s_\mu(1, q^{-1}, \ldots, q^{1-k})}$$

Define

$$F = \sum_{\mu \in \mathbb{GT}_k^+} a_\mu \frac{s_\mu^*(x_1, x_2, \ldots, x_k; q)}{s_\mu^*(0, \ldots, 0; q)}. \tag{3.19}$$

Then the series on the right side of (3.19) *converges uniformly on any ball* $B(0,r)$ *with radius* $0<r<1$ *and in every point* $q^{\lambda-\delta}$. *Furthermore,*

$$F=\mathcal{S}(x_1,\dots,x_k;P_k).$$

If the support of the measure P_k is finite, then both $\mathcal{S}^*(x_1,\dots,x_k;P_k)$ and F are polynomials and Lemma 3.5.12 immediately follows from the symmetry between formulas (3.8) and (3.7). The complete proof of the lemma is given in Section 3.6.1.

Proof of Proposition 3.5.11. Let us expand $\mathcal{S}^*(x_1,\dots,x_k;\mathcal{E}_k^\nu)$ in Taylor series:

$$\mathcal{S}^*(x_1,\dots,x_k;\mathcal{E}_k^\nu)=\sum_{\mu\in\mathbb{GT}_k^+}a_\mu\frac{s_\mu(x_1,\dots,x_k)}{s_\mu(1,q^{-1},\dots,q^{1-k})}.$$

Using Lemma 3.5.12 we conclude that

$$\mathcal{S}(x_1,\dots,x_k;\mathcal{E}_k^\nu)=\sum_{\mu\in\mathbb{GT}_k^+}a_\mu\frac{s_\mu^*(x_1,x_2,\dots,x_k;q)}{s_\mu^*(0,\dots,0;q)}.$$

Comparing with Theorem 3.5.1 we see that

$$a_\mu=(-1)^{|\mu|}q^{n(\mu)-n(\mu')}\mathrm{Spec}_\mu(s_\mu)s_\mu^*(0,\dots,0;q).$$

The combinatorial formulas for polynomials s_λ and s_λ^* imply that

$$(-1)^{|\mu|}q^{n(\mu)-n(\mu')}\frac{s_\mu^*(0,\dots,0;q)}{s_\mu(1,q^{-1},\dots,q^{1-k})}=1.$$

Thus,

$$\begin{aligned}&\mathcal{S}^*(x_1,\dots,x_k;\mathcal{E}_k^\nu)\\&\quad=\sum_{\mu\in\mathbb{GT}_k^+}(-1)^{|\mu|}q^{n(\mu)-n(\mu')}\mathrm{Spec}_\mu(s_\mu)s_\mu^*(0,\dots,0;q)\frac{s_\mu(x_1,\dots,x_k)}{s_\mu(1,q^{-1},\dots,q^{1-k})}\\&\qquad\qquad=\sum_{\mu\in\mathbb{GT}_k^+}\mathrm{Spec}_\mu(s_\mu)s_\mu(x_1,\dots,x_k).\end{aligned}$$

It remains to prove that

$$H^\nu(x_1)\cdots H^\nu(x_k)=\sum_{\mu\in\mathbb{GT}_k^+}\mathrm{Spec}_\nu(s_\mu)s_\mu(x_1,\dots,x_k),\tag{3.20}$$

where

$$H^\nu(x)=\sum_{j=0}^{\infty}\mathrm{Spec}_\nu(h_j)x^j.$$

The formula (3.20) is a particular case of the well-known Cauchy identity for symmetric functions (see e.g. [58, Chapter 1, Section 4]) and we leave its proof to the reader. □

3.5.4 Some properties of measures $\mathcal{E}^\nu$

In this section we discuss various properties of the measures $\mathcal{E}_k^\nu$.

For any nondecreasing infinite sequence of integers $\nu = (\nu_1, \nu_2, \dots)$ set

$$A_\ell(\nu) = (\nu_1 + \ell, \nu_2 + \ell, \dots)$$

Proposition 3.5.13. *For any nondecreasing sequence of nonnegative integers $\nu = (\nu_1, \nu_2, \dots)$ any any $\ell = 1, 2, \dots$ we have*

$$A_\ell(\mathcal{E}^\nu) = \mathcal{E}^{A_\ell(\nu)}.$$

Proof. There exists a sequence of signatures $\lambda(i)$, such that $\mathcal{E}_k^\nu = \lim P_k^{\lambda(i)}$ for any k. Using Theorem 3.5.1 we conclude that

$$A_\ell(\mathcal{E}_k^\nu) = \lim A_\ell P_k^{\lambda(i)} = \lim P_k^{A_\ell(\lambda_i)} = \mathcal{E}_k^{A_\ell(\nu)}.$$

□

Let us introduce a partial order on $\mathcal{N}$. We write $\nu \le \nu'$ if $\nu_i \le \nu_i'$ for every i.

Proposition 3.5.14. *(i). For any signature $\mu \in \mathbb{GT}_k$, $\mathcal{E}_k^\nu(\mu) = 0$ unless $\mu \ge (\nu_k, \nu_{k-1}, \dots, \nu_1)$. Furthermore, $\mathcal{E}_k^\nu((\nu_k, \dots, \nu_1)) > 0$.*

(ii). If $\nu' > \nu$, then

$$\mathcal{E}_k^\nu((\nu_k, \dots, \nu_1)) > \mathcal{E}_k^{\nu'}((\nu_k, \dots, \nu_1)).$$

Proof. Without loss of generality assume that $\nu_1 \ge 0$. For μ not belonging to $\mathbb{GT}_k^+$, we have $\mathcal{E}_k^\nu(\mu) = 0$. For $\mu \in \mathbb{GT}_k^+$, $\mathcal{E}_k^\nu(\mu)$ is found from the representation:

$$H^\nu(x_1) \cdots H^\nu(x_k) = \sum_{\mu \in \mathbb{GT}_k^+} \mathcal{E}_k^\nu(\mu) \frac{s_\mu^*(q^{k-1}x_1, \dots, q^{k-1}x_k; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})}. \tag{3.21}$$

To find $\mathcal{E}_k^\nu(\mu)$ we substitute $x = q^{-(k-1+\lambda-\delta)}$ into (3.21) and use interpolation property of polynomials s^* (which is stated in Proposition 3.3.4). We start from the empty diagram $\lambda = \emptyset$ and then add boxes to it and find the numbers $\mathcal{E}_k^\nu(\mu)$ inductively. (We discuss this procedure in more details later. See Proposition 3.6.1.) It follows from the definition of function $H^\nu(t)$ that

$$H^\nu(q^{-(\lambda_1+k-1)}) \cdots H^\nu(q^{-(\lambda_k)}) = 0,$$

unless $(\lambda_1, \dots, \lambda_k) \ge (\nu_k, \dots, \nu_1)$. Thus, $\mathcal{E}_k^\nu(\mu) = 0$ unless $\mu \ge (\nu_k, \nu_{k-1}, \dots, \nu_1)$.

Next, substitute $x = (q^{-(\nu_k+k-1)}, \dots, q^{-\nu_1})$ in (3.21). We obtain

$$\begin{aligned} &H^\nu(q^{-(\nu_k+k-1)}) \cdots H^\nu(q^{-(\nu_1)}) \\ &\qquad = \mathcal{E}_k^{(\nu_k,\dots,\nu_1)}((\nu_k, \dots, \nu_1)) \frac{s^*_{(\nu_k,\dots,\nu_1)}(q^{-\nu_k}, \dots, q^{-\nu_1+k-1}; q^{-1})}{s^*_{(\nu_k,\dots,\nu_1)}(0, \dots, 0; q^{-1})}. \end{aligned}$$

Observe that for $i = 1, \dots, k$ we have $H^\nu(q^{-(\nu_i+i-1)}) > 0$. It follows that

$$\mathcal{E}_k^\nu((\nu_k, \dots, \nu_1)) > 0.$$

Now, let us prove the second part of Proposition 3.5.14. If for some $i \in \{1, \dots, k\}$, $\nu_i' > \nu_i$, then

$$\mathcal{E}_k^{\nu'}((\nu_k, \dots, \nu_1)) = 0.$$

Otherwise note that for $i = 1, \dots, k$ we have $H^\nu(q^{-(\nu_i+i-1)}) > H^{\nu'}(q^{-(\nu_i+i-1)})$. In both cases

$$\mathcal{E}_k^\nu((\nu_k, \dots, \nu_1)) > \mathcal{E}_k^{\nu'}((\nu_k, \dots, \nu_1)).$$

□

Actually we can say more, i.e. $\mathcal{E}_k^\nu((\nu_k, \dots, \nu_1)) > c > 0$ for some constant c depending solely on k. Let us prove this fact for $k = 1$.

Lemma 3.5.15. *We have*

$$\mathcal{E}_1^\nu(\nu_1) \geq \prod_{i=1}^{\infty}(1 - q^i).$$

Proof. Assume without loss of generality that $\nu_1 \geq 0$. Repeating the argument of Proposition 3.5.14 we conclude that

$$\begin{aligned}
\mathcal{E}_1^\nu(\nu_1) &= \frac{H^\nu(q^{-\nu_1})}{(1 - q^{-\nu_1}) \cdots (1 - q^{-1})} \\
&= \frac{\prod\limits_{j \geq 0,\, j \neq \nu_1} (1 - q^{-\nu_1} q^j)}{\prod\limits_{j=2}^{\infty} (1 - q^{-\nu_1} q^{\nu_j + j - 1})} \frac{1}{(1 - q^{-\nu_1}) \cdots (1 - q^{-1})} \\
&= \frac{\prod\limits_{j=\nu_1+1}^{\infty} (1 - q^{-\nu_1} q^j)}{\prod\limits_{j=2}^{\infty} (1 - q^{-\nu_1} q^{\nu_j + j - 1})} \geq \prod_{j=\nu_1+1}^{\infty} (1 - q^{-\nu_1} q^j) = \prod_{i=1}^{\infty}(1 - q^i)
\end{aligned}$$

□

The set $\mathcal{N} \subset \mathbb{Z}^\infty$ has a natural topology as a subset of the direct product of discrete spaces $\mathbb{Z}$. A sequence $\theta(i)$ converges to ν in this topology if and only if for every k there exist i_0 such that $\theta(i)_k = \nu_k$ for $i > i_0$.

Proposition 3.5.16. *Sequence of probability measures $\mathcal{E}^{\theta(i)}$ weakly converges to a probability measure P if and only if $P = \mathcal{E}^\nu$ for some ν, and $\theta(i) \to \nu$.*

Proof. Suppose that $\theta(i) \to \nu$. Without loss of generality assume that $\nu_1 \geq 0$. Then for $i > i_i$, $\theta(i)_1 \geq 0$. Thus, for $i > i_1$ we may use Proposition 3.5.11. One proves that

$$H^{\theta(i)}(t) \rightrightarrows H^\nu(t)$$

uniformly on compact subsets of $\mathbb{C}$. It follows that for any k

$$\mathcal{S}^*(x_1,\dots,x_k;\mathcal{E}^{\theta(i)}) = H^{\theta(i)}(x_1)\cdots H^{\theta(i)}(x_k) \rightrightarrows H^{\nu}(x_1)\cdots H^{\nu}(x_k) \\ = \mathcal{S}^*(x_1,\dots,x_k;\mathcal{E}^{\nu}).$$

Using Proposition 3.4.12 and Proposition 3.4.9 we conclude that measures $\mathcal{E}^{\theta(i)}$ weakly converge to $\mathcal{E}^{\nu}$.

Now suppose that $\mathcal{E}^{\theta(i)}$ is a weakly convergent sequence. Lemma 3.5.15 implies that $\mathcal{E}_1^{\theta(i)}(\theta(i)_1) > c$ for some constant $c > 0$. Thus, $\theta(i)_1$ should be bounded from below. (Otherwise, measures $\mathcal{E}_1^{\theta(i)}$ "escape to infinity".) Choose $l \geq \max(-\theta(i)_1)$. Applying, if necessary, A_ℓ we may assume without loss of generality that $\theta(i)_1 \geq 0$ for all i. Then Proposition 3.4.12 yields that q–interpolation Schur generating functions of measures $\mathcal{E}_1^{\theta(i)}$ converge as $i \to \infty$ to q–interpolation Schur generating function of measure P uniformly on compact subsets of $\mathbb{C}$. Recall that

$$\mathcal{S}^*(t;\mathcal{E}_1^{\theta(i)}) = H^{\theta(i)}(t) = \prod_{x\in X(\theta(i))}(1-q^x t).$$

Due to Rouché's theorem, uniform convergence of analytic functions implies the convergence of positions of their simple zeros. Since zeros of $H^{\theta(i)}(t)$ are precisely q^{-x}, $x \in X(\theta(i))$, we conclude that the sets $X(\theta(i))$ converge (in a sense that for any n, $X(\theta(i)) \cap [0,\dots,n]$ stabilizes as $i \to \infty$) to a set $\hat{X}$. It remains to prove that $\hat{X}$ has infinite complement. Indeed, otherwise there exists k such that $\theta(i)_k \to \infty$. But $\mathcal{E}_k^{\theta(i)}$ is concentrated on signatures λ such that $\lambda_i \geq \theta(i)_{k+1-i}$, thus, measures $\mathcal{E}_k^{\theta(i)}$ can not converge to a probability measure on $\mathbb{GT}_k$.

Consequently, $\hat{X} = X(\nu)$ for a certain $\nu \in \mathcal{N}$. Therefore, $\mathcal{E}^{\theta(i)} \to \mathcal{E}^{\nu}$. □

Next we prove that measures E^{ν} are linearly independent in the following sense:

Proposition 3.5.17. *If for some $\theta \in \mathcal{N}$ and for some probability measure π defined on σ–algebra of Borel sets in $\mathcal{N}$ we have*

$$\mathcal{E}^{\theta} = \int_{\mathcal{N}} \mathcal{E}^{\nu} d\pi,$$

i.e. for any cylinder set C_τ

$$\mathcal{E}^{\theta}(C_\tau) = \int_{\mathcal{N}} \mathcal{E}^{\nu}(C_\tau) d\pi,$$

then π is a delta measure on θ, in other words $\pi(\theta) = 1$.

Proof. Let us prove that $\pi(\{\nu : \nu \geq \theta\}) = 1$. Assume the opposite. Then there exist k and $\tilde{\nu}_1,\dots,\tilde{\nu}_k$, such that $\pi(\{\nu : \nu_i = \tilde{\nu}_i \text{ for } i = 1,\dots,k\}) > 0$ and $\nu_i < \theta_i$ for some $i \in \{1,\dots,k\}$. But we have

$$\mathcal{E}_k^{\theta} = \int_{\mathcal{N}} \mathcal{E}_k^{\nu} d\pi,$$

in particular

$$\mathcal{E}_k^\theta((\tilde{\nu}_1, \dots, \tilde{\nu}_k)) = \int_{\mathcal{N}} \mathcal{E}_k^\nu((\tilde{\nu}_k, \dots, \tilde{\nu}_1)) d\pi, \tag{3.22}$$

The first part of Proposition 3.5.14 implies that the left side of (3.22) vanishes, while the right side is positive. This contradiction proves that π–almost surely $\nu \geq \theta$.

On the other hand

$$\mathcal{E}_k^\theta((\theta_k, \dots, \theta_1)) = \int_{\mathcal{N}} \mathcal{E}_k^\nu((\theta_k, \dots, \theta_1)) d\pi, \tag{3.23}$$

Using the second part of Proposition 3.5.14 we conclude that if $\pi(\{\nu : \nu > \theta\}) > 0$ then the right side of (3.23) should be strictly less than the left side. Thus, $\pi(\theta) = 1$. □

3.5.5 Proof of Theorem 3.1.1

We start with 3 propositions which hold not only for the for the q–Gelfand–Tsetlin graph, but in a much larger generality.

Let Ω_q denote the convex set of q–central probability measures on $\mathcal{T}$. There is a natural topology on Ω_q, i.e. a minimal topology such that for any cylinder set C_τ the map

$$O_\tau : \Omega_q \to \mathbb{R}, \quad O_\tau(P) = P(C_\tau)$$

is continuous. Convergence in this topology coincides with weak convergence of probability measures.

Recall that the minimal boundary of the q–Gelfand-Tsetlin graph $\operatorname{Ex}\Omega_q$ is the set of extremal points of Ω_q. We denote elements of $\operatorname{Ex}\Omega_q$ by ω.

Proposition 3.5.18. *Ω_q is a simplex, i.e. for any $P \in \Omega_q$ there is a unique measure π on $\operatorname{Ex}\Omega_q$ such that*

$$P = \int_{\operatorname{Ex}\Omega_q} \omega d\pi.$$

Proposition 3.5.19. *The minimal boundary is a subset of the Martin boundary of the q–Gelfand–Tsetlin graph. More precisely, if $P \in \operatorname{Ex}\Omega_q$ and P_k is a q–coherent system corresponding to P, then P_k belongs to the Martin boundary of the q–Gelfand–Tsetlin graph.*

Remark. For the most non-degenerate examples the Martin boundary of the graph coincides with the minimal boundary. However, there exist graphs for which the minimal boundary is strictly less than the Martin boundary.

Proposition 3.5.20. *Let $P \in \operatorname{Ex}\Omega_q$ and let $\tau \in \mathcal{T}$. P-almost surely the sequence of signatures $\tau(N)$ is regular and*

$$P_k^{\tau(N)} \to P_k$$

for every k.

For the proofs see [71, Theorem 9.2], [65, Section 6] and [71, Proposition 10.8] respectively. Similar propositions were proved by Diaconis and Freedman in the framework of *partial exchangeability.* See [22, Theorem 1.1].

Theorem 3.5.21. *The set Ω_q of all q–central probability measures on $\mathcal{T}$ is a simplex with extreme points $\mathcal{E}^\nu$, i.e. for any q–central probability measure $P \in \Omega_q$ there exists a unique probability measure π on $\mathcal{N}$ such that*

$$P = \int_{\mathcal{N}} \mathcal{E}^\nu d\pi.$$

Proof. Proposition 3.5.18 implies that Ω_q is a simplex. Proposition 3.5.19 and Theorem 3.5.1 imply that

$$\operatorname{Ex}\Omega_q \subset \{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}},$$

where $\mathcal{E}^\nu$ is a q–central measure corresponding to q–coherent system $\mathcal{E}^\nu_k$. It follows from Proposition 3.5.16 that $\{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}$ with topology induced from Ω_q is isomorphic to $\mathcal{N}$. Finally, Proposition 3.5.17 implies that $\operatorname{Ex}\Omega_q = \{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}$. Indeed, if $Q \in \mathcal{E}^\nu$, then

$$Q = \int_{\operatorname{Ex}\Omega_q} \omega d\pi = \int_{\{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}} \omega d\pi',$$

where π' is a probability measure on $\{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}$ such that

$$\pi'(\operatorname{Ex}\Omega_q) = 1.$$

But then π' is δ–measure on a single element $Q = \mathcal{E}^\nu$. Thus, $\mathcal{E}^\nu \in \operatorname{Ex}\Omega_q$. □

The proved theorem is readily seen to be equivalent to Theorem 3.1.1. Now Proposition 3.1.2 is a straightforward corollary of Theorem 3.1.1 and Proposition 3.5.20

3.6 Some proofs

3.6.1 Relations between Schur and interpolation Schur functions

In this section we aim to prove Proposition 3.5.8 and Lemma 3.5.12. To do that we need some preparations.

Denote by $\mathcal{F}_N$ a class of q–Schur generating functions of probability measures supported on $\mathbb{GT}^+_N$. In other words, $F(x_1,\dots,x_N) \in \mathcal{F}_N$ if F is a symmetric analytic functions $F(x_1,\dots,x_N)$ on D_N such that

$$F(x_1,x_2,\dots,x_N) = \sum_{\mu\in\mathbb{GT}^+_N} \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})} c_\mu, \tag{3.24}$$

for a certain sequence of numbers, c_μ, $\mu \in \mathbb{GT}_N^+$, such that $c_\mu \geq 0$ and $\sum\limits_{\mu \in \mathbb{GT}_N^+} c_\mu = 1$.

Clearly, (3.24) is essentially just a Taylor series decomposition, thus, if such decomposition exists, then it is unique.

We also want to consider decompositions of symmetric functions into interpolation Schur polynomials:

$$F(x_1, \ldots, x_N) = \sum_\mu a_\mu s_\mu^*(x_1, x_2 \cdot, \ldots, x_N; q) \tag{3.25}$$

Proposition 3.6.1. *There exist coefficients K_μ^λ, ($\mu, \lambda \in \mathbb{GT}_N^+$), such that for every function $F(x_1, \ldots, x_N)$ defined in points $q^{\lambda-\delta}$ for every $\lambda \in \mathbb{GT}_N^+$, there is a unique decomposition*

$$F(x_1, \ldots, x_N) = \sum_\mu a_\mu s_\mu^*(x_1, x_2 \cdot, \ldots, x_N; q), \tag{3.26}$$

where the series converges in points $x = q^{\lambda-\delta}$ for every $\lambda \in \mathbb{GT}_N^+$. We have

$$a_\mu = \sum_{\lambda \in \mathbb{GT}_N^+} K_\mu^\lambda F(q^{\lambda-\delta}). \tag{3.27}$$

Furthermore, $K_\mu^\lambda = 0$ unless $\lambda \subset \mu$, thus, all sums in (3.27) *are finite.*

Remark. Note that this proposition is valid not only for $0 < q < 1$ but also for $q > 1$.

Proof of Proposition 3.6.1. Substitute $x = q^{\lambda-\delta}$ for every $\lambda \in \mathbb{GT}_N^+$ in (3.26). We obtain a system of linear equations

$$F(q^{\lambda-\delta}) = \sum_{\mu \in \mathbb{GT}_N^+} a_\mu L_\lambda^\mu, \tag{3.28}$$

where

$$L_\lambda^\mu = s_\mu^*(q^{\lambda-\delta}).$$

Proposition 3.3.4 implies that $L_\lambda^\mu = 0$, unless $\mu \subset \lambda$, and $L_\mu^\mu \neq 0$. Thus, the matrix L_λ^μ has a triangular structure (with respect to the partial order on signatures defined in Section 3.2) and all the sums in (3.28) are finite. Consequently, one may solve the system (3.28) inductively, starting from c_μ with $|\mu| = 0$, then proceeding to c_μ with $|\mu| = 1$ and so on. In other words, there exists a matrix K_μ^λ (which also has a triangular structure), such that

$$\sum_{\lambda \in \mathbb{GT}_N^+} K_{\mu_1}^\lambda L_\lambda^{\mu_2} = \begin{cases} 1, & \mu_1 = \mu_2, \\ 0. & \textit{otherwise}. \end{cases}$$

The coefficients of the matrix K_μ^λ are the desired ones. □

For a general function F we can not claim that the series (3.26) converges in any points other that $q^{\lambda-\delta}$. However, for functions from $\mathcal{F}_n$ the following lemma holds:

Lemma 3.6.2. *Suppose that function F belong to $\mathcal{F}_N$. There exists a unique decomposition*

$$F(x_1,\dots,x_N)=\sum_{\mu} a_\mu s^*_\mu(x_1,x_2,\dots,x_N;q),$$

such that the series converges uniformly on any ball $B(0,r)$ with radius $0<r<1$ and in every point $q^{\lambda-\delta}$.

Proof. To simplify the notations let us consider the case $N=1$. The proof in the general case follows the same steps.

Equality (3.7) yields

$$x^l=\sum_{m\le l}\left[\frac{(q^l-1)\dots(q^l-q^{m-1})}{(q^m-1)\dots(q^m-q^{m-1})}\right](x-1)\dots(x-q^{m-1}) \tag{3.29}$$

By the definition of $\mathcal{F}_1$ we have

$$F(x)=\sum_{l\ge 0} c_l x^l,$$

where $c_l\ge 0$ and $\sum c_l=1$.

Substituting (3.29) we get

$$F(x)=\sum_{l\ge 0} c_l \sum_{m\le l}\left[\frac{(q^l-1)\dots(q^l-q^{m-1})}{(q^m-1)\dots(q^m-q^{m-1})}\right](x-1)\dots(x-q^{m-1}) \tag{3.30}$$

Observe that the coefficients $\frac{(q^l-1)\dots(q^l-q^{m-1})}{(q^m-1)\dots(q^m-q^{m-1})}$ are uniformly bounded in l,m. If $x=q^k$, then in (3.29) only first $k+1$ terms are nonzero and if $|x|<r<1$, then the series in (3.29) converges exponentially fast. In both cases (3.30) absolutely converges and we may change the order of summation.

We obtain

$$F(x)=\sum_{m\ge 0}(x-1)\dots(x-q^{m-1})\cdot\sum_{l\ge m} c_l\left[\frac{(q^l-1)\dots(q^l-q^{m-1})}{(q^m-1)\dots(q^m-q^{m-1})}\right], \tag{3.31}$$

which is the required decomposition.

The uniqueness of the decomposition follows from Proposition 3.6.1. □

Proposition 3.6.3. *Suppose that functions F^n and F belong to $\mathcal{F}_N$, $F^n \rightrightarrows F$ on D_N. Then the coefficients a^n_μ of the interpolation Schur polynomials expansion* (3.26) *for the functions F_n converge to the corresponding coefficients a_μ of the function F.*

Proof. Uniform convergence on D_N implies that $F^n(q^{\lambda-\delta}) \to F(q^{\lambda-\delta})$. Applying Proposition 3.6.1 we conclude that $a_\mu^n \to a_\mu$. $\square$

Combining Lemma 3.6.2 with Proposition 3.6.3 we arrive at Proposition 3.5.8.

Now let us turn to q–interpolation Schur generating functions. Let $\mathcal{F}_N^*$ denote a class of q–interpolation Schur generating functions of probability measures supported on $\mathbb{GT}_N^+$. In other words, $F(x_1, \dots, x_N) \in \mathcal{F}_N^*$ if F is a symmetric analytic functions $F(x_1, \dots, x_N)$ on $\mathbb{C}^N$ such that

$$F(x_1, \dots, x_N) = \sum_{\mu \in \mathbb{GT}_N^+} c_\mu \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})} \tag{3.32}$$

for a certain sequence of numbers, c_μ, $\mu \in \mathbb{GT}_N^+$, such that $c_\mu \geq 0$ and $\sum_{\mu \in \mathbb{GT}_N^+} c_\mu = 1$.

Note that if decomposition (3.32) exists, then it is unique. To prove this fact we repeat the argument of Proposition 3.6.1.

Next, we want to study the relation between q–Schur generating function and q–interpolation Schur generating function of the same probability measure.

Let Sym(N) denote the space of symmetric polynomials in N variables $x_1, \dots, x_N$. Consider a linear map G : Sym(N) $\to$ Sym(N) defined on Schur polynomials' basis through

$$G\left(\frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})}\right) = \frac{s_\mu^*(x_1, x_2, \dots, x_N; q)}{s_\mu^*(0, \dots, 0; q)}, \tag{3.33}$$

or, equivalently,

$$G(s_\mu(x_1, \dots, x_N)) = (-1)^{|\mu|} q^{n(\mu)-n(\mu')} s_\mu^*(x_1, \dots, x_N; q)$$

Note that $s_\lambda^*(x_1, \dots, x_N; 1) = s_\lambda(x_1 - 1, \dots, x_N - 1)$. Thus for $q = 1$ the map G becomes a simple change of variables:

$$G_{q=1} f(x_1, \dots, x_N) = f(1 - x_1, \dots, 1 - x_N).$$

Let us also consider another map G' : Sym(N) $\to$ Sym(N) defined on q–interpolation polynomials through:

$$G'\left(\frac{s_\mu^*(q^{k-1}x_1, q^{k-1}x_2, \dots, q^{k-1}x_k; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})}\right) = \frac{s_\mu(x_1, \dots, x_k)}{s_\mu(1, q^{-1}, \dots, q^{1-k})} \tag{3.34}$$

Lemma 3.6.4. *$G = G'$ on all finite degree polynomials.*

Remark. In one-dimensional case we have:

$$G(x^k) = (1 - x)(1 - xq^{-1}) \dots (1 - xq^{1-k}),$$

$$G'((1 - x)(1 - xq) \dots (1 - xq^{k-1})) = x^k$$

The fact that $G = G'$ follows from the q–binomial theorem.

Proof of Lemma 3.6.4. Take the equality (3.8) with q replaced by q^{-1} and with x replaced by $q^{k-1}x$. We get:

$$s^*_\lambda(q^{k-1}x_1,\dots,q^{k-1}x_N;q^{-1}) \\ = \sum_\mu \frac{s^*_\mu(q^{\lambda-\delta};q)}{s^*_\mu(q^{\mu-\delta});q)}\frac{s^*_\lambda(0,\dots,0;q^{-1})}{s^*_\mu(0,\dots,0;q^{-1})}q^{(N-1)|\mu|}s_\mu(x_1,\dots,x_N).$$

Apply G in the sense of formula (3.33) to the righthand-side and G' in the sense of formula (3.34) to the lefthand-side. We get the equality

$$s_\lambda(x_1,\dots,x_N)\frac{s^*_\lambda(0,\dots,0;q^{-1})}{s_\lambda(1,q^{-1},\dots,q^{1-N})} \stackrel{?}{=} \sum_\mu \frac{s^*_\mu(q^{\lambda-\delta};q)}{s^*_\mu(q^{\mu-\delta});q)}\frac{s^*_\lambda(0,\dots,0;q^{-1})}{s^*_\mu(0,\dots,0;q^{-1})} \\ \times \frac{s_\mu(1,q^{-1},\dots,q^{1-N})}{s^*_\mu(0,\dots,0;q)}q^{(N-1)|\mu|}s^*_\mu(x_1,x_2,\dots,x_Nk;q). \quad (3.35)$$

Since

$$\frac{q^{(N-1)|\mu|}s_\mu(1,q^{-1},\dots,q^{1-N})}{s^*_\mu(0,\dots,0;q^{-1})s^*_\mu(0,\dots,0;q^{-1})} = \frac{1}{s_\mu(1,q^{-1},\dots,q^{1-N})},$$

(3.35) is equivalent to

$$s_\lambda(x_1,\dots,x_N) \stackrel{?}{=} \sum_\mu \frac{s^*_\mu(q^{\lambda-\delta};q)}{s^*_\mu(q^{\mu-\delta};q)}\frac{s_\lambda(1,q^{-1},\dots,q^{1-N})}{s_\mu(1,q^{-1},\dots,q^{1-N})}s^*_\mu(x_1,x_2,\dots,x_N;q),$$

which is exactly (3.7). Hence, the last equality is true and G coincides with G' on all q–interpolation Schur polynomials. Consequently, they coincide on all polynomials by linearity. □

Next, we want to extend the domain of definition of the maps G and G'.

For any function $f \in \mathcal{F}^*_N$ we can define $G'(f)$ as follows:

$$f(x_1,\dots,x_N) = \sum_{\mu\in\mathbb{GT}^+_N} c_\mu \frac{s^*_\mu(q^{N-1}x_1,q^{N-1}x_2,\dots,q^{N-1}x_N;q^{-1})}{s^*_\mu(0,\dots,0;q^{-1})},$$

$$G'(f) = \sum_{\mu\in\mathbb{GT}^+_N} c_\mu \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})}.$$

It is clear that G' is a bijection between $\mathcal{F}^*_N$ and $\mathcal{F}_N$. More precisely,

$$G'(\mathcal{S}^*(x_1,\dots,x_N;P)) = \mathcal{S}(x_1,\dots,x_N;P)$$

for any probability measure P on $\mathbb{GT}_N$ with $\mathrm{supp}(P) \subset \mathbb{GT}^+_N$.

Proof of Lemma 3.5.12. Recall that we want to prove the following statement: Suppose that $F \in \mathcal{F}_N^*$ has the following Taylor series expansion

$$F(x_1, \dots, x_N) = \sum_{\mu \in \mathbb{GT}_N^+} a_\mu \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})}$$

Define

$$G(f) = \sum_{\mu \in \mathbb{GT}_N^+} a_\mu \frac{s_\mu^*(x_1, x_2, \dots, x_N; q)}{s_\mu^*(0, \dots, 0; q)}. \tag{3.36}$$

Then the series on the right side of (3.36) converges uniformly on any ball $B(0, r)$ with radius $0 < r < 1$ and in every point $q^{\lambda - \delta}$. Furthermore, $G(f) = G'(f)$.

Let us start the proof. Lemma 3.6.2 yields that $G'(f)$ can be represented as a linear combination of q–interpolation Schur polynomials:

$$G'(f) = \sum_{\mu \in \mathbb{GT}_N^+} b_\mu \frac{s_\mu^*(x_1, x_2, \dots, x_N; q)}{s_\mu^*(0, \dots, 0; q)}.$$

This series converges uniformly on any ball $B(0, r)$ with radius $0 < r < 1$ and in every point $q^{\lambda - \delta}$. To prove the proposition we should check that $a_\mu = b_\mu$.

Suppose that

$$F(x_1, \dots, x_N) = \sum_{\mu \in \mathbb{GT}_N^+} c_\mu \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})}.$$

Set

$$f^m(x_1, \dots, x_N) = \sum_{|\mu| \le m} c_\mu \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})}$$

and $\widehat{f}^m = F - f^m$. Let a_μ^m be the corresponding coefficient of the Taylor series expansion of f^m and let $\widehat{a}_\mu^m$ be the corresponding coefficient of the Taylor series expansion of $\widehat{f}^m$. Clearly, $a_\mu = a_\mu^m + \widehat{a}_\mu^m$. Represent $G'(f^m)$ and $G'(\widehat{f}^m)$ as linear combinations of q–interpolation Schur polynomials and let b_μ^m, $\widehat{b}_\mu^m$ be the corresponding coefficients:

$$G'(f^m) = \sum_{\mu \in \mathbb{GT}_k^+} b_\mu^m \frac{s_\mu^*(x_1, x_2, \dots, x_k; q)}{s_\mu^*(0, \dots, 0; q)},$$

$$G'(\widehat{f}^m) = \sum_{\mu \in \mathbb{GT}_k^+} \widehat{b}_\mu^m \frac{s_\mu^*(x_1, x_2, \dots, x_k; q)}{s_\mu^*(0, \dots, 0; q)},$$

It is clear that $b_\mu = b_\mu^m + \widehat{b}_\mu^m$.

Lemma 3.6.4 implies that $b_\mu^m = a_\mu^m$. It remains to prove that both $\widehat{a}_\mu^m$ and $\widehat{b}_\mu^m$ tend to zero as m tends to infinity.

It follows from Proposition 3.4.7 that $\widehat{f}^m \rightrightarrows 0$ uniformly on compact sets as $m \to \infty$. Uniform convergence of analytical functions implies convergence of their Taylor expansion coefficients. Thus, $\widehat{a}_\mu^m \to 0$ as $m \to \infty$. By the definition

$$G'(f^m) = \sum_{|\mu|>m} c_\mu \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})}.$$

Proposition 3.4.5 implies that $G'(f^m) \rightrightarrows 0$ uniformly on D_N. Repeating the argument in the proof of Proposition 3.6.3 we conclude that $\widehat{b}_\mu^m \to 0$ as $m \to \infty$. □

3.6.2 q–Central measures and probability generating functions

Proof of Proposition 3.4.3. We want to prove the following statement: if measure P on $\mathcal{T}_N$ is such that

$$P(\tau(1) \prec \dots \tau(N)) = \frac{q^{|\tau(1)|+\dots+|\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))} P_N(\tau(N)),$$

for any path $\tau \in \mathcal{T}_N$, then $P_1, P_2, \dots, P_N$ is a q–coherent system.

We proceed by induction in N. Case $N = 1$ is trivial. For general N we have

$$\begin{aligned} P_{N-1}(\mu) &= \sum_{\tau \in \mathcal{T}_N,\, \tau(N-1)=\mu} P_N(\tau(N)) \frac{q^{|\tau(1)|+\dots+|\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))} \\ &= \sum_\lambda \frac{P_N(\lambda) q^{|\mu|}}{\mathrm{Dim}_q(\lambda)} \sum_{\tau \in \mathcal{T}_N,\, \tau(N)=\lambda, \tau(N-1)=\mu} q^{|\tau(1)|+\dots+|\tau(N-2)|} \\ &= P_N(\lambda) q^{|\mu|} \frac{\mathrm{Dim}_q(\mu)}{\mathrm{Dim}_q(\lambda)} = \sum_\lambda P_N(\lambda) P(\lambda \to \mu) \end{aligned}$$

Thus, P_{N-1} and P_N are q–coherent.

Next, projection of measure P on $\mathcal{T}_{N-1}$ defines a probability measure that we denote by $\tilde{P}$. If $\tau' \in \mathcal{T}_{N-1}$, then

$$\tilde{P}(\tau') = \sum_\lambda P(\tau'(1) \prec \dots \prec \tau'(N-1) \prec \lambda) = q^{|\tau(1)|+\dots+|\tau(N-1)|} \sum_{\lambda | \tau'(N-1) \prec \lambda} \frac{P_N(\lambda)}{\mathrm{Dim}_q(\lambda)},$$

It follows that if $\tau^1, \tau^2 \in \mathcal{T}_{N-1}$ and $\tau^1(N-1) = \tau^2(N-1)$, then

$$\frac{\tilde{P}(\tau^1)}{\tilde{P}(\tau^2)} = \frac{q^{|\tau^1(1)|+\dots+|\tau^1(N-2)|}}{q^{|\tau^2(1)|+\dots+|\tau^2(N-2)|}},$$

Thus,

$$\tilde{P}(\tau') = \frac{q^{|\tau'(1)|+\cdots+|\tau'(N-2)|}}{\sum\limits_{\tau\in\mathcal{T}_{N-1},\,\tau(N-1)=\tau'(N-1)} q^{|\tau(1)|+\cdots+|\tau(N-2)|}} \sum_{\tau\in\mathcal{T}_{N-1},\,\tau(N-1)=\tau'(N-1)} \tilde{P}(\tau)$$
$$= \frac{q^{|\tau'(1)|+\cdots+|\tau'(N-2)|}}{\mathrm{Dim}_q(\tau'(N-1))}\tilde{P}_{N-1}(\tau'(N-1))$$

Consequently, it follows by induction that $\tilde{P}_1,\ldots,\tilde{P}_{N-1}$ form a q–coherent system. Since $P_i = \tilde{P}_i$ for $i = 1,\ldots,N-1$ and measures P_N and P_{N-1} are q–coherent, $P_1,\ldots P_N$ is a q–coherent system. □

Proof of Proposition 3.4.4. We want to prove that for any q–coherent system $P_1, P_2, \ldots$, there exists a unique q–central measure P such that P_k is a projection of P on $\mathbb{GT}_k$ for every $k = 1, 2, \ldots$.

For any N let $P^{(N)}$ be a probability measure on $\mathcal{T}_N$ defined through

$$P^{(N)}(\tau) = \frac{q^{|\tau(1)|+\cdots+|\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))}P_N(\tau(N)).$$

Proposition 3.4.3 yields that projections $P_k^{(N)}$ of $P^{(N)}$ on $\mathbb{GT}_k$ form a q–coherent system. Since $P_N^{(N)} = P_N$ and $P_1,\ldots,P_N$ also form a q–coherent system, thus, $P_k^{(N)} = P_k$.

Repeating the argument of Proposition 3.4.3 we conclude that projection of $P^{(N)}$ on $\mathcal{T}_{N-1}$ coincides with $P^{(N-1)}$.

Let P be projective limit of the measures $P^{(N)}$ as $N \to \infty$. Clearly, P is a q–central measure on $\mathcal{T}$ and projections of P on $\mathbb{GT}_k$ are exactly P_k. □

Proof of Proposition 3.4.6. We want to prove that two probability measures P_N and P_{N+1} on $\mathbb{GT}_N$ and $\mathbb{GT}_{N+1}$, respectively, are q–coherent if and only if

$$\mathcal{S}(x_1,\ldots,x_N;P_N) = \mathcal{S}(x_1,\ldots,x_N,q^{-N};P_{N+1}).$$

Using Proposition (3.3.1) we obtain

$$\frac{s_\lambda(x_1,\ldots,x_N,q^{-N})}{s_\lambda(1,q^{-1},\ldots,q^{-N})} = \frac{\sum\limits_{\mu\prec\lambda} s_\mu(x_1,\ldots,x_N)q^{-N(|\lambda|-|\mu|)}}{q^{-N|\lambda|}s_\lambda(1,q,\ldots,q^N)}$$
$$= \sum_{\mu\prec\lambda}\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}q^{|\mu|}\frac{s_\mu(1,q,\ldots,q^{N-1})}{s_\lambda(1,q,\ldots,q^N)}$$
$$= \sum_{\mu\in\mathbb{GT}_N}\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}P(\lambda\to\mu).$$

Thus,

$$\mathcal{S}(x_1,\dots,x_N,q^{-N};P_{N+1}) = \sum_{\lambda\in\mathbb{GT}_{N+1}} \frac{s_\lambda(x_1,\dots,x_N,q^{-N})}{s_\lambda(1,q^{-1},\dots,q^{-N})} P_{N+1}(\lambda)$$
$$= \sum_{\mu\in\mathbb{GT}_N} \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})} \sum_{\lambda\in\mathbb{GT}_{N+1}} P(\lambda\to\mu)P_{N+1}(\lambda),$$

On the other hand,

$$\mathcal{S}(x_1,\dots,x_N;P_N) = \sum_{\mu\in\mathbb{GT}_N} \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})} P_N(\mu).$$

Comparing the last two expressions we conclude that

$$\mathcal{S}(x_1,\dots,x_N,q^{-N};P_{N+1}) = \mathcal{S}(x_1,\dots,x_N;P_N)$$

if and only if

$$P_N(\mu) = \sum_{\lambda\in\mathbb{GT}_{N+1}} P(\lambda\to\mu)P_{N+1}(\lambda)$$

for all $\mu\in\mathbb{GT}_N$. □

Proof of Proposition 3.4.8. We want to prove that two probability measures P_N and P_{N+1} on $\mathbb{GT}_N^+$ and $\mathbb{GT}_{N+1}$, respectively, are q–coherent if and only if

$$\mathcal{S}^*(x_1,\dots,x_N;P_N) = \mathcal{S}^*(x_1,\dots,x_N,0;P_{N+1}).$$

To prove this proposition we need a lemma.

Lemma 3.6.5 (The branching rule for q–interpolation Schur polynomials)**.** *For any $\lambda\in\mathbb{GT}_{N+1}^+$ we have*

$$\frac{s_\lambda^*(q^Nx_1,q^Nx_2,\dots,q^Nx_N,0;q^{-1})}{s_\lambda^*(0,\dots,0;q^{-1})}$$
$$= \sum_{\mu\in\mathbb{GT}_N} \frac{s_\mu^*(q^{N-1}x_1,q^{N-1}x_2,\dots,q^{N-1}x_N;q^{-1})}{s_\mu^*(0,\dots,0;q^{-1})} P(\lambda\to\mu).$$

Proof. Using the combinatorial formula (see Proposition 3.3.5) we obtain

$$s_\lambda^*(q^Nx_1,\dots,q^Nx_N,q^Nx_{N+1};q^{-1}) = \sum_T \prod_{(i,j)\in\lambda} \left(q^Nx_{T(i,j)} - q^{i-j-T(i,j)+N+1}\right),$$

where the sum is taken over all semistandard Young tableaux $T(i,j)$ of shape λ filled with numbers $1,\dots,N+1$. Note that the part of T filled with numbers $1,\dots,N$ is a Young diagram of shape μ such that $\mu\prec\lambda$. Consequently,

substituting $x_{N+1} = 0$ we get

$$s^*_\lambda(q^N x_1, \dots, q^N x_N, 0; q^{-1})$$
$$= \sum_{\mu \prec \lambda} \left(\prod_{(i,j) \in \lambda \setminus \mu} (0 - q^{i-j}) \right) \sum_T \prod_{(i,j) \in \mu} \left(q^N x_{T(i,j)} - q^{i-j-T(i,j)+N+1} \right),$$

where the sums are taken over semistandard Young tableaux $T(i,j)$ of shape μ filled with numbers $1, \dots, N$. Consequently,

$$s^*_\mu(q^N x_1, \dots, q^N x_N, 0; q^{-1})$$
$$= \sum_{\mu \prec \lambda} \left(\prod_{(i,j) \in \lambda \setminus \mu} (-q^{i-j}) \right) q^{|\mu|} s^*_\mu(q^{N-1} x_1, \dots, q^{N-1} x_N; q^{-1}).$$

It remains to prove that

$$\prod_{(i,j) \in \lambda \setminus \mu} (-q^{i-j})\, q^{|\mu|} = \frac{s^*_\lambda(0, \dots, 0; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})} P(\lambda \to \mu). \tag{3.37}$$

We claim that for any $\lambda \in \mathbb{GT}_N$

$$s^*_\lambda(0, \dots, 0; q^{-1}) = s_\lambda(1, q^{-1}, \dots, q^{1-N}) q^{(N-1)|\lambda|} \prod_{(i,j) \in \lambda} (-q^{i-j})$$
$$= s_\lambda(1, q, \dots, q^{N-1}) \prod_{(i,j) \in \lambda} (-q^{i-j})$$

The claim follows from the combinatorial formulas for Schur polynomials and for q–interpolation Schur polynomials. Using the last relation one immediately proves (3.37). □

To finish the proof of Proposition 3.4.8 we use the last lemma and repeat the argument of Proposition 3.4.6. □

Proof of Proposition 3.4.10. We want to prove that if a sequence of probability measure P^i_N on $\mathbb{GT}_N$. weakly converge to P_N, then

$$\mathcal{S}(x_1, \dots, x_N; P^i_N) \rightrightarrows \mathcal{S}(x_1, \dots, x_N; P_N)$$

uniformly on T_N. Furthermore, if P_N, P^i_N are supported on $\mathbb{GT}^+_N$, then the convergence is uniform on D_N.

Since P_N is a probability measure on $\mathbb{GT}_N$, there exists k such that $P_N(\{\mu : -k < \mu_i < k\}) > 1 - \varepsilon$. Observe that $\{\mu : -k < \mu_i < k\}$ is a finite set with less than $(2k)^N$ elements. Thus, weak convergence of measures P^i_N implies that for all $i > i_0$ and all $\lambda \in \mathbb{GT}_N$ such that $-k < \lambda_i < k$ we have

$$\left| P^i_N(\lambda) - P_N(\lambda) \right| < \frac{\varepsilon}{(2k)^N}$$

and

$$P_N^i(\{\mu : |\mu| < k\}) > 1 - 2\varepsilon.$$

Consequently, since

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, \dots, q^{1-N})} \right| \le 1$$

on T_N, we have for $i > i_0$ and $(x_1, \dots, x_N) \in T_N$

$$\begin{aligned} &\left|\mathcal{S}(x_1, \dots, x_N; P_N^i) - \mathcal{S}(x_1, \dots, x_N; P_N)\right| \\ &\qquad \le \sum_{|\mu|<k} |P_N^i(\mu) - P_N(\mu)| \left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, \dots, q^{1-N})} \right| \\ &\quad + \sum_{|\mu|\ge k} (P_N^i(\mu) + P_N(\mu)) \left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, \dots, q^{1-N})} \right| \le k^N \frac{\varepsilon}{k^N} + \varepsilon + 2\varepsilon = 4\varepsilon \end{aligned}$$

It follows, that $\mathcal{S}(x_1, \dots, x_N; P_N^i)$ converges uniformly on T_N.

If P_N, P_N^i are supported on $\mathbb{GT}_N^+$, then the argument is the same with T_N replaced by D_N. □

Proof of Proposition 3.4.11. We want to prove the following statement: if P_N^i is a sequence of probability measures on $\mathbb{GT}_N$ such that the functions

$$\mathcal{S}(x_1, \dots, x_N; P_N^i)$$

converge uniformly on T_N to a function $S(x_1, \dots, x_N)$, then there exists a probability measure P_N such that

$$S(x_1, \dots, x_N) = \mathcal{S}(x_1, \dots, x_N; P_N)$$

and P_N^i weakly converge to P_N.

Recall that (by its definition) a rational Schur function is the ratio of the alternating sum of monomials and the Vandermonde determinant:

$$s_\lambda(x_1, \dots, x_N) = \frac{\mathrm{Alt}(x^{\lambda_1+N-1} \cdots x^{\lambda_N})}{V(x_1, \dots, x_N)},$$

where

$$V(x_1, \dots, x_N) = \prod_{1\le i<j\le N} (x_i - x_j)$$

and

$$\mathrm{Alt}(x_1^{k(1)} \dots x_N^{k(N)}) = \sum_\sigma (-1)^\sigma x_1^{k(\sigma(1))} \cdots x_N^{k(\sigma(N))},$$

sum is taken over all permutations of length N and $(-1)^\sigma$ is the sign of the permutation σ.

The following estimates are useful in the sequel:

Lemma 3.6.6. *There exist three constants $C_1(N)$, $C_2(N)$, $C_3(N)$ such that for any $(x_1, \dots, x_N) \in T_N$ and any $\lambda \in \mathbb{GT}_N$ we have*

$$C_1(N) < |V(x_1, \dots, x_N)| < C_2(N)$$

and absolute values of the coefficients of the monomials in the alternating sum

$$\frac{\mathrm{Alt}(x^{\lambda_1+N-1} \cdots x^{\lambda_N})}{s_\lambda(1, q^{-1}, \dots, q^{1-N})}$$

are bounded by $C_3(N)$.

We leave the proof of this lemma to the reader.

Denote

$$R^i(x_1, \dots, x_N) = V(x_1, \dots, x_N)\mathcal{S}(x_1, \dots, x_N; P_N^i)$$

and

$$R(x_1, \dots, x_N) = V(x_1, \dots, x_N)S(x_1, \dots, x_N).$$

Clearly, $R^i(x_1, \dots, x_N)$ converges to $R(x_1, \dots, x_N)$ uniformly on T_N.

We have

$$R^i(x_1, \dots, x_N) = \sum_{\lambda \in \mathbb{GT}_N} P_N^i(\lambda) \frac{\mathrm{Alt}(x^{\lambda_1+N-1} \cdots x^{\lambda_N})}{s_\lambda(1, q^{-1}, \dots, q^{1-N})}. \tag{3.38}$$

Note that if we expand (3.38) in a single sum of monomials, then we get a Fourier series expansion of $R^i(x_1, \dots, x_N)$ in the conventional sense. Furthermore, estimates of Lemma 3.6.6 guarantee that this Fourier series uniformly converges on T_N. It is well known that uniform convergence of the continuous functions implies the convergence of their Fourier coefficients. (This fact follows from the integral formula for the Fourier coefficients.) Consequently, $P_N^i(\lambda)$ converges to a certain number $P_N(\lambda)$ for every λ. Since $P_N^i(\lambda) \geq 0$ and $\sum_\lambda P_N^i(\lambda) = 1$, we have

$$P_N(\lambda) \geq 0, \qquad \sum_{\lambda \in \mathbb{G}T_N} P_N(\lambda) \leq 1. \tag{3.39}$$

If the last sum equals to 1, then the numbers $P_N(\lambda)$ define a probability measure P_N. Measures P_N^i weakly converge to P_N and we are done. However, the proof of the fact that $\sum_\lambda P_N(\lambda) = 1$ needs an additional argument. Define

$$\widehat{R}(x_1, \dots x_N) = \sum_{\lambda \in \mathbb{GT}_N} P_N(\lambda) \frac{\mathrm{Alt}(x^{\lambda_1+N-1} \cdots x^{\lambda_N})}{s_\lambda(1, q^{-1}, \dots, q^{1-N})}.$$

Inequalities (3.39) and estimates of Lemma 3.6.6 guarantee that $\widehat{R}(x_1, \dots x_N)$ is well-defined on T_N. Now observe that by the construction all the Fourier coefficients of R and $\widehat{R}$ coincide. Thus, all Fourier coefficients of $R - \widehat{R}$ are

equal to zero. Consequently, (since Laurent polynomials are dense in the space of continuous functions of T_N) $R = \widehat{R}$. Therefore

$$\sum_{\lambda \in \mathbb{G}T_N} P_N(\lambda) = \frac{\widehat{R}(1, q^{-1}, \dots, q^{-N})}{V(1, q^{-1}, \dots, q^{-N})} = \frac{R(1, q^{-1}, \dots, q^{-N})}{V(1, q^{-1}, \dots, q^{-N})}$$
$$= \lim_{i \to \infty} \frac{R^i(1, q^{-1}, \dots, q^{-N})}{V(1, q^{-1}, \dots, q^{-N})} = \lim_{i \to \infty} 1 = 1.$$

□

Proof of Proposition 3.4.12. We want to prove that a sequence of probability measures P_N^i on $\mathbb{GT}_N^+$ weakly converges to P_N if and only if

$$\mathcal{S}^*(x_1, \dots, x_N; P_N^i) \rightrightarrows \mathcal{S}^*(x_1, \dots, x_N; P_N)$$

uniformly on compact subsets of $\mathbb{C}^N$.

If functions $\mathcal{S}^*(x_1, \dots, x_N; P_N^i)$ converge, then using the argument of Proposition 3.6.1 we conclude that the coefficients of their q–interpolation Schur expansions converge. Thus, measures P_N^i weakly converge.

The second part of the proof repeats the argument of Proposition 3.4.10 with inequality

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, \dots, q^{1-N})} \right| \le 1$$

replaced by the bound

$$\left| \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})} \right| < A(M)$$

valid for $|x_i| \le M$, as was explained in the proof of Proposition 3.4.7. □

3.6.3 Tightness (Proof of Proposition 3.5.7)

We want to prove that if $\lambda(i)$ is a sequence of signatures stabilizing to ν, then the family of functions

$$g_i(x_1, \dots, x_k) = \frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_\lambda(i)(1, q, \dots, q^{1-N(i)})}$$

is a relatively compact subset of the set of continuous functions on k–dimensional torus T_k with uniform convergence topology.

We need the following lemma.

Lemma 3.6.7. *Let $\lambda(i)$ be a sequence of positive signatures stabilizing to ν with $\nu_1 = 0$. For any integer $k \ge 0$ set*

$$V_{k,i}(\phi) = \frac{s_{\lambda(i)}(q^{-k}e^{-i\phi}, q^{-1}, \dots, q^{1-N(i)})}{s_\lambda(i)(1, q^{-1}, \dots, q^{1-N(i)})}.$$

There exists a function $c(k,\nu)$ and a number i_0 such that

$$\left|\frac{\partial V_{k,i}(\phi)}{\partial \phi}\right| \le c(k,\nu)$$

for $i > i_0$.

Proof. Assume without lost of generality that $\lambda(i)_{N(i)} = 0$. Denote $x = q^{-k}e^{-i\phi}$. The branching rule for Schur polynomials yields

$$V_{k,i}(\phi) = \sum_{\mu \prec \lambda(i)} x^{|\lambda|-|\mu|} \frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}$$
$$= \frac{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})} \sum_{m\ge 0} x^m \sum_{\mu\prec\lambda(i),\, |\lambda(i)|-|\mu|=m} \frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}.$$

(Here we used the notation $\lambda(i)$ both for $(\lambda(i)_1,\dots,\lambda(i)_{N(i)})$ and $(\lambda(i)_1,\dots,\lambda(i)_{N(i)-1})$.) Thus,

$$\left|\frac{\partial V_{k,i}(\phi)}{\partial \phi}\right| \le \frac{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}$$
$$\times \sum_{m\ge 0} m|x|^m \sum_{\mu\prec\lambda(i),\, |\lambda(i)|-|\mu|=m} \frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}. \quad (3.40)$$

The branching rule for Schur polynomials applied to $s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})$ yields that the first fraction in the last formula is not greater than 1. To estimate the double sum, we first consider the terms with $m = 1$. Using [58, Example 3.1] we obtain

$$\frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})} = q^{|\mu|(1-N(i))-|\lambda(i)|(1-N(i))} \frac{s_\mu(1,q,\dots,q^{N(i)-2})}{s_{\lambda(i)}(1,q,\dots,q^{N(i)-2})}$$
$$= q^{1-N(i)} q^{n(\mu)-n(\lambda(i))} \prod_{p<q} \frac{1-q^{\mu_p-\mu_q-p+q}}{1-q^{\lambda(i)_p-\lambda(i)_q-p+q}}$$

Suppose that μ differs from $\lambda(i)$ in row $N(i)-j$, then

$$q^{1-N(i)} q^{n(\mu)-n(\lambda(i))} = q^j$$

and

$$\prod_{p<q} \frac{1-q^{\mu_p-\mu_q-p+q}}{1-q^{\lambda(i)_p-\lambda(i)_q-p+q}} \le \prod_{\ell=1}^{\infty} (1-q^\ell)^{-1}.$$

It follows that the terms corresponding to $m = 1$ in the double sum in (3.40) are bounded from above by

$$\sum_{j=1}^{\infty} q^j \prod_{\ell=1}^{\infty} (1-q^\ell)^{-1}.$$

For general m let $\mu_n = \lambda(i)_n - f_{N(i)-n}$, $n = 1, 2, \ldots, N(i) - 1$. Similarly to $m = 1$ case we get

$$\frac{s_\mu(q^{-1}, \ldots, q^{1-N(i)})}{s_{\lambda(i)}(q^{-1}, \ldots, q^{1-N(i)})} \leq q^{f_1+2f_2+\cdots+(N(i)-1)f_{N(i)-1}} \prod_{\ell=1}^{\infty} (1-q^\ell)^{-(f_1+f_2+\cdots+f_n)}.$$

Consider the generating function

$$a_i(t) = \sum_{f_1, f_2, \ldots} q^{f_1+2f_2+3f_3\cdots} \cdot t^{f_1+f_2+f_3+\cdots},$$

where the sum is taken over all finite collections of integers $\{f_n\}$ satisfying $0 \leq f_n \leq \lambda_{N(i)-n} - \lambda_{N(i)-n+1}$. We will use $a_i'(t)$ for $\frac{\partial}{\partial t} a_i(t)$. The above arguments prove that

$$\left| \frac{\partial V_{k,i}(\phi)}{\partial \phi} \right| \leq a_i' \left(q^{-k} \prod_{\ell=1}^{\infty} (1-q^\ell)^{-1} \right).$$

Now choose r such that $q^{-k} \prod_{\ell=1}^{\infty} (1-q^\ell)^{-1} < q^{-r}$. Suppose that i is large enough, so that $\lambda(i)_{N(i)-n+1} = \nu_n$ for $n = 1, 2, \ldots, r+1$. Let

$$b(t) = \prod_{j=1}^{r} \left(\sum_{u=0}^{\nu_{j+1}-\nu_j} \left(tq^j\right)^u \right) \cdot \prod_{j=r+1}^{\infty} \left(\sum_{u=0}^{\infty} \left(tq^j\right)^u \right).$$

Clearly, $a_i(t) \leq b(t)$, furthermore, $a_i'(t) \leq b'(t)$. But

$$b(t) = \prod_{j=1}^{r} \left(\sum_{u=0}^{\nu_{j+1}-\nu_j} \left(tq^j\right)^u \right) \cdot \prod_{j=r+1}^{\infty} \left(1 - \left(tq^j\right)\right)^{-1}.$$

Consequently, $b(t)$ is an analytic function in ball $\{t : t \leq q^{-r}\}$. Thus,

$$a_i' \left(q^{-k} \prod_{\ell=1}^{\infty} (1-q^\ell)^{-1} \right) < b' \left(q^{-k} \prod_{\ell=1}^{\infty} (1-q^\ell)^{-1} \right) < \infty.$$

□

Proof of Proposition 3.5.7. First, let $\nu_1 = 0$. Observe that $g_i(x_1, \ldots, x_k)$ is a symmetric polynomial with positive coefficients (for large enough i). Thus, for any $(x_1, \ldots, x_k) \in T_k$ and any $1 \leq \ell \leq k$ we have

$$\begin{aligned}
&\left| \frac{\partial}{\partial \phi} g_i(\ldots, x_{\ell-1}, e^{i\phi} x_\ell, x_{\ell+1}, \ldots) \right| \\
&\qquad \leq \left| \frac{\partial}{\partial \phi} g_i(1, q^{-1}, \ldots, q^{2-\ell}, q^{1-\ell} e^{i\phi}, q^{-\ell}, \ldots, q^{1-k}) \right| \\
&\qquad = \left| \frac{\partial}{\partial \phi} g_i(q^{1-\ell} e^{i\phi}, 1, q^{-1}, \ldots, q^{2-\ell}, q^{-\ell}, \ldots, q^{1-k}) \right| \\
&\qquad\qquad \leq \left| \frac{\partial}{\partial \phi} g_i(q^{1-\ell} e^{i\phi}, q^{-1}, q^{-2}, \ldots, q^{1-k}) \right| < const,
\end{aligned}$$

where the last inequality is Lemma 3.6.7. The above uniform estimate for the derivatives yields that the family of functions g_i is equicontinuous on T_n. We also have $|g_i| \leq 1$ on T_n. Thus, by the Arzelà–Ascoli theorem the set of functions $\{g_i\}$ is relatively compact.

For general ν we note that if $\lambda(i)$ stabilizes to ν then $A_{-\nu_1}(\lambda(i))$ stabilizes to $\nu' = A_{-\nu_1}(\nu)$ with $\nu'_1 = 0$. Since for any λ and any ℓ

$$s_{A_\ell(\lambda)}(x_1, \ldots, x_N) = (x_1 \cdots x_N)^\ell s_\lambda(x_1, \ldots, x_N),$$

the case of general ν reduces to the case $\nu_1 = 0$. □

3.6.4 Analyticity (proof of Proposition 5.10)

We want to prove that for every ν the series

$$\sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\mu) s^*_\mu(x_1, \ldots, x_k)$$

converges for all $x_1, \ldots x_k$ and defines an entire function.

The combinatorial formula for s^*_μ (Proposition 3.3.5) implies that for every M there exists a constant $C(M)$ such that for every $x_1, \ldots, x_k$ with $|x_i| < M$, we have

$$\left|s^*_\mu(x_1, \ldots, x_k)\right| < C(M)^{|\mu|} < \left(C(M)^k\right)^{\mu_1}.$$

Let us fix ν and estimate $q^{n(\mu)-n(\mu')}\mathrm{Spec}_\nu(s_\lambda)$.

Lemma 3.6.8. *There exists a constant W such that*

$$|\mathrm{Spec}_\nu(h_k)| \leq W^k q^{k^2/2}$$

for any k. Furthermore, for any $V > 0$ there exists k_0 such that for every $k > k_0$ we have

$$|\mathrm{Spec}_\nu(h_k)| \leq V^k q^{k^2/2}.$$

Proof. Recall that

$$\sum_{k=0}^{\infty} \mathrm{Spec}_\nu(h_k) t^k = H^\nu(t) = \prod_{x \in X(\nu)} (1 - q^x t).$$

(See Section 3.5.2 for the definition of $X(\nu)$.) If $X(\nu)$ is finite, then the statement of the lemma is obvious. Otherwise,

$$\mathrm{Spec}_\nu(h_k) = (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k} q^{X(\nu,i_1)+\cdots+X(\nu,i_k)},$$

where $X(\nu, i)$ stays for the ith element in $X(\nu)$. Clearly, $i_\ell \geq \ell$, consequently

$$|\mathrm{Spec}_\nu(h_k)| \leq \sum_{i_1 \geq 1,\, i_2 \geq 2, \ldots, i_k \geq k} q^{X(\nu,i_1)+\cdots+X(\nu,i_k)} = q^{X(\nu,1)+\cdots+X(\nu,k)}(1-q)^{-k}.$$

Since $X(\nu, i) \geq i - 1$, the first part of Lemma 3.6.8 is proved. Now recall that the complement of the set $X(\nu)$ in $\mathbb{Z}_{\geq 0}$ is infinite. Thus, for any r there exists k_0 such that for every $k > k_0$,

$$X(\nu, 1) + \cdots + X(\nu, k) > (0 + 1 + \cdots + k - 1) + rk.$$

Choosing r large enough we prove the second part of Lemma 3.6.8 $\square$

The Jacobi–Trudy formula for Schur polynomials (see e.g. [58, 3.4]) implies that for $\lambda \in \mathbb{GT}_k^+$, we have

$$\mathrm{Spec}_\nu(s_\lambda) = \det\left[\mathrm{Spec}_\nu(h_{\lambda_i - i + j})\right]_{i,j=1,\ldots,k},$$

where we agree that $\mathrm{Spec}_\nu(h_{-1}) = \mathrm{Spec}_\nu(h_{-2}) = \cdots = 0$. Since k is fixed, writing determinant as an alternating sum of products and applying Lemma 3.6.8 we conclude that for any $V > 0$ there exists n_0 such that for every $\lambda \in \mathbb{GT}_k^+$ with $\lambda_1 > n_0$ we have

$$|\mathrm{Spec}_\nu(s_\lambda)| \leq V^{\lambda_1} \prod_{i=1}^{k} q^{\lambda_i^2/2}.$$

For any $\lambda \in \mathbb{GT}_k^+$ a simple computation proves that $n(\lambda') = \sum_{i=1}^k \lambda_i(\lambda_i - 1)/2$. Thus for $x_1, \ldots, x_k$ with $|x_i| < M$ we have

$$\left| \sum_{\mu \in \mathbb{GT}_k^+,\, \mu_1 \geq n_0} (-1)^{|\mu|} q^{n(\mu) - n(\mu')} \mathrm{Spec}_\nu(s_\mu) s_\mu^*(x_1, \ldots, x_k) \right|$$
$$\leq \sum_{\mu \in \mathbb{GT}_k^+,\, \mu_1 \geq n_0} V^{\mu_1} |s_\mu^*(x_1, \ldots, x_k)| \leq \sum_{\mu_1 = n+0}^{\infty} V^{\mu_1} \left((\mu_1 + 1)^{k-1}\right) \left(C(M)^k\right)^{\mu_1}.$$

(Here we used the rough estimate that the number of signatures $\mu \in \mathbb{G}T_k$ with $\mu_1 = n$ is at most $(n+1)^{k-1}$.) If V is small enough, then the above series converges. Consequently,

$$\sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu) - n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \ldots, x_k)$$

converges and defines an entire function.

3.7 q–Toeplitz matrices (Proof of Proposition 3.1.4)

In this section we prove the following theorem:

Theorem 3.7.1. *Let c_l, $l \geq 0$ be a sequence of non-negative numbers and suppose that $\sum_l c_l = 1$. Let $d_{i,j}$, $i > 0$, $j > 0$ be a unique q–Toeplitz matrix such that*

$$d[i,1] = c_{i-1}, \quad i = 1, 2, \ldots.$$

Set

$$\phi(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1} (1 - tq^i)$$

an define coefficients c_λ ($\lambda \in \mathbb{GT}_N^+$) as the coefficients of the expansion

$$\phi(x_1) \cdots \phi(x_N) = \sum_{\lambda \in \mathbb{GT}_N^+} c_\lambda (-1)^{|\lambda|} s_\lambda^*(q^{N-1}x_1, \ldots, q^{N-1}x_N; q^{-1}). \tag{3.41}$$

Then

$$c_\lambda = q^{(N-1)|\lambda|} \det \Big[d[\lambda_{N-i+1} + i, j] \Big]_{i,j=1,\ldots,N}.$$

Remark. If $\phi(t)$ is a polynomial, then the series (3.41) is finite. For general $\phi(t)$ Proposition 3.6.1 yields that the series (3.41) converges at least in points $q^{1-N-\lambda+\delta}$, and such convergence is enough for our purposes.

Before proving Theorem 3.7.1 let us obtain Proposition 3.1.4 as its corollary.

Proof of Proposition 3.1.4. Every initial column minor of the matrix $d^\nu[i,j]$ is

$$\det \Big[d[\lambda_{N-i+1} + i, j] \Big]_{i,j=1,\ldots,N}$$

for a certain N and $\lambda \in \mathbb{GT}_N^+$. Theorem 3.7.1 and Theorem 3.1.1 imply that

$$\det \Big[d_{\lambda_{N-i+1}+i,j} \Big]_{i,j=1,\ldots,N} = q^{-(N-1)|\lambda|} c_\lambda = q^{-(N-1)|\lambda|} (-1)^{|\lambda|} \frac{\mathcal{E}_N^\nu(\lambda)}{s_\lambda^*(0, \ldots, 0; q^{-1})}.$$

Observe that

$$\frac{(-1)^{|\lambda}}{s_\lambda^*(0, \ldots, 0; q^{-1})} > 0,$$

thus, all initial column minors of $d^\nu[i,j]$ are non-negative.

To finish the proof note that since $d^\nu[i,j]$ is a triangular matrix all its initial row minors (i.e. minors corresponding to the first N rows and arbitrary N columns) are either zero or equal to certain initial column minors. □

To prove Theorem 3.7.1 we need two general formulas involving factorial Schur polynomials.

Proposition 3.7.2. *We have*

$$\prod_{i=1}^{N} \prod_{j=1}^{m} (y_j - x_i) = \sum_{\lambda \subset m^N} (-1)^{|\lambda|} s_\lambda(x_1, \ldots, x_N \mid a) s_{\hat{\lambda}'}(y_1, \ldots, y_n \mid a),$$

where λ' is the transpose diagram and $\widehat{\lambda}'$ is the complement of λ' in diagram N^m

Proof. See [59, Proof of (6.17)]. □

Denote

$$e_k(y_1,\dots,y_m \mid a) = \begin{cases} s_{1^k}(y \mid a), & 0 < k \le m, \\ 1, & k = 0, \\ 0, & k < 0 \text{ or } k > m. \end{cases}$$

Proposition 3.7.3. *Factorial Schur polynomials admit a determinantal formula:*

$$s_\lambda(y_1,\dots,y_m \mid a) = \det[e_{\lambda'_i - i + j}(y_1,\dots,y_m \mid \tau^{j-1}a)]_{i,j=1,\dots,m},$$

where m is an arbitrary integer greater or equal than λ_1, λ'_i are row lengths of the transposed diagram (equivalently, they are column lengths of λ), and $\tau^{j-1}a$ stands for the sequence $(\tau^{j-1}a)_i = a_{j-1+i}$.

Proof. See [59, (6.7)]. □

Now let $\widehat{a}$ be the following sequence:

$$\widehat{a}_j = -q^{1-j}, \quad j \in \mathbb{Z}.$$

Now fix m arbitrary numbers $y_1,\dots,y_m$ and let $d[i,j]_{i,j=1,2,\dots}$ be an infinite matrix given by

$$d[i,j] = e_{m-i+j}(y_1,\dots,y_m \mid \tau^{j-1}\widehat{a}).$$

Lemma 3.7.4. *$d[i,j]$ is a q–Toeplitz matrix, i.e.*

$$d[i,j+1] = d[i-1,j] + (q^{1-j} - q^{1-i})d[i,j].$$

(Here we agree that $d[i,j] = 0$ is either $i < 1$ or $j < 1$.)

Proof. Setting $N = 1$ in Proposition 3.7.2, we get

$$\prod_{i=1}^m (y_i - t) = \sum_{k=0}^m (-1)^k h_k(t \mid a) e_{m-k}(y_1,\dots,y_m \mid a),$$

where

$$h_k(t \mid a) = s_k(t \mid a) = (t+a_1)(t+a_2)\dots(t+a_k).$$

We see that the numbers $e_{m-k}(y \mid a)$ are the coefficients of the decomposition of the function $\prod_{i=1}^m (y_i - t)$ into the sum of polynomials $(-1)^k h_k(t \mid a)$. We can use any sequence a. In particular, if we set $a_i = -q^{2-i-\ell}$ then we get

$$\prod_{i=1}^m (y_i - t) = \sum_{k=0}^m d[k+\ell,\ell](q^{1-\ell} - t)(q^{1-\ell-1} - t)\dots(q^{2-k-\ell} - t). \tag{3.42}$$

Observe that the left side of (3.42) does not depend on ℓ. Comparing (3.42) for ℓ and $\ell+1$ and using the fact that

$$\begin{aligned}(q^{1-\ell}-t)(q^{1-\ell-1}-t)\dots(q^{2-k-\ell}-t)\\ &= (q^{1-(\ell+1)}-t)(q^{1-(\ell+1)-1}-t)\dots(q^{2-k-(\ell+1)}-t)\\ &+ (q^{1-\ell}-q^{2-k-(\ell+1)})(q^{1-(\ell+1)}-t)(q^{1-(\ell+1)-1}-t)\dots(q^{2-k-\ell}-t),\end{aligned}$$

we get

$$d[k+\ell+1,\ell+1] = d[k+\ell,\ell] + (q^{1-\ell}-q^{-k-\ell})d[k+\ell+1,\ell].$$

To finish the proof substitute $j=\ell$, $i=k+\ell+1$. □

Proposition 3.7.5. *Let $H(t)$ be a polynomial of degree m with $H(0)=1$. Let the coefficients c_ℓ, $i=0,\dots,m$ be defined from the expansion*

$$H(t) = \sum_{\ell=0}^{m} c_\ell \prod_{i=0}^{\ell-1}(1-tq^i).$$

More generally, define coefficients c_λ ($\lambda\in\mathbb{GT}_N^+$) as the coefficients of the expansion

$$H(x_1)\cdots H(x_N) = \sum_{\lambda\in\mathbb{GT}^+} c_\lambda(-1)^{|\lambda|}s^*_\lambda(q^{N-1}x_1,\dots,q^{N-1}x_N;q^{-1}). \tag{3.43}$$

Let $\widetilde{d}[i,j]$ denote a unique q–Toeplitz matrix such that

$$\widetilde{d}[i,1] = c_{i-1}.$$

Then we have

$$c_\lambda = q^{(N-1)|\lambda|}\det\left[\widetilde{d}[\lambda_{N-i+1}+i,j]\right]_{i,j=1,\dots,N}.$$

Proof. Let y_i be the roots of $H(t)$, i.e.

$$H(t) = \prod_{i=1}^{m}(1-y_i^{-1}t).$$

Let, as above, define

$$\widehat{a}_j = -q^{1-j}, \quad j\in\mathbb{Z}$$

and

$$d[i,j] = e_{m-i+j}(y_1,\dots,y_m \mid \tau^{j-1}\widehat{a}).$$

Setting $N=1$ in Proposition 3.7.2, we conclude that

$$d[i,1] = c_{i-1}\prod_{\ell=1}^{m} y_\ell^{-1}.$$

Lemma 3.7.4 yields that $d[i,j]$ is a q–Toeplitz matrix, thus,

$$d[i,j] = \widetilde{d}[i,j] \prod_{\ell=1}^{m} y_\ell^{-1}.$$

Recall that

$$s^*_\lambda(x_1,\dots,x_N;q^{-1}) = s_\lambda(x_1,\dots,x_N \mid a)$$

with

$$a_j = -q^{N-j}.$$

Definition of factorial Schur functions $s_\lambda(x \mid a)$ implies that

$$s_\lambda(qy \mid qa) = q^{|\lambda|} s_\lambda(y \mid a).$$

(Here qa stands for the sequence with $(qa)_j = q \cdot a_j$.)

It follows that

$$s^*_\lambda(q^{N-1}x_1,\dots,q^{N-1}x_N;q^{-1}) = q^{(N-1)|\lambda|} s_\lambda(x_1,\dots,x_N \mid \widehat{a}).$$

Now let b_λ be the coefficient of the expansion

$$\prod_{i=1}^{N}\prod_{j=1}^{m}(y_j - x_i) = \sum_{\lambda \in m^N} (-1)^{|\lambda|} s_\lambda(x_1,\dots,x_N \mid \widehat{a}) b_\lambda.$$

Comparing the last formula with the definition (3.43) of c_λ we see that

$$c_\lambda = q^{(N-1)|\lambda|} \prod_{\ell=1}^{m} y_\ell^{-N} b_\lambda.$$

On the other hand, Proposition 3.7.2 yields that

$$b_\lambda = s_{\widehat{\lambda}'}(y_1,\dots,y_m \mid \widehat{a}).$$

Applying Proposition 3.7.3 we conclude that

$$b_\lambda = \det[e_{\widehat{\lambda}_i - i + j}(y_1,\dots,y_m \mid \tau^{j-1}\widehat{a})]_{i,j=1,\dots,N}.$$

It is clear that

$$\widehat{\lambda}_i - i = m - (\lambda_{N-i} + i).$$

Thus,

$$e_{\widehat{\lambda}_i - i + j}(y_1,\dots,y_m \mid \tau^{j-1}\widehat{a}) = d[\lambda_{N-i} + i, j].$$

Consequently,

$$c_\lambda = q^{(N-1)|\lambda|} \prod_{\ell=1}^{m} y_\ell^{-N} \det\Big[d[\lambda_{N-i} + i, j]\Big]_{i,j=1,\dots,N}$$
$$= q^{(N-1)|\lambda|} \det\{\widetilde{d}[\lambda_{N-i} + i, j]\}_{i,j=1,\dots,N}.$$

□

Now we are ready to prove Theorem 3.7.1.

Proof of Theorem 3.7.1. Recall that c_l, $l \geq 0$ is a sequence of non-negative numbers and $\sum_l c_l = 1$.

Let us denote

$$c_l^{(m)} = \begin{cases} \frac{c_l}{\sum_{i=0}^m c_i}, & 0 \leq l \leq k, \\ 0, \text{ otherwise.} \end{cases}$$

And let $d^{(m)}[i,j]$, $i > 0$, $j > 0$ be a unique q–Toeplitz matrix such that

$$d^{(m)}[i,1] = c_{i-1}^{(m)}, \quad i = 1, 2, \ldots.$$

Observe that

$$\lim_{m\to\infty} d^{(m)}[i,j] = d[i,j].$$

Consequently, all minors of $d[i,j]$ are limits of the corresponding minors of $d^{(m)}[i,j]$.

Next, let $H^{(m)}(t)$ be a degree m polynomial such that

$$H^{(m)}(t) = \sum_{\ell=0}^{m} c_\ell \prod_{i=0}^{\ell-1}(1 - tq^i).$$

Let $c_\lambda^{(k)}$ be the coefficients of the expansion

$$H^{(m)}(x_1)\cdots H^{(m)}(x_N) = \sum_{\lambda\in\mathbb{GT}^+} c_\lambda^{(k)}(-1)^{|\lambda} s_\lambda^*(q^{N-1}x_1, \ldots, q^{N-1}x_N; q^{-1}).$$

Observe that

$$H^{(m)}(t) \rightrightarrows \phi(t)$$

uniformly on compact subsets of $\mathbb{C}$. Hence,

$$H^{(m)}(x_1)\cdots H^{(m)}(x_N) \rightrightarrows \phi(x_1)\cdots\phi(x_N).$$

Then Proposition 3.6.1 implies that

$$c_\lambda^{(m)} \to c_\lambda.$$

Applying Proposition 3.7.5 we conclude that

$$c_\lambda = \lim_{m\to\infty} c_\lambda^{(m)} = \lim_{m\to\infty} \det\Big[d^{(m)}[\lambda_{N-i}+i,j]\Big]_{i,j=1,\ldots,N}$$
$$= \det\Big[d[\lambda_{N-i}+i,j]\Big]_{i,j=1,\ldots,N}.$$

□

Chapter 4

Disjointness of representations arising in harmonic analysis on the infinite-dimensional unitary group

4.1 Introduction

Harmonic analysis on a group studies the regular representation and another natural representations connected with this group. Chapter 4 deals with the infinite-dimensional unitary group $U(\infty)$, that is, the union of the classical groups $U(N)$ naturally embedded in one another. The conventional definition of the regular representation does not apply to $U(\infty)$ because this group is not locally compact. In his pioneering paper [74] Pickrell proposed a construction for circumventing this difficulty in a similar situation. This paper served as a starting point for [71] where Olshanski introduced a natural generalization of the regular representation for the the "big" group $U(\infty)$.

The space of the representations introduced by Olshanski is an inductive limit of the spaces $L_2(U(N))$. For this limit to be well defined one should choose embeddings $L_2(U(N)) \hookrightarrow L_2(U(N+1))$. It turns out that this choice is far from being unique. Different possible choices lead to a family of representations $T_{z,w}$ which depend on two complex parameters z and w. The representation $T_{z,w}$ does not change if z or w is replaced by $\overline{z}$ or $\overline{w}$, respectively. The structure of this representation strongly depends on whether the parameters are integer or not. In this paper we handle the latter case.

Since all representations $T_{z,w}$ are inductive limits of the same sequence of

spaces, these arises a very natural question: Are these representations really different? We answer this question by proving that the representations $T_{z,w}$ are pairwise disjoint. Recall that two representations T and T' are said to be disjoint if they have no equivalent nonzero subrepresentations. In Chapter 4 we prove the following theorem:

Theorem 4.1.1. *Suppose that the parameters z, w, z' and w' are noninteger, $\{z, \overline{z}\} \neq \{z', \overline{z'}\}$, $\{w, \overline{w}\} \neq \{w', \overline{w'}\}$ (all pairs are unordered), and* $\mathrm{Re}(z+w) > -1/2$. *Then representations $T_{z,w}$ and $T_{z',w'}$ are disjoint.*

Decompositions of representations $T_{z,w}$ into irreducible representations are determined by certain measures, which we call spectral. To prove the disjointness of representations $T_{z,w}$, it suffices to prove that of the corresponding spectral measures. These measures were fully described in [10]. There exist explicit formulas for the correlation functions of spectral measures. But, as far as the author knows, there is no no direct method for proving the disjointness of measures by looking at their correlation functions. Thus, we have to use a different approach. We reduce the disjointness of spectral measures to that of certain probability measures $\rho_{z,w}$ on the space of paths in the Gelfand-Tsetlin graph; it is these measures $\rho_{z,w}$ which are the main object of study in this chapter.

The disjointness of representations and measures associated with "big" groups has been discussed in numerous papers, and still, many related problems remain unsolved.

The authors of [49] studied a family of representations T_z of the infinite symmetric group and obtained a result similar to Theorem 4.1.1. We use some ideas from that paper, although the case of the unitary group turned out to be much more complicated and requires new arguments.

Another closely related example was given in [11], where Borodin and Olshanski studied the so-called Hua-Pickrell s–measures on the space of infinite Hermitian matrices and, in particular proved the pairwise disjointness of these measures.

In [61] Neretin generalized the original construction of Pickrell to the ten classical series of compact Riemannian symmetric spaces. The disjointness problem can be posed in each of these ten cases. Perhaps, disjointness can be proved by methods similar to those used in this paper.

Note that Theorem 4.1.1 imposes a constraint on the parameters z and w, namely: $\mathrm{Re}(z+w) > -1/2$. Actually, the construction of the representations $T_{z,w}$ makes sense for all $z, w \in \mathbb{C}$. But for $\mathrm{Re}(z+w) \leq -1/2$, we obtain representations without so-called distinguished K–invariant vector. It would be interesting to study these representations too, but our methods do not apply to them.

In Section 4.7 we present a result of a different kind. A path in the Gelfand-Tsetlin graph is a sequence $\{\lambda(N)\}$ of dominant weights of irreducible representations of the unitary groups $U(N)$, and each $\lambda(N)$ can be regarded as the pair $(\lambda^+(N), \lambda^-(N))$ of Young diagrams. Thus, each measure $\rho_{z,w}$ generates two random sequences, $\{\lambda^+(N)\}$ and $\{\lambda^-(N)\}$, of Young diagrams which form a Markov growth process. As yet another application of the method described

in Section 4.4, we prove that the length of the diagonal in $\{\lambda^{\pm}(N)\}$ grows at most logarithmically in N.

4.2 Representations of $U(\infty)$, Gelfand-Tsetlin graph, reduction to spectral measures

In this section we collect some general facts about representations of the group $U(\infty)$; most of them can be found in the paper [71].

Consider the chain of the compact classical groups $U(N)$, $N = 1, 2, \dots$, which are embedded one into another in a natural way. Let $U(\infty)$ be their union. Following the philosophy of [72], we form a (G, K)–pair, where G is the group $U(\infty) \times U(\infty)$ and K is the diagonal subgroup, isomorphic to $U(\infty)$. We are dealing with unitary representations T of the group G possessing a distinguished cyclic K–invariant vector ξ. Such representations are called *spherical representations* of the pair (G, K). They are completely determined by the corresponding matrix coefficients $\psi(\cdot) = (T(\cdot)\xi, \xi)$. The ψ's are called *spherical functions*. They are K–biinvariant functions on G, which can be converted (via restriction to the subgroup $U(\infty) \times \{e\} \subset G$) to central functions χ on $U(\infty)$ (that are functions which are constant on conjugacy classes). Irreducible representations T are in one-to-one correspondence with *extreme characters* (i.e. extreme points in the convex set of all characters, which are central positive definite continuous functions on $U(\infty)$ taking on a value of 1 at the unity element of the group).

Irreducible spherical representations of (G, K) and extreme characters of $U(\infty)$ admit a complete description. They depend on countably many continuous parameters. There is a bijective correspondence $\chi^{(\omega)} \leftrightarrow \omega$ between extreme characters and points ω of the infinite-dimensional domain

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^{\pm} = (\alpha_1^{\pm} \geq \alpha_2^{\pm} \geq \dots \geq 0) \in \mathbb{R}^{\infty}, \quad \beta^{\pm} = (\beta_1^{\pm} \geq \beta_2^{\pm} \geq \dots \geq 0) \in \mathbb{R}^{\infty},$$

$$\sum_{i=1}^{\infty} (\alpha_i^{\pm} + \beta_i^{\pm}) \leq \delta^{\pm}, \quad \beta_1^+ + \beta_1^- \leq 1.$$

D. Voiculescu discovered an explicit formula for the functions $\chi^{(\omega)}(U)$, where $U \in U(\infty)$. We do not need it in the present paper, this formula can be found in [93] or [71].

It was proved in [71] that any character χ corresponds to a unique probability measure σ on Ω, such that

$$\chi(U) = \int_\Omega \chi^{(\omega)}(U)\sigma(d\omega), \quad U \in U(\infty).$$

We call σ the *spectral measure* of the character χ. The inverse statement is also true, every probability measure on Ω corresponds to a certain character of the group $U(\infty)$.

According to the general theory, the disjointness of two spherical representations T and T' is equivalent to the disjointness of the spectral measures σ and σ' corresponding to their characters. Our aim is to prove the disjointness of these measures.

The Gelfand-Tsetlin graph $\mathbb{GT}$(also known as the graph of signatures) is a convenient tool for describing characters of $U(\infty)$. The vertices of the graph symbolize the irreducible representations of groups $U(N)$ while the edges encode the inclusion relations between irreducible representations of $U(N)$ and $U(N+1)$. The N–th level of the graph, denoted by $\mathbb{GT}_N$, corresponds to the irreducible representations of $U(N)$. Vertices of $\mathbb{GT}_N$ can be identified with dominant weights for $U(N)$, i.e., these are N–tuples of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$ which are also called *signatures*. We join two signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N+1}$ by an edge and write $\lambda \prec \mu$ if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \lambda_N \geq \mu_{N+1}.$$

In what follows it is convenient for us to represent signature λ by two Young diagrams λ^+ and λ^-: the row lengths of λ^+ are the positive coordinates λ_i, while the row lengths of λ^- are the absolute values of the negative coordinates. We call λ^+ and λ^- *a positive Young diagram* and a *negative Young diagram*, respectively.

If $\tilde{\lambda}$ and $\tilde{\mu}$ are two Young diagrams such that $\tilde{\lambda} \subset \tilde{\mu}$ then their set-theoretical difference is called *skew diagram* and we denote it by $\tilde{\mu}/\tilde{\lambda}$ (see [58, Chapter 1]). Recall that a horizontal strip is a skew Young diagram containing at most one box in each column. Note that $\lambda \prec \mu$ if and only if both μ^+/λ^+ and μ^-/λ^- are *horizontal strips*.

Recall that number $c(\square) = j - i$ is called the *content* of the box $\square$ lying at the intersection of the ith row and jth column.

We denote by χ^λ the irreducible character of $U(N)$, indexed by the signature $\lambda \in \mathbb{GT}_N$. Given an arbitrary character χ of the group $U(\infty)$ we may expand the restriction of χ on $U(N)$ into a convex combination of the functions $\chi^\lambda(\cdot)/\chi^\lambda(e)$. The coefficients of this expansion determine a probability distribution $M_N(\lambda)$ on the discrete set $\mathbb{GT}_N$. In this way, we get a bijection between characters and certain sequences of probability distributions $\{M_N\}_{N=1,2,\ldots}$, which are called *coherent systems* (because of certain coherency relations connecting M_N and M_{N+1}). Additional information can be found in [71].

A path in the graph of signatures $\mathbb{GT}$ is a finite or infinite sequence $\tau = (\tau(1), \tau(2), \ldots)$, such that $t(N) \in \mathbb{GT}_N$, and for every $N = 1, 2, \ldots$, $\tau(N)$ and

$\tau(N+1)$ are joined by an edge. Usually $\tau_i(N)$ stands for the ith coordinate of the signature $\tau(N)$.

We say that a box $\square$ is added to the positive Young diagram at level N while moving along a path τ if $\square \in \tau^+(N+1)/\tau^+(N)$.

We can define a topology on the set $\mathcal{T}$ of all infinite paths in $\mathbb{GT}$. A base of this topology consists of *cylindrical sets* C_τ, where $\tau = (\tau(1), \dots, \tau(N))$ is a finite path and

$$C_\tau = \{\zeta \in \mathcal{T} | \zeta(1) = \tau(1), \dots, \zeta(N) = \tau(N)\}.$$

Now we may take σ–algebra of Borel sets generated by this topology and consider measures on this σ–algebra.

A measure on $\mathcal{T}$ is called *central* if the measure of a cylindrical set C_τ, $\tau = (\tau(1), \dots, \tau(N))$ depends solely on $\tau(N)$.

Consider an arbitrary signature $\lambda \in \mathbb{GT}_N$. The set of finite paths terminating at λ consists of exactly $\mathrm{Dim}_N(\lambda) = \chi^\lambda(e)$ elements. Given a coherent system on the graph of signatures we may construct a certain central measure ρ on the set $\mathcal{T}$ by setting

$$\rho(C_\tau) = \frac{M_N(\tau(N))}{\mathrm{Dim}_N(\tau(N))}$$

where C_τ is an arbitrary cylindrical set as above and $\tau(N)$ is the endpoint of τ. Correctness of this definition follows from the properties of coherent systems.

Thus, we obtain a sequence of bijections connecting characters of the group $U(\infty)$, probability measures σ on Ω (spectral measures), coherent systems $\{M_N\}$, and central measures ρ on $\mathcal{T}$.

Let us show that the disjointness of spectral measures σ and σ' follows from the disjointness of the corresponding central measures ρ and ρ' (the similar proposition in the case of the infinite symmetric group was proved in [49]).

Proposition 4.2.1. *Assume that central measures ρ and ρ' on $\mathcal{T}$ are disjoint. Then the corresponding spectral measures σ and σ' on Ω are also disjoint.*

Proof. Assume the contrary. Denote by $\tilde{\sigma} = \sigma \wedge \sigma'$ the greatest lower bound of the measures σ and σ' (we say that a measure μ is less than a measure ν if $\mu(A) \le \nu(A)$ for an arbitrary measurable set A). The existence of such measure can be easily verified. It is evident that if σ and σ' are not disjoint, then $\tilde{\sigma}$ is a non-zero measure.

Next, observe that the correspondence $\sigma \leftrightarrow \rho$ between probability spectral measures on Ω and central measures on $\mathcal{T}$ can be extended to arbitrary finite measures, which are not necessarily probability measures.

Denote by $\tilde{\rho}$ the central measure corresponding to $\tilde{\sigma}$.

There is an integral representation for the value of this measure on a cylindrical set. Let $\omega \in \Omega$. Let ρ_ω be the central measure on $\mathcal{T}$ corresponding to the extreme character $\chi^{(\omega)}(U)$. Denote by $f_{\tau(N)}(\omega)$ the value of ρ_ω on the cylindrical set C_τ, where $\tau(N)$ is the endpoint of τ. We have

$$\tilde{\rho}(C_\tau) = \int_\Omega f_{\tau(N)}(\omega)\tilde{\sigma}(d\omega).$$

From the last formula, nonnegativity of $f_\lambda(\omega)$, and inequalities $\tilde{\sigma} \le \sigma$, $\tilde{\sigma} \le \sigma'$ it follows that $\tilde{\rho} \le \rho$ and $\tilde{\rho} \le \rho'$. Consequently, $\rho \wedge \rho' \ne 0$ and the measures ρ and ρ' are not disjoint. $\square$

In what follows we are going to prove disjointness of the central measures on the space of paths in the graph of signatures corresponding to the characters of the representations $T_{z,w}$ and $T_{z',w'}$. Let us denote these measures by $\rho_{z,w}$ and $\rho_{z',w'}$.

It was shown in [71] that the coherent system corresponding to the representation $T_{z,w}$ is given by

$$M_N^{z,w}(\lambda) = (S_N(z,w))^{-1} \cdot \prod_{i=1}^{N} \left| \frac{1}{\Gamma(z-\lambda_i+i)\Gamma(w+N+1+\lambda_i-i)} \right|^2 \cdot \mathrm{Dim}_N^2(\lambda),$$

where $S_N(z,w)$ is the normalization constant whose explicit value is not important for us. Denote by $P_N^{z,w}(\lambda)$ the measure $\rho_{z,w}$ of an arbitrary cylindrical set C_τ, such that λ is the endpoint of the path τ. The following formula holds:

$$\rho_{z,w}(C_\tau) = P_N^{z,w}(\tau(N)) = M_N^{z,w}(\tau(N))/\,\mathrm{Dim}_N(\tau(N)).$$

The measure $\rho_{z,w}$ is called the (z,w)measure below.

From this point on we forget about representations $T_{z,w}$, all further arguments are based on the last two formulas.

4.3 Scheme of the proof

Our aim is to prove the pairwise disjointness of (z,w)-measures.

Take two pairs (z,w) and (z',w'), such that all of the four parameters are noninteger, $z \ne z'$, $z \ne \overline{z'}$, and $w \ne w'$, $w \ne \overline{w'}$. Consider the ratio $\frac{P_N^{z,w}}{P_N^{z',w'}}$. Let $\mathfrak{Q}$ be the set of infinite paths along which this ratio tends to a finite non-zero limit. Clearly, this is a Borel set. The disjointness of the measures follows from the equalities $\rho_{z,w}(\mathfrak{Q}) = \rho_{z',w'}(\mathfrak{Q}) = 0$ and the following proposition.

Proposition 4.3.1 ([82, Chapter 7, Section 6, Theorem 2]). *Let X be a measurable space with a filtration $\{A_n\}$. Denote by P_n and $\tilde{P}_n$ restrictions of two arbitrary probability measures P and $\tilde{P}$ on the σ–algebra A_n. Suppose that the measure $\tilde{P}_n$ is absolutely continuous with respect to the measure P_n and Radon-Nikodym derivative $\frac{d\tilde{P}_n}{dP_n}$ tends to zero (as n tends to ∞) almost everywhere with respect to P. Then th measures P and $\tilde{P}$ are disjoint.*

In order to show that the set $\mathfrak{Q}$ has measure zero, we introduce an auxiliary set G (it will be defined later on). We introduce the temporary notations $\mathfrak{Q}_0 = G \cap \mathfrak{Q}$. We prove below that the measure of $\mathfrak{Q}_0$ is equal to zero (see Sections 4 and 5 of this chapter) and then we show that the (z,w)–measure of the set G is equal to 1 (see Section 6 of this chapter). Clearly, these two facts imply the vanishing of the (z,w)measure of the set $\mathfrak{Q}$.

The first part of the proof in which bounds for the measure of $\mathfrak{Q}_0$) are derived contains technical details; we preface it by some of the main ideas. It turns out that we can pass from $\mathfrak{Q}_0$ to a family of sets $\mathfrak{Q}_0^l$. For every $\mathfrak{Q}_0^l$, we construct a collection of maps $\{f_i\}$ which take $\mathfrak{Q}_0^l$ to a collection $\{\mathfrak{Q}_i^l\}$ of pairwise disjoint sets. The measure of each set $\mathfrak{Q}_i^l$ is equal to that of the set$\mathfrak{Q}_0^l$. Since there exist infinitely many such sets $\mathfrak{Q}_i^k$ we conclude that each of them has measure zero. Hence, the measure of $\mathfrak{Q}_0$ is also equal to zero.

The maps f_i are essentially minor small reorganizations of paths. It turns out, that changing only one level of a path may generate a rather big fluctuation of the ratio $\frac{P_N^{z,w}}{P_N^{z',w'}}$ from the hypothetical limit value . This observation is the basic idea underlying the proof of the theorem.

Each map f_i is very simple. We fix a large integer k in advance. Each f_i modifies every path by of adding a box with content k to the positive Young diagram at level $N+1$ rather than N. Although, not every path can be modified in this way, we show later that for almost all paths in the set $\mathfrak{Q}_0^l$ maps f_i can be well defined.

An important point in the argument is the observation that the probability of the addition of a horizontal strip containing m boxes with contents $k, k+1, \dots, k+m-1$ at level N does not exceed $c(k,m)/N^m$ (we use this statement only in the case $m=1$, but it also holds in a more general case).

4.4 Modification of paths

We proceed to the detailed proof.

Let $\tau = (\tau(1), \tau(2), \dots)$ be a path in the graph of signatures. Then $\tau^+(N)$ is an increasing sequence of Young diagrams. (To simplify notations we usually write $\tau^+(N)$ instead of $(\tau(N))^+$.) Let $\overline{\tau}^+ = \bigcup\limits_N \tau^+(N)$. Clearly, $\overline{\tau}^+$ is either the whole quadrant or only some its part called *a thick hook*. Let G^+ denote the set of all paths τ such that $\overline{\tau}^+$ is the whole quadrant. In a similar way we set $\overline{\tau}^- = \bigcup\limits_N \tau^-(N)$ and define G^- to be the set of all paths τ for which $\overline{\tau}^-$ is the whole quadrant. In Section 5 of this chapter we show that $\rho_{z,w}(G^+ \cup G^-) = 1$. Our immediate goal is to prove the following theorem.

Theorem 4.4.1. *If $\{z, \overline{z}\} \neq \{z', \overline{z'}\}$, then*

$$\rho_{z,w}(\mathfrak{Q} \cap G^+) = \rho_{z',w'}(\mathfrak{Q} \cap G^+) = 0.$$

If $\{w, \overline{w}\} \neq \{w', \overline{w'}\}$, then

$$\rho_{z,w}(\mathfrak{Q} \cap G^-) = \rho_{z',w'}(\mathfrak{Q} \cap G^-) = 0.$$

This theorem implies, in particular, that if $z \neq z'$, $z \neq \overline{z'}$, $w \neq w'$ and $w \neq \overline{w'}$, then the corresponding measures are disjoint.

Note that both the Gelfand-Tsetlin graph and the (z,w)-measures are invariant under the symmetry

$$z \leftrightarrow w, \quad (\lambda_1, \dots, \lambda_N) \leftrightarrow (-\lambda_N, \dots, -\lambda_1).$$

Thus, it is sufficient to prove only the first part of the theorem.

Denote

$$h_N(\tau) = \frac{P_{N+1}^{z,w}(\tau(N+1))}{P_{N+1}^{z',w'}(\tau(N+1))} : \frac{P_N^{z,w}(\tau(N))}{P_N^{z',w'}(\tau(N))}.$$

Note that if $P_N^{z,w}(\tau(N))/P_N^{z',w'}(\tau(N))$ tends to a finite nonzero limit, then $h_N(\tau) \to 1$.

Denote by $\mathfrak{A}$ the set of paths along which $h_N \to 1$. Our aim is to prove that the (z,w)measure of the set $\mathfrak{A} \cap G^+$ is equal to zero ($\rho_{z,w}(\mathfrak{A} \cap G^+) = 0$).

Take $\delta > 0$. We say that a δ–fluctuation, or simply a fluctuation occurs at level N if $|h_N - 1| > \delta$. Let A_t^δ denote the set of those paths for which no δ-fluctuations occur starting from level t. We shall show that if δ is sufficiently small, then $\rho_{z,w}(A_t^\delta \cap G^+) = 0$ for every t. Clearly, this implies our theorem.

Theorem 4.4.2. *There exists $\delta > 0$ such that for every t*

$$\rho_{z,w}(A_t^\delta \cap G^+) = 0.$$

Proof. Recall once again that the content of the box lying at the intersection of the ith row and the jth column of a Young diagram is the integer $c(\square) = j - i$.

It is evident that for any path from G^+ and for every k a box with the content k is added to the positive Young infinitely many times.

Our further argument is base on the following idea. Adding a box with relatively small positive content may cause a δfluctuation. Suppose that $\overline{\tau}$ is a finite fragment of a path in which no δ-fluctuations occur, and a box with content k is added while moving along $\overline{\tau}$. Then for almost any such fragment $\overline{\tau}$ there exists another fragment $\overline{\tau}'$, such that $\overline{\tau}$ and $\overline{\tau}'$ have the same starting point and endpoint, but there is a δ-fluctuation in $\overline{\tau}'$. Since infinitely many boxes with the content k are added along a typical path, we see that the last claim also implies a stronger assertion. If a fragment $\overline{\tau}$ is long enough, than we can construct arbitrary many such fragments $\overline{\tau}'$. In turn, this implies that the set of paths without δ–fluctuations has measure zero.

So, take a positive integer k; its value will be specified later on .

Given an integer k and a sequence of positive integers $N_1 < N_2 < \ldots$, let $B(k, \{N_i\})$ denote a set of all paths $\tau \in G^+$ such that for every $m > 0$ there exist a positive integer $n(m)$ and a box $\square(m)$ with coordinates (i,j) which meet the following conditions:

(i). $n(m)$ is a minimal number from $[N_m, N_{m+1})$ such that $\tau^+(n+1)/\tau^+(n)$ contains a box with content k

(ii). $\square(m) \in \tau^+(n(m)+1))/\tau^+(n(m))$ and $c(\square(m)) = j - i = k$

(iii). $\tau^+(n(m)+1))/\tau^+(n(m))$ does not contain a box with content $k+1$ (i.e. it does not contain a box with coordinates $(i, j+1)$)

(iv). $\tau^+(n(m)+2))/\tau^+(n(m)+1)$ does not contain a box with content $k-1$ (consequently, it does not contain a box with coordinates $(i+1, j)$)

In what follows we construct certain maps f_i on these sets which "create δ-fluctuations"

Proposition 4.4.3 (about admissible paths)**.** *Assume that k is so large that* $\mathrm{Re}(k+w) > 0$ *then for an arbitrary $\varepsilon > 0$ there exists a sequence of integers $N_1 < N_2 < \dots$ such that $\rho_{z,w}(G^+ \setminus B(k, \{N_i\}) < \varepsilon$.*

We give the proof of this proposition in the next section. Below we apply this proposition to prove Theorem 4.4.2.

Let H denote the intersection of the sets $B(k, \{N_i\})$ and A_t^δ. Clearly, it is sufficient to prove that for a sufficiently small δ the set H has measure zero (the choice if δ will not depend on ε).

Any path from the set H can be informally described in the following way. Starting from some level it has no δfluctuations but boxes with content k, which were defined in the proposition, are still added to the positive Young diagram.

Take a set H and the corresponding sequence of integers $\{N_i\}$. Choose sufficiently large number N_p among the N_i's. For an integer $q > p$, consider the levels from N_p to N_q of the graph of signatures. We refer to the sequence $(\tau(N_p), \tau(N_p+1), \dots, \tau(N_q))$ as the $[N_p, N_q]$-*fragment* of the path $\tau = (\tau(1), \tau(2), \dots)$.

Take an arbitrary signature λ^p at level N_p and an arbitrary signature λ^q at level N_q. Consider all $[N_p, N_q]$-fragments of paths τ for which $\tau(N_p) = \lambda^p$ and $\tau(N_q) = \lambda^q$. Clearly, the number of such $[N_p, N_q]$–fragments is finite. Since (z, w)-measures are central, it follows that these fragments have uniform distribution. Studying the fraction of the $[N_p, N_q]$-fragments of paths in H among all such $[N_p, N_q]$-fragments, we will prove that this fraction tends to zero when $q \to \infty$ (uniformly in λ^p and λ^q). Obviously, this implies the vanishing of the measure of H.

Then we choose signatures at levels $N_{p+1}, N_{p+2} \dots N_{q-1}$ and examine the $[N_p, N_q]$-fragments of paths τ for which $\tau(N_p) = \lambda^p, \tau(N_{p+1}) = \lambda^{p+1}, \dots, \tau(N_q) = \lambda^q$. Again, it suffices to show that the fraction of such $[N_p, N_q]$-fragments, obtained from the paths in H, among all of these $[N_p, N_q]$-fragments tends to zero, provided that q tends to infinity.

Thus, take $[N_p, N_q]$-fragment, satisfying the above conditions. Consider a $[N_s, N_{s+1}]$-fragment which is a subfragment of the above $[N_p, N_q]$-fragment, and define a path modification f_s in the following way.

We know that a box with content k is added to the diagram at some level $n(s) \in [N_s, N_s+1, \dots, N_{s+1})$. Consider the modification f_s of the part τ which consists in adding this box at level $n(s)+1$ rather than at level $n(s)$, but (i.e., we remove this box from the skew diagram $\tau^+(n(s)+1)/\tau^+(n(s))$ and add the very same box to the skew diagram $\tau^+(n(s)+2)/\tau^+(n(s)+1)$) Such a modification is possible thanks to conditions 3 and 4 on the set $B(k, \{N_i\})$. Thus, we obtain a map defined on H. Clearly, this map is injective. (To determine the preimage of a path we must find the first addition of a box with content k and simply move this addition to the preceding level)

Lemma 4.4.4. *It is possible to choose k, δ and p such that for any path $\tau \in H$, $f_s(\tau)$ does not belong to H.*

Proof. Consider

$$h_{n(s)}(\tau) = \frac{P^{z,w}_{n(s)+1}(\tau(n(s)+1))}{P^{z',w'}_{n(s)+1}(\tau(n(s)+1))} : \frac{P^{z,w}_{n(s)}(\tau(n(s)))}{P^{z',w'}_{n(s)}(\tau((n(s))))}.$$

Note that $\tau(n(s))$ does not change under the map f_s while $\tau(n(s)+1)$ does. Set $N := n(s)+1$ and examine the level N. The essence of our modification is that in the new path for a single i the number λ_i (which is the length of the ith row of the diagram $\tau^+(N)$) decreased by 1. Recall that

$$P^{z,w}_N(\lambda) = (S_N(z,w))^{-1} \cdot \prod_{i=1}^{N} \left| \frac{1}{\Gamma(z-\lambda_i+i)\Gamma(w+N+1+\lambda_i-i)} \right|^2 \cdot \mathrm{Dim}_N(\lambda)$$

Consequently, under the transform $\lambda_i \mapsto \lambda_i - 1$ this density is multiplied by

$$\left| \frac{w+N+1+k}{z-k} \right|^2 \cdot (\text{factors not depending on } z, w).$$

We conclude that the ratio of densities $P^{z,w}_N / P^{z',w'}_N$ is multiplied by

$$\left| \frac{z'-k}{z-k} \right|^2 \cdot \left| \frac{w+N+1+k}{w'+N+1+k} \right|^2 \tag{$*$}$$

Examine the first factor. Note that if $z \neq z'$ and $z \neq \overline{z'}$, then one can find $k > 0$ and $\nu > 0$ such that

$$\left| \left| \frac{z'-k}{z-k} \right|^2 - 1 \right| > \nu$$

and $\mathrm{Re}(k+w) > 0$. Let us fix such k and ν.

Now consider the second factor in $(*)$. It is clear that this factor approaches 1 as N tends to infinity. Assume that N_p is so large that for any $N > N_p$ the factor under consideration differs from 1 by less than $\nu/100$.

Then the product $(*)$ differs from 1 by more than $\nu/2$.

Choose $\delta \leq \nu/100$. It is evident that for such δ the ratios of densities before and after applying f_s can not both belong to the δ–neighborhood of 1 (as f_s multiplies the ratio of densities by the number $(*)$ that is far enough from 1).

Choose p such that the inequality $N_p \geq t$ also holds. From all that has been said above it follows that in any path from H there are no δ-fluctuations at levels starting from N_p, but after applying f_s such δ-fluctuations inevitably appear. Thus the image with respect to f_s of a path from the set H does not belong to H. □

Lemma 4.4.5. *Suppose that the parameters are chosen as in Lemma 4.4.4. If $s \neq s'$, then the sets $f_s(H)$ and $f_{s'}(H)$ are mutually disjoint.*

Proof. It follows from the proof of the last lemma that in the paths from $f_s(H)$ and $f_{s'}(H)$ δfluctuations appear at distinct levels, thus, these paths are distinct. □

Now suppose that there are precisely E distinct $[N_p, N_q]$-fragments of paths in the set H. Every f_s produces E additional fragments and all f_s together produce $E(q-p)$ distinct fragments. Consequently, the fraction of the $[N_p, N_q]$0fragments of the paths from the set H among all $[N_p, N_q]$-fragments (hence also the measure of the set H) does not exceed $\frac{E}{E+E(q-p)}$. Choosing a sufficiently large number q large we can make the last ration arbitrarily small. This completes the proof of Theorem 4.4.2.

□

4.5 Proof of the proposition about admissible paths

Proof of Proposition 4.4.3. Let us treat the (z, w)-measure as a Markov process on the Gelfand-Tsetlin graph with transition probabilities

$$p_{z,w}(\mu|\lambda) = \frac{P_{N+1}^{z,w}(\mu)}{P_N^{z,w}(\lambda)}, \quad \mu \in \mathbb{GT}_{N+1}, \quad \lambda \in \mathbb{GT}_N.$$

Thus, from now on we talk about the random path τ or and about the Markov chain $\tau(N)$.

Lemma 4.5.1. *Assume k is so large that* $\mathrm{Re}(k+w) > 0$. *Fix an arbitrary signature μ at level N (thus also fix the corresponding "positive" Young diagram). Then the conditional probabilities satisfy the inequalities*

$$\mathrm{Prob}\{\exists \square \in \tau^+(N+1)/\tau^+(N) : c(\square) = k \mid \tau(N) = \mu\} < \frac{c(k)}{N},$$

$$\mathrm{Prob}\{\exists \square \in \tau^+(N+1)/\tau^+(N) : c(\square) = k \\ \mid \tau(N) = \mu, \exists \square \in \tau^+(N+1)/\tau^+(N) : c(\square) = k-1\} < \frac{c(k)}{N},$$

Remark. Instead of μ we may fix the whole path $(\mu^1, \ldots, \mu^N)$ from the first level to the signature μ^N, and replace the condition $\tau(N) = \mu$ by the condition $\tau(1) = \mu^1, \ldots, \tau(N) = \mu^N$. The centrality property of the measure implies that this change does not affect either our statement or its proof.

Proof of Lemma 4.5.1. Note that for the fixed signature μ there is at most one box with content k that can be added. Denote the coordinates of this box by (i,j). Let us compare the conditional probability of the event $\tau_i(N+1)=j-1$, given that $\tau(N)=\mu$, with the conditional probability of the event $\tau_i(N+1)\geq j$. We will show that the latter differs from the former by the factor $c(k)/N$, which clearly implies the lemma.

Besides fixing signature μ fix additionally all row lengths at level $N+1$, except for the ith row, in an arbitrary way. Thus, the condition takes form $\tau(N)=\mu$, $\tau_1(N+1)=\lambda_1,\dots,\tau_{i-1}(N+1)=\lambda_{i-1},\tau_{i+1}(N+1)=\lambda_{i+1},\dots,\tau_{N+1}(N+1)=\lambda_{N+1}$.

Let p_m denote the ratio of the conditional probability of the event $\tau_i(N+1)=j-1+m$ to the conditional probability of the event $\tau_i(N+1)=j-1$. The problem reduces to verifying the inequality $\sum_{m=1}^{\infty}p_m<c(k)/N$. Using the definition of the transition probability and Weyl's dimension formula we obtain

$$\mathrm{Dim}_{N+1}(\lambda)=\prod_{1\leq u<v\leq N+1}\frac{\lambda_u-\lambda_v+v-u}{v-u},$$

$$p_m=\left|\frac{\Gamma(z-k+1)\Gamma(w+N+1+k)}{\Gamma(z-k+1-m)\Gamma(w+N+1+k+m)}\right|^2 \cdot\prod_{a=1}^{i-1}\frac{\lambda_a-a-(k-1+m)}{\lambda_a-a-(k-1)}\cdot\prod_{a=i+1}^{N+1}\frac{(k-1+m)-(\lambda_a-a)}{(k-1)-(\lambda_a-a)}.$$

The first product is less than 1 and we can simply omit it, for the second one we use the inequality

$$\frac{(k-1+m)-(\lambda_a-a)}{(k-1)-(\lambda_a-a)}=\frac{m+(k-1-\lambda_a+a)}{k-1-\lambda_a+a}\leq\frac{m+(a-i)}{a-i}.$$

Consequently,

$$\begin{aligned}p_m&\leq\left|\frac{\Gamma(z-k+1)\Gamma(w+N+1+k)}{\Gamma(z-k+1-m)\Gamma(w+n+1+k+m)}\right|^2\prod_{a=i+1}^{N+1}\frac{m+(a-i)}{a-i}\\&\leq\left|\frac{\Gamma(z-k+1)\Gamma(w+N+1+k)}{\Gamma(z-k+1-m)\Gamma(w+N+1+k+m)}\right|^2\prod_{a=1}^{N}\frac{m+a}{a}\\&=\frac{|(k-z)_m|^2}{|(w+N+1+k)_m|^2}\cdot\frac{(m+N)!}{N!m!}=\frac{|(k-z)_m|^2(N+1)_m}{|(w+N+1+k)_m|^2m!}\\&\leq\frac{((|\mathrm{Re}(k-z)|+|\mathrm{Im}z|)_m)^2}{|\mathrm{Re}(w+N+1+k)|_m m!}\cdot\frac{(N+1)_m}{|\mathrm{Re}(w+N+1+k)|_m}\end{aligned}$$

Note that, if $\mathrm{Re}(k+w)>0$, then the second factor is less than 1, thus,

$$\begin{aligned} p_m &\le \frac{\left((|\mathrm{Re}(k-z)|+|\mathrm{Im}z|)_m\right)^2}{|\mathrm{Re}(w+N+1+k)|_m m!} \\ &= \frac{(|\mathrm{Re}(k-z)|+|\mathrm{Im}z|)^2}{|\mathrm{Re}(w+N+1+k)|m}\frac{\left((|\mathrm{Re}(k-z)|+|\mathrm{Im}z|+1)_{m-1}\right)^2}{|\mathrm{Re}(w+N+2+k)|_{m-1}(m-1)!} \\ &\le \frac{c(k)}{N}\frac{\left((|\mathrm{Re}(k-z)|+|\mathrm{Im}z|+1)_{m-1}\right)^2}{|\mathrm{Re}(w+N+2+k)|_{m-1}(m-1)!} \end{aligned}$$

It means that

$$\sum_{m=1}^{\infty} p_m \le \frac{c(k)}{N} F(|\mathrm{Re}(k-z)|+|\mathrm{Im}z|+1, |\mathrm{Re}(k-z)|+|\mathrm{Im}z|+1, |\mathrm{Re}(w+N+2+k)|; 1)$$

Here $F(a,b,c;1)$ is the hypergeometric function ${}_2F_1$ with parameters a,b,c and the argument 1.

In the case under consideration $a=b=|\mathrm{Re}(k-z)|+|\mathrm{Im}z|+1$ and $c=|\mathrm{Re}(w+N+2+k)|$.

Using Gauss' formula for the value of the hypergeometric function at 1 (see, e.g., [24]) we obtain

$$\sum_{m=1}^{\infty} p_m \le \frac{c(k)}{N}\frac{\Gamma(c)\Gamma(c-2a)}{\Gamma(c-a)\Gamma(c-a)}$$

When $N\to\infty$, the parameter c tends to infinity, while a does not change. Thus, $\frac{\Gamma(c)\Gamma(c-2a)}{/}(\Gamma(c-a)\Gamma(c-a))\to 1$. Consequently,

$$\sum_{m=1}^{\infty} p_m \le \frac{c_1(k)}{N}.$$

□

Lemma 4.5.2. *Fix k and choose two sufficiently large integers m and n. Consider the following event: There exists l, $n<l<m$, such that two conditions hold*

(i). There exist a box $\square$ such that $c(\square)=k$, $\square\in\tau^+(l+1)/\tau^+(l)$

(ii). Either there exists a box $\square : c(\square)=k+1,\quad \square\in\tau^+(l+1)/\tau^+(l)$ or there exist a box $\square : c(\square)=k-1,\quad \square\in\tau^+(l+2)/\tau^+(l+1)$.

The probability of this event does not exceed $c_1(k)/n$

Proof. Let us represent the event "There exists l, $n<l<m$, such that two conditions hold" as a disjoint union of the events "The minimal l, $n<l<m$ for which two conditions hold is i" $(n<i<m)$. Denote these events by L^i. Let

τ^i be a path going from the first level of the graph of signatures to level i, such that the minimal l, $n < l < m$, such that two conditions holds is exactly i. Then let us write L^i as the disjoint union of the elementary events $C^i_{\tau^i}$ corresponding to the various τ^i.

Applying the previous lemma for every $C^i_{\tau^i}$ and then summing up all the estimates completes the proof. □

Now we continue the proof of Proposition 4.4.3.

Note that Lemma 4.5.2 implies the following: If N_1 is sufficiently large and the N_i's increase at least as fast as 2^i, then the third condition and the fourth condition in the definition of $B(k, \{N_i\})$ hold on a set with the measure arbitrarily close to 1 automatically.

Given ε we choose N_1 so large that the measure of the set, where then the third condition or the fourth condition in the definition of $B(k, \{N_i\})$ might not hold, does not exceed $\varepsilon/2$. Denote the complement of this set by S.

Let $A_1 = S \cap G^+$. Obviously, $\rho_{z,w}(G^+ \setminus A_1) \leq \varepsilon/2$.

Denote by $A_2^R \subset A_1$ $(R > N_1)$ the subset containing paths such that a box with content k is added at a level from $\{N_1, N_1 + 1, \dots, R\}$. It is clear that $A_2^R \subset A_2^{R+1}$. We know that along every path from the set $A_1 \subset G^+$ boxes with content k are added infinitely many times. Consequently, the sets A_2^R exhaust A_1. Thus, we can choose a number $N_2 > 2 \cdot N_1$ such that

$$\rho_{z,w}(A_1 \setminus A_2^{N_2}) < \varepsilon/4.$$

Let $A_2 = A_2^{N_2}$

Further we choose a number $N_3 > 2 \cdot N_2$ and the set $A_3 \subset A_2$ in a similar way. The inequality $\rho_{z,w}(A_2 \setminus A_3) < \varepsilon/8$ holds. And so on.

The intersection of all sets A_i is the desired set B_ε.

□

4.6 The measure of the set of paths contained in a thick hook

We say that both positive and negative Young diagrams of a path $\tau = (\tau(1), \tau(2), \dots)$ are contained in a thick hook if both $\bigcup_N \tau^+(N)$ and $\bigcup_N \tau^-(N)$ are not the whole quadrant (i.e. these sets are so-called thick hooks). Denote the set of all such paths by $\overline{G}$. Clearly $\overline{G} = \mathcal{T} \setminus (G^+ \cup G^-)$.

Theorem 4.6.1. *The (z, w)measure of the set $\overline{G}$ is equal to zero.*

Proof. Denote by $R(i, j, l, m, t)$ the set of paths τ satisfying for every $N \geq t$

(i). $\tau_i(N) = j - 1$,

(ii). $\tau_{i-1}(N) \geq j$,

(iii). $\tau_{N-l}(N) = m + 1$,

(iv). $\tau_{N-l+1}(N) \le m$.

Clearly, $\overline{G}$ is a union of various $R(i,j,l,m,t)$. Thus, to verify the claim it is sufficient to prove that for any quintuple (i,j,l,m,t), $\rho_{z,w}(R(i,j,l,m,t)) = 0$ holds.

It is convenient to view $\tau(N)$ as a Markov chain again.

Lemma 4.6.2. *Suppose N is sufficiently large. Let p_N denote the conditional probability of the event "$\tau_i(N+1) = j-1$ and $\tau_{N+1-l}(N+1) = m+1$" given that $\tau_i(N) = j-1$, $\tau_{i-1}(N) \ge j$, $\tau_{N-l}(N) = m+1$, $\tau_{N-l+1}(N) \le m$*

Then p_N does not exceed $1 - \varepsilon/N$.

Proof. Consider all possible signatures at level $N+1$ which are joined by an edge with a given signature μ at level N. For every signature λ at level $N+1$, such that $\lambda_i = j-1$, we introduce a new signature λ' as follows. All row lengths of λ' except for the ith row, are the same as in λ, while $\lambda_i = j$.

Let us compare the conditional probabilities of the events $\tau(N+1) = \lambda'$ and $\tau(N+1) = \lambda$, given that $\tau(N) = \mu$.

This conditional probabilities differ by the factor

$$\frac{P_N^{z,w}(\lambda')}{P_N^{z,w}(\lambda)} = \left| \frac{z-j+i}{w+N+1+j-i} \right|^2 \cdot \frac{\mathrm{Dim}_{N+1}(\lambda')}{\mathrm{Dim}_{N+1}(\lambda)}.$$

If N is large enough, then the first factor in the right-hand side is greater than const/N^2.

By Weyl's dimension formula

$$\mathrm{Dim}_{N+1}(\lambda) = \prod_{1 \le i < j \le N+1} \frac{\lambda_i - \lambda_j + j - i}{j-i}.$$

Consequently,

$$\frac{\mathrm{Dim}_{N+1}(\lambda')}{\mathrm{Dim}_{N+1}(\lambda)} = \prod_{1 \le p < i} \frac{\lambda_p - p - j + i}{\lambda_p - p - (j-1) + i} \cdot \prod_{N+1 \ge p > i} \frac{j - i - (\lambda_p - p)}{j - i - 1 - (\lambda_p - p)}. \quad (**)$$

Note that here the first product is bounded from below by the constant $2^{-(i-1)}$.

Further, all factors in the second product are greater than 1 and less than 2. Moreover, if $i < p < N-l$, then $m < \lambda_p < j$. Thus, $\lambda_p - p$ attains all values from $\{j-i-N, \dots, j-i-2\}$, except for s values; and s can be bounded from above by a number not depending on N. We conclude that $(**)$ can be bounded from below by the following expression:

$$\mathrm{const} \cdot \prod_{g=j-i-2}^{j-i-N} \frac{j-i-g}{j-i-g-1} = \mathrm{const} \cdot \frac{2}{1}\frac{3}{2} \cdots \frac{N}{N-1} = \mathrm{const} \cdot N$$

Therefore, the conditional probabilities differ by a factor which is more than const/N. Consequently, $p_N \le 1 - \frac{\varepsilon}{N}$. □

We conclude that

$$\rho_{z,w}(R(i,j,l,m,t)) \leq \prod_{N=t+1}^{\infty} p_N \leq \prod_{N=t+1}^{\infty} \left(1 - \frac{\varepsilon}{N}\right) \to 0.$$

□

The author believes that the theorem about the disjointness of the measures remains valid in a more general case. It seems quite plausible that the following stronger statement holds:the (z,w)-measure of the set of paths contained in a left thick hook is equal to zero. (Recall, that a path τ is contained in a left thick hook, if there exist two such numbers i and j, that $\tau_i(N) < j$ for every N)

However, at the moment there is no simple and direct proof of this statement. It seems likely that it could be proved using the results of the paper [13] about limit behavior of the intermediate Frobenius coordinates for the positive and negative Young diagrams, corresponding to a signature $\tau(N)$.

If this conjecture is true, Theorem 4.1.1 can be extended to some additional cases. For example, we can state that the representations $T_{z,w}$ and $T_{z,w'}$, where $\{w, \overline{w}\} \neq \{w', \overline{w'}\}$ are also disjoint.

4.7 Appendix. Growth of a diagonal

Let τ be an infinite path in the graph of signatures. Recall that, there exists a unique Young diagram $\tau^+(N,\tau)$, corresponding to the signature $\tau(N)$. We call this diagram a positive Young diagram of $\tau(N)$.

Below we prove as a corollary from Lemma 4.5.1 that for almost every (with respect to the (z,w)-measure) path the length of the diagonal of the positive Young diagram grows at most logarithmically in the number of the level.

Let $\tilde{s}_N(\tau)$ denote the length of the main diagonal of $\lambda^+(N,\tau)$.

If the (z,w)–measure is treated as a probability measure on the set of all paths $\mathcal{T}$, then $\tilde{s}_N$ is a sequence of random variables.

Theorem 4.7.1. *There exists a positive constant $c(z,w)$ such that almost surely with respect to the (z,w)–measure*

$$\overline{\lim_{N\to\infty}} \frac{\tilde{s}_N}{\log(N)} \leq c(z,w).$$

Proof. First we collect some necessary general definitions and facts.

Let X denote the set $\{0,1\}^\infty$ of infinite sequences of 0's and 1's. We equip X with the topology of the direct product. Consider the sigma-algebra of Borel sets on X.

We denote the ith term of the sequence $A \in X$ by A_i. We also use the notation $A_i = q_i(A)$.

There is a natural partial order on X: $A \leq B$, if $A_i \leq B_i$ for all i.

We say that a real-valued function f on the space of sequences is *increasing*, if $A \leq B$ implies $f(A) \leq f(B)$. Denote by $\mathbb{M}$ the family of all continuous increasing functions.

Consider two probability measures μ_1 and μ_2 on X. We write $\mu_1 \leq \mu_2$ if $E_{\mu_1} f \leq E_{\mu_2} f$ for all functions $f \in \mathbb{M}$. Here $E_\mu f$ is the expectation of the function f with respect to the measure μ.

Suppose $\mathcal{A} \subset X$. In what follows we identify $\mathcal{A}$ with the event "$x \in \mathcal{A}$" and denote by $\mathrm{Prob}_\mu \mathcal{A}$ the probability of this event with respect to the probability measure μ. We also denote by $\mathrm{Prob}_\mu\{\mathcal{A}|\mathcal{B}\}$ a conditional probability of the event $\mathcal{A}$ given the event $\mathcal{B}$.

Lemma 4.7.2. *Suppose μ and ν are two probability measures on X, such that ν is a Bernoulli measure (i.e. product–measure) and the following estimates hold*

$$\mathrm{Prob}_\nu\{q_n = 1\} \geq \mathrm{Prob}_\mu\{q_n = 1 \mid \mathcal{A}\},$$

for all events $\mathcal{A}$, which belong to the σalgebra generated by the first $n-1$ coordinate functions $q_1, \ldots, q_{n-1}$.

Then $\mu \leq \nu$.

Such lemma was stated in [55, Lemma 1.1]. The proof of a more general fact can be found in [78, Lemma 1].

Now we return to our theorem.

The main diagonal of a Young diagram is formed by boxes of the diagram such that $c(\square) = 0$. Consider a secondary diagonal, corresponding to content of a box k, i.e. the set of boxes with $c(\square) = k$.

Choose k such that $\mathrm{Re}(k + w) > 0$. Denote the length of the secondary diagonal, corresponding to the box content k, by s_N. It is clear that $\tilde{s}_N \leq s_N + k$. Therefore, it is sufficient to prove the theorem for the random variables s_N instead of $\tilde{s}_N$.

Note that since $\mathrm{Re}(k + w) > 0$ we can apply Lemma 4.5.1 to estimate the growth of the diagonal.

Introduce a family of random variables $\xi_N = \xi_N(\tau)$ on $\mathcal{T}$. If a box with content k is added to the "positive" Young diagram when one moves from level $N-1$ to level N of the Gelfand-Tsetlin graph along the path τ (i.e., if the length of the secondary diagonal increases by one), then $\xi_N(\tau) = 1$. Otherwise, $\xi_N(\tau) = 0$. It is evident that $s_N = \sum_{i=1}^N \xi_i$.

The joint distribution of the random variables ξ_i defines in a natural way a probability measure on the set of sequences X. Denote this measure by μ.

The claim of Theorem 12 can be reformulated as a property of μ, as follows. We replace the probability space $\mathcal{T}$ by X and the (z, w)measure by μ. Then the random variables ξ_i turn into the coordinate functions q_i . The length of the secondary diagonal s_N turns into the sum $q_1 + q_2 + \cdots + q_N$.

Proposition 4.7.3. *There exists a constant $c_1(z, w)$ such that for any collection $\{a_1, \ldots, a_{N-1}\}$ of 0's and 1's the following estimate on the conditional*

probability holds

$$\mathrm{Prob}\{\xi_N = 1 | \xi_1 = a_1, \dots, \xi_{N-1} = a_{N-1}\} \le \frac{c_1(z,w)}{N}$$

Proof. Immediately follows from Lemma 4.5.1. □

Consider the following family $\{\nu_N\}$ of probability measures on $\{0,1\}$:

$$\nu_N(\{1\}) = \min(1, c_1(z,w)/N),$$

$$\nu_N(\{0\}) = 1 - \min(1, c_1(z,w)/N).$$

Here $c_1(z,w)$ is the constant from Proposition 4.7.3.

Now denote by ν the direct product of the measures ν_N.

Applying Proposition 4.7.3 and Lemma 4.7.2 we conclude that $\mu \le \nu$.

Next, we prove that an analogue of the strong law of large numbers holds for the measure ν. Thus ν satisfies the estimate similar to the one of Theorem 4.7.1.

Proposition 4.7.4. *Almost surely with respect to the measure ν*

$$\lim_{N\to\infty} \frac{q_1 + q_2 + \dots + q_N}{\log(N)} = c_1(z,w)$$

Proof. It is clear that $E_\nu q_i = c_1(z,w)/i$. Consequently,

$$\lim_{N\to\infty} E_\nu \left(\frac{q_1 + q_2 + \dots + q_N}{\log(N)} \right) = c_1(z,w)$$

Denote $p_i = q_i - E_\nu(q_i)$. We have to prove that almost surely with respect to the measure ν

$$\lim_{N\to\infty} \frac{p_1 + p_2 + \dots + p_N}{\log(N)} = 0$$

Lemma 4.7.5. *Suppose b_i is an arbitrary numerical sequence such that the series $\sum\limits_{i=1}^{\infty} b_i$ converges. Then*

$$\lim_{N\to\infty} \frac{b_1 \cdot \log(1) + b_2 \cdot \log(2) + \dots + b_N \cdot \log(N)}{\log(N)} = 0$$

A proof can be found in, e.g., [82, Chapter 4, Section 3, Lemma 2].

Introduce $t_i = p_i/\log(i)$. The last lemma implies that it is sufficient to prove almost sure convergence of the series $\sum_i = 1^\infty t_i$. Let us estimate the variance of the random variable t_i. We denote the variance by D_ν. Evidently, $D_\nu q_i \le c/i$. Therefore, $D_\nu t_i \le c/(i \log^2(i))$. Hence, the series $\sum_{i=1}^{\infty} D_\nu t_i$ converges. It remains to apply the following lemma:

Lemma 4.7.6. *Suppose that ϕ_i is a sequence of independent centered random variables such that the series $\sum\limits_{i=1}^{\infty} D\phi_i$ converges. Then the series $\sum\limits_{i=1}^{\infty} \phi_i$ converges almost surely.*

See [82, Chapter 4, Section 2, Theorem 3] for a proof.

This concludes the proof of Proposition 4.7.4. □

It follows from Proposition 4.7.4 that for almost every with respect to the measure ν sequence in X, there exists such N that for every $n > N$:

$$\frac{\sum_{i=1}^{n} q_i}{\log(n)} < c, \qquad (***)$$

where c is a constant depending solely on z, w, and q_i is the ith coordinate function. Now we show that this statement also holds for the measure μ.

Let $\mathcal{A}_N \subset X$ be the set where the inequality $(***)$ holds for all $n > N$. It is clear that for every N, $\mathcal{A}_N \subset \mathcal{A}_{N+1}$. By Proposition 4.7.4

$$\lim_{N\to\infty} \mathrm{Prob}_\nu \mathcal{A}_N = 1.$$

Lemma 4.7.7.

$$Prob_\mu \mathcal{A}_N \geq \mathrm{Prob}_\nu \mathcal{A}_N$$

Proof. Denote by $\mathcal{B}_N^i \subset X$ the set where inequality $(***)$ holds for $n = N, N+1, \ldots, N+i-1$. The set $\mathcal{A}_N$ coincides with the intersection of the decreasing sequence of the sets $\mathcal{B}_N^i$

$$\mathcal{A}_N = \bigcap_{i=1}^{\infty} \mathcal{B}_N^i.$$

Note that if the inequality $(***)$ holds for some sequence A and $B \leq A$ (with respect to the partial order on X defined above), then the inequality also holds for B. Consequently, the indicator of the set $\mathcal{B}_N^i$ is a decreasing function. Thus, $\nu \geq \mu$ implies that $\mathrm{Prob}_\mu \mathcal{B}_N^i \geq \mathrm{Prob}_\nu \mathcal{B}_N^i$. Therefore

$$\mathrm{Prob}_\mu \mathcal{A}_N = \lim_{i\to\infty} \mathrm{Prob}_\mu \mathcal{B}_N^i \geq \lim_{i\to\infty} \mathrm{Prob}_\nu \mathcal{B}_N^i = Prob_\nu \mathcal{A}_N.$$

□

It follows from the lemma that

$$\lim_{N\to\infty} \mathrm{Prob}_\mu \mathcal{A}_N = 1.$$

Hence, for almost every with respect to the measure μ, sequence in X there exists a number N such that for every $n > N$ $(***)$ holds. The last statement is equivalent to Theorem 4.7.1.

□

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Samenvatting

Het onderwerp van dit proefschrift is gerelateerd aan (mathematische) statistische mechanica en kansrekening enerzijds, en representatietheorie van "grote" groepen anderzijds. Een typisch voorbeeld van een "grote" groep is de vereniging van unitaire groepen $U(N)$ natuurlijk ingebed in elkaar; dit heet de oneindig–dimensionale unitaire groep $U(\infty)$.

In statistische mechanica behandelen we modellen van stochastische discrete oppervlakken in $\mathbb{R}^3$ of, equivalent, stochastische Young-diagrammen. Het verband tussen stochastische Young diagrammen en representatietheorie is bekend sinds de jaren '70: in 1977 bewezen Logan, Shepp en, onafhankelijk, Vershik en Kerov de wet van de grote getallen voor (2-D) Young-diagrammen verdeeld volgens de Plancherel-maat, welk een onderwerp is van representatietheoretische oorsprong; van een andere kant suggereerden Vershik en Kerov in hun artikelen een benadering van representatietheorie van de oneindige symmetrische groep $S(\infty)$ en de oneindig-dimensionale unitaire groep $U(\infty)$, welke de classificatie van karakters (d.w.z. positief-definiete centraal genormaliseerde continue functies) van de groep reduceert tot de studie van bepaalde Markovketens op de verzameling Young diagrammen.

Het proefschrift begint met de studie van objecten van puur combinatorische oorsprong: we onderzoeken diverse maten op 3-D Young diagrammen beperkt tot een $a \times b \times c$-balk en het limietgedrag van deze maten als $a, b, c \to \infty$. Het meest bekende voorbeeld van een dergelijke maat, uniforme distributie, wordt al meer dan tien jaar bestudeerd door diverse auteurs. In 1998 bewezen Cohn, Larsen en Propp de wet van de grote getallen voor uniform verdeelde 3-D Young diagrammen in een $a \times b \times c$-balk. In andere woorden, ze lieten zien dat als $a, b, c \to \infty$ zodanig dat de verhoudingen onveranderd blijven, geschaalde getrapte oppervlakken (de rand van het 3-D Young diagram) convergeren naar een deterministische limietvorm. Later generaliseerden Cohn, Kenyon en Propp (zie ook artikelen van Destainville) deze resultaten tot getrapte oppervlakken met algemenere randvoorwaarden of, gelijk hieraan, tot 3-D Young diagrammen beperkt tot ingewikkeldere ruimtelijke objecten dan een balk. Kenyon, Okounkov en Sheffield gaven een beschrijving van de hieruit voortvloeiende limietvormen en vonden een boeiende relatie tussen de limietvormen en zekere algebraïsche krommen.

Van de andere kant zijn er artikelen die vrije (d.w.z. niet beperkt tot een balk) 3-D Young diagrammen behandelen. Uiteraard is er geen uniforme kansmaat op de verzameling van *alle* 3-D Young diagrammen. Men kan echter de uniforme kansmaat vervangen door een andere natuurlijke kansverdeling die het gewicht q^{vol} toewijst aan een Young diagram met volume vol.

In het eerste deel van het proefschrift bestuderen we een algemenere 2–parameter familie van maten die de uniforme kansmaat vervormt op 3-D Young diagrammen in een balk. Een distributie van deze familie kan worden gezien als een interpolatie tussen de maten q^{vol} en q^{-vol}. We bewijzen diverse resultaten over deze maten: we construeren een familie van Markovketens welke overeenkomen met de maten en, in het bijzonder, leiden tot een efficiënt as-

elect steekproef-algoritme; we vinden de overeenkomstige generalisatie van de bekende MacMahon productformule voor het aantal 3-D Young diagrammen in een gegeven balk; we berekenen de correlatie functies die de lokale meetkunde beschrijven van stochastische 3-D Young diagrammen en bestuderen de asymptoten als $a, b, c \to \infty$ en $q \to 1$.

Het tweede deel van het proefschrift behandelt combinatorische structuren gerelateerd aan de groep $U(\infty)$. Vershik en Kerov merkten in een van hun artikelen op dat er een bijectie is tussen karakters van $U(\infty)$ en kansmaten op paden in de zogenaamde Gelfand–Tsetlin graaf die aan een zekere Gibbs-eigenschap voldoen. Op deze manier kan het probleem van de classificatie van de karakters worden geherformuleerd als de classificatie van Gibbs-maten op paden. Laatstgenoemd vraagstuk heeft een lange geschiedenis (zie artikelen van Voiculescu, Boyer, Vershik-Kerov, Okounkov-Olshanski) en blijkt verwant te zijn aan, in het bijzonder, diverse deformatieklassen van representaties van de groep $U(\infty)$ en de studie van totaal positieve Toeplitz-matrices geclassificeerd door Edrey.

Met behulp van combinatorische bijecties kunnen paden in Gelfand-Tsetlin-grafen worden gezien als getrapte oppervlakken of 3-D Young diagrammen (ditmaal met een oneindige inhoud). Dan verandert de Gibbs-eigenschap in een uniformiteit van zekere voorwaardelijke maten op 3-D Young diagrammen. In dit proefschrift bestuderen we een 1-parameter vervorming van deze Gibbs-eigenschap. Onze vervorming houdt sterk verband met de maat q^{vol} op 3-D Young diagrammen en is, op zeker manier, geïnspireerd door de studie naar die maat. Hoewel de vervorming puur combinatorisch is, zijn er aanwijzingen dat deze nieuwe q–Gibbs-maten toch verband houden met representatietheorie, maar met de unitaire groepen vervangen door hun q-vervormingen — quantumgroepen. We bewijzen een classificatiestelling voor de q–Gibbs-maten en behandelen hun relaties met zekere klassen van matrices met niet-negatieve minoren en met limieten van symmetrische polynomen als het aantal variabelen naar oneindig gaat.

In het laatste deel van het proefschrift bestuderen we een vraagstuk van puur representatietheoretische aard. We behandelen een 2-parameter familie van representaties van $U(\infty)$ (om precies te zijn, we bestuderen een familie van bolvormige representaties van het Gelfandpaar $(U(\infty) \times U(\infty), diag(U(\infty)))$) ingevoerd door Olshanski als een vervanging voor de niet-bestaande reguliere representatie van $U(\infty)$. We zoeken naar een antwoord op de vraag of deze representaties disjunct zijn (d.w.z. ze bevatten geen isomorfe deelrepresentaties). Om deze disjunctheid te bewijzen reduceren we dit probleem tot de analyse van zekere maten op de Gelfand–Tsetlin-graaf of, in andere woorden, bestuderen we opnieuw de stochastische Young diagrammen.

Het proefschrift bestaat uit een introductiehoofdstuk waarin we elk van de opgeloste problemen melden en de nodige details geven en 3 hoofdstukken met belangrijke resultaten. In principe is elk van deze hoofdstukken een autonoom artikel dat los kan worden gelezen.

Hoofdstuk 2 is gebaseerd op een gezamelijk artikel met A. Borodin en

E. Rains gepubliceerd in Selecta Mathematica, [6]. Hoofdstuk 3 is gebaseerd op een nog ongepubliceerd artikel [37]. Hoofdstuk 4 is gebaseerd op een artikel gepubliceerd in Functional Analysis and its applications, [34].

Curriculum Vitae

Vadim Gorin was born in 1986 in Moscow. In 2003 he graduated from Moscow school 57. In the years 2003-2008 he was studying mathematics at Moscow State University. Vadim graduated from the University with master's diploma cum laude. By that time he had written 3 research papers. Afterwards Vadim continued his research at Moscow State University, Independent University of Moscow, Institute for Information Transmission Problems and the University of Utrecht. Since 2007 Vadim was engaged in an international cooperation project "Combinatorial stochastic processes" supported by the Utrecht University and its partners. In 2011 Vadim got a research position at the Institute for Information Transmission Problems. Also, since 2006 Vadim is affiliated with the Russian Hydrological and Meteorological Centre, since 2009 he was teaching mathematics at Moscow school 57 and during the summer of 2009 he was an intern at Microsoft Research, Theory Group (Redmond, USA).

In the school years Vadim was a winner of a number of mathematical competitions. Later he was awarded prizes and scholarships for his research papers (2nd prize of The Leonhard Eulers Foundation Contest for students, 1st and 2nd prizes of Moebius Contest for students and graduate students, A.N. Kolmogorov scholarship, scholarship of the government of Russia, scholarship of the "Dynasty" foundation and IUM-Simons Foundation scholarship). Vadim has published 5 mathematical papers, one paper on meteorology, and has a number of papers submitted or in preparation. After the defence he plans to do a postdoc at MSRI, Berkeley.