INTEGRABILITY OF LIE BRACKETS

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Abstract. In this paper we present the solution to a longstanding problem of differential geometry: Lie's third theorem for Lie algebroids. We show that the integrability problem is controlled by two computable obstructions. As applications we derive, explain and improve the known integrability results, we establish integrability by local Lie groupoids, we clarify the smoothness of the Poisson sigma-model for Poisson manifolds, and we describe other geometrical applications. Our approach also puts into a new perspective the work of Cattaneo and Felder for the special case of Poisson manifolds and the "new" proof of Lie's third theorem given by Duistermaat and Kolk.

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0. Introduction

This paper is concerned with the general problem of integrability of geometric structures. The geometric structures we consider are always associated with local Lie brackets $[\ ,\ ]$ on sections of some vector bundles, or what one calls *Lie algebroids*. A Lie algebroid can be thought of as the appropriate replacement for the tangent bundle, the locus where infinitesimal geometry takes place. Roughly speaking, the general integrability problem asks for the existence of a "space of arrows" and a product which unravels the infinitesimal structure. These global objects are usually known as *Lie groupoids* (or *differentiable groupoids*) and in this paper we shall give the precise obstructions to integrate a Lie algebroid to a Lie groupoid. For an introduction to this problem and a brief historical account we refer the reader to the recent monograph [3].

To describe our results, let us start by recalling that a *Lie algebroid* over a manifold M consists of a vector bundle A over M, endowed with a Lie bracket $[\ ,\ ]$ on the space of sections $\Gamma(A)$, together with a bundle map $\# : A \to TM$, called *the anchor*. One requires the induced map $\# : \Gamma(A) \to \mathcal{X}^1(M)$ ([1]) to be a Lie algebra map, and also the Leibniz identity

$$[\alpha, f\beta] = f[\alpha, \beta] + \#\alpha(f)\beta,$$

to hold.

For any $x \in M$, there is an induced Lie bracket on

$$\mathfrak{g}_x \equiv \mathrm{Ker}\,(\#_x) \subset A_x$$

which makes it into a Lie algebra. In general, the dimension of $\mathfrak{g}_x$ varies with x. The image of $\#$ defines a smooth generalized distribution in M, in the sense of Sussmann ([24]), which is integrable. When we restrict to a leaf L of the associated foliation, the $\mathfrak{g}_x$'s are all isomorphic and fit into a Lie algebra bundle $\mathfrak{g}_L$ over L (see [17]). In fact, there is an induced Lie algebroid

$$A_L = A|_L$$

which is transitive (i.e. the anchor is surjective), and $\mathfrak{g}_L$ is the kernel of its anchor map. A general Lie algebroid A can be thought of as a singular foliation on M, together with transitive algebroids A_L over the leaves L, glued in some complicated way.

The integrability problem for Lie algebroids can be illustrated by looking at some basic examples:

- For algebroids over a point (i.e. Lie algebras) the integrability problem is solved by Lie's third theorem on the integrability of (finite dimensional) Lie algebras by Lie groups;
- For algebroids with zero anchor map (i.e. bundles of Lie algebras), it is Douady-Lazard [10] extension of Lie's third theorem which ensures that the Lie groups integrating each Lie algebra fiber fit into a smooth bundle of Lie groups;
- For algebroids with injective anchor map (i.e. involutive distributions $\mathcal{F} \subset TM$), the integrability problem is solved by Frobenius' integrability theorem.

Other fundamental examples come from E. Cartan infinite continuous groups (Singer and Sternberg, [23]), the integrability of infinitesimal actions of Lie algebras on manifolds (Palais, [22]), of Poisson manifolds (Weinstein, [25]), algebras of vector fields

[1] We denote by $\Omega^r(M)$ and $\mathcal{X}^r(M)$, respectively, the spaces of differential r-forms and r-multivector fields on a manifold M. If E is a bundle over M, $\Gamma(E)$ will denote the space of global sections.

(Nistor, [20]) and of abstract Atiyah sequences (Almeida and Molino, [2]). These, together with various other examples will be discussed in the forthcoming sections.

Let us look closer at the most trivial example. A vector field $X \in \mathcal{X}^1(M)$ is the same as Lie algebroid structure on the trivial line bundle $\mathbb{L} \to M$: the anchor is just multiplication by X, while the Lie bracket on $\Gamma(\mathbb{L}) \simeq C^\infty(M)$ is given by $[f, g] = X(f)g - fX(g)$. The integrability result here states that a vector field is integrable to a local flow. It may be useful to think of the flow Φ^t_X as a collection of arrows $x \longrightarrow \Phi^t_X(x)$ between the different points of the manifold, which can be composed by the rule $\Phi^t_X\Phi^s_X = \Phi^{s+t}_X$. The points which can be joined by such an arrow with a given point x form the orbit of Φ_X (or the integral curve of X) through x.

The general integrability problem is similar: it asks for the existence of a "space of arrows" and a partially defined multiplication, which unravels the infinitesimal structure $(A, [\ ,\ ], \#)$. In a more precise fashion, a *groupoid* is a small category $\mathcal{G}$ all of whose arrows are invertible. If the set of objects (points) is M, we say that $\mathcal{G}$ is a groupoid over M. We shall denote by the same letter $\mathcal{G}$ the space of arrows, and write

$$\begin{array}{c} \mathcal{G} \\ {\scriptstyle \mathbf{s}}\downarrow\downarrow{\scriptstyle \mathbf{t}} \\ M \end{array}$$

where $\mathbf{s}$ and $\mathbf{t}$ are the source and target maps. If $g, h \in \mathcal{G}$ the product gh is defined only for pairs (g, h) in the set of composable arrows

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} | \mathbf{t}(h) = \mathbf{s}(g)\} ,$$

and we denote by $g^{-1} \in \mathcal{G}$ the inverse of g, and by $1_x = x$ the identity arrow at $x \in M$. If $\mathcal{G}$ and M are topological spaces, all the maps are continuous, and $\mathbf{s}$ and $\mathbf{t}$ are open surjections, we say that $\mathcal{G}$ is a *topological groupoid.* A *Lie groupoid* is a groupoid whose space of arrows $\mathcal{G}$ and space of objects M are smooth manifolds, whose source and target maps $\mathbf{s}, \mathbf{t}$ are submersions, and with all the other structure maps smooth. We require M and the $\mathbf{s}$-fibers $\mathcal{G}(x, -) = \mathbf{s}^{-1}(x)$, where $x \in M$, to be Hausdorff manifolds, but it is important to allow the total space $\mathcal{G}$ of arrows to be non-Hausdorff. This is dictated by very simple examples: the monodromy groupoid of a foliation is non-Hausdorff if there are vanishing cycles.

As in the case of Lie groups, any Lie groupoid $\mathcal{G}$ has an associated Lie algebroid $A = A(\mathcal{G})$. As a vector bundle, it is the restriction to M of the bundle $T^{\mathbf{s}}\mathcal{G}$ of $\mathbf{s}$-vertical vector fields on M. Its fiber at $x \in M$ is the tangent space at 1_x of the $\mathbf{s}$-fibers $\mathcal{G}(x, -) = \mathbf{s}^{-1}(x)$, and the anchor map is just the differential of the target map $\mathbf{t}$. To define the bracket, one shows that $\Gamma(A)$ can be identified with $\mathcal{X}^{\mathbf{s}}_{inv}(\mathcal{G})$, the space of $\mathbf{s}$-vertical, right-invariant, vector fields on $\mathcal{G}$. The standard formula of Lie brackets in terms of flows shows that $\mathcal{X}^{\mathbf{s}}_{inv}(\mathcal{G})$ is closed under $[\cdot, \cdot]$. This induces a Lie bracket on $\Gamma(A)$, which makes A into a Lie algebroid.

We say that a Lie algebroid A is integrable if there exists a Lie groupoid $\mathcal{G}$ inducing A. The extension of Lie's theory (Lie's first and second theorem) to Lie algebroids has a promising start.

Theorem (Lie I). *If A is an integrable Lie algebroid, then there exists a (unique) $\mathbf{s}$-simply connected Lie groupoid integrating A.*

This has been proved in [16] (see also [17] for the transitive case). A different argument, which is just an extension of the construction of the smooth structure on the universal cover of a manifold (cf. Theorem 1.13.1 in [9]), will be presented below. Here $\mathbf{s}$-simply connected means that the $\mathbf{s}$-fibers $\mathbf{s}^{-1}(x)$ are 1-connected. The Lie groupoid in the theorem is often called the *monodromy groupoid* of A, and will be denoted by $\mathrm{Mon}\,(A)$. For the simple examples above, $\mathrm{Mon}\,(TM)$ is the homotopy groupoid of M, $\mathrm{Mon}\,(\mathcal{F})$ is the monodromy groupoid of the foliation $\mathcal{F}$, while $\mathrm{Mon}\,(\mathfrak{g})$ is the unique simply-connected Lie group integrating $\mathfrak{g}$.

The following result is standard (we refere to [18, 16], although the reader may come across it in various other places). See also section 2 below.

Theorem (Lie II). *Let $\phi : A \to B$ be a morphism of integrable Lie algebroids, and let $\mathcal{G}$ and H be integrations of A and B. If $\mathcal{G}$ is $\mathbf{s}$-simply connected, then there exists a (unique) morphism of Lie groupoids $\Phi : \mathcal{G} \to H$ integrating ϕ.*

In contrast with the case of Lie algebras or foliations, there is no Lie's third theorem for general Lie algebroids. Examples of non-integrable Lie algebroids are known (we will see several of them in the forthcoming sections) and, up to now, no good explanation for this failure was known. For instance, the various integrability criteria one finds in the literature are (apparently) non-related: some require a nice behavior of the Lie algebras $\mathfrak{g}_x$, some require a nice topology of the leaves of the induced foliation, and most of them require regular algebroids. A good understanding of this failure should shed some light on the following questions:

- *Is there a (computable) obstruction to the integrability of Lie algebroids?*
- *Is the integrability problem a local one?*
- *Are Lie algebroids locally integrable?*

In this paper we will provide answers to these questions. In particular, we will show that the obstruction to integrability comes from the relation between the topology of the leaves of the induced foliation and the Lie algebras defined by the kernel of the anchor map.

We will now outline our integrability result. Given an algebroid A and $x \in M$, we will construct certain (monodromy) subgroups

$$N_x(A) \subset A_x,$$

which lie in the center of the Lie algebra $\mathfrak{g}_x = Ker(\#_x)$: they consist of those elements $v \in Z(\mathfrak{g}_x)$ which are *homotopic* to zero (see section 1). As we shall explain, these groups arise as the image of a second order monodromy map

$$\partial : \pi_2(L_x) \to \mathcal{G}(\mathfrak{g}_x)$$

which relates the topology of the leaf L_x through x with the simply connected Lie group $\mathcal{G}(\mathfrak{g}_x)$ integrating the Lie algebra $\mathfrak{g}_x = Ker(\#_x)$. From a conceptual point of view, the monodromy map can be viewed as an analogue of a boundary map of the homotopy long exact sequence of a fibration (namely $0 \to \mathfrak{g}_{L_x} \to A_{L_x} \to TL_x \to 0$). In order to measure the discreteness of the groups $N_x(A)$ we let

$$r(x) = d(0, N_x(A) - \{0\})$$

where the distance is computed with respect to a (arbitrary) norm on the vector bundle A. Here we adopt the convention $d(0, \emptyset) = +\infty$. We will see that r is *not* a continuous function. Our main result is:

Theorem (Obstructions to Lie III). *For a Lie algebroid A over M, the following are equivalent:*

(i) *A is integrable;*
(ii) *For all $x \in M$, $N_x(A) \subset A_x$ is discrete and $\liminf_{y\to x} r(y) > 0$;*

We stress that these obstructions are computable in many examples. First of all, the definition of the monodromy map is explicit. Moreover, given a splitting $\sigma : TL \to A_L$ of $\#$ with $Z(\mathfrak{g}_L)$-valued curvature 2-form Ω_σ, we will see that

$$N_x(A) = \{\int_\gamma \Omega_\sigma : \gamma \in \pi_2(L, x)\} \subset Z(\mathfrak{g}_x).$$

With this information at hand the reader can already jump to the examples (see sections 3.3, 3.4, 4.1 and 5).

As is often the case, the main theorem is just an instance of a more fruitful approach. In fact, we will show that a Lie algebroid A always admits an "integrating" *topological* groupoid $\mathcal{G}(A)$. Although it is not always smooth (in general it is only a leaf space), it does behave like a Lie groupoid. This immediately implies the integrability of Lie algebroids by "local Lie groupoids", a result which has been assumed to hold since the original works of Pradines in the 1960's.

The main idea of our approach is as follows: Suppose $\pi : A \to M$ is a Lie algebroid which can be integrated to a Lie groupoid $\mathcal{G}$. Denote by $P(\mathcal{G})$ the space of $\mathcal{G}$-*paths*, with the C^2-topology:

$$P(\mathcal{G}) = \left\{g : [0,1] \to \mathcal{G} |\ g \in C^2,\ \mathbf{s}(g(t)) = x,\ g(0) = 1_x\right\}$$

(paths lying in $\mathbf{s}$-fibers of $\mathcal{G}$ starting at the identity). Also, denote by $\sim$ the equivalence relation defined by C^1-homotopies in $P(\mathcal{G})$ with fixed end-points. Then we have a standard description of the monodromy groupoid as

$$\mathrm{Mon}\,(A) = P(\mathcal{G})/\sim .$$

The source and target maps are the obvious ones, and for two paths $g, g' \in P(\mathcal{G})$ which are composable (i. e. $\mathbf{t}(g(1)) = \mathbf{s}(g'(0))$) we define

$$g' \cdot g(t) \equiv \begin{cases} g(2t), & 0 \le t \le \frac{1}{2} \\ g'(2t-1)g(1), & \frac{1}{2} < t \le 1 \end{cases}$$

Note that any element in $P(\mathcal{G})$ is equivalent to some g such that $\dot{g}(0) = \dot{g}(1)$, and if g and g' have this property, then $g' \cdot g \in P(\mathcal{G})$. Therefore, this multiplication is associative up to homotopy, so we get the desired multiplication on the quotient space which makes $\mathrm{Mon}\,(A)$ into a (topological) groupoid. The construction of the smooth structure on $\mathrm{Mon}\,(A)$ is similar to the construction of the smooth structure on the universal cover of a manifold (see e. g. Theorem 1.13.1 in [9]).

Now, any $\mathcal{G}$-path g defines an A-path a, i. e. a curve $a : I \to A$ defined on the unit interval $I = [0,1]$, with the property that

$$\#a(t) = \frac{d}{dt}\pi(a(t)).$$

The A-path a is obtained from g by differentiation and right translations. This defines a bijection between $P(\mathcal{G})$ and the set $P(A)$ of A-paths and, using this bijection, we can transport homotopy of $\mathcal{G}$-paths to an equivalence relation (*homotopy*) of A-paths. Moreover, this equivalence can be expressed using the infinitesimal data only (section 1, below). It follows that a monodromy type groupoid $\mathcal{G}(A)$ can be constructed without any integrability assumption. This construction of $\mathcal{G}(A)$, suggested by Alan Weinstein, in general only produces a *topological* groupoid (section 2). Our main task will then be to understand when does the Weinstein groupoid $\mathcal{G}(A)$ admit the desired smooth structure, and that is where the obstructions show up. We first describe the second order monodromy map which encodes these obstructions (section 3) and we then show that these are in fact the only obstructions to integrability (section 4). In the final section, we derive the known integrability criteria from our general result and we give two applications.

Acknowledgments

The construction of the groupoid $\mathcal{G}(A)$ was suggested to us by Alan Weinstein, and is inspired in a "new" proof of Lie's third theorem in the recent monograph [9] by Duistermaat and Kolk. We are in debt to him for this suggestion as well as many comments and discussions. The same type of construction, for the special case of Poisson manifolds, appears in the work of Cattaneo and Felder [4]. Though they do not discuss integrability obstructions, their paper was also a source of inspiration for the present work.

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1. A-paths and homotopy

In this section A is a Lie algebroid over M, $\# : A \to TM$ denotes the anchor, and $\pi : A \to M$ denotes the projection.

In order to construct our main object of study, the groupoid $\mathcal{G}(A)$ that plays the role of the monodromy groupoid $\mathrm{Mon}\,(A)$ for a general (non-integrable) algebroid, we need the appropriate notion of paths on A. These are known as *A-paths* or admissible paths, and we shall discuss them in this section.

1.1. **A-paths.** We call a C^1 curve $a : I \to A$ an *A-path* if

$$\#a(t) = \frac{d}{dt}\gamma(t).$$

where $\gamma(t) = \pi(a(t))$ is the base path (necessarily of class C^2). We let $P(A)$ denote the space of A-paths, endowed with the topology of uniform convergence.

We emphasize that this is the right notion of paths in the world of algebroids. From this point of view, one should view a as a bundle map

$$a\, dt : TI \to A$$

which covers the base path $\gamma : I \to M$ and this gives a algebroid morphism $TI \to A$.

Obviously, the base path of an A-path sits inside a leaf L of the induced foliation, and so can be viewed as an A_L-path. The key remark is:

Proposition 1.1. *If $\mathcal{G}$ integrates the Lie algebroid A, then there is a homeomorphism $D^R : P(\mathcal{G}) \to P(A)$ between the space of $\mathcal{G}$-paths, and the space of A-paths (D^R is called the differentiation of $\mathcal{G}$-paths, and its inverse is called the integration of A-paths).*

Proof. Any $\mathcal{G}$-path $g : I \to \mathcal{G}$ defines an A-path $D^R(g) : I \to A$ by the formula

$$(D^R g)(t) = (dR_{g(t)^{-1}})_{g(t)}\dot{g}(t)\ ,$$

where, for $h : x \to y$ arrow in $\mathcal{G}$, $R_h : \mathbf{s}^{-1}(y) \to s^{-1}(x)$ is the right multiplication by h. Conversely, any A-path a arises in this way, by integrating (using Lie II) the Lie algebroid morphism $TI \to A$ defined by a. You should then notice that any Lie groupoid homomorphism $\phi : I \times I \to \mathcal{G}$ from the pair groupoid into $\mathcal{G}$ is of the form $\phi(s,t) = g(s)g^{-1}(t)$ for some $\mathcal{G}$-path g.

A more explicit argument, avoiding Lie II, and which also shows that the inverse of D^R is continuous, is as follows. Given a, we choose a time dependent section α of A above a, i. e. so that

$$a(t) = \alpha(t, \gamma(t)).$$

If we let $\varphi_\alpha^{t,0}$ be the flow of the right-invariant vector field that corresponds to α, then $g(t) = \varphi_\alpha^{t,0}(\gamma(0))$ is the desired $\mathcal{G}$-path. Indeed, right-invariance guarantees that this flow is defined for all $t \in [0,1]$ and also implies that

$$(D^R g)(t) = (dR_{g(t)^{-1}})_{g(t)}(\alpha(t, g(t))) = \alpha(t, \gamma(t)) = a(t).$$

□

1.2. **A-paths and connections.** Given an A-connection on a vector bundle E over M, most of the classical construction (which we recover when $A = TM$) extend to Lie algebroids, provided we use A-paths. This is explained in detail in [11, 12], and here we recall only the results we need.

A *A-connection* on a vector bundle E over M can be defined by a covariant derivative operator $\Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(\alpha, u) \mapsto \nabla_\alpha u$ satisfying $\nabla_{f\alpha} u = f \nabla_\alpha u$, and $\nabla_\alpha(fu) = f\nabla_\alpha u + \#\alpha(f) u$. The curvature of ∇ is given by the usual formula

$$R_\nabla(\alpha, \beta) = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha,\beta]},$$

and ∇ is called flat if $R_\nabla = 0$. For an A-connection ∇ on the vector bundle A, the torsion of ∇ is also defined as usual by:

$$T_\nabla(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta].$$

Given an A-path a with base path $\gamma : I \to M$, and $u : I \to E$ a path in E above γ, then the derivative of u along a, denoted $\nabla_a u$, is defined as usual: choose a time dependent section ξ of E such that $\xi(t, \gamma(t)) = u(t)$, then

$$\nabla_a u(t) = \nabla_a \xi^t(x) + \frac{d\xi^t}{dt}(x), \quad \text{at} \quad x = \gamma(t) \ .$$

One has then the notion of parallel transport along a, $T_a^t : E_{\gamma(0)} \to E_{\gamma(t)}$, and for the special case $E = A$, we can talk about the geodesics of ∇. Geodesics are A-paths a with the property that $\nabla_a a(t) = 0$. As in the classical case, one has existence and uniqueness of geodesics with given initial base point $x \in M$ and "initial speed" $a_0 \in A_{x_0}$.

Example 1.2. If L is a leaf of the foliation induced by A, then $\mathfrak{g}_L = Ker(\#|_L)$ is a representation of A_L, with $\nabla_\alpha \beta = [\alpha, \beta]$. In particular, for any A-path a, the induced parallel transport defines a linear map, called *the linear holonomy* of a,

$$\mathrm{Hol}\,(a) : \mathfrak{g}_x \to \mathfrak{g}_y \ ,$$

where x, y are the initial and the end point of the base path. For more on linear holonomy we refer to [11].

Most of the connections that we will use are induced by a standard TM-connection ∇ on the vector bundle A. Associated with ∇ there is an obvious A-connection on the vector bundle A

$$\nabla_\alpha \beta \equiv \nabla_{\#\alpha} \beta.$$

A bit more subtle are the following two A-connections on A and on TM, respectively, (see [6]):

$$\overline{\nabla}_\alpha \beta \equiv \nabla_{\#\beta} \alpha + [\alpha, \beta], \qquad \overline{\nabla}_\alpha X \equiv \#\nabla_X \alpha + [\#\alpha, X].$$

Note that $\overline{\nabla}_\alpha \#\beta = \#\overline{\nabla}_\alpha \beta$, so in the terminology of [11] this means that $\overline{\nabla}$ is a basic connection on A. These connections play a fundamental role in the theory of characteristic classes (see [5, 6, 11]).

1.3. **Homotopy of A-paths.** As we saw above, if A is integrable, A-paths are in a bijective correspondence with $\mathcal{G}$-paths. Let us see now how one can transport the notion of homotopy to $P(A)$, so that it only uses the infinitesimal data, i. e., on Lie algebroid data.

Let us fix

$$a_\epsilon(t) = a(\epsilon, t) : I \times I \to M$$

a *variation of A-paths*, that is a family of A-paths a_ϵ which is of class C^2 on ϵ, with the property that the base paths $\gamma_\epsilon(t) = \gamma(\epsilon, t) : I \times I \to M$ have fixed end points. If A came from a Lie groupoid $\mathcal{G}$, and a_ϵ came from $\mathcal{G}$-paths g_ϵ, then g_ϵ is not necessarily a homotopy between g_0 and g_1, because the end-points $g_\epsilon(1)$ may vary. The following lemma describe two distinct ways of controlling the variation $\frac{d}{d\epsilon}g_\epsilon(1)$: one way uses a connection on A, and the other uses flows of sections of a A (see Appendix A). They both depend only on infinitesimal data.

Proposition 1.3. *Let A be an algebroid and $a = a_\epsilon$ a variation of A-paths.*

(i) *If ∇ is an TM-connection on A with torsion T_∇, the solution $b = b(\epsilon, t)$ of the differential equation*

$$\partial_t b - \partial_\epsilon a = T_\nabla(a, b), \qquad b(\epsilon, 0) = 0, \tag{1}$$

does not depend on ∇. Moreover, $\# b = \frac{d}{d\epsilon}\gamma$.

(ii) *If ξ_ϵ are time depending sections of A such that $\xi_\epsilon(t, \gamma_\epsilon(t)) = a_\epsilon(t)$, then $b(\varepsilon . t)$ is given by*

$$b(\epsilon, t) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon}{d\epsilon}(s, \gamma_\epsilon(s)) ds. \tag{2}$$

(iii) *If $\mathcal{G}$ integrates A and g_ϵ are the $\mathcal{G}$-paths satisfying $D^R(g_\epsilon) = a_\epsilon$, then $b = D^R(g^t)$, where g^t are the $\mathcal{G}$-paths $\varepsilon \to g^t(\epsilon) = g(\epsilon, t)$.*

This motivates the following definition:

Definition 1.4. We say that two A-paths a_0 and a_1 are equivalent (or homotopic), and write $a_0 \sim a_1$, if there exists a variation a_ϵ with the property that b insured by Proposition 1.3 satisfies $b(\epsilon, 1) = 0$ for all $\epsilon \in I$.

When A admits an integration $\mathcal{G}$, then the isomorphism $D^R : P(\mathcal{G}) \to P(A)$ of Proposition 1.1 transforms the usual homotopy into the homotopy of A-paths. Note also that, as A-paths should be viewed as algebroid morphisms, the pair (a, b) defining the equivalence of A-paths should be viewed as a true homotopy

$$a dt + b d\epsilon : TI \times TI \to A.$$

in the world of algebroids. In fact, equation (1) is just an explicit way of saying that this is a morphism of Lie algebroids (see [14]).

Proof of Proposition 1.3: Let ξ_ϵ be as in the statement, and let η be given by

$$\eta(\epsilon, t, x) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon}{d\epsilon}(s, \Phi_{\#\xi_\epsilon}^{s,t}(x)) ds \in A_x$$

We note that η coincides with the solution of the equation

$$\frac{d\eta}{dt} - \frac{d\xi}{d\epsilon} = [\eta, \xi] \ . \tag{3}$$

with $\eta(\epsilon, 0) = 0$. Indeed, since

$$\eta(\epsilon, t, -) = \int_0^t (\phi_{\xi_\epsilon}^{s,t})^* (\frac{d\xi_\epsilon^s}{d\epsilon}) ds \in \Gamma(A),$$

equation (3) immediately follows from the basic formula (A.2) for flows. Also, $X = \#\xi$ and $Y = \#\eta$ satisfy a similar equation on M, and since we have $X(\epsilon, t, \gamma_\epsilon(t)) = \frac{d\gamma}{dt}$, it follows that $Y(\epsilon, t, \gamma_\epsilon(t)) = \frac{d\gamma}{d\epsilon}$. In other words, $b(\epsilon, t) = \eta(\epsilon, t, \gamma(\epsilon, t))$ satisfies $\#b = \frac{d\gamma}{d\epsilon}$. We now have

$$\partial_t b = \nabla_{\frac{d\gamma}{dt}}\eta + \frac{d\eta}{dt} = \nabla_{\#\xi}\eta + \frac{d\eta}{dt}$$

at $x = \gamma_\epsilon(t)$. Subtracting from this the similar formula for $\partial_\epsilon a$ and using (3) we get

$$\partial_t b - \partial_\epsilon a = \nabla_{\#\xi}\eta - \nabla_{\#\eta}\xi + [\eta, \xi] = T_\nabla(\xi, \eta).$$

We are now left proving (iii). Assume that $\mathcal{G}$ integrates A and g_ϵ are the $\mathcal{G}$-paths satisfying $D^R(g_\epsilon) = a_\epsilon$. The formula of variation of parameters applied to the right-invariant vector field ξ_ϵ shows that

$$\begin{aligned}\frac{\partial g(\varepsilon, t)}{\partial \varepsilon} &= \int_0^t (d\varphi^{t,s}_{\xi_\epsilon})_{g(\varepsilon,s)} \frac{d\xi^s_\epsilon}{d\varepsilon}(g(\varepsilon, s))ds \\ &= (dR_{g(\varepsilon,t)})_{\gamma_\varepsilon(t)} \int_0^t \phi^{t,s}_{\xi_\epsilon} \frac{d\xi^s_\epsilon}{d\varepsilon}(\gamma_\varepsilon(s))ds.\end{aligned}$$

But then:

$$D^R(g^t) = \int_0^t \phi^{t,s}_{\xi_\epsilon} \frac{d\xi^s_\epsilon}{d\varepsilon}(\gamma_\varepsilon(s))ds = b(\varepsilon, t).$$

□

The next lemma gives elementary properties of homotopies of A-paths:

Lemma 1.5. *Let A be a Lie algebroid.*

(i) *If $\tau : I \to I$, with $\tau(0) = 0$, $\tau(1) = 1$ is a smooth change of parameter, then any A-path a is equivalent to its reparametrization $a^\tau(t) \equiv \tau'(t)a(\tau(t))$.*
(ii) *Any A-path a_0 is equivalent to a smooth (i. e. of class C^∞) A-path.*
(iii) *If two smooth A-paths a_0, a_1 are equivalent, then there exists a smooth homotopy between them.*

Proof. To prove (i), we consider the variation

$$a_\epsilon(t) = ((1-\epsilon) + \epsilon\tau'(t))a((1-\epsilon)t + \epsilon\tau(t)).$$

and we check that the associated b satisfies $b(\epsilon, 1) = 0$. In fact, one can compute by any of the methods of Proposition 1.3:

$$b(\epsilon, t) = (\tau(t) - t)a((1-\epsilon)t + \epsilon\tau(t)).$$

For example, if we let α be a time-dependent section which extends the path a, and define a 1-parameter family of time-dependent sections ξ_ϵ by:

$$\xi_\epsilon(t, x) = ((1-\epsilon) + \epsilon\tau'(t))\alpha((1-\epsilon)t + \epsilon\tau(t), x),$$

then ξ_ϵ extends a_ϵ and the family

$$\eta(\epsilon, t, x) = (\tau(t) - t)\alpha((1-\epsilon)t + \epsilon\tau(t), x)$$

satisfies (3). Hence, we must have $b(\epsilon, t) = \eta(\epsilon, t, \gamma(\epsilon, t))$ as claimed.

For (ii), note that from the similar claim for ordinary paths on manifolds (see e. g. Theorem 1.13.1 in [9]), we can find a C^r-homotopy γ_ϵ between the base path γ_0 of a_0 and a smooth path γ_1. Clearly, we can do it so that γ_ϵ stays in the same leaf L as γ_0, and so that $\gamma_\epsilon(t)$ is smooth in the domain $t \in [0, 1]$, $\epsilon \in [c, 1]$ for some constant $0 < c < 1$. We now choose a smooth splitting $\sigma : TL \to A|_L$ of the anchor map, and put $b(\epsilon, t) = \sigma(\frac{d}{d\epsilon}\gamma_\epsilon(t))$. Let a be the solution of the differential equation (1), with the initial conditions $a(0, t) = a_0(t)$. Clearly a is smooth on the domain on which b is, hence it defines a homotopy between a_0 and the smooth

A-path a_1. Part (iii) is just a degree-one higher version of part (ii), and can be proven similarly replacing the path a_0 by the given homotopy between a_0 and a_1 (a similar argument will be presented in detail in the proof of Proposition 3.5). $\square$

1.4. **Representations and A-paths.** A flat A-connection on a vector bundle E defines a *representation* of A on E. The terminology is inspired by the case of Lie algebras. There is also an obvious notion of representation of a Lie groupoid $\mathcal{G}$: this is a vector bundle E over the space M of objects, together with smooth actions $g : E_x \to E_y$ defined for arrows g from x to y in $\mathcal{G}$, satisfying the usual identities. By differentiation, any such representation becomes a representation of the Lie algebroid A of $\mathcal{G}$ (see e. g. [5, 14]). Moreover, when $\mathcal{G} = \mathrm{Mon}\,(A)$ is the unique $\mathbf{s}$-simply connected Lie groupoid integrating A, this construction induces a bijection

$$\mathrm{Rep}\,(\mathrm{Mon}\,(A)) \cong \mathrm{Rep}\,(A)$$

between the (semi-rings of equivalence classes of) representations. This is explained in [13], but it follows also from our construction of $\mathcal{G}(A)$ (see next section) since we have:

Proposition 1.6. *If a_0 and a_1 are equivalent A-paths from x to y. Then for any representation E of A, parallel transports $E_x \to E_y$ along a_0 and a_1 coincide.*

Proof. We first claim that for any A-connection ∇ on E, $adt + bd\epsilon$ and homotopy between a_0 and a_1:

$$\nabla_{a_\epsilon}\nabla_{b_t}u - \nabla_{b_t}\nabla_{a_\epsilon}u = R_\nabla(a, b)u$$

for all paths $u : I \times I \to E$ above $\gamma(\epsilon, t)$. To see this, let us assume that ϵ, η are as in the proof of Proposition 1.3, and let α be a family of time-dependent sections of E so that $u(\epsilon, t) = \alpha(\epsilon, t, \gamma(\epsilon, t))$. Then

$$\nabla_{b_t}u = \nabla_\eta\alpha + \frac{d\alpha}{d\epsilon}$$

at $x = \gamma(\epsilon, t)$. Hence

$$\nabla_{a_\epsilon}\nabla_{b_t}u = \nabla_\xi\nabla_\eta\alpha + \nabla_\xi(\frac{d\alpha}{d\epsilon}) + \nabla_\eta(\frac{d\alpha}{dt}) + \frac{d^2\alpha}{d\epsilon dt} + \nabla_{\frac{d\eta}{dt}}\alpha.$$

Subtracting the analogous formula for $\nabla_{b_t}\nabla_{a_\epsilon}u$ and using (3) the claim follows.

When ∇ is flat, this formula applied $u(\epsilon, t) = T^t_{a_\epsilon}(u_0)$, where $T^t_{a_\epsilon}$ denotes parallel transport, gives $\nabla_{a_\epsilon}\nabla_{b_t}u = 0$. But $\nabla_{b_t}u = 0$ at $t = 0$, hence $\nabla_{b_t}u = 0$ for all t's. Since $u(0, t) = T^t_{a_0}(u_0)$ it follows that $u(\epsilon, t) = T^\epsilon_{b_t}T^t_{a_0}(u_0)$. Thererefore $T^t_{a_\epsilon} = T^\epsilon_{b_t}T^t_{a_0}$, for all ϵ, t and, in particular, for $\epsilon = t = 1$ we get $T^1_{a_1} = T^1_{a_0}$. $\square$

Recalling the notion of linear holonomy (cf. Example 1.2) we have:

Corollary 1.7. *If a_0 and a_1 are equivalent A-paths from x to y, they induce the same linear holonomy maps*

$$\mathrm{Hol}\,(a_0) = \mathrm{Hol}\,(a_1) : \mathfrak{g}_x \to \mathfrak{g}_y.$$

2. The Weinstein groupoid

We are now ready to define the Weinstein groupoid $\mathcal{G}(A)$ of a general Lie algebroid, which in the integrable case will be the unique $\mathbf{s}$-simply connected groupoid integrating A.

2.1. **The groupoid $\mathcal{G}(A)$.** Let a_0, a_1 be two composable A-paths, i. e. so that $\pi(a_0(1)) = \pi(a_1(0))$. We define their concatenation

$$a_1 \odot a_0(t) \equiv \begin{cases} 2a_0(2t), & 0 \le t \le \frac{1}{2} \\ 2a_1(2t-1), & \frac{1}{2} < t \le 1 \end{cases}$$

This is essentially the multiplication that we need. However, $a_1 \odot a_0$ is only *piecewise* smooth. One way around this difficulty is allowing for A-paths which are *piecewise* smooth. Instead, let us fix a cutoff function $\tau \in C^\infty(\mathbb{R})$ with the following properties

(a) $\tau(t) = 1$ for $t \ge 1$ and $\tau(t) = 0$ for $t \le 0$;
(b) $\tau'(t) > 0$ for $t \in]0,1[$.

For an A-path a we denote, as above, by a^τ its reparametrization $a^\tau(t) = \tau'(t)a(\tau(t))$. We now define the multiplication by

$$a_1 a_0 \equiv a_1^\tau \odot a_0^\tau \in P(A).$$

According to Lemma 1.5 (i), a_0a_1 is equivalent to $a_0 \odot a_1$ whenever $a_0(1) = a_1(0)$. We also consider the natural structure maps: source and target $\mathbf{s}, \mathbf{t} : P(A) \to M$ which map a to $\pi(a(0))$ and $\pi(a(1))$, respectively, the identity section $\varepsilon : M \to P(A)$ mapping x to the constant path above x, and the inverse $\iota : P(A) \to P(A)$ mapping a to $\overline{a}$ given by $\overline{a}(t) = -a(1-t)$.

Theorem 2.1. *Let A be a Lie algebroid over M. Then the quotient*

$$\mathcal{G}(A) \equiv P(A)/\sim$$

is a $\mathbf{s}$*-simply connected topological groupoid independent of the choice of cuttoff function. Moreover, whenever A is integrable, $\mathcal{G}(A)$ admits a smooth structure which makes it into the unique* $\mathbf{s}$*-simply connected Lie groupoid integrating A.*

Proof. If we take the maps on the quotient induced from the structure maps defined above, then $\mathcal{G}(A)$ is clearly a groupoid. Note that the multiplication on $P(A)$ was defined so that, whenever $\mathcal{G}$ integrates A, the map D^R of Proposition 1.1 preserves multiplications. Hence the only thing we still have to prove is that $\mathbf{s}, \mathbf{t} : \mathcal{G}(A) \to M$ are open maps.

Given $D \subset \mathcal{G}(A)$ open, we will show that its saturation $\tilde{D}$ w.r.t. the equivalence relation $\sim$ is still open. This follows from the fact, to be shown later in Theorem 4.8, that the equivalence relation can be defined by a foliation on $P(A)$. A more direct argument is to show that for any two equivalent A-paths a_0 and a_1, there exists a homeomorphism of $T : P(A) \to P(A)$ such that $T(a) \sim a$ for all a's, and $T(a_0) = a_1$. To construct such a T we let $\eta = \eta(\epsilon, t)$ be a family of time dependent sections of A which determines the equivalence $a_0 \sim a_1$ (see Proposition 1.3), so that $\eta(\epsilon, 0) = \eta(\epsilon, 1) = 0$ (we may assume η has compact support, so that all the flows involved are everywhere defined). Given an A-path b_0, we consider a time dependent section ξ_0 so that $\xi_0(t, \gamma_0(t)) = b(t)$ and denote by ξ the solution of equation (3) with initial condition ξ_0. If we set $\gamma_\epsilon(t) = \Phi^{\epsilon,0}_{\#\eta_t}\gamma_0(t))$ and $b_\epsilon(t) = \xi_\epsilon(t, \gamma_\epsilon(t))$, then $T_\eta(b_0) \equiv b_1$ is homotopic to b_0 via b_ϵ, and maps a_0 into a_1.

□

2.2. **Homomorphisms.** Note that, although $\mathcal{G}(A)$ is not always smooth, in many aspects it always behave like in the smooth (i. e. integrable) case. For instance, we can call a representation of $\mathcal{G}(A)$ *smooth* if the action becomes smooth when pull-backed to $P(A)$. Similarly one can talk about smooth functions on $\mathcal{G}(A)$, about its tangent space, etc. This subsection and the next are variations on this theme.

Proposition 2.2. *Let A and B be Lie algebroids. Then:*

(i) *Every algebroid homomorphism $\phi : A \to B$ determines a smooth groupoid homomorphism $\Phi : \mathcal{G}(A) \to \mathcal{G}(B)$ of the associated Weinstein groupoids. If A and B are integrable, then $\Phi_* = \phi$;*

(ii) *Every representation $E \in Rep(A)$ determines a smooth representation of $\mathcal{G}(A)$, which in the integrable case is the induced representation.*

Proof. For (i) we define Φ in the only possible way: If $a \in P(A)$ is an A-path then $\phi \circ a$ is an A-path in $P(B)$. Moreover, it is easy to see that if $a_1 \sim a_2$ are equivalent A-paths then $\phi \circ a_1 \sim \phi \circ a_2$, so we get well-defined smooth map $\Phi : \mathcal{G}(A_1) \to \mathcal{G}(A_2)$ by setting

$$\Phi([a]) \equiv [\phi \circ a].$$

This map is clearly a groupoid homomorphism.

Part (ii) follows easily from Proposition 1.6. □

In particular we see that, as in the smooth case, there is a bijection between the representations of A and the (smooth) representations of $\mathcal{G}(A)$:

$$\mathrm{Rep}\,(\mathcal{G}(A)) \cong \mathrm{Rep}\,(A).$$

2.3. **The exponential map.** Assume first that $\mathcal{G}$ is a Lie groupoid integrating A, and ∇ is a TM-connection on A. Then the pull-back of ∇ along the target map $\mathbf{t}$ defines a family of (right invariant) connections ∇_x on the manifolds $\mathbf{s}^{-1}(x)$. The associated exponential maps $\mathrm{Exp}_{\nabla_x} : A_x = T_x^{\mathbf{s}}\mathcal{G} \to \mathbf{s}^{-1}(x)$ fit together into a global exponential map [21]

$$\overline{\mathrm{Exp}}_\nabla : A \to \mathcal{G}$$

(defined only on an open neighborhood of the zero section). By standard arguments, $\overline{\mathrm{Exp}}_\nabla$ is a diffeomorphism on a small enough neighborhood of M.

Now if A is not integrable, we still have the exponential map associated to a connection ∇ on A. It is defined as usual, so $\mathrm{Exp}_\nabla(a)$ is the value at time $t = 1$ of the geodesic (A-path) with the initial condition a. By a slight abuse of notation we view it as a map

$$\mathrm{Exp}_\nabla : A \to P(A).$$

Of course, Exp_∇ is only defined on an open neighborhood of M inside A consisting of elements whose geodesics are defined for all $t \in [0, 1]$. Passing to the quotient, we have an induced exponential map

$$\overline{\mathrm{Exp}}_\nabla : A \to \mathcal{G}(A).$$

For integrable A, this coincides with the exponential map above.

Note that the exponential map we have discussed so far depends on the choice of connection. To get an exponential independent of the connection recall from [17] that an *admissible section* of a Lie groupoid $\mathcal{G}$ is a differentiable map $\sigma : M \to \mathcal{G}$ such that $\mathbf{s}\circ\sigma(x) = x$ and $\mathbf{t}\circ\sigma : M \to M$ is a diffeomorphism. Also, each admissible section $\sigma \in \Gamma(\mathcal{G})$ determines diffeomorphisms

$$\begin{aligned} \mathcal{G} \ni g \mapsto \sigma g &\equiv \sigma(x)g, &\quad& \text{where } x = \mathbf{t}(g),\\ \mathcal{G} \ni g \mapsto g\sigma &\equiv g\sigma(y), && \text{where } \mathbf{t}\circ\sigma(y) = \mathbf{s}(g). \end{aligned}$$

Now, each section $\alpha \in \Gamma(A)$ can identified with a right-invariant vector field on $\mathcal{G}$, and we denote its flow by φ^t_α. We define an admissible section $\exp(\alpha)$ of $\mathcal{G}$ by setting:

$$\exp(\alpha)(x) \equiv \varphi^1_\alpha(x).$$

This gives an exponential map $\exp : \Gamma(A) \to \Gamma(\mathcal{G})$ which, in general, is defined only for sections α sufficiently close to the zero section (e. g., sections with compact support).

In the non-integrable case, we can also define an exponential map $\exp : \Gamma(A) \to \Gamma(\mathcal{G}(A))$ to the admissible smooth sections of the Weinstein groupoid as follows. First of all notice that

$$a_\alpha(x)(t) = \alpha(t, \phi^{t,0}_{\#\alpha}(x))$$

defines an A path $a_\alpha(x)$ for any $x \in M$ and for any time depending section α of A whose flow is defined up to $t = 1$ (e. g., if α is sufficiently close to zero, or if it is compactly supported). This defines a smooth map $a_\alpha : M \to P(A)$. For $\alpha \in \Gamma(A)$ close enough to the identity section we set

$$\exp(\alpha)(x) = [a_\alpha(x)].$$

Notice that $a = a_\alpha(x)$ is the unique A-path with $a(0) = \alpha(x)$ and $a(t) = \alpha(\pi(a(t)))$, for all $t \in I$.

In the integrable case these two constructions coincide. Moreover, for a general Lie algebroid, we have the following

Proposition 2.3. *Let A be a Lie algebroid and $\alpha, \beta \in \Gamma(A)$. Then, as admissible sections,*

$$\exp(t\alpha)\exp(\beta)\exp(-t\alpha) = \exp(\phi^t_\alpha \beta),$$

where ϕ^t_α denotes the infinitesimal flow of α (see Appendix A).

Proof. First we make the following remark concerning functoriality of exp: Let $\phi : A_1 \to A_2$ a morphism of Lie algebroids and let $\Phi : \mathcal{G}(A_1) \to \mathcal{G}(A_2)$ be the corresponding morphism of groupoids (Proposition 2.2 (i)). If one denotes by $\tilde{\phi}$ (resp. $\tilde{\Phi}$) the corresponding homomorphism of sections (resp. admissible sections), then we obtain the following commutative diagram:

$$\begin{tikzcd}
\Gamma(\mathcal{G}(A_1)) \arrow[r, "\tilde{\Phi}"] & \Gamma(\mathcal{G}(A_2)) \\
\Gamma(A_1) \arrow[u, "\exp"] \arrow[r, "\tilde{\phi}"'] & \Gamma(A_2) \arrow[u, "\exp"']
\end{tikzcd}$$

To proof the proposition is therefore enough to proof that for the homomorphism $\Phi^t_\alpha : \mathcal{G}(A) \to \mathcal{G}(A)$ associated to $\phi^t_\alpha : A \to A$ we have:

$$\Phi^t_\alpha(g) = \exp(t\alpha) g \exp(-t\alpha),$$

or, equivalently, that

$$[\phi^t_\alpha \circ a] = \exp(t\alpha)[a]\exp(-t\alpha).$$

for any A-path $a \in \mathcal{G}(A)$. To prove this, one considers the variation of A-paths $a_\varepsilon = \exp(-\varepsilon t\alpha) \cdot (\phi^{\varepsilon t}_\alpha \circ a) \cdot \exp(-\varepsilon t\alpha)$, and checks that this realizes an equivalence of A -paths using proposition 1.3. □

Remark 2.4. Hence $\mathcal{G}(A)$ behaves in many respects like a smooth manifold, even if A is not integrable. This might be important in various aspects of non-commutative geometry and its applications to singular foliations and analysis: one might expect that the algebras of pseudodifferential operators and the C^*-algebra of $\mathcal{G}(A)$ (see [21]) can be constructed even in the non-integrable case. A related question is when $\mathcal{G}(A)$ is a measurable groupoid.

Although the exponential map does exist even in the non-integrable case, its injectivity on a neighborhood of M only holds if A is integrable. One could say that this is *the* difference between the integrable and the non-integrable cases, as we will see in the next sections. However, our main job is to relate the kernel of the exponential and the geometry of A, and this is the origin of our obstructions: the monodromy groups described in the next section consist of the simplest elements which belong to this kernel. It turns out that these elements are enough to control the entire kernel.

3. Monodromy

Let A be a Lie algebroid over M, $x \in M$. In this section we give several descriptions of the (second order) monodromy groups of A at x, which control the integrability of A.

3.1. **Monodromy groups.** There are several possible ways of introducing the monodromy groups. Our first description is as follows:

Definition 3.1. We define $N_x(A) \subset A_x$ as the set of those elements $v \in Z(\mathfrak{g}_x)$ with the property that the constant A-path v is equivalent to the trivial A-path.

Let us denote by $\mathcal{G}(\mathfrak{g}_x)$ the simply-connected Lie group integrating $\mathfrak{g}_x$ (equivalently, the Weinstein groupoid associated to $\mathfrak{g}_x$). Also, let $\mathcal{G}(A)_x$ be the isotropy groups of the Weinstein groupoid $\mathcal{G}(A)$:

$$\mathcal{G}(A)_x \equiv \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x) \subset \mathcal{G}(A) \ .$$

Closely related to the groups $N_x(A)$ are the following:

Definition 3.2. We define the subgroup $\tilde{N}_x(A)$ of $\mathcal{G}(\mathfrak{g}_x)$ consisting on the equivalence classes $[a] \in \mathcal{G}(\mathfrak{g}_x)$ of $\mathfrak{g}_x$-paths with the property that, as an A-path, a is equivalent to the trivial A-path.

The precise relation is as follows:

Lemma 3.3. *For any Lie algebroid A, and any $x \in M$, $\tilde{N}_x(A)$ are subgroups of $\mathcal{G}(\mathfrak{g}_x)$ contained in the center $Z(\mathcal{G}(\mathfrak{g}_x))$, and their intersection with the connected component $Z(\mathcal{G}(\mathfrak{g}_x))^0$ of the center is isomorphic to $N_x(A)$.*

Proof. Given $g \in \tilde{N}_x(A) \subset \mathcal{G}(\mathfrak{g}_x)$ represented by a $\mathfrak{g}_x$-path a, Proposition 1.6 implies that parallel transport $T_a : \mathfrak{g}_x \to \mathfrak{g}_x$ along a is the identity. On the other hand, since a sits inside $\mathfrak{g}_x$, it is easy to see that $T_a = ad_g$, the adjoint action by the element $g \in \mathcal{G}(\mathfrak{g}_x)$ represented by a. This shows that $g \in Z(\mathcal{G}(\mathfrak{g}_x))$. The last part follows from the fact that the exponential map induces an isomorphism $\exp : Z(\mathfrak{g}_x) \to Z(\mathcal{G}(\mathfrak{g}_x))^0$ (cf. e. g. 1.14.3 in [9]), and $N_x(A) = \exp^{-1}(\tilde{N}_x(A))$. ☐

Corollary 3.4. *For any Lie algebroid A, and any $x \in M$, the following are equivalent:*

(i) $\tilde{N}_x(A)$ *is closed;*
(ii) $\tilde{N}_x(A)$ *is discrete;*
(iii) $N_x(A)$ *is closed;*
(iv) $N_x(A)$ *is discrete.*

We remark that a special case of our main theorem shows that the previous assertions are in fact equivalent to the integrability of $A|_{L_x}$, the restriction of A to the leaf through x.

3.2. **A second order monodromy map.** Let $L \subset M$ denote the leaf through x. We define a homomorphism $\partial : \pi_2(L, x) \to \mathcal{G}(\mathfrak{g}_x)$ whose image is precisely the group $\tilde{N}_x(A)$. This second order monodromy map relates the topology of the leaf through x with the Lie algebra $\mathfrak{g}_x$.

Let $[\gamma] \in \pi_2(L, x)$ be represented by a smooth path $\gamma : I \times I \to L$ which maps the boundary into x. We choose a morphism of algebroids

$$adt + bd\epsilon : TI \times TI \to A_L$$

(i. e. (a, b) satisfies equation (1)) which lifts $d\gamma : TI \times TI \to TL$ via the anchor, and such that $a(0, t)$, $b(\epsilon, 0)$, and $b(\epsilon, 1)$ vanish. This is always possible: for example, we can put $b(\epsilon, t) = \sigma(\frac{d}{d\epsilon}\gamma(\epsilon, t))$ where $\sigma : TL \to A_L$ is a splitting of the anchor map, and take a to be the unique solution of the differential equation (1) with the initial conditions $a(0, t) = 0$. Since γ is constant on the boundary, $a_1 = a(1, -)$ stays inside the Lie algebra $\mathfrak{g}_x$, i. e. defines a $\mathfrak{g}_x$-path

$$a_1 : I \to \mathfrak{g}_x \ .$$

Its integration (cf. [9], or our Proposition 1.1 applied to the Lie algebra $\mathfrak{g}_x$) defines a path in $\mathcal{G}(\mathfrak{g}_x)$, whose end point is denoted by $\partial(\gamma)$.

Proposition 3.5. *The element $\partial(\gamma) \in \mathcal{G}(\mathfrak{g}_x)$ does not depend on the auxiliary choices we made, and only depends on the homotopy class of γ. Moreover, the resulting map*

$$\partial : \pi_2(L, x) \to \mathcal{G}(\mathfrak{g}_x) \tag{4}$$

is a morphism of groups whose image is precisely $\tilde{N}_x(A)$.

Notice the similarity between the construction of ∂ and the construction of the boundary map of the homotopy long exact sequence of a fibration: if we view $0 \to \mathfrak{g}_L \to A_L \to TL \to 0$ as analogous to a fibration, the first few terms of the associated long exact sequence will be

$$\pi_2(L, x) \stackrel{\partial}{\to} \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}(A)_x \to \pi_1(L, x).$$

The exactness at $\mathcal{G}(\mathfrak{g}_x)$ is precisely the last statement of the proposition. We leave it to the reader the (easy) check of exactness at $\mathcal{G}(A)_x$.

Proof of Proposition 3.5: From the definitions it is clear that Im $\partial = \tilde{N}_x(A)$ so all we have to check is that ∂ is well defined. For that we assume that

$$\gamma^i = \gamma^i(\epsilon, t) : I \times I \to L, \quad i \in \{0, 1\}$$

are homotopic relative to the boundary, and that

$$a^i dt + b^i d\epsilon : TI \times TI \to A_L. \ \ i \in \{0, 1\}$$

are lifts of $d\gamma^i$ as above. We prove that the paths $a^i(1, t)$ $(i \in \{0, 1\})$ are homotopic as $\mathfrak{g}_x$-paths).

By hypothesis, there is a homotopy $\gamma^u = \gamma^u(\epsilon, t)$ $(u \in I)$ between γ^0 and γ^1. We choose a family $b^u(\epsilon, t)$ joining b^0 and b^1, such that $\#(b^u(\epsilon, t)) = \frac{d\gamma^u}{d\epsilon}$ and $b^u(\epsilon, 0) = b^u(\epsilon, 1) = 0$. We also choose a family of sections η depending on u, ϵ, t such that

$$\eta^u(\epsilon, t, \gamma^u(\epsilon, t)) = b^u(\epsilon, t), \text{ with } \eta = 0 \text{ when } t = 0, 1.$$

As in the proof of Proposition 1.3, let ξ and θ be the solutions of

$$\begin{cases} \frac{d\xi}{d\epsilon} - \frac{d\eta}{dt} = [\xi, \eta], \text{ with } \xi = 0 \text{ when } \epsilon = 0, 1, \\ \\ \frac{d\theta}{d\epsilon} - \frac{d\eta}{du} = [\theta, \eta], \text{ with } \theta = 0 \text{ when } \epsilon = 0, 1. \end{cases}$$

Setting $u = 0, 1$ we get

$$a^i(\epsilon, t) = \xi^i(\epsilon, t, \gamma^i(\epsilon, t)), \quad i = 0, 1.$$

On the other hand, setting $t = 0, 1$ we get

$$\theta = 0 \text{ when } t = 0, 1.$$

A brief computation shows that $\phi \equiv \frac{d\xi}{du} - \frac{d\theta}{dt} - [\xi, \theta]$ satisfies

$$\frac{d\phi}{d\epsilon} = [\phi, \eta],$$

and since $\phi = 0$, when $\epsilon = 0$, it follows that

$$\frac{d\xi}{du} - \frac{d\theta}{dt} = [\xi, \theta].$$

If in this relation we choose $\epsilon = 1$, and use $\theta^u(1, t) = 0$ when $t = 0, 1$, we conclude that $a^i(1, t) = \xi^i(1, t, \gamma^i(1, t))$, $i = 0, 1$, are equivalent. □

3.3. **Computing the monodromy.** Let us indicate briefly how the monodromy groups (Definition 3.1 or, alternatively, Definition 3.2), can be explicitly computed in many examples. We consider the short exact sequence

$$0 \to \mathfrak{g}_L \to A_L \xrightarrow{\#} TL \to 0$$

and a linear splitting $\sigma : TL \to A_L$ of $\#$. The *curvature* of σ is the element $\Omega_\sigma \in \Omega^2(L; \mathfrak{g}_L)$ defined by:

$$\Omega_\sigma(X, Y) \equiv \sigma([X, Y]) - [\sigma(X), \sigma(Y)] \ .$$

Then computation of monodromy can be reduced to the following

Lemma 3.6. *If there is a splitting σ with the property that its curvature Ω_σ is $Z(\mathfrak{g}_L)$-valued, then*

$$N_x(A) = \{\int_\gamma \Omega_\sigma : [\gamma] \in \pi_2(L, x)\} \subset Z(\mathfrak{g}_x)$$

for all $x \in L$.

Before we give a proof some explanations are in order.

First of all, $Z(\mathfrak{g}_L)$ is canonically a flat vector bundle over L. The corresponding flat connection can be expressed with the help of the splitting σ as

$$\nabla_X \alpha = [\sigma(X), \alpha],$$

and it is easy to see that the definition does not depend on σ. In this way Ω_σ appears as a 2-cohomology class with coefficients in the local system defined by $Z(\mathfrak{g}_L)$ over L, and then the integration is just the usual pairing between cohomology and homotopy. In practice one can always avoid working with local coefficients: if $Z(\mathfrak{g}_L)$ is not already trivial as a vector bundle, one can achieve this by pulling back to the universal cover of L (where parallel transport with respect to the flat connection gives the desired trivialization).

Proof of Lemma 3.6. We may assume that $L = M$, i.e. A is transitive. In agreement with the comments above, we also assume for simplicity that $Z(\mathfrak{g})$ is trivial as a vector bundle. The formula above defines a connection ∇^σ on the entire $\mathfrak{g}$. We use σ to identify A with $TM \oplus \mathfrak{g}$ so the bracket becomes

$$[(X, v), (Y, w)] = ([X, Y], [v, w] + \nabla^\sigma_X(w) - \nabla^\sigma_Y(v) - \Omega_\sigma(X, Y)).$$

We choose a connection ∇^M on M, and we consider the connection $\nabla = (\nabla^M, \nabla^\sigma)$ on A. Note that

$$T_\nabla((X, v), (Y, w)) = (T_{\nabla^M}(X, Y), \Omega_\sigma(X, Y) - [v, w])$$

for all $X, Y \in TM$, $v, w \in \mathfrak{g}$. This shows that two A-paths a and b as in Proposition 1.3 will be of the form $a = (\frac{d\gamma}{dt}, \phi)$, $b = (\frac{d\gamma}{d\epsilon}, \psi)$ where ϕ, ψ are paths in $\mathfrak{g}$ satisfying

$$\partial_t \psi - \partial_\epsilon \phi = \Omega_\sigma(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}) - [\phi, \psi].$$

Now we only have to apply the definition of ∂: Given $[\gamma] \in \pi_2(M, x)$, we choose the lift $adt + bd\epsilon$ of $d\gamma$ with $\psi = 0$ and

$$\phi = -\int_0^\varepsilon \Omega_\sigma(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}).$$

Then ϕ takes values in $Z(\mathfrak{g}_x)$, and we obtain $\partial[\gamma] = [\int_\gamma \Omega_\sigma]$.

□

Example 3.7. Recall (e. g. [17]) that any two-form $\omega \in \Omega^2(M)$ induces an algebroid $A_\omega = TM \oplus \mathbb{L}$ with anchor $(X, \lambda) \mapsto X$ and Lie bracket

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f) + \omega(X, Y)).$$

Using the obvious splitting of A, Lemma 3.3 tells us that

$$N_x(A_\omega) = \left\{ \int_\gamma \omega : [\gamma] \in \pi_2(M, x) \right\} \subset \mathbb{R}$$

is the group of periods of ω. Other examples will be discussed in the next sections.

3.4. **Measuring the monodromy.** In order to measure the size of the monodromy groups $N_x(A)$, we fix some norm on the Lie algebroid A and for $x \in M$ we set

$$r(x) \equiv d(0, N_x(A) - \{0\}),$$

where we adopt the convention that $d(0, \emptyset) = +\infty$.

When x varies on a leaf L this function varies continuously, since the norm on A is assumed to vary continuously and the groups $N_x(A)$ are all isomorphic for $x \in L$. On the other hand, when x varies in a transverse direction the behaviour of $r(x)$ is far from being continuous as illustrated by the following examples:

Example 3.8. We take for A the trivial 3-dimensional vector bundle over $M = \mathbb{R}^3$, with basis $\{e_1, e_2, e_3\}$. The Lie bracket on A is defined by

$$[e_2, e_3] = ae_1 + bx^1\bar{n}$$
$$[e_3, e_1] = ae_2 + bx^2\bar{n}$$
$$[e_1, e_2] = ae_3 + bx^3\bar{n}$$

where $\bar{n} = \sum_i x^i e_i$ is a central element, and depends on two (arbitrary) smooth functions a and b of the radius R, with $a(R) > 0$ whenever $R > 0$. The anchor is given by

$$\#(e_i) = av_i, \qquad i = 1, 2, 3$$

where v^i is the infinitesimal generator of a rotation about the i-axis:

$$v_1 = x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3}, \quad v_2 = x^1\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^1}, \quad v_3 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}.$$

The leaves of the foliation induced on $\mathbb{R}^3$ are the spheres S_R^2 centered at the origin, and the origin is the only singular point.

We now compute the function r using the obvious metric on A. We restrict to a leaf S_R^2 with $R > 0$, and as splitting of $\#$ we choose the map defined by

$$\sigma(v_i) = \frac{1}{a}(e_i - \frac{x^i}{R^2}\bar{n}).$$

Then we obtain the center-valued 2-form (cf. section 3.3)

$$\Omega_\sigma = \frac{bR^2 - a}{a^2 R^2}\omega \bar{n}$$

where $\omega = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$. Since $\int_{S^2_R} \omega = 4\pi R^3$ it follows that

$$N(A) \simeq 4\pi \frac{bR^2 - a}{a^2 R}\mathbb{Z}\bar{n} \subset \mathbb{R}\bar{n}.$$

This shows that

$$r(x,y,z) = \begin{cases} +\infty & \text{if } R = 0 \text{ or } a = bR^2, \\ \\ 4\pi\frac{bR^2-a}{a^2} & \text{otherwise.} \end{cases}$$

So the monodromy might vary in a non-trivial fashion, even nearby regular leaves.

In the previous example the function r is *not* upper semi-continuous. In the next example we show that r, in general, is also *not* lower semi-continuous.

Example 3.9. Let $\mathcal{F}$ be the Reeb foliation in $\mathbb{S}^3$, and consider the central extension Lie algebroid $A = \mathcal{F} \oplus \mathbb{L}$ associated with a closed 2-form $\omega \in \Omega^2(\mathbb{S}^3)$ (as in Example 3.7). We obtain a regular Lie algebroid A whose leaves are the leaves of $\mathcal{F}$.

Now choose ω so that its pullback to the compact leaf $\mathbb{S}^2 \subset \mathbb{S}^3$ has a nontrivial cohomology class. Then, the monodromy group above a leaf L is formed by the integrals of $\omega|_L$ over the classes $[\gamma] \in \pi_2(L)$. Since all the leaves other than the sphere are contractible, we get:

$$r(x) = \begin{cases} r_0 & \text{if } x \in \mathbb{S}^2 \\ \\ +\infty & \text{otherwise} \end{cases}$$

Note that r_0 can take any value in the interval $(0, +\infty)$.

One might hope that if the anchor is injective in a "large set" then the monodromy groups can be controlled in a very precise way. Our next example shows that this is not the case.

Example 3.10. This example is in fact a variation of Example 3.8, and we use the same notation. We let $M = \mathbb{S}^2 \times \mathbb{H}$, where $\mathbb{H}$ denotes the quaternions. The Lie algebroid $\pi : A \to M$ is trivial as a vector bundle, has rank 3, and relative to a basis of sections $\{e_1, e_2, e_3\}$ the Lie bracket is defined by $[e_1, e_2] = e_3$ and cyclic permutations. To define the anchor, we let v_1, v_2, v_3 be the vector fields on $\mathbb{S}^2$ obtain by restricting the infinitesimal generators of rotations, and we let w_1, w_2, w_3 be the vector fields on $\mathbb{H}$ corresponding to multiplication by $\vec{i}, \vec{j}, \vec{k}$. The anchor of the algebroid is then defined by setting $\# e_i \equiv (v_i, w_i)$, $i = 1, 2, 3$. For this Lie algebroid one has:

- the anchor is injective on a dense open set;
- there is exactly one singular leaf, namely the sphere $\mathbb{S}^2 \times \{0\}$.

Now observe that the monodromy above the singular leaf is non-trivial, since the restriction of A to this singular leaf is the central extension algebroid $T\mathbb{S}^2 \oplus \mathbb{L}$ defined by the area form on $\mathbb{S}^2$. For the function r we have again:

$$r(x) = \begin{cases} r_0 & \text{if } x \in \mathbb{S}^2 \times \{0\} \\ \\ +\infty & \text{otherwise} \end{cases}$$

Note that in this case $r_0 > 0$. We will show later (Section 5.2.4) that when the anchor is *almost injective* we have $r(x) > 0$ for all $x > 0$.

4. Obstructions to integrability

In this section we first state our main result which give the obstructions to integrability, and give a few examples. We then give another description of the Weinstein groupoid which is more suitable for proving the theorem.

4.1. **The main theorem.** Let A be a Lie algebroid over M. Using the notations introduced above, our main result is the following:

Theorem 4.1. *A Lie algebroid A over M is integrable if and only if*

(i) *Longitudinal obstruction: $N_x(A) \subset A_x$ is discrete (i. e., $r(x) \neq 0$),*
(ii) *Transverse obstruction: $\liminf_{y\to x} r(y) > 0$,*

for all $x \in M$.

The next examples illustrate this result and show that these two obstructions are independent.

Example 4.2. In this example, non-integrability is forced by the first obstruction. We simply take the central extension Lie algebroid $A_\omega = TM \oplus \mathbb{L}$ associated with a closed 2-form on M whose group of periods is not cyclic (cf. Example 3.7). Then $r(x) = 0$ so the first obstruction ensures us that A_ω is non-integrable. We point out that this is a well-known counter-example to integrability (cf. e. g. [3] pp. 118) which is usually approach through the theory of transversely parallelizable foliations (see also Section 5.3 below).

Example 4.3. Let us give an example of a regular Lie algebroid whose first obstruction is trivial, while the second one is not. Take $\mathcal{F}$ to be the trivial foliation of $M = N \times T$ with leaves $N \times \{t\}$, $t \in T$. Also we choose a closed 2-form ω on N whose group of periods is cyclic and we set $\omega_t = \phi(t)\omega$, where ϕ is some smooth function on T. Since the pull-back of ω_t to any leave is closed, we obtain the central extension Lie algebroid $A_{\omega_t} = \mathcal{F} \oplus \mathbb{L}$, as in Example 3.7, whose leaves are the leaves of $\mathcal{F}$.

The "first obstruction" is satisfied for all leaves, but clearly the "second obstruction" is not satisfied at the points $t_0 \in T$ with the property that $\phi(t_0) = 0$ and ϕ is not locally constant at t_0.

Example 4.4. Consider the Lie algebroid A over $\mathbb{R}^3$ discussed in Example 3.8. Then A satisfies the first obstruction, but it does not satisfy the second obstruction at points where $aR^2 - b$ vanishes (without vanishing identically in some neighborhood of the point) and also at the origin if $\liminf_{R\to 0} \frac{bR^2-a}{a^2} = 0$.

For example, choosing $a = R^2$, $b = R^3 + 1$, the resulting Lie algebroid A over $\mathbb{R}^3$ has the following two properties:

(a) Its restriction to $\mathbb{R}^3 - 0$ is integrable;
(b) Its restriction to any disc around the origin is not integrable (because of the second obstruction at $x = 0$).

Example 4.5. Let us explain Weinstein's example of a non-integrable regular Poisson manifold given in [25] (see also [4], section 6). He takes $M = \mathbb{R}^3 - \{0\} \cong \mathfrak{su}(2)^* - \{0\}$ with the Kirillov Poisson structure scaled by a function $f(R)$ depending on the radius. The associated algebroid is in fact $T^*M = A|_{\mathbb{R}^3-\{0\}}$, where A is the Lie algebroid of Examples 3.8 and 4.4 with $a = f$, $b = \frac{1}{R}f'$. Its integrability is then controlled by

$$r(R) = 4\pi\frac{Rf' - f}{f^2} = -A'(R),$$

where $A(R) = \frac{4\pi R}{f}$ is the symplectic area.

We refer to section 5 for various integrability criteria that can be deduced from the theorem, including all criteria that have appeared before in the literature.

4.2. **The Weinstein groupoid as a leaf space.** Before we can proceed with the proof of our main result, we need a better control on the equivalence relation defining the Weinstein groupoid $\mathcal{G}(A)$. In this section we will show that $\mathcal{G}(A)$ is the leaf space of a foliation $\mathcal{F}(A)$ on $P(A)$, of finite codimension, whose leaves are precisely the equivalence classes of the homotopy relation $\sim$ of A-paths.

As before, A is a fixed Lie algebroid over M. We will use the following notations when working in local coordinates: we let $x = (x^1, \ldots, x^n)$ denote local coordinates on M, and we denote by $\{e_1, \ldots, e_k\}$ a (local) basis of A over this chart. The anchor and the bracket of A decompose as

$$\#e_p = \sum b_p^i \frac{\partial}{\partial x^i}, \qquad [e_p, e_q] = \sum_r c_{pq}^r e_r,$$

and an A-path a can be written $a(t) = \sum_p a^p(t) e_p$.

Let us first describe the smooth structure on $P(A)$. We consider the larger space $\tilde{P}(A)$ of all C^1-curves $a : I \to A$ whose base path $\gamma = \pi \circ a$ is of class C^2. It has an obvious structure of Banach manifold, whose tangent space $T_a(\tilde{P}(A))$ consists of curves $U : I \to TA$ s. t. $U(t) \in T_{a(t)}A$. Using a TM-connection ∇ on A, such curves can be viewed as pairs (u, ϕ) formed by a curve $u : I \to A$ over γ and a curve $\phi : I \to TM$ over γ (namely, the vertical and horizontal component of U).

Lemma 4.6. *$P(A)$ is a (Banach) submanifold of $\tilde{P}(A)$. Moreover, given a connection ∇ on A, the tangent space $T_a P(A)$ consists of those paths $U = (u, \phi)$ with the property that*

$$\#u = \overline{\nabla}_a \phi.$$

Proof. We consider the smooth map $F : \tilde{P}(A) \to \tilde{P}(TM)$ given by

$$F(a) = \#a - \frac{d}{dt}\pi \circ a.$$

Clearly $P(A) = F^{-1}(Q)$, where Q is the subspace of $\tilde{P}(TM)$ consisting of zero paths. Fix $a \in P(A)$, with base path $\gamma = \pi \circ a$, and let us compute the image of $U = (u, \phi) \in T_a \tilde{P}(A)$ by the differential

$$(dF)_a : T_a \tilde{P}(A) \to T_{0_\gamma} \tilde{P}(TM) \ .$$

The result will be some path $t \mapsto (dF)_a \cdot U(t) \in T_{0_{\gamma(t)}} TM$, hence, using the canonical splitting $T_{0_x} TM \cong T_x M \oplus T_x M$, it will have a horizontal and vertical component. We claim that for any connection ∇, if (u, ϕ) are the components of U, then

$$((dF)_a \cdot U)^{\text{hor}} = \phi, \quad ((dF)_a \cdot U)^{\text{ver}} = \#u - \overline{\nabla}_a \phi.$$

Note that this immediately implies that F is transverse to Q, so the assertion of the proposition follows. Since this decomposition is independent of the connection ∇ and it is local (we can look at restrictions of a to smaller intervals), we may assume that we are in local coordinates, and that ∇ is the standard flat connection. We now use the notations above, and we denote by $\frac{\partial}{\partial x^i}$ the horizontal basis of $T_{0_x} TM$, and by $\frac{\delta}{\delta x^i}$ the vertical basis. A simple computation shows that the horizontal component of $(dF)_a(u, \phi)$ is $\sum \phi^i \frac{\partial}{\partial x^i}$, while its vertical component is

$$\sum_j \left(-\dot{\phi}^j(t) + \sum_p u^p(t) b_p^j(\gamma(t)) + \sum_{p,i} a^p(t) \phi^i(t) \frac{\partial b_p^j}{\partial x^i}(\gamma(t)) \right) \frac{\delta}{\delta x^j}.$$

That this is precisely $\#u - \overline{\nabla}_a\phi$ immediately follows by computing

$$\overline{\nabla}_{e_p}\frac{\partial}{\partial x^i} = \#\nabla_{\frac{\partial}{\partial x^i}} e_p - \left[\frac{\partial}{\partial x^i}, \#e_p\right]$$
$$= -\sum_j \frac{\partial b_p^j}{\partial x^i}\frac{\partial}{\partial x^j}$$

□

We now construct an involutive sub-bundle $\mathcal{F}(A)$ of $TP(A)$, i. e. a foliation on $P(A)$. Let us fix a connection ∇ on A, and let a be an A-path with base path γ. We denote by $\mathcal{P}_{0,\gamma}(A)$ the space of all C^2-paths $b: I \to A$ such that $b(t) \in A_{\gamma(t)}$ and $b(0) = b(1) = 0$. For any such b we have a tangent vector $X_{b,a} \in T_aP(A)$ whose components (u, ϕ) with respect to the connection ∇ are

$$u = \overline{\nabla}_a b, \quad \phi = \#b.$$

Lemma 4.6 shows that these are indeed tangent to $P(A)$, and we set:

$$\mathcal{F}_a(A) \equiv \{X_{b,a} \in T_aP(A) : b \in \mathcal{P}_{0,\gamma}(A)\}.$$

Some geometric insight to this sub-bundle can be obtain by considering the Lie algebra of time depending sections of A vanishing at the end-points:

$$P_0\Gamma(A) = \left\{I \ni t \mapsto \eta_t \in \Gamma(A) : \eta_0 = \eta_1 = 0, \eta \text{ is of class } C^2 \text{ in } t\right\}$$

For any such section η we consider the induced path $b(t) = \eta(t, \gamma(t))$ and put $X_{\eta,a} \equiv X_{b,a}$. The resulting map

$$P_0\Gamma(A) \to \mathcal{X}(P(A)), \quad \eta \mapsto X_\eta$$

is an action of the Lie algebra $P_0\Gamma(A)$ on $P(A)$.

Remark 4.7. The spaces $\mathcal{P}_{0,\gamma}(A)$ fit into a vector bundle $\mathcal{P}_0(A)$ over the path space $P(M)$. Its space of sections is $\Gamma(\mathcal{P}_0(A)) = P_0\Gamma(A)$, and there is an obvious map $\mathcal{P}_{0,\gamma}(A) \to T_\gamma P(M)$ induced by the anchor $\#$ of A. Hence $\mathcal{P}_0(A)$ is an algebroid over $P(M)$. Given $\eta \in P_0\Gamma(A)$, $\gamma \in P(M)$, then $t \mapsto \eta(t, \gamma(t))$ is precisely the evaluation $ev_\gamma(\eta) \in \mathcal{P}_{0,\gamma}(A)$. The map

$$\mathcal{P}_{0,\gamma}(A) \to T_aP(A), \quad b \mapsto X_{b,a}$$

can then be viewed as an action (see e. g. [14]) of the Lie algebroid $\mathcal{P}_0(A)$ over $P(M)$ on the space $P(A)$.

We now show that this foliation is in fact the same as the partition of $P(A)$ into equivalent classes of A-paths:

Proposition 4.8. *For a Lie algebroid A:*

(i) *The spaces $\mathcal{F}_a(A)$ do not depend on the choice of connection ∇. More precisely, for any $\eta \in P_0\Gamma(A)$,*

$$X_{\eta,a}(t) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_{\eta_t}^{\epsilon,0} a(t) + \frac{d\eta_t}{dt}(\gamma(t)).$$

(ii) *$\mathcal{F}(A)$ is a foliation on $P(A)$ of finite codimension equal to $n + k$ where $n = \dim M$ and $k = \operatorname{rank} A$.*

(iii) *Two A-paths are equivalent (homotopic) if and only if they are in the same leaf of $\mathcal{F}(A)$.*

(iv) *For any (local) connection ∇ on A, the exponential map $\operatorname{Exp}_\nabla : A \to P(A)$ is transverse to $\mathcal{F}(A)$.*

Proof. We first assume that η_ϵ is a family of elements of $P_0\Gamma(A)$ of class C^1 on $\epsilon \in I$, and we will see that it induces a vector field X_η on $P(A)$ tangent to $\mathcal{F}(A)$ and whose flow preserves the equivalence of paths. This is only a reformulation of Proposition 1.3. Hence, let $a_0 \in P(A)$ with base path γ_0, and let ξ_0 be a time dependent section of A such that $\xi_0(t, \gamma_0(t)) = a_0(t)$. We denote by ξ the solution of (3) with the initial condition $\xi(0,t) = \xi_0(t)$. Then, as in the proof of Proposition 1.3,

$$\xi(\epsilon, t) = \int_0^\epsilon (\phi_{\eta_t}^{\epsilon',\epsilon})^* \frac{d\eta}{dt}(\epsilon', t)d\epsilon' + (\phi_{\eta_t}^{0,\epsilon})^*\xi_0.$$

Now consider the base path

$$\gamma_\epsilon(t) = \Phi^{\epsilon,0}_{\#\eta_t}\gamma_0(t)$$

and the paths above it

$$a_\epsilon(t) = \xi_\epsilon(t, \gamma_\epsilon(t)),\ b_\epsilon(t) = \eta_\epsilon(t, \gamma_\epsilon(t))\ .$$

We can view $\epsilon \mapsto a_\epsilon$ as a curve in $P(A)$ starting at a_0, and defining a tangent vector

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_\epsilon \in T_{a_0}P(A).$$

Given some connection ∇, Proposition 1.3 shows that this tangent vector has vertical component

$$\partial_\epsilon a = \partial_t b - T(a, b) = \overline{\nabla}_a(b)$$

at $\epsilon = 0$, while the horizontal component is

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \gamma_\epsilon(t) = \#b_0(t).$$

In other words,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_\epsilon = X_{b_0,a_0} \in \mathcal{F}_{a_0}(A).$$

On the other hand, the formula above describing ξ shows that

$$a_\epsilon(t) = \int_0^\epsilon \phi_{\eta_t}^{\epsilon,\epsilon'} \frac{d\eta_t}{dt}(\epsilon', \phi^{\epsilon',0}_{\#\eta_t}(\gamma_0(t))d\epsilon' + \phi_{\eta_t}^{\epsilon,0}(\gamma_0(t)).$$

The derivative at $\epsilon = 0$ is precisely the expression given in (i) and this also shows that (iii) holds.

To determine the codimension of $\mathcal{F}_a(A)$ we note that given (u, ϕ) satisfying $\#u = \overline{\nabla}_a\phi$ (i. e., a vector tangent to $P(A)$) and lying in $\mathcal{F}_a(A)$, we have

(a) $\phi(0) = 0$;
(b) If we consider the solution b of the equation $\overline{\nabla}_a(b) = u$ with initial condition $b(0) = 0$ (which can be expressed in terms of the parallel transport along a with respect to $\overline{\nabla}$), we must have $b(1) = 0$.

Conversely, if (a) and (b) hold, we have that $\overline{\nabla}_a(\#b - \phi) = 0$ and $\#b - \phi$ vanishes at $t = 0$. It follows that $\phi = \#b$ and $u = \overline{\nabla}_a b$, so (u, ϕ) is a tangent vector in $\mathcal{F}_a(A)$. This shows that $\operatorname{codim}\mathcal{F} = \dim M + \operatorname{rank} A$.

Finally, to prove (iv), we assume for simplicity that we are in local coordinates and that ∇ is the trivial flat connection (this is actually all we will use for the proof of the main theorem, and this in turn will imply the full statement of (iv)). Also, only we need to show is that $\operatorname{Exp}_\nabla(A)$ is transverse to $\mathcal{F}(A)$ at any trivial A-path $a = O_x$ over $x \in M$. Now, the equations for the geodesics show that if (u, ϕ) is a tangent vector to $\operatorname{Exp}_\nabla(A)$ at a then we must have:

$$\dot{\phi}^i = b^i_p(x)u^p, \qquad \dot{u}^p = 0.$$

Therefore, we see that:

$$T_a \mathrm{Exp}_{\nabla}(A) = \{(u,\phi) \in T_a P(A) : u(t) = u_0, \phi(t) = \phi_0 + t\# u_0\}\,.$$

Suppose that a tangent vector (u,ϕ) belongs to this $n+k$ dimensional space and is also tangent to $\mathcal{F}(A)$. Then (a) above implies that $\phi_0 = 0$, while (b) says that the solution b of $\frac{db}{dt} = u_0$ with $b(0) = 0$ satisfies $b(1) = 0$. Therefore, we must have $\phi_0 = 0$ and $u_0 = 0$, so (u,ϕ) is the null tangent vector. This shows that $\mathrm{Exp}_{\nabla}(A)$ is transverse to $\mathcal{F}(A)$ at 0_x, for any x. □

Remark 4.9. In [4], Cattaneo and Felder obtain the Weinstein groupoid for the special case of Poisson manifolds by a Hamiltonian reduction procedure. The Lie algebraic interpretation given above for the foliation $\mathcal{F}(A)$ shows that our construction of $\mathcal{G}(A)$ for general A is also obtain by a kind of reduction procedure for Lie algebroid actions.

4.3. **Proof of the main theorem.** In this section we prove our main theorem (for notations, see Section 3).

To prove that both conditions are necessary, choose some connection ∇ on A, and let $\overline{\mathrm{Exp}}_{\nabla} : A \to \mathcal{G}(A)$ be the associated exponential map. Clearly the restriction of $\overline{\mathrm{Exp}}_{\nabla}$ to $\mathfrak{g}_x$ is the composition of the exponential map of $\mathfrak{g}_x$ with the obvious map $i : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}(A)_x$, which shows that $\overline{\mathrm{Exp}}_{\nabla}(v_x) = 1_x$ for all $v_x \in N_x(A)$ in the domain of the exponential map. On the other hand, if A is integrable, we know that $\overline{\mathrm{Exp}}_{\nabla}$ will be a diffeomorphism on a small neighborhood of M on A. Hence there must exist an open $\mathcal{U} \subset A$ such that $\mathcal{U} \cap N(A) = M$, where $N(A) = \cup_x N_x(A)$. But this is obviously equivalent to the conditions in the statement.

We now show that these conditions also guarantee the integrability of A. First we prove that the two conditions together imply that $\mathcal{F}(A)$ is a simple foliation:

Lemma 4.10. *For each $a \in P(A)$, there exists $S_a \subset P(A)$ transverse to $\mathcal{F}(A)$, which intersects each leaf of $\mathcal{F}(A)$ in at most one point.*

Proof. The proof is a sequence of reductions and careful choices, and is divided into several steps. So let $a \in P(A)$ and denote by x the initial point of its base path.

Claim 1. We may assume that $a = 0_x$.

To see this, we choose a compactly supported, time-dependent, section ξ of A so that $\xi(t,\gamma(t)) = a(t)$. If $\sigma_\xi = \exp(\xi)$ is the associated admissible section (see section 2.3), left multiplication $T : P(A) \to P(A)$, $T(b) = \sigma_\xi(\mathbf{t}(b))b$ defines a smooth injective map with $T(0_x) = a$. If there is a section S_x around 0_x, as in the statement of the Lemma, it then follows that $T : S_x \hookrightarrow P(A)$ intersects each leaf in at most one point. Since S_x has the same dimension as the codimension of $\mathcal{F}(A)$, $S_a := T(S_x)$ will have the desired properties.

From now on we fix $x \in M$ and we are going to prove the Lemma for $a = 0_x$. We also fix local coordinates around x, and let ∇ be the canonical flat connection on the coordinate neighborhood. We also choose a small neighborhood U of 0_x in A so that the exponential map $\mathrm{Exp}_{\nabla} : U \to P(A)$ is defined and is transverse to $\mathcal{F}(A)$. We are going to show that it intersects each leaf of $\mathcal{F}(A)$ in at most one point, provided U is chosen small enough.

Claim 2. We may choose U such that for any $v \in U \cap \mathfrak{g}_y$ $(y \in M)$ with the property that $\mathrm{Exp}_{\nabla}(v)$ is homotopic to 0_y, we must have $v \in Z(\mathfrak{g}_y)$.

Given a norm $|\cdot|$ on A, the set $\{|[v,w]| : v, w \in \mathfrak{g}_y \text{ with } |v| = |w| = 1\}$, where $y \in M$ varies in a neighborhood of x, is bounded. Rescaling $|\cdot|$ if necessary, we find a neighborhood D of x in M, and a norm $|\cdot|$ on $A_D = \{v : \pi(v) \in D\}$, such that

$|[v,w]| \leq |v||w|$ for all $v, w \in \mathfrak{g}_y$ with $y \in D$. We now choose U so that $U \subset A_D$ and $|v| \leq \pi$ for all $v \in U$. If v is as in the claim, it follows from Proposition 1.6 (see also the proof of Lemma 3.3) that parallel transport $T_v : \mathfrak{g}_y \to \mathfrak{g}_y$ along the constant A-path v is the identity. But T_v is precisely the exponential of the linear map $ad_v : \mathfrak{g}_y \to \mathfrak{g}_y$. Since $|ad_v| \leq |v| \leq \pi$, it follows (see e. g. [10]) that $ad_v = 0$ so $v \in Z(\mathfrak{g}_y)$.

Claim 3. We may choose U such that, if $v \in U \cap \mathfrak{g}_y$ ($y \in M$) has the property that $\operatorname{Exp}_\nabla(v)$ is homotopic to 0_y then $v = 0_y$.

Obviously this is just a restatement of the obstruction assumptions, combined with the previous claim.

Claim 4. We may choose U such that, if $v \in U$ has the property that the base path of $\operatorname{Exp}_\nabla(v)$ is closed, then $v \in \mathfrak{g}_y$.

To see this, we note that the equations for the geodesics in local coordinates reduce to:

$$\begin{cases} \dot{x}^i = \sum_p b^i_p(x(t))a^p, \\ \dot{a}^p = 0. \end{cases}$$

By the period bounding lemma ([1] and [8], Appendix A), any non-trivial periodic orbit of this system with initial condition on a open set D has period

$$T \geq \frac{2\pi}{M_D} \qquad \text{where} \qquad M_D = \sup_{\substack{1 \leq j,k \leq m \\ x \in D}} \left\| \sum_p \frac{\partial b^j_p}{\partial x^k}(x) a^p \right\|.$$

Hence it suffices to make sure that $U \subset A_D$ where D is chosen small enough so that $M_D < 2\pi$.

Now, for any open $O \subset P(A)$, we consider the plaques in O of $\mathcal{F}(A)$, or, equivalently, the leaves of $\mathcal{F}(A)|_O$. For $a, b \in O$, we write $a \sim_O b$ if a and b lie in the same plaque. From now on we fix U satisfying all the conditions above, and we choose an open O so that $\operatorname{Exp}_\nabla : U \to P(A)$ intersects each plaque inside O exactly in one point. This is possible since $\operatorname{Exp}_\nabla$ is transversal to $\mathcal{F}(A)$. Apart from the pair (O, U), we also choose similar pairs (O_i, U_i), $i = 1, 2$, such that $O_1O_1 \subset O$, $O_2O_2 \subset O_1$ and $O_i^{-1} = O_i$.

Claim 5. It is possible to choose a neighborhood V of x in U_2 so that, for all $v \in V$,

$$0_y \cdot \operatorname{Exp}_\nabla(v) \sim_O \operatorname{Exp}_\nabla(v).$$

We know that for any v there is a natural homotopy between the two elements above. This homotopy can be viewed as a smooth map $h : I \times U \to P(A)$ with $h(0, v) = 0 \cdot \operatorname{Exp}_\nabla(v)$, $h(1, v) = \operatorname{Exp}_\nabla(v)$, $h(t, 0_y) = 0_y$. Since I is compact and O is open, we can find V around x such that $h(I \times V) \subset O$. Obviously V has the desired property.

Claim 6. It is possible to choose V so that, for all $v, w \in V$,

$$(\operatorname{Exp}_\nabla(v) \cdot \overline{\operatorname{Exp}_\nabla(w)}) \cdot \operatorname{Exp}_\nabla(w) \sim_O \operatorname{Exp}_\nabla(v)$$

This is proved exactly as the previous claim.

Claim 7. $\operatorname{Exp}_\nabla : V \to P(A)$ intersects each leaf of $\mathcal{F}(A)$ in at most one point.

To see this, let us assume that $v, w \in V$ have $\operatorname{Exp}_\nabla(v) \sim \operatorname{Exp}_\nabla(w)$. Then $a_1 := \operatorname{Exp}_\nabla(v) \cdot \overline{\operatorname{Exp}_\nabla(w)} \in O_1$ will be homotopic to the trivial A-path 0_y. On the other hand, by the choice of the pair (O_1, U_1), $a_1 \sim_{O_1} \operatorname{Exp}_\nabla(u)$ for an unique

$u \in U_1$. Since $\mathrm{Exp}_{\nabla}(u)$ is equivalent to 0_y, its base path must be closed, hence, by claim 4 above, $u \in \mathfrak{g}_y$. Using Claim 3, it follows that $u = 0$, hence $a_1 \sim_{O_1} 0_y$. Since $O_1 O_1 \subset O$, this obviously implies that

$$a_1 \cdot \mathrm{Exp}_{\nabla}(w) \sim_O 0_y \cdot \; Exp_{\nabla}(w).$$

Since V satisfies Claim 5 and Claim 6, we get $\mathrm{Exp}_{\nabla}(v) \sim_O \mathrm{Exp}_{\nabla}(w)$. Hence, by the construction of O, $v = w$. This also concludes the proof of the lemma. □

Note that the previous lemma implies that $\mathcal{G}(A)$ has a natural quotient differentiable structure: the charts are just the S_a's, and the change of coordinates is smooth since it is just the holonomy of $\mathcal{F}(A)$. Hence we can complete the proof of Theorem 4.1 by showing that:

Lemma 4.11. *For the quotient differentiable structure $\mathcal{G}(A)$ is a Lie groupoid with Lie algebroid A.*

Proof. It is clear from the definitions that A can be identified with $T^{\mathbf{s}}_M \mathcal{G}(A)$ and that under this identification # coincides with the differential of the target $\mathbf{t}$. So we need only to check that the bracket of right-invariant vector fields on $\mathcal{G}(A)$ is identified with the bracket of sections of A. For this we note that on one hand, the bracket is completely determined by the infinitesimal flow of sections through the basic formula (A.2). On the other hand, we now know that the exponential $\exp : \Gamma(A)) \to \Gamma(\mathcal{G}(A))$ is injective in a neighborhood of the zero section, and so Proposition 2.3 shows that the infinitesimal flow of a section α is the infinitesimal flow of the right-invariant vector field on $\mathcal{G}(A)$ determined by α. Hence, we must have $A(\mathcal{G}(A)) = A$. □

Remark 4.12. The proof above (namely an argument similar to Claim 1 above) shows that, in the main theorem, it suffices to require that for each leaf L, there exists $x \in L$ satisfying the two obstructions.

5. Examples and Applications

In this section we review the known integrability criteria, we derive them from Theorem 4.1, an present an application to the theory of transversely parallelizable foliations.

5.1. Local integrability.

Regarding the local nature of integrability, note that

- From Examples 4.3 and 4.5 we learn that a Lie algebroid can be locally integrable (i.e. each point has a neighborhood U so that $A|_U$ is integrable), and not globally integrable. This shows that the integrability problem is not a local one.
- Example 4.4 shows that there are algebroids which are not even locally integrable.

However, a general "local integrability" result has long been assumed to be true, namely the integrability by *local groupoids*. This result was first annonuced by Pradiness, but a proof has never been published. One of the main difficulties is that, if one tries to extend the known result from Lie groups (see e.g. [9]), one faces the problem of finding a CBH-formula. However, with the Weinstein groupoid at hand (and its description as a leaf space) this result can be proved quite easily.

For a *local Lie groupoid* the structure maps are only defined on (and the usual properties only hold for) elements which are close enough to the space M of units (these are obvious generalizations of Cartan's local Lie groups, as explained in Section 1.8 of [9]).

Corollary 5.1. *Any Lie algebroid is integrable by a local Lie groupoid.*

Proof. One uses exactly the same arguments as in the proof of Claim 2 and Claims 4 through 6 in Lemma 4.10, namely: We choose a connection ∇ on A, a neighborhood U of M in A and an open O in $P(A)$, with $O = O^{-1}$, such that $\mathrm{Exp}_{\nabla} : U \to P(A)$ intersects each plaque of $\mathcal{F}(A)$ in O in exactly one point. Eventually choosing smaller pairs (O, U) (similar to the (O_i, U_i) in the cited proof), the structure of local groupoid will be defined on U: the inverse of $v \in U$ is the unique $\bar{v} \in U$ with the property that $\overline{\mathrm{Exp}_{\nabla}(v)} \sim_O \mathrm{Exp}_{\nabla}(\bar{v})$; the multiplication $v \cdot w$ of $v, w \in U$ is defined only for pairs (u, v) for which $\mathrm{Exp}_{\nabla}(v)\mathrm{Exp}_{\nabla}(w) \in O$, and is the unique element with the property that the last product of exponentials is $\sim_O \mathrm{Exp}_{\nabla}(v \cdot w)$. The associativity around the units is proved exactly as Claims 5 and 6 of the cited lemma, while the fact that the resulting local groupoid integrates A is a variation of Lemma 4.11. □

5.2. **Integrability Criteria.** We start with following general integrability criterion which is an obvious consequence of our main result which implies most of the known results (and even much stronger versions of them):

Corollary 5.2. *If $N_x(A)$ is trivial for all $x \in M$, then A is integrable. In particular, A is integrable if any of the following three conditions holds for all $x \in M$:*

(i) *the Lie algebras $\mathfrak{g}_x$ are semi-simple (more generally, if they have trivial center);*
(ii) *the leaves L_x are 2-connected (more generally, if $\pi_2(L_x)$ have only elements of finite order);*
(iii) *there is a splitting $\sigma : TL_x \to A|_{L_x}$ of the anchor compatible with the Lie bracket;*

We now briefly deduce the known integrability results.

5.2.1. *Transitive algebroids.* In the case of Lie algebroids the main theorem (see also Remark 4.12) becomes:

Corollary 5.3. *Let A be a transitive Lie algebroid over M. Then A is integrable if and only if $N_x(A)$ is discrete in A_x for one (or, equivalently, all) $x \in M$.*

We mention in passing that this is strongly related to Mackenzie's criteria [17], and we urge the interested reader to find the precise relation.

There are some obvious consequences of this result. For example:

Corollary 5.4. *Every transitive Lie algebroid A over a 2-connected base M is integrable.*

Note also that since $\mathrm{s}^{-1}(x)$ is a principal $\mathcal{G}(A)_x$-bundle over M, it follows that if M is contractible then A is in fact isomorphic to a direct sum $TM \oplus \mathfrak{g}$ (compatible with the Lie brackets), where $\mathfrak{g} = \mathfrak{g}_x$. Hence:

Corollary 5.5. *Any transitive Lie algebroid over a contractible base M is isomorphic to $TM \oplus \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$.*

In Mackenzie's approach this result is first obtained in order to to construct his obstruction.

5.2.2. *Regular Lie algebroids.* Although many of the known integrability criteria require regular algebroids, it turns out that regularity is superfulous (see below). This is the case, for example, with Dazord-Hector ([7]) integrability criteria for totally aspherical regular Poisson manifolds, and with Nistor's results [20] on the integrability of regular algebroids whose anchor has either a splitting compatible with the Lie bracket, or semi-simple kernels.

Let us mentioned, however, a result which fails in the non-regular case as shown by Example 4.4:

Corollary 5.6. *Any regular Lie algebroid is locally integrable.*

This follows because regular foliations are locally trivial. As in the transitive case, it is possible to describe explicitly the local structure of regular algebroids. Choosing local coordinates in M so that the foliation becomes the obvious p-dimensional foliation on $\mathbb{R}^p \times \mathbb{R}^q$ then, locally, the algebroid is $T\mathbb{R}^p \times \mathfrak{g}$ where $\mathfrak{g}$ is a bundle of Lie algebras over $\mathbb{R}^q$.

5.2.3. *Semi-direct products.* Closely related to Palais' integrability [22] of infinitesimal actions of Lie algebras $\mathfrak{g}$ on manifolds M is the integrability of the transformation Lie algebroid $A = \mathfrak{g} \times M$. Recall that, as a vector bundle, A is just the trivial vector bundle with fiber $\mathfrak{g}$, the anchor is the infinitesimal action, while the bracket on $C^\infty(M;\mathfrak{g})$ is uniquely determined by the Leibniz rule and the Lie bracket of $\mathfrak{g}$.

Since $N_x(A)$ sits inside $N(\mathfrak{g})$ for all $x \in M$, the conditions of the main theorem are satisfied, hence

Corollary 5.7. *For any infinitesimal action of the Lie algebra $\mathfrak{g}$ on M, $\mathfrak{g} \times M$ is integrable.*

This is known as Dazord's criterion (cf. [3]), but it also appears implicitly in Palais' work [22]. Implicit in Palais' work is also the precise relation between this result and the integrability of infinitesimal actions. This relation has been clearly explained by Moerdijk-Mrčun in [16], where the reader can find various extensions to semi-direct products of algebroids. Let us point out that exactly the same argument as above shows that the semi-direct product of an integrable algebroid by a regular foliation is integrable, and this is one of the main results of [16].

5.2.4. *Algebras of vector fields and quasi-foliations.* For a Lie algebroid A over M we say that the the anchor is *almost injective* at $x_0 \in M$ if there is a neigborhood U of x_0 in M and an open dense subset $O \subset U$ such that $\#_x$ is injective for all $x \in O$. Note that if the anchor is injective at x_0 then it is almost injective at x_0. We say that the anchor is almost injective if it is almost injective at every point.

Any Lie subalgebra $\Gamma \subset \mathcal{X}(M)$ which is a finitely generated projective $C^\infty(M)$-module is the space of sections of an algebroid whose anchor is almost injective. This produces a large class of examples of Lie algebroids, including all regular foliations. As explained in [20], such Γ's arise naturally in the analysis on manifolds with corners as algebras of vector fields with a certain behavior on the faces of M. Their integrability is relevant to various aspects of analysis and quantization (see [21] for details). Such algebroids were also studied by Claire Debord on her Ph. D. Thesis ([8]), and they give rise to *quasi-foliations* of M.

Our main result implies the following integrability criterion due to Debord:

Corollary 5.8. *A Lie algebroid whose anchor is injective on a dense open set is integrable.*

To prove this result we need the following lemma.

Lemma 5.9. *Let X_n be a sequence of complete vector fields on $\mathbb{R}^n$ with flows ϕ_n^t and assume for some open set U one has:*

- (a) *$\phi_n^1(x) = x$, for all $x \in V_n$ in some open subset $V_n \subset U$;*
- (b) *$||\mathcal{J}(X_n)|| \to 0$ as $n \to +\infty$ where $\mathcal{J}(X)$ denotes the jacobian of X and the norm is the sup norm over $\overline{U}$.*

Then there exists a $n_0 \in \mathbb{N}$ such that $X_n \equiv 0$ in V_n for all $n > n_0$.

Proof. Suppose not. By (a), each X_n has a nontrivial periodic orbit with initial condition in $V_n \subset U$, with period $T \leq 1$. But by the period bounding lemma ([1]

and [8], Appendix A), any non-trivial periodic orbit of X_n with initial condition in U has period

$$T \geq \frac{2\pi}{||\mathcal{J}(X_n)||}.$$

This contradicts (b). $\square$

Sketch of proof of Corollary 5.8. We saw above (see Corollary 1.7) that equivalent A-paths have the same linear holonomy. In fact more is true: equivalent A-paths have the same *non-linear* holonomy. The proof is similar, except that now one needs to use non-linear connections defined by horizontal lifts (see [12]). Hence, if we fix $x_0 \in M$ and let $a(t) = v$ be a constant A-path with $v \in N_{x_0}(A)$, then a has trivial holonomy.

Now fix some transverse section S to the leaf L through x_0. We dentote by A_S the transverse Lie algebroid over S, so

$$A_S|_x = \{a \in A_x : \#a \in T_x S\}.$$

It follows from the construction of holonomy given in [12], Section 3.1, that we can choose a neighborhood U of x_0 in S such that for all sufficiently small $v \in N_{x_0}(A)$ there is a section $\alpha \in \Gamma(A_S)$ defined over U, with $\alpha(x_0) = v$, and the time-1 flow of α is the holonomy of the A-path $a(t) = v$. Since this holonomy is trivial, the time-1 flow of the vector field $\#\alpha$ on U is the identity map.

Now, assume that $\#$ is almost injective at x_0 and let $\{v_n\} \subset N_{x_0}(A)$ be a sequence such that $v_n \to 0$ as $n \to +\infty$. For the associated sections $\alpha_n \in \Gamma(A_S)$, the vector fields $X_n = \#\alpha_n$ satisfy conditions (a) and (b) of the lemma above with $V_n = U$. Hence there exists a $n_0 \in \mathbb{N}$ such that $X_n \equiv 0$ in U for all $n > n_0$. By almost injectivity, we must have $\alpha_n = 0$ in a neighborhood of x_0, so we conclude that $v_n = 0$ for all $n > n_0$. This shows that $r(x_0) > 0$, so the first obstruction is satisfied.

To show that $\liminf_{y \to x_0} r(y) > 0$ we proceed as follows. Fix some open set U containing x_0 where $\#$ is injective on a dense open set. Suppose we have some sequence of base points $x_n \in U$ converging to x_0, and let $\{v_n\} \subset N_{x_n}(A) - 0$ be a sequence converging to 0. At each x_n we choose a transverse section S_n and take $V_n = S_n \times L_{x_n}$ as a neighborhood of x_n. We extend first the vector field X_n to V_n, by taking X_n to be zero along the leave direction, and then we extend X_n to U such that the norm satisfies:

$$||\mathcal{J}(X_n)||_U \leq C\,||\mathcal{J}(X_n)||_{V_n}$$

for some constant C independent of n. Clearly, the sequence X_n satisfies the conditions of the lemma, so there exists a $n_0 \in \mathbb{N}$ such that $X_n \equiv 0$ in V_n for all $n > n_0$. By almost injectivity, we conclude that $v_n = 0$ for all $n > n_0$. This shows that $\liminf_{y \to x_0} r(y) > 0$, so the second obstruction is also satisfied. $\square$

5.2.5. *Poisson manifolds.* The Weinstein groupoid of the algebroid associated to a Poisson manifold (the cotangent bundle T^*M) is precisely the phase space $\mathcal{G}$ of the Poisson sigma-model studied by Cattaneo and Felder in [4]. Our constructions explain the constructions in [4], while our main result clarifies the smoothness of the Poisson-sigma model $\mathcal{G}$.

The following obvious application of our general criteria, is the main positive result of [4].

Corollary 5.10. *Any Poisson structure on a domain in $\mathbb{R}^2$ is integrable.*

The result is certainly not true in higher dimension, as shown by Weinstein's example of a non-integrable regular Poisson structure in $\mathbb{R}^3 - 0$ (Example 4.5).

In general, our main result applied to this context describes the precise obstructions for the integrability of Poisson manifolds. Let us point out the following simple integrability result:

Corollary 5.11. *All Poisson manifolds whose symplectic leaves have vanishing second homotopy groups are integrable.*

The integrability criterion of Dazord and Hector ([7]) is in fact this result specialized to the case of a regular Poisson manifold.

Note also that the monodromy groups of the regular symplectic leaves L (i. e. around which the rank is locally maximal) of a Poisson manifold M are particularly simple, as it is the associated monodromy map

$$\partial : \pi_2(L, x) \to \mathcal{N}^*_x(L).$$

Indeed, since the kernel $\mathcal{N}^*(L)$ of $\#$ over L is abelian, by Lemma 3.6 we can use any linear splitting σ. The resulting cohomology class

$$\Omega_L = [\Omega_\sigma] \in H^2(L; \mathcal{N}^*(L))$$

is independent of the splitting σ, ∂ is just the integration of Ω_L over elements in $\pi_2(L, x)$, and its image defines the monodromy groups

$$N_x \subset \mathcal{N}^*_x(L).$$

Notice also that if M is regular and $\mathcal{F}$ is its symplectic foliation, then using a global splitting σ for $\#$ one gets a globally defined cohomology class $\Omega \in H^2(\mathcal{F}; \mathcal{N}^*)$ which lies in the foliated cohomology with coefficients in the kernel of $\#$. Clearly, $\Omega_L = \Omega|_L$ for each L.

5.2.6. *Van Est argument.* Probably the most elegant proof of the integrability of Lie algebras is Van Est's cohomological argument which we briefly recall. Given a Lie algebra $\mathfrak{g}$, we form the exact sequence $0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to ad(\mathfrak{g}) \to 0$. Here $ad(\mathfrak{g})$ is easily seen to be integrable (it is a Lie sub-algebra of $\mathfrak{gl}(\mathfrak{g})$). Also recall that simply connected Lie groups are automatically 2-connected. The core of Van Est's argument is then the following result for the particular case of Lie algebras

Corollary 5.12. *If B fits into an exact sequence of Lie algebroids*

$$0 \to E \to B \xrightarrow{\pi} A \to 0$$

with E abelian, and A integrable by a groupoid with 2-connected $\mathbf{s}$-fibers, then B is integrable.

This result for Lie algebroids is Theorem 5 of [5]. Interesting enough, it shows that the integrability criterion of Dazord and Hector [7] mentioned above is actually Van Est's argument applied to regular Poisson manifolds.

The proof in [5] is an extension of Van Est's cohomological methods. Using a splitting σ of π we obtain an action of A on E, and a 2-cocycle Ω_σ on A with values on E. This is well known (see e. g. [17]), and can also be viewed as an extension of the constructions in section 3.3. We can then form the group of periods $Per_x \subset E_x$ of Ω_σ. The cohomological proof actually shows that B is integrable provided A is, and the groups P_x vanish (cf. Remark 5 and Corollary 2 in [5]).

Let us briefly point out how our result implies (and further clarifies) the previous corollary. Let $x \in M$ sitting in a singular leaf L. The necessary information is organized in the following diagram

$$\begin{tikzcd}
 & \pi_2(\mathbf{s}_A^{-1}(x)) \arrow[rr, "\mathbf{t}_A"] \arrow[d, "\partial_E"] & & \pi_2(L) \arrow[rr, "\partial"] \arrow[dl, "\partial_B"'] \arrow[dr, "\partial_A"] & & \pi_1(\mathcal{G}(A)_x) \arrow[dl, "j"'] & \\
0 \arrow[r] & E_x \arrow[r] & \mathcal{G}(\mathfrak{g}_x(B)) \arrow[rr] & & \mathcal{G}(\mathfrak{g}_x(A)) \arrow[rr] & & 0
\end{tikzcd}$$

Here $\mathbf{s}_A$, $\mathbf{t}_A$, $\mathfrak{g}(A)$, ∂_A are respectively the source and target map, the kernel of the anchor, and the monodromy map of A, and we use analogous notations for B. Also, ∂ is the boundary map in homotopy associated to $\mathbf{s}_A^{-1}(x) \to L$ with fiber $G(A)_x$ and j is the obvious inclusion whose image is precisely $\tilde{N}_x(A)$. Finally, ∂_E denotes the monodromy map associated to the exact sequence in the corollary (constructed exactly as the monodromy map of section 3.2), and whose image is precisely the group of periods Per_x.

Lemma 5.13. *There is a short exact sequence of abelian groups:*

$$0 \to Per_x \to \tilde{N}_x(B) \to \tilde{N}_x(A) \to 0.$$

Proof. These follows by diagram chasing since the two horizontal sequences above are exact. □

Therefore $\tilde{N}_x(B)$ appears as a twisted semi-direct product of $\tilde{N}_x(A)$ and Per_x. The simplest case where our main theorem applies is when Per_x vanishes. This gives precisely the corollary (and its stronger version) above.

5.3. **Transversely parallelizable foliations.** Historically, the first examples of non-integrable Lie algebroids [2] came from Molino's treatement (see [19]) of transversaly parallelizable foliations which we now briefly recall.

Given a foliation $\mathcal{F}$ of M, let us denote by $l(M, \mathcal{F})$ the algebra of transversal vector fields, i. e. sections of the normal bundle which can be locally projected along submersions which locally define the foliation. Then $(M, \mathcal{F})$ is *transversally paralelizable* if its normal bundle admits a global frame consisting of transversal vector fields. In this case the Lie algebra $l(M, \mathcal{F})$ is free as a module over the space $\Omega_b^0(M, \mathcal{F})$ of basic functions, on which it acts by derivations.

Let us see that the Lie bracket on $l(M, \mathcal{F})$ is of the type studied in this paper. We assume for simplicity that M is compact. Then the closures of the leaves of $\mathcal{F}$ form a new foliation $\bar{\mathcal{F}}$ on M, whose leaf space is a smooth (Hausdorff) manifold $W = M/\bar{\mathcal{F}}$, and is called the *basic manifold* of the foliation. Since $\mathcal{F}$ and $\bar{\mathcal{F}}$ have the same basic functions, $l(M, \mathcal{F})$ is the space of sections of a transitive Lie algebroid over W, which we denote by $A(M, \mathcal{F})$. Its anchor $\#$ is just the action of $l(M, \mathcal{F})$ on $\Omega_b^0(M, \mathcal{F}) \cong C^\infty(W)$, and the kernel of $\#$ has the following geometric interpretation. For each leaf L of $\mathcal{F}$, the foliation $(\bar{L}, \mathcal{F}|_{\bar{L}})$ is transversally parallelizable with dense leaves. It follows that $l(\bar{L}, \mathcal{F}|_{\bar{L}})$ is a finite dimensional Lie algebra, and moreover, $(\bar{L}, \mathcal{F}|_{\bar{L}})$ is a Lie foliation induced by a canonical $l(\bar{L}, \mathcal{F}|_{\bar{L}})$-valued Maurer-Cartan form. Denoting by $w \in W$ the point defined by $\bar{L}$, $l(\bar{L}, \mathcal{F}|_{\bar{L}})$ is canonically isomorphic to $Ker(\#_w)$. This shows that all the Lie algebras $l(\bar{L}, \mathcal{F}|_{\bar{L}})$ are isomorphic. The resulting Lie algebra $\mathfrak{g}(M, \mathcal{F})$ (defined up to isomorphisms) is usually called the *structural Lie algebra* of the foliation.

The main result of Almeida and Molino in [2] says that $(M, \mathcal{F})$ is developable (i. e. its lift to the universal cover of M is simple) if and only if the Lie algebroid $A(M, \mathcal{F})$ is integrable. This discussion extends to transversally complete foliations $(M, \mathcal{F})$ without any compactness assumption on M (see [19]).

Now, our constructions produce a monodromy map $\partial : \pi_2(W) \to G(M, \mathcal{F})$ with values in the simply connected Lie group integrating the structural Lie algebra $\mathfrak{g}(M, \mathcal{F})$, which controls the developability of the foliation:

Corollary 5.14. *A transversally parallelizable foliation $(M, \mathcal{F})$ on a compact manifold M is developable if and only if the image of the monodromy map*

$$\partial : \pi_2(W) \to G(M, \mathcal{F})$$

is discrete.

A simple consequence of this result is:

Corollary 5.15. *Let $(M,\mathcal{F})$ be a transversally parallelizable manifold on a compact manifold M. Then $(M,\mathcal{F})$ is developable provided one of the following conditions hold:*

(i) *the structural Lie algebra $\mathfrak{g}(M,\mathcal{F})$ has trivial center;*
(ii) *$\pi_2(W)$ has only elements of finite order.*

This result should be compared with Corollary 1 pp. 301, and Corollary 1 pp. 303 in [19].

Appendix A - Flows

In this appendix we discuss the flows associated to sections of Lie algebroids, which generalize the ordinary flows of vector fields (sections of $A = TM$). This is used throughout the paper, most notably for defining the equivalence relation on A-paths (section 1.3). As in the main body of the paper, A denotes a Lie algebroid over M, $\# : A \to TM$ denotes its anchor and $\pi : A \to M$ the projection.

A.1. **Flows and infinitesimal flows.** Given a time dependent vector field X on M, we denote by $\Phi_X^{t,s}$ its flow from time s to time t. Hence

$$\frac{d}{dt}\Phi_X^{t,s}(x) = X(t,\Phi_X^{t,s}(x)), \quad \Phi_X^{s,s}(x) = x \ .$$

We have $\Phi^{t,s}\Phi^{s,u} = \Phi^{t,u}$ and when X is autonomous $\Phi_X^{t,s} = \Phi_X^{t-s}$ only depends on $t-s$. Differentiating, we obtain the infinitesimal flow of X:

$$\phi_X^{t,s}(x) \equiv (d\Phi_X^{t,s})_x : T_xM \to T_{\Phi_X^{t,s}(x)}M.$$

Let us assume now that $\mathcal{G}$ is a Lie groupoid integrating the algebroid A. Given a time-dependent section α of A, we denote by the same letter the right invariant (time-dependent) vector field on $\mathcal{G}$ induced by α and by $\varphi_\alpha^{t,s} : \mathcal{G} \to \mathcal{G}$ its flow. If $x = \mathbf{s}(g)$ and $y = \mathbf{t}(g)$, then $\varphi_\alpha^{t,s}(g)$ is the arrow

$$\varphi_\alpha^{t,s}(g) : x \longrightarrow \Phi_{\#\alpha}^{t,s}(y)$$

and also satisfies the right-invariance property:

$$\varphi_\alpha^{t,s}(g) = \varphi_\alpha^{t,s}(y)g.$$

The *infinitesimal flow* of α,

$$\phi_\alpha^{t,s}(x) : A_x \to A_{\Phi_{\#\alpha}^{t,s}(y)},$$

is defined as

$$\phi_\alpha^{t,s}(x) \equiv (dR_{\varphi_\alpha^{s,t}(x)})_{\varphi_\alpha^{t,s}(x)}(d\varphi_\alpha^{t,s})_x. \tag{A.1}$$

The classical relation between Lie brackets and flows translates at this level to

$$\left.\frac{d}{dt}\right|_{t=s}(\phi_\alpha^{t,s})^*\beta = [\alpha^s,\beta], \tag{A.2}$$

where we have set

$$(\phi_\alpha^{t,s})^*(\beta)(x) = \phi_\alpha^{s,t}\beta(\phi_{\#\alpha}^{t,s}(x)).$$

We wish to extend the infinitesimal flow to sections of general Lie algebroids, not necessarily integrable. For this we can use the following general construction of (infinitesimal) flows. Let us assume that E is a vector bundle over M. A derivation on E is a pair (D,X) where $D : \Gamma(E) \to \Gamma(E)$ is a differential operator, X is a vector field on M, satisfying the Leibniz rule

$$D(f\alpha) = fD(\alpha) + X(f)\alpha, \quad \forall\ f \in C^\infty(M), \alpha \in \Gamma(E).$$

Now, any time-dependent derivation (D, X) on E has an associated (infinitesimal) flow: It is a family of linear isomorphisms

$$\phi_D^{t,s}(x) : E_x \to E_{\Phi_X^{t,s}(x)},$$

which is characterized uniquely by the properties

(a) $\phi_D^{t,s}\phi_D^{s,u} = \phi_D^{t,u}$, $\phi_D^{t,t} = Id$;
(b) $\frac{d}{dt}\big|_{t=s}(\phi_D^{t,s})^*\beta = D^s(\beta)$, for all sections $\beta \in \Gamma(E)$.

Here D^t is D at the fixed time t, and $(\phi_D^{t,s})^*\beta = \phi_D^{s,t}\beta\Phi_X^{t,s}$. This follows by the standard arguments.

Alternatively, one can use the groupoid $Aut(E)$ over E, whose arrows from x to y are all linear isomorphisms $E_x \to E_y$. Its Lie algebroid is usually denoted by $DO(E)$, and its sections are precisely derivations of E (cf. [14, 17]). Hence (D, E) can be viewed as a time-dependent section of $DO(E)$, and then $\phi_D^{t,s}$ is just the associated flow on $Aut(E)$. Both definitions of $\phi_D^{t,s}(x)$ show they are defined whenever $\Phi_X^{t,s}(x)$ is defined.

Most flows in differential geometry (e. g. the flows of vector fields, parallel transport) are obtained in this way.

A.2. **The infinitesimal flow of a section.** We we apply the previous construction to a time dependent section α of the Lie algebroid A, where $X = \#\alpha$ and $D = [\alpha, -] : \Gamma(A) \to \Gamma(A)$. The resulting flow

$$\phi_\alpha^{t,s}(x) : A_x \to A_{\Phi_{\#\alpha}^{t,s}(x)}$$

is uniquely determined by $\phi_\alpha^{t,s}\phi_\alpha^{s,u} = \phi_\alpha^{t,u}$, $\phi_\alpha^{t,t} = Id$, and the formula (A.2) above. In particular, if A is integrable, then $\phi_\alpha^{t,s}$ coincides with (A.1) above. As in the case of vector fields, if α is autonomous, then $\phi_\alpha^{t,s} = \phi_\alpha^{t-s}$ only depends on $t - s$.

Let us indicate an alternative description. Recall that on A^* one has a Poisson bracket $\{\ ,\ \}_A$ which is linear on the fibers. A section α of A defines in a natural way a function $f_\alpha : A^* \to \mathbb{R}$ which is linear on the fibers ("evaluation"), and we denote by X_α the Hamiltonian vector field associated with f_α. It is easy to see (cf. [11]) that:

(a) The assignment $\alpha \mapsto f_\alpha$ defines a Lie algebra homomorphism $(\Gamma(A), [\ ,\ ]) \to (C^\infty(A^*), \{\ ,\ \}_A)$;
(b) X_α is π-related to $\#\alpha$: $\pi_* X_\alpha = \#\alpha$, where $\pi : A^* \to M$ is the natural projection;

For each t, the flow $\Phi^{s,t}$ of X_α defines a Poisson automorphism of A^* (wherever defined), which maps linearly fibers to fibers of A^*. So, in fact, $\Phi^{s,t} : A^* \to A^*$ is a bundle map and from (b) we have that it covers $\Phi_{\#\alpha}^{s,t}$, the flow of $\#\alpha$. By transposition we obtain the infinitesimal flow $\phi_\alpha^{t,s}(x) : A_x \to A_{\Phi_{\#\alpha}^{t,s}(x)}$.

Example A.1. As a simple example, consider a Lie algebra $A = \mathfrak{g}$ as a Lie algebroid over a point, and $\alpha \in \mathfrak{g}$ (a constant section). The Poisson bracket on the dual $\mathfrak{g}^*$ is the Kirillov Poisson structure and so the hamiltonian flow on $\mathfrak{g}^*$ of the evaluation function f_α is given by the co-adjoint action. It follows that the infinitesimal flow of α is then $\phi_\alpha^t = Ad(\exp(t\alpha))$.

This example shows that one can think of the infinitesimal flow of a section as a generalization of the adjoint action, although for a general Lie algebroid it does not make sense to speak of the adjoint representation!

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