

# On an optimal consumption problem for $p$ -integrable consumption plans

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**Summary.** A generalization is presented of the existence results for an optimal consumption problem of Aumann and Perles [4] and Cox and Huang [10]. In addition, we present a very general optimality principle.

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## 1 Introduction

In a seminal paper [4] Aumann and Perles gave existence results for optimal consumption problems with linear inequality and equality constraints that are special cases of two problems,  $(IC_p)$  and  $(EC_p)$ , to be formulated in section 2. These are variational problems in a space of  $p$ -integrable functions, either for  $p = 0$  (0-integrability being interpreted as mere measurability) or for  $p \geq 1$ , as is the case in [4, 10]. Problem  $(EC_p)$  generalizes a problem studied in [4] in the Main Theorem (p. 489) and in Theorem 6.2 (the former has  $p = 1$  and the latter is for  $p > 1$ ). A version of  $(IC_1)$  was considered in [4, Theorem 6.1]. More recently, Cox and Huang continued this work in [10], where they gave existence results for a dynamic consumption-portfolio problem. They did so by using the well-known fact [13] that such problems can be transformed into a static problem of the type  $(IC_p)$ ,  $p \geq 1$ , using Ito's calculus. The existence results in [10] show several differences with the results in [4]. As one practical limitation of the version of  $(IC_p)$  used in [10] we point out that it only allows for a single consumption good and one inequality constraint. This restriction plays an important technical role in [10]. Closer inspection of [10] *vis à* [4] reveals a number of other substantial technical differences between [4] and [10] that affect certain comparisons with [4] that were claimed in [10]. Next to the already cited fact that [4] deals with a multi-good model, these differences are as follows. (i) In all of [10] the utility function  $u(z, \omega)$  is concave in the decision variable  $z$ , but it is not so in any of the three above-mentioned existence results in [4]. (ii) On the other hand, in all of [4] the underlying measure space is nonatomic, whereas in [10] it is general. (iii) In all of [10],  $u(z, \omega)$  is required to be increasing (by this we mean *strictly* increasing) in  $z$ , but this is not so in [4, Theorem 6.1] (which has no monotonicity requirement at all) and [4, Main Theorem] (which only requires  $u(z, \omega)$  to be nondecreasing in  $z$ ); however, Theorem 6.2 in [4] requires  $u(z, \omega)$  to be increasing in  $z$ .

For these reasons, the totality of the results in [4] and [10] is intransparent. To subsume all of the cited results in [4] and [10] and to go beyond them, this work presents three central existence results. These offer several considerable improvements, in particular for the utility functions. For  $p = 0$  (and also for  $p = 1$  under additional conditions that turn out to be valid in [4] but not in [10]) our main existence results are Propositions 2.5 and 2.6, respectively for the inequality- and the equality-constrained problems. These propositions are immediate consequences of [6, Corollary 2], a result recapitulated here as Theorem 3.1. A growth property  $(\gamma_1)$  from [8] is used, as well as its logical extension  $(\gamma_p)$ . We show that this unifies the different growth conditions used by Aumann and Perles [4] and Cox and Huang [10]. Our main existence result is Theorem 2.8; this is new, but it is obtained along the lines set out by Aumann and Perles in their proof of [4, Theorem 6.2]. First, for  $(IC_0)$  the propositions mentioned above yield existence of an optimal solution  $x_*$  in a space of measurable functions. Next, in Theorem 3.2 optimality is characterized by a pointwise optimality principle, which comes from [1, 2, 11] (see [4, Theorem 5.1]). It is essential that all

Lagrange multipliers of this optimality principle be strictly positive (Corollary 3.3); this forces  $x_*$  to be  $p$ -integrable, as a consequence of the optimality principle and the growth conditions for  $u(z, \omega)$ . In addition, such strict positivity causes the optimal solutions of  $(IC_p)$  and  $(EC_p)$  to coincide, because of complementary slackness.

## 2 Existence results

For  $p = 0$  and  $p \geq 1$  we consider the following optimal consumption problem with linear inequality constraints

$$(IC_p) \quad \sup_{x \in \mathcal{L}_Z^p} \{U(x) : \int_{\Omega} x(\omega) \cdot \xi_i(\omega) \mu(d\omega) \leq \alpha_i, i = 1, \dots, m\}$$

and its equality-constrained counterpart

$$(EC_p) \quad \sup_{x \in \mathcal{L}_Z^p} \{U(x) : \int_{\Omega} x(\omega) \cdot \xi_i(\omega) \mu(d\omega) = \alpha_i, i = 1, \dots, m\}.$$

As we shall see in section 5, this model can easily incorporate consumption over time as well. Here  $(\Omega, \mathcal{F}, \mu)$  is a finite measure space and  $\mathcal{L}_Z^p$  is shorthand for the set of all  $p$ -integrable consumption functions on  $(\Omega, \mathcal{F}, \mu)$  with values in  $Z := \mathbb{R}_+^d$ . Here  $d$  is a fixed, given dimension. For  $p = 0$  this definition has to be understood as follows:  $\mathcal{L}_Z^0$  is the set of all *measurable* functions from  $\Omega$  into  $\mathbb{R}_+^d$ . Also,  $\alpha_1, \dots, \alpha_m > 0$  are given constants. Further  $\xi_1, \dots, \xi_m$  are given functions in  $\mathcal{L}_Z^0$ ,  $\xi_i = (\xi_{i,1}, \dots, \xi_{i,d})$ , with

$$\hat{\xi}(\omega) := \min_{1 \leq j \leq d} \sum_{i=1}^m \xi_{i,j}(\omega) > 0 \text{ for every } \omega \text{ in } \Omega. \quad (1)$$

By nonnegativity of  $x \cdot \xi_i$ , the meaning of  $\int_{\Omega} x \cdot \xi_i d\mu$  is always clear (the integral is allowed to be  $+\infty$ ).<sup>1</sup> Finally, above we denote

$$U(x) := \int_{\Omega} u(x(\omega), \omega) \mu(d\omega),$$

where  $u : \mathbb{R}_+^d \times \Omega \rightarrow [-\infty, \infty)$  is a  $\mathcal{B}(\mathbb{R}_+^d) \times \mathcal{F}$ -measurable utility function. Of course, the integrand  $\omega \mapsto u(x(\omega), \omega)$  is  $\mathcal{F}$ -measurable for every  $x \in \mathcal{L}_Z^p$ , but it is not necessarily summable. However, growth property  $(\gamma_p)$  that is to follow will hold for all our existence results. This implies that  $\int_{\Omega} \max(u(x(\omega), \omega), 0) \mu(d\omega)$  is finite for all  $x \in \mathcal{L}_Z^p$ , so, by allowing for  $U(x) = -\infty$ , the meaning of the integral is never in doubt; this means that we interpret the integral in the definition of  $U(x)$  as a *quasi-integral* [15].

Extensions, examples and special cases of this model are discussed in sections 4 and 5. As one particular economic example of  $(IC_p)$  one could, for instance, think of a consumer, facing uncertainty about the true state of nature, who consults  $m$  experts. Each expert  $i$  suggests a random variable  $\xi_i \in \mathcal{L}_Z^0$  to describe expert  $i$ 's best guess for stochastic price behavior: should state  $\omega$  in  $\Omega$  arise under  $\mu$ , then expert  $i$  predicts that this results in the price vector  $\xi_i(\omega) \in \mathbb{R}^d$ . If the consumer takes all expert opinions seriously, he/she could wish to use only state-contingent consumption plans  $x \in \mathcal{L}_Z^p$  for which for each  $i$  the expectation  $\int_{\Omega} \xi_i \cdot x d\mu$  does not exceed a certain budget value. As illustrated by Example 4.12, mechanical problems of the type  $(EC_p)$  were already studied by Newton.

The following special conditions will sometimes be imposed on  $(\xi_{i,j})$ . Of these, order-equivalence works in connection with  $p \geq 1$ , both for  $(IC_p)$  and  $(EC_p)$ , and diagonal dominance serves to make all problems  $(EC_p)$ ,  $p = 0$  or  $p \geq 1$ , automatically feasible.

<sup>1</sup> Thus, we dispense with the condition  $\xi_i \in \mathcal{L}^q$  of [10], with  $q$  as specified in footnote 2. In retrospect, this justifies Cox and Huang's use of both  $p = 1$  and  $p > 1$  in [10], although their own restriction  $\xi_i \in \mathcal{L}^q$  effectively rules out  $p = 1$  (i.e.,  $q = \infty$ ) because of their formula (8).

**Definition 2.1** (i) The matrix function  $(\xi_{i,j})$  is said to be *order-equivalent to*  $\hat{\xi}$  if there exists  $C > 0$  such that

$$\max_{1 \leq j \leq d} \sum_{i=1}^m \xi_{i,j}(\omega) \leq C \hat{\xi}(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

(ii) The matrix function  $(\xi_{i,j})$  is said to have *diagonal structure* if  $m = d$  and  $\xi_{i,j} \equiv 0$  whenever  $i \neq j$ ,  $i, j = 1, \dots, d$ .

Observe already that diagonal structure implies  $\xi_{i,i} > 0$  for every  $i$ , in view of (1). Note also that Aumann and Perles [4] use diagonal structure, with  $\xi_i$  identically equal to the  $i$ -th unit vector  $e_i$ . Hence, they also have order-equivalence with  $\hat{\xi} \equiv 1$ . In [10] one simply has  $m = d = 1$ , whence  $\hat{\xi} = \xi_{1,1}$ . The growth condition for  $u$  mentioned above is as follows; it is an obvious extension to  $p \geq 1$  of the property introduced in [8] to unify the three different growth conditions used in [4].

**Definition 2.2**  $u$  has *growth property*  $(\gamma_p)$  if for every  $\epsilon > 0$  there exists  $\psi_\epsilon \in \mathcal{L}_+^p$  such that for a.e.  $\omega \in \Omega$

$$u(z, \omega) \leq \epsilon \hat{\xi}(\omega) |z| + \hat{\xi}(\omega) \psi_\epsilon(\omega) \text{ for all } z \in \mathbb{R}_+^d.$$

In connection with the existence results for  $p \geq 1$  the following nonsatiation condition is important:

**Definition 2.3** The function  $u$  is said to be *essentially nonsatiated with respect to*  $\xi_1, \dots, \xi_m$  if there do not exist  $j$ ,  $1 \leq j \leq m$ , and  $\lambda_i \geq 0$ ,  $i \neq j$ , for which

$$\operatorname{argmax}_{z \in \mathbb{R}_+^d} [u(z, \omega) - \sum_{i, i \neq j} \lambda_i z \cdot \xi_i(\omega)] \neq \emptyset \text{ for a.e. } \omega \text{ in } \Omega.$$

**Remark 2.4** Obviously, if  $(\xi_{i,j})$  has diagonal structure, then  $u$  is nonsatiated with respect to  $\xi_1, \dots, \xi_m$  if and only if there do not exist  $j$ ,  $1 \leq j \leq m$ , and  $\lambda_i \geq 0$ ,  $i \neq j$ , for which

$$\operatorname{argmax}_{z \in \mathbb{R}_+^d} u(z, \omega) - \sum_{i, i \neq j} \lambda_i \xi_{i,i}(\omega) z^i \neq \emptyset \text{ a.e.}$$

So the above certainly holds if for every  $\omega$  in some non-null subset  $B$  of  $\Omega$  (i.e.,  $\mu(B) > 0$ ) and every  $j$  the function  $u(z, \omega)$  is nonsatiated in each coordinate  $z^j$  of  $z$  (i.e.,  $\operatorname{argmax}_{z^j \geq 0} u(z, \omega) = \emptyset$  for every  $z^1, \dots, z^{j-1}, z^{j+1}, \dots, z^d \geq 0$ ). In particular, this holds when  $u(z, \omega)$  is strictly increasing in each coordinate  $z^j$  for all  $\omega$  in some subset non-null subset of  $\Omega$ .

**Proposition 2.5 (existence of optimal measurable plans)** *Suppose that  $u(z, \omega)$  is upper semi-continuous in  $z$  for a.e.  $\omega$  in  $\Omega$ . Suppose also that  $u$  satisfies growth condition  $(\gamma_1)$ . Then problem  $(IC_0)$ , provided that it is feasible, has an optimal solution  $x_*$  with  $\int_\Omega |x_*| \hat{\xi} d\mu < +\infty$ .*

**Proposition 2.6 (existence of optimal measurable plans)** *Suppose that  $u(z, \omega)$  is upper semi-continuous and nondecreasing in  $z$  for a.e.  $\omega$  in  $\Omega$  and that  $(\xi_{i,j})$  has diagonal structure. Suppose also that  $u$  satisfies growth condition  $(\gamma_1)$ . Then problem  $(EC_0)$  has an optimal solution  $x_*$ , with  $\int_\Omega |x_*| \hat{\xi} d\mu < +\infty$ , that is simultaneously an optimal solution of  $(IC_0)$ .*

Observe that Proposition 2.6 contains no explicit feasibility condition. Here  $u(z, \omega)$  is said to be nondecreasing in  $z$  if  $z' \geq z$  (coordinatewise) in  $\mathbb{R}^d$  implies  $u(z', \omega) \geq u(z, \omega)$ .

**Remark 2.7** Of course, if  $\operatorname{ess\,inf}_\Omega \hat{\xi}$ , the essential infimum of  $\hat{\xi}$  over  $\Omega$ , is strictly positive, the additional property  $\int_\Omega |x_*| \hat{\xi} d\mu < +\infty$  of  $x_*$  implies  $x_* \in \mathcal{L}_Z^1$ , which causes the existence results for  $(IC_0)$  and  $(IC_1)$ , as well as those for  $(EC_0)$  and  $(EC_1)$ , to coincide. This observation applies in particular to [4], where  $\hat{\xi} \equiv 1$ ; cf. section 4.

The following theorem is the main result of this work. It gives sufficient conditions for the existence of an optimal solution of  $(IC_p)$  and of  $(EC_p)$ .

**Theorem 2.8 (existence of optimal  $p$ -integrable plans)** *Suppose for  $p \geq 1$  that  $u(z, \omega)$  is upper semicontinuous in  $z$  for a.e.  $\omega$  in  $\Omega$ , that  $u(z, \omega)$  is concave in  $z$  for a.e.  $\omega$  in the purely atomic part  $\Omega^{pa}$  of  $(\Omega, \mathcal{F}, \mu)$  and that  $u$  is essentially nonsatiated with respect to  $\xi_1, \dots, \xi_m$ . Suppose also that  $u$  satisfies growth condition  $(\gamma_p)$ , that  $(\xi_{i,j})$  is order-equivalent to  $\hat{\xi}$  and that there exists some  $\tilde{x} \in \mathcal{L}_Z^p$  for which the function  $\omega \mapsto u(\tilde{x}(\omega), \omega)/\hat{\xi}(\omega)$  is  $p$ -integrable. Then problem  $(IC_p)$ , provided that it is feasible, has an optimal solution that is simultaneously an optimal solution of problem  $(EC_p)$ .*

Recall here [14] that  $\Omega$  can always be partitioned into a *purely atomic* part  $\Omega^{pa}$  (this is the union of at most countably many non-null atoms) and a *nonatomic* part  $\Omega^{na}$ .

**Remark 2.9** Suppose that  $(\xi_{i,j})$  has diagonal structure with  $\hat{\xi}^{-1} \in \mathcal{L}^p$ . Then in Theorem 2.8 problem  $(IC_p)$  is feasible. By  $\hat{\xi}^{-1} \in \mathcal{L}^p$  and  $\hat{\xi} \leq \xi_{i,i} \leq C\hat{\xi}$  we have  $\hat{\xi}_{i,i}^{-1} \in \mathcal{L}_+^p$  for  $i = 1, \dots, d$ . Hence,  $(\alpha_1 \hat{\xi}_{1,1}^{-1}, \dots, \alpha_d \hat{\xi}_{d,d}^{-1})$  in  $(\mathcal{L}_+^p)^d$  defines a feasible solution of  $(EC_p)$ , whence of  $(IC_p)$ .

Even when  $m = d = 1$  the essential nonsatiation condition that we use constitutes a considerable improvement over [4, Theorem 6.2] and [10], where  $u(z, \omega)$  is required to be strictly increasing in each coordinate of  $z$  for all (or at least a.e.)  $\omega$  in  $\Omega$ . See Examples 4.8 and 4.9 below.

### 3 Auxiliary results and proofs

The proof of Proposition 2.5 is an immediate application of the following result from [6], where it was shown to extend [8, Proposition 1, p. 155] and the existence results of [3, 5]) to a general underlying measure space (all those references use a nonatomic measure).

**Theorem 3.1 ([6, Corollary 2])** *Let  $g_0, g_1, \dots, g_{m+1} : \mathbb{R}_+^d \times \Omega \rightarrow (-\infty, +\infty]$  be product measurable functions. Also, let  $\beta_1, \dots, \beta_{m+1}$  be given real numbers. Suppose that  $g_1(z, \omega), \dots, g_m(z, \omega)$  are lower semicontinuous in the variable  $z$  and suppose that  $g_{m+1}(z, \omega)$  is inf-compact in the variable  $z$  and nonnegative. Suppose also that for every  $\epsilon > 0$  there exists  $\psi_\epsilon \in \mathcal{L}^1$  such that for  $i = 0, \dots, m$*

$$\max(-g_i(z, \omega), 0) \leq \epsilon g_{m+1}(z, \omega) + \psi_\epsilon(\omega) \text{ for a.e. } \omega. \quad (2)$$

*Then the optimization problem*

$$\inf_{x \in \mathcal{L}_Z^0} \left\{ \int_{\Omega} g_0(x(\omega), \omega) \mu(d\omega) : \int_{\Omega} g_i(x(\omega), \omega) \mu(d\omega) \leq \beta_i, i = 1, \dots, m+1 \right\}$$

*has an optimal solution, provided that this problem is feasible.*

In [6, Corollary 2] a more general space is taken instead of  $\mathbb{R}_+^d$ . Above the integrals  $\int_{\Omega} g_i(x(\omega), \omega) \mu(d\omega)$ ,  $x \in \mathcal{L}_Z^0$ , should be interpreted as quasi-integrals (concretely, they can have values  $+\infty$ , but not  $-\infty$ ).

**PROOF OF PROPOSITION 2.5.** Let  $\alpha_{m+1} := \sum_i \alpha_i$  and consider the following auxiliary optimization problem:

$$(Q) \quad \inf_{x \in \mathcal{L}_Z^0} \left\{ \int_{\Omega} -u(x(\omega), \omega) \mu(d\omega) : \int_{\Omega} x \cdot \xi_i d\mu \leq \alpha_i, i = 1, \dots, m, \int_{\Omega} \hat{\xi} |x| d\mu \leq \alpha_{m+1} \right\}.$$

Let us show that this problem is equivalent with  $(IC_0)$ . First, the  $m+1$ -st constraint of the optimization problem is redundant (it is only introduced because it is formally required). To see its redundance, just observe that the elementary inequality  $\hat{\xi}|z| \leq \hat{\xi} \sum_j z^j \leq \sum_i \xi_i \cdot z$  for all  $z \in \mathbb{R}_+^d$  causes the first  $m$  constraints in  $(Q)$  to imply the  $m+1$ -st one. Secondly, the change into a minimization problem is explained by the additional minus sign. So  $(IC_0)$  is equivalent to  $(Q)$ ; hence, it is enough to prove existence of an optimal solution of  $(Q)$ . We do this by a direct application of Theorem 3.1, setting  $g_0(z, \omega) := -u(z, \omega)$ ,  $g_{m+1}(z, \omega) := \hat{\xi}(\omega)|z|$ ,  $g_i(z, \omega) := z \cdot \xi_i(\omega)$  and  $\beta_i := \alpha_i$

for  $i = 1, \dots, m + 1$ . Before invoking Theorem 3.1 it remains to verify (2). For  $i = 1, \dots, m$  this is trivial by  $g_i \geq 0$  and for  $i = 0$  it is an immediate consequence of  $(\gamma_1)$ . QED

**PROOF OF PROPOSITION 2.6.** Because of the additional diagonal structure,  $(\alpha_1 \xi_{1,1}^{-1}, \dots, \alpha_d \xi_{d,d}^{-1})$  in  $\mathcal{L}_Z^0$  defines a feasible solution of  $(EC_0)$ , whence of  $(IC_0)$ . So Proposition 2.5 can be applied. This guarantees existence of an optimal solution  $x_{**}$  of  $(IC_0)$ . Define  $x_* \in \mathcal{L}_Z^0$  by

$$x_*^i(\omega) := x_{**}^i(\omega) + (\alpha_i - \alpha'_i) \xi_{i,i}^{-1}(\omega), \quad (3)$$

for  $\alpha'_i := \int_{\Omega} x_{**} \cdot \xi_i d\mu = \int_{\Omega} x_{**}^i \xi_{i,i} d\mu \leq \alpha_i$ . Then  $U(x_*) = U(x_{**})$ , which causes  $x_*$  to be an optimal solution of both  $(EC_0)$  and  $(IC_0)$ . The identity holds, because on the one hand  $x_*$  is obviously feasible for  $(EC_0)$  (whence for  $(IC_0)$ , which implies  $U(x_*) \leq U(x_{**})$ ) and on the other hand  $x_*(\omega) \geq x_{**}(\omega)$  (coordinatewise), causing  $u(x_*(\omega), \omega) \geq u(x_{**}(\omega), \omega)$  for all  $\omega$ , whence  $U(x_*) \geq U(x_{**})$ . QED

We now prepare the proof of Theorem 2.8. We shall need the following theorem, which comes from [1, 2, 11]. Essentially, it is based on an application of Lyapunov's theorem (convexity of the range of a nonatomic vector measure) and the separating hyperplane theorem in  $\mathbb{R}^{m+1}$ , plus some measurable selection arguments.

**Theorem 3.2 (optimality principle)** *Suppose for any  $p$ ,  $p = 0$  or  $p \geq 1$ , that  $u(z, \omega)$  is concave in  $z$  for for a.e.  $\omega$  in the purely atomic part  $\Omega^{pa}$ . Then  $x_* \in \mathcal{L}_Z^p$  is an optimal solution of  $(IC_p)$  if and only if  $x_*$  is feasible for  $(IC_p)$  and there exist  $\lambda_1, \dots, \lambda_m \geq 0$  such that the following two conditions hold:*

$$x_*(\omega) \in \operatorname{argmax}_{z \in \mathbb{R}_+^d} u(z, \omega) - \sum_{i=1}^m \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega \text{ (pointwise maximum principle).}$$

$$\lambda_i \left( \int_{\Omega} x_* \cdot \xi_i d\mu - \alpha_i \right) = 0, \quad i = 1, \dots, m \text{ (complementary slackness).}$$

Under additional conditions for  $\xi_1, \dots, \xi_m$ , a similar result can be also given for  $(EC_p)$  [1], but for the present paper this is not very relevant.<sup>2</sup> The following Corollary 3.3 of the above theorem will play an essential role in establishing existence for  $p \geq 1$ . Its essential nonsatiatedness condition alone is responsible for the (strict) positivity of its multipliers; cf. Examples 4.8 and 4.9.

**Corollary 3.3** *Suppose for any  $p$ ,  $p = 0$  or  $p \geq 1$ , that  $u(z, \omega)$  is concave in  $z$  for for a.e.  $\omega$  in  $\Omega^{pa}$  and that  $u$  is essentially nonsatiated with respect to  $\xi_1, \dots, \xi_m$ . Then  $x_* \in \mathcal{L}_Z^p$  is an optimal solution of  $(IC_p)$  if and only if  $x_*$  is feasible for  $(EC_p)$  and there exist  $\lambda_1, \dots, \lambda_m > 0$  such that*

$$x_*(\omega) \in \operatorname{argmax}_{z \in \mathbb{R}_+^d} u(z, \omega) - \sum_{i=1}^m \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega \text{ (pointwise maximum principle).}$$

In the above corollary any optimal solution of  $(IC_p)$  is also an optimal solution of  $(EC_p)$ , but the converse implication need not hold:

**Example 3.4** Let  $\Omega$  be the unit interval, equipped with Lebesgue measure  $\mu$ . Let  $m = d = 1$ ,  $\eta \in [0, 1)$  and define the utility function as follows:

$$u(z, \omega) := \begin{cases} \frac{1}{z} & \text{if } 0 < z \leq 1 \\ 0 & \text{if } z = 0 \\ \infty & \text{if } z > 1 \end{cases}$$

Consider the problems  $(IC_p)$  and  $(EC_p)$  with  $\xi_1 \equiv 1$  and  $\alpha_1 = 1$ . Then, apart from null sets,  $x \equiv 1$  is the only feasible element of  $(EC_p)$  for which  $U(x) > -\infty$ . Hence,  $x_* \equiv 1$  is the (essentially) unique optimal solution of  $(EC_p)$ . However,  $(IC_p)$  clearly has no optimal solution, even though  $u$  meets all conditions of Corollary 3.3 (here  $\Omega^{pa} = \emptyset$ ).

<sup>2</sup>By [1, 4.3.3] an analogous characterization holds for  $(EC_p)$  if  $\xi_1, \dots, \xi_m$  are additionally  $q$ -integrable, with  $q := p/(1-p)$  if  $p > 1$  and  $q := \infty$  if  $p = 0$ .

PROOF OF THEOREM 3.2. When  $\Omega^{na}$  is equipped with  $\mathcal{F} \cap \Omega^{na}$  and  $\mu(\cdot \cap \Omega^{na})$ , it forms a nonatomic measure space. Denote by  $\mathcal{V} [\mathcal{W}]$  the space of all  $p$ -integrable functions from  $\Omega^{na}$  [ $\Omega^{pa}$ ] into  $\mathbb{R}_+^d$ . Every  $x \in \mathcal{L}_Z^p$  can be identified with the pair  $(v, w)$  in  $\mathcal{V} \times \mathcal{W}$ , where  $v := x|_{\Omega^{na}}$  is the restriction of  $x$  to the nonatomic part  $\Omega^{na}$  and where  $w := x|_{\Omega^{pa}}$  is the restriction of  $x$  to the purely atomic part  $\Omega^{pa}$ . Then  $x_*$  is an optimal solution of  $(IC_p)$  if and only if  $(v_*, w_*)$ , with  $v_* := x_*|_{\Omega^{na}}$  and  $w_* := x_*|_{\Omega^{pa}}$ , is an optimal solution of the following optimization problem

$$(L) \quad \inf_{v \in \mathcal{V}, w \in \mathcal{W}} \left\{ - \int_{\Omega^{na}} u(v(\omega), \omega) \mu(d\omega) + a_0(w) : \int_{\Omega^{na}} v \cdot \xi_i d\mu + a_i(w) \leq \alpha_i, i = 1, \dots, m \right\}.$$

Here  $a_0(w) := - \int_{\Omega^{pa}} u(w(\omega), \omega) \mu(d\omega)$  is convex in the variable  $w$  (by the given concavity property of  $u$ ). Each  $a_i(w) := \int_{\Omega^{pa}} w \cdot \xi_i d\mu$  is also obviously convex in  $w$ . In the terminology of [1, 4.3.3], problem (L) is a *Lyapunov-type optimization problem*. By the main theorem of section 4.3.3 in [1, p. 240-241] it follows that, corresponding to the optimal pair  $(v_*, w_*)$ , there exist nonnegative multipliers  $\lambda_0, \lambda_1, \dots, \lambda_m$ , not all zero, such that the two minimum principles

$$v_*(\omega) \in \operatorname{argmin}_{z \in \mathbb{R}_+^d} - \lambda_0 u(z, \omega) + \sum_{i=1}^m \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega^{na}$$

and

$$w_* \in \operatorname{argmin}_{w \in \mathcal{W}} \sum_{i=0}^m \lambda_i a_i(w),$$

hold, as well as the complementary slackness relationships

$$\lambda_i \left( \int_{\Omega^{na}} v_* \cdot \xi_i d\mu + a_i(w_*) - \alpha_i \right) = 0, i = 1, \dots, m.$$

Writing out the definition of the  $a_i(w)$  immediately gives that the above complementary slackness relationship is equivalent to the one stated in Theorem 3.2. Since  $\alpha_1, \dots, \alpha_m > 0$ , a Slater type constraint qualification holds, which causes  $\lambda_0 \neq 0$ . This can also be seen directly: if  $\lambda_0$  were 0, then obviously  $\sum_i \lambda_i x_*(\omega) \cdot \xi_i(\omega) = 0$  for a.e.  $\omega$  (set  $z = x_*(\omega)/2$  and  $z = 2x_*(\omega)$  respectively). This would result in  $\sum_i \lambda_i \int_{\Omega} x_* \cdot \xi_i d\mu = 0$ . By complementarity,  $\sum_i \lambda_i \int_{\Omega} x_* \cdot \xi_i d\mu = \sum_i \lambda_i \alpha_i$ , so we would have  $\sum_i \lambda_i \alpha_i = 0$ . By  $\lambda_i \geq 0$  and  $\alpha_i > 0$  for all  $i$ , this would mean that also the multipliers  $\lambda_1, \dots, \lambda_m$  are zero. This gives a contradiction. So  $\lambda_0 \neq 0$ , and, rather than dividing all  $\lambda_i$  by  $\lambda_0$ , we can suppose without loss of generality  $\lambda_0 = 1$ . Also, because  $\Omega^{pa}$  consists of at most countably many atoms and because each function in  $\mathcal{W}$  is a.e. constant on such an atom, it is easy to see that the second minimum principle is equivalent to the following:

$$w_*(\omega) \in \operatorname{argmin}_{z \in \mathbb{R}_+^d} - u(z, \omega) + \sum_{i=1}^m \lambda_i z \cdot \xi_i(\omega) \text{ for a.e. } \omega \text{ in } \Omega^{pa}$$

Combined, the above two minimum principles (with  $\lambda_0 = 1$ ) are precisely equivalent to the pointwise maximum principle that is stated in Theorem 3.2. QED

Some comments should be added to justify the application above of the main theorem of [1, section 4.3.3]. Formally speaking, the conditions of [1] require  $\Omega$  to be a Lebesgue interval of  $\mathbb{R}$  and the functions  $u(z, \omega)$  and  $z \cdot \xi_i(\omega)$  to be jointly continuous in  $(z, \omega)$ . However, from the proof in [1] it is evident that the only reason for this is the rather crude Lemma D) on p. 244, which is known to hold in much more general forms for functions  $u(z, \omega)$  and  $z \cdot \xi_i(\omega)$  that are just jointly measurable in  $(z, \omega)$  and for a *decomposable* class of measurable functions, such as  $\mathcal{L}_Z^p$ . This is the so-called reduction theorem; e.g., see [16, Theorem 3A], [7, Theorem B.1] and [12]. Actually, the approach taken in [1] can already be found in more general terms in [2].

PROOF OF COROLLARY 3.3. Clearly, all that has to be done is to demonstrate that  $\lambda_i > 0$  for  $i = 1, \dots, m$  in the pointwise maximum principle of Theorem 3.2 (because complementary slackness then implies feasibility for  $(EC_p)$ ). If there were  $j$  with  $\lambda_j = 0$ , then the pointwise maximum

principle would imply that  $x_*(\omega)$  belongs to  $\operatorname{argmax}_{z \in \mathbb{R}_+^d} [u(z, \omega) - \sum_{i, i \neq j} \lambda_i z \cdot \xi_i(\omega)]$  for a.e.  $\omega$ . But this contradicts the definition of essential nonsatiation. QED

**PROOF OF THEOREM 2.8.** Since the conditions of Proposition 2.5 clearly hold, we certainly have existence of an optimal solution  $x_* \in \mathcal{L}_Z^0$  of  $(IC_0)$ . We can apply Corollary 3.3 (for  $p = 0$ ) to  $(IC_0)$ . Observe already that this already gives feasibility of  $x_*$  for  $(EC_0)$ . Setting  $z := \tilde{x}(\omega)$  in the pointwise maximum principle, we obtain

$$u(x_*(\omega), \omega) - \sum_{i=1}^m \lambda_i x_*(\omega) \cdot \xi_i(\omega) \geq u(\tilde{x}(\omega), \omega) - \sum_{i=1}^m \lambda_i \tilde{x}(\omega) \cdot \xi_i(\omega) \text{ a.e.}$$

where  $\tilde{x} \in \mathcal{L}_Z^p$  is as postulated in Theorem 2.8. Let  $\epsilon := \min_{1 \leq i \leq m} \lambda_i$ ; then  $\epsilon > 0$  by Corollary 3.3. By using  $(\gamma_p)$  and order-equivalence of  $(\xi_{i,j})$  we obtain from the above

$$\frac{\epsilon}{2} \hat{\xi}(\omega) |x_*(\omega)| + \hat{\xi}(\omega) \psi_{\epsilon/2}(\omega) - u(\tilde{x}(\omega), \omega) + C\sqrt{d} \max_i \lambda_i \hat{\xi}(\omega) |\tilde{x}(\omega)| \geq \epsilon \hat{\xi}(\omega) |x_*(\omega)|.$$

Here we use the elementary inequalities  $\epsilon |x_*| \hat{\xi} \leq \sum_i \lambda_i x_* \cdot \xi_i \leq Cd^{1/2} \max_i \lambda_i \hat{\xi} |x_*|$ . After division by  $\hat{\xi}(\omega)$ , the resulting majorization of  $\epsilon |x_*|/2$  by the  $p$ -integrable expression  $\psi_{\epsilon/2} - u(\tilde{x}(\cdot), \cdot)/\hat{\xi} + Cd^{1/2} \max_i \lambda_i |\tilde{x}|$  immediately implies the  $p$ -integrability of  $|x_*|$ . Finally,  $(EC_p)$ -feasibility of  $x_*$  now follows simply from our earlier observation about its  $(EC_0)$ -feasibility. So  $x_*$  is also an optimal solution of problem  $(EC_p)$ . QED

## 4 Applications

In this section we show how the existence results in [4] and [10] all follow from the results developed in section 2. We also give some examples to show that Theorem 2.8 also applies to new situations, not covered by [4, 10]. To begin with, we prepare the conversion of the following growth properties used in [4, 10] for  $p \geq 1$ :

**Definition 4.1** (i)  $u$  has *growth property*  $(\delta_p)$  if for every  $\epsilon > 0$  there exists  $\phi_\epsilon \in \mathcal{L}_+^p$  such that for a.e.  $\omega$

$$u(z, \omega) \leq \epsilon \hat{\xi}(\omega) |z| \text{ for all } z \in \mathbb{R}_+^d \text{ with } |z| \geq \phi_\epsilon(\omega).$$

(ii)  $u$  has *growth property*  $(\delta'_p)$  if for every  $\epsilon > 0$  there exists  $\phi'_\epsilon \in \mathcal{L}_+^p$  such that for a.e.  $\omega$

$$u(z, \omega) \leq \epsilon \hat{\xi}(\omega) |z| \text{ for all } z \in \mathbb{R}_+^d \text{ with } \min_{1 \leq i \leq d} z_i \geq \phi'_\epsilon(\omega).$$

Because of  $d = 1$ , in [10] one has  $|z| = z$  for all  $z \in \mathbb{R}_+$ , which causes the growth properties  $(\delta_p)$  and  $(\delta'_p)$  to be indistinguishable. Growth property  $(\delta'_p)$ , for  $p \geq 1$ , can already be found in [4], and also property  $(\delta_1)$ . Growth property  $(\delta'_p)$  is also used (but just for  $m = d = 1$ ) in [10, Definition 4.1, Lemma 4.2, ff.], as can be seen by means of the following example.

**Example 4.2** (i) Suppose that there exist  $b \in (0, 1)$ ,  $\beta_1 \geq 0$  and  $\beta_2 > 0$  such that for a.e.  $\omega$

$$u(z, \omega) \leq \beta_1 + \beta_2 |z|^{1-b} \text{ for all } z \in \mathbb{R}_+^d.$$

Suppose also that  $\hat{\xi}^{-1}$  belongs to  $\mathcal{L}^{p/b}$ . Then growth condition  $(\delta_p)$  holds: similar to [10, Lemma 4.2], we simply observe that  $u(z, \omega) \leq \beta_1 + \epsilon \hat{\xi}(\omega) |z|$  for a.e.  $\omega$  and for all  $z$  with  $|z| \geq (\epsilon \hat{\xi}(\omega) / \beta_2)^{-1/b}$ . Hence,  $u(z, \omega) \leq 2\epsilon \hat{\xi}(\omega) |z|$  if  $|z| \geq \phi_{2\epsilon}(\omega)$ , where  $\phi_{2\epsilon} := \max[(\epsilon \hat{\xi} / \beta_2)^{-1/b}, \beta_1 \hat{\xi}^{-1}]$  defines a function in  $\mathcal{L}_+^p$ . This shows  $(\delta_p)$  to hold.

(ii) If  $\beta_2 = 0$  in part (i), then condition  $(\gamma_p)$  holds trivially. This implies that condition  $(\delta_p)$  then holds as well (by Proposition 4.3a below), without the above condition for  $\hat{\xi}^{-1}$ .

**Proposition 4.3** *a. For any  $p \geq 1$ ,  $(\gamma_p)$  implies  $(\delta_p)$  implies  $(\delta'_p)$ .  
b. Suppose that*

$$u(z, \omega) \text{ is nondecreasing in } z \text{ for a.e. } \omega \text{ in } \Omega.$$

*Then for any  $p \geq 1$  the three growth properties  $(\gamma_p)$ ,  $(\delta_p)$  and  $(\delta'_p)$  are equivalent.*

**PROOF** *a.  $((\gamma_p) \Rightarrow (\delta_p))$ : For any  $\epsilon > 0$  we have  $u(z, \omega)/\hat{\xi}(\omega) \leq \epsilon|z|/2 + \psi_{\epsilon/2}(\omega)$  for a.e.  $\omega$  and all  $z$ . Define  $\phi_\epsilon := 2\epsilon^{-1}\psi_{\epsilon/2} \in \mathcal{L}_+^p$ . Then  $|z| \geq \phi_\epsilon(\omega)$  is easily seen to imply  $u(z, \omega)/\hat{\xi}(\omega) \leq \epsilon|z|$ .*

*$((\delta_p) \Rightarrow (\delta'_p))$ : This follows simply from the implication  $\min_i z_i \geq \phi_\epsilon(\omega) \Rightarrow |z| \geq \phi_\epsilon(\omega)$ .*

*b.  $((\delta'_p) \Rightarrow (\delta_p))$ : For any  $\epsilon > 0$  let  $\phi'_\epsilon$  be as in the definition of  $(\delta'_p)$ . Set  $\phi_\epsilon := d\phi'_\epsilon$ , with  $\epsilon' := \epsilon/d^{1/2}$ . Then, for any  $z \in \mathbb{R}_+^d$ , let  $z' := (\hat{z}, \dots, \hat{z})$ , where  $\hat{z} := \max_i z_i$ . Then  $|z| \geq \phi_\epsilon(\omega) \Rightarrow |z'| = d^{1/2}\hat{z} \geq \phi'_\epsilon(\omega)$ , which causes  $u(z', \omega)/\hat{\xi}(\omega) \leq \epsilon'|z'| = \epsilon\hat{z} \leq \epsilon|z|$ . Finally, observe that  $u(z, \omega) \leq u(z', \omega)$  by monotonicity of  $u$ , since obviously  $z' \geq z$ .*

*$((\delta_p) \Rightarrow (\gamma_p))$ : Define  $\psi_\epsilon := d^{1/2}(\phi_1 + \phi_\epsilon) \in \mathcal{L}_+^p$ , with  $\phi_1$  (for  $\epsilon := 1$ ) and  $\phi_\epsilon$  as in the definition of condition  $(\delta_p)$ . Then  $\psi_\epsilon(\omega) = |z_\epsilon(\omega)|$ , where  $z_\epsilon(\omega) \in \mathbb{R}_+^d$  is the vector all of whose components are equal to  $\phi_1(\omega) + \phi_\epsilon(\omega)$ . Observe that  $u(z_\epsilon(\omega), \omega) \leq \hat{\xi}(\omega)\psi_\epsilon(\omega)$  by  $(\delta_p)$  (for  $\epsilon := 1$ ), in view of  $\psi_\epsilon(\omega) = |z_\epsilon(\omega)| \geq \phi_1(\omega)$ . Let  $\omega \in \Omega$  be arbitrary and nonexceptional and let  $z \in \mathbb{R}_+^d$  be arbitrary. Now either  $z \leq z_\epsilon(\omega)$  (i.e., componentwise) or not. In the latter case one has  $|z| \geq \phi_\epsilon(\omega)$  (since at least one coordinate must be greater than  $\psi_\epsilon(\omega)$ ), which implies  $u(z, \omega) \leq \epsilon\hat{\xi}(\omega)|z|$ . In the former case one has  $u(z, \omega) \leq u(z_\epsilon(\omega), \omega)$  by monotonicity of  $u$ , which gives  $u(z, \omega) \leq \hat{\xi}(\omega)\psi_\epsilon(\omega)$  when it is combined with the earlier inequality for  $u(z_\epsilon(\omega), \omega)$ . We conclude that in either case  $u(z, \omega) \leq \epsilon\hat{\xi}(\omega)|z| + \hat{\xi}(\omega)\psi_\epsilon(\omega)$ . That is to say,  $(\gamma_p)$  has been shown to hold. QED*

It is intuitively obvious that the global growth control of  $u$ , as exercised by  $(\gamma_p)$ , cannot be maintained under  $(\delta_p)$  and  $(\delta'_p)$ , which only exercise such control outside a certain radius from the origin. This is confirmed by the following example, which shows that the implications in Proposition 4.3a cannot be reverted without additional conditions such as monotonicity.

**Example 4.4** Let  $d = 1$  and consider  $\Omega = (0, 1)$  with the Lebesgue measure. Let  $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be as follows:

$$u(z, \omega) := \begin{cases} \sqrt{z-1} & \text{if } z \geq 1, \\ \omega^{-1}(1-z) & \text{if } z < 1 \end{cases}$$

Also, let  $\hat{\xi} \equiv 1$ . Then  $(\gamma_p)$  cannot hold (since  $1/\omega = u(0, \omega) \leq \psi_\epsilon(\omega)$  would force non-integrability of  $\psi_\epsilon$ ). However, for  $z \geq \phi_\epsilon(\omega) := \max(1, \epsilon^{-2})$  one has  $u(z, \omega) \leq \epsilon|z|$ .

We begin to apply our results of section 2 to situations – rather they are *generalizations* of such situations – considered in [4].

**Corollary 4.5** ([4, Main Theorem]) *Suppose that  $(\xi_{i,j})$  has diagonal structure with  $\text{ess inf}_\Omega \hat{\xi} > 0$  and that  $u(z, \omega)$  is upper semicontinuous and nondecreasing in  $z$  for a.e.  $\omega$ . Suppose also that  $u$  has growth property  $(\delta'_1)$ . Then problem  $(EC_1)$  has an optimal solution that is also an optimal solution of  $(IC_1)$ .*

**PROOF.** Proposition 4.3 implies that  $(\gamma_1)$  holds. We may now apply Proposition 2.6, which gives the existence of an optimal solution  $x_*$  of  $(EC_0)$ , with  $\int_\Omega x_* \hat{\xi} d\mu < +\infty$ , that is optimal for  $(IC_0)$  at the same time. By remark 2.7,  $x_*$  is also an optimal solution of  $(EC_1)$  and  $(IC_1)$ . QED

**Corollary 4.6** ([4, Theorem 6.1]) *Suppose that  $u(z, \omega)$  is upper semicontinuous in  $z$  for a.e.  $\omega$  in  $\Omega$ . Suppose also that  $\text{ess inf}_\Omega \hat{\xi} > 0$  and that  $u$  has growth property  $(\delta_1)$ , together with the following additional property: for every  $\eta \in \mathcal{L}_+^1$  there exists  $\zeta \in \mathcal{L}_+^1$  such that  $|z| \leq \eta(\omega)$  implies  $u(z, \omega) \leq \zeta(\omega)\hat{\xi}(\omega)$ . Then problem  $(IC_1)$  has an optimal solution.*

**PROOF.** To prove that  $u$  has growth property  $(\gamma_1)$ , let  $\epsilon > 0$  be arbitrary. By  $(\delta_1)$  there exists  $\phi_\epsilon \in \mathcal{L}_+^1$  such that  $|z| \geq \phi_\epsilon(\omega)$  implies  $u(z, \omega)/\hat{\xi}(\omega) \leq \epsilon|z|$ . By the additional property

there exists  $\zeta_\epsilon \in \mathcal{L}_+^1$  such that  $|z| < \phi_\epsilon(\omega)$  implies  $u(z, \omega)/\hat{\xi}(\omega) \leq \zeta_\epsilon(\omega)$ . Together, this means that  $u(z, \omega)/\hat{\xi}(\omega) \leq \epsilon|z| + \zeta_\epsilon(\omega)$  for all  $z$ . This proves  $(\gamma_1)$ . All conditions of Proposition 2.5 are now fulfilled, so there exists an optimal solution  $x_*$  of problem  $(IC_0)$ , with  $\int_\Omega x_* \hat{\xi} d\mu < +\infty$ . By Remark 2.7  $x_*$  is also an optimal solution of  $(IC_1)$ . QED

**Corollary 4.7** ([4, Theorem 6.2]) *Suppose that  $(\Omega, \mathcal{F}, \mu)$  is nonatomic, that  $u(z, \omega)$  is upper semicontinuous and nondecreasing in  $z$  for a.e.  $\omega$  in  $\Omega$  and that  $u(z, \omega)$  is increasing in  $z$  for all  $\omega$  in some non-null subset of  $\Omega$ . Suppose also that  $(\xi_{i,j})$  has diagonal structure, is order equivalent to  $\hat{\xi}$ , with  $\hat{\xi}^{-1} \in \mathcal{L}^p$ . Suppose further that  $u$  is nonnegative and has growth property  $(\delta'_p)$ . Then problem  $(EC_p)$  has an optimal solution that is also an optimal solution of  $(IC_p)$ .*

**PROOF.** Let us check that the conditions of Theorem 2.8 hold. Here we have  $\Omega^{p^a} = \emptyset$ , so that the concavity condition holds vacuously. Also, by Remark 2.4,  $u$  is clearly nonsatiated with respect to  $(\xi_{i,j})$ . By Proposition 4.3,  $u$  has property  $(\gamma_p)$ , since  $u(z, \omega)$  is certainly nondecreasing in  $z$ . By  $(\gamma_p)$ , we get for  $\tilde{x} \equiv 0$  that  $0 \leq u(\tilde{x}(\cdot), \cdot)/\hat{\xi} \leq \psi_1$  (take  $\epsilon = 1$ ). By  $u \geq 0$ , this proves that  $u(\tilde{x}(\cdot), \cdot)$  belongs to  $\mathcal{L}^p$ . So all conditions of Theorem 2.8 hold. It follows that there exists an optimal solution of  $(EC_p)$  that is also an optimal solution of  $(IC_p)$ . QED

Even as specializations of Theorem 2.8, the above corollaries still improve the corresponding results in [4] in a number of respects. For instance, Corollaries 4.5 and 4.6 do not require  $(\Omega, \mathcal{F}, \mu)$  to be nonatomic, Corollary 4.7 does not require  $u(z, \omega)$  to be increasing for a.e.  $\omega$  and none of the three corollaries requires  $\xi_i \equiv e_i$ . Besides, they allow for easy improvements that have not been considered in [4]. For instance, in Corollary 4.7 one could also consider a general measure space instead of a nonatomic one by introducing for  $\omega \in \Omega^{p^a}$  extra concavity for  $u(z, \omega)$  in the variable  $z$ , just as in Theorem 2.8. Also, in that same corollary, one could omit the nondecreasingness of  $u(z, \omega)$  in  $z$  for most  $\omega$  (except for those  $\omega$  that are in the non-null set mentioned in the statement) by requiring  $(\gamma_p)$  to hold instead of  $(\delta'_p)$ . This is illustrated by the following examples:

**Example 4.8** Let  $\Omega$  be the unit interval, equipped with Lebesgue measure  $\mu$ . Let  $m = d = 1$ ,  $\eta \in (0, 1]$  and define the utility function as follows:

$$u(z, \omega) := \begin{cases} -z^2 & \text{if } \omega \leq 1 - \eta \\ \sqrt{z\omega} & \text{if } \omega > 1 - \eta \end{cases}$$

[Here one could think of  $1 - \eta$  as some critical value; if the state of nature  $\omega$  is less than this value, the benefit of consumption is completely reversed.] Consider the problems  $(IC_p)$  and  $(EC_p)$  with  $\xi_1 \equiv 1$ . It is obvious that  $u$  satisfies growth condition  $(\gamma_p)$  for any  $p \geq 1$  and that  $u(\tilde{x}(\omega), \omega) = 0$  on  $\Omega$  for  $\tilde{x} \equiv 0$ . Even though  $u(z, \omega)$  is *decreasing* in  $z$  for  $\omega \in [0, 1 - \eta]$ , the conditions of Theorem 2.8, and in particular essential nonsatiation, are valid. This theorem therefore establishes existence of an optimal solution of  $(IC_p)$  and  $(EC_p)$  for every  $p \geq 1$  (note that  $(IC_p)$  always has  $x \equiv \alpha_1$  as a feasible solution – cf. Remark 2.9). It is illuminating to inspect this result by a more complete analysis of this example, based on an application of Theorem 3.2 (or Corollary 3.3). By this result the optimal solution  $x_*$  of  $(IC_p)$  must be feasible and must satisfy  $x_*(\omega) \in \operatorname{argmax}_{z \geq 0} u(z, \omega) - \lambda_1 z$  a.e. for some  $\lambda_1 \geq 0$ . If  $\lambda_1 = 0$ , then for  $\omega > 1 - \eta$  the above “argmax set” would be empty, which would give a contradiction. So the only possibility is  $\lambda_1 > 0$  (note that this is in agreement with Corollary 3.3). For a.e.  $\omega \in [0, 1 - \eta]$  this gives  $x_*(\omega) = 0$ . For a.e.  $\omega \in (1 - \eta, 1]$  the above pointwise maximum principle gives  $x_*(\omega) = \omega/4\lambda_1^2$ . To satisfy complementary slackness we also need  $\int_0^1 x_* = \alpha_1$ , and this is easily seen to be solved for  $\lambda_1 = [(2\eta - \eta^2)/8\alpha_1]^{1/2}$ . The sufficiency part of Theorem 3.2 now also guarantees that the above  $x_*$  is an optimal solution of  $(IC_p)$ . In fact, the above derivation shows that it is essentially (i.e., apart from null sets) the unique optimal solution of  $(IC_p)$  and  $EC_p)$ .

**Example 4.9** Let  $\Omega$  be the unit interval, equipped with Lebesgue measure  $\mu$ . Let  $m = d = 1$ ,  $\eta \in [0, 1]$  and define the utility function as follows:

$$u(z, \omega) := \begin{cases} \min(z\sqrt{\omega}, 1) & \text{if } \omega \leq 1 - \eta \\ \sqrt{z\omega} & \text{if } \omega > 1 - \eta \end{cases}$$

Consider the problems  $(IC_p)$  and  $(EC_p)$  with  $\xi_1 \equiv 1$ . It is not hard to check that  $u$  satisfies growth condition  $(\gamma_p)$  for any  $p \geq 1$  and that  $u(\tilde{x}(\omega), \omega) = 0$  on  $\Omega$  for  $\tilde{x} \equiv 0$ . However, in Case 1 below the essential satiation condition is violated:

*Case 1:*  $\eta = 0$ ,  $\alpha_1 = 2$ . This is precisely the example stated in [4, p. 502]. Although in this case the problem is completely elementary, we give a formal derivation for reasons of comparison with case 2 below. First of all, because  $u(z, \omega)$  is nondecreasing in  $z$ , any optimal solution of  $(IC_p)$  also leads to an optimal solution of  $(EC_p)$  (see the proof of Proposition 2.6 – it turns out that this time we cannot use complementary slackness). So it makes sense to start looking for an optimal solution of  $(IC_p)$ . By Theorem 3.2, to find an optimal solution  $x_*$  of  $(IC_p)$  we must find a multiplier  $\lambda_1 \geq 0$  such that  $x_*(\omega) \in \operatorname{argmax}_{z \geq 0} u(z, \omega) - \lambda_1 z$  a.e. If  $\lambda_1 > 0$ , then the pointwise maximum principle implies  $x_*(\omega) = 0$  if  $\sqrt{\omega} < \lambda_1$  and  $x_*(\omega) = 1/\sqrt{\omega}$  if  $\sqrt{\omega} > \lambda_1$ . This clearly violates  $\int_0^1 x_* = 2$ , which must hold by complementary slackness in this case. So  $\lambda_1 > 0$  is impossible, and we are left with  $\lambda_1 = 0$ . In this case the pointwise maximum principle implies  $x_*(\omega) \geq 1/\sqrt{\omega}$  a.e. Together with the feasibility constraint  $\int_0^1 x_* \leq 2$ , this implies  $x_*(\omega) = 1/\sqrt{\omega}$  a.e. Observe that  $x_* \in \mathcal{L}_Z^1$ , but  $x_* \notin \mathcal{L}_Z^2$ . So, by the sufficiency part of Theorem 3.2,  $x_*$  is the essentially unique optimal solution of  $(IC_0)$ ,  $(EC_0)$ ,  $(IC_1)$  and  $(EC_1)$ , but *not* of  $(IC_2)$  or  $(EC_2)$ . In fact, it follows that  $(IC_2)$  does not have an optimal solution at all, since the preceding application of the necessary conditions in Theorem 3.2 gave us the above  $x_*$  as its only candidate for optimality. Similar nonexistence can be proven for  $(EC_2)$  by considering an analogue of Theorem 3.2, mentioned in footnote 2.

*Case 2:*  $\eta = 0.19$ ,  $\alpha_1 = 5.89875$ . This time the essential nonsatiation condition is valid (see Remark 2.4), so Theorem 2.8 applies: we know in advance that there exists an optimal solution of  $(IC_p)$  and  $(EC_p)$  for any  $p \geq 1$ . This is confirmed by determining the optimal solution explicitly. Again, by Theorem 3.2, the optimal solution  $x_*$  of  $(IC_p)$  must be feasible and satisfy  $x_*(\omega) \in \operatorname{argmax}_{z \geq 0} u(z, \omega) - \lambda_1 z$  a.e. for some  $\lambda_1 \geq 0$ . If  $\lambda_1 = 0$ , then for  $\omega > 0.81$  the pointwise maximum principle would be self-contradictory, its “argmax set” being empty. So we are left with  $\lambda_1 > 0$ . For  $\omega > 0.81$ , the set  $\operatorname{argmax}_{z \geq 0} \sqrt{z\omega} - \lambda_1 z$  is the singleton  $\{\omega/4\lambda_1^2\}$  (see Example 4.8). For  $\omega \leq .81$ , the set  $\operatorname{argmax}_{z \geq 0} \min(z\sqrt{\omega}, 1) - \lambda_1 z$  is the singleton  $\{1/\sqrt{\omega}\}$  if  $\lambda_1 < \sqrt{\omega}$ , but if  $\lambda_1 > \sqrt{\omega}$  it is the singleton  $\{0\}$ . We now distinguish (a)  $\lambda_1 \geq 0.9$  and (b)  $0 < \lambda_1 < 0.9$ . In case (a) we find, by the pointwise maximum principle,  $x_*(\omega) = 0$  for a.e.  $\omega \leq 0.81$ , by  $\omega < \lambda_1^2$ . In case (b) we find (a.e.), by the same principle,  $x_*(\omega) = 0$  if  $\omega \in [0, \lambda_1^2)$  and  $x_*(\omega) = 1/\sqrt{\omega}$  if  $\omega \in (\lambda_1^2, 0.81]$ . In both cases the equation  $\int_0^1 x_* = 5.89875$  is forced by complementary slackness, since  $\lambda_1 > 0$ . In case (a) this equation gives immediately  $\lambda_1 = 0.0853\dots$ , which is in conflict with the underlying inequality (a). In case (b) that same equation is the cubic equation  $1.8 - 2\lambda_1 + 0.0429875\lambda_1^{-2} = 5.89875$ , of which  $\lambda_1 = 0.1$  is the only root complying with (b). By the sufficiency part of Theorem 3.2,  $x_*(\omega) = 0$  if  $\omega \in [0, 0.01)$ ,  $x_*(\omega) = 1/\sqrt{\omega}$  if  $\omega \in (0.01, 0.81]$  and  $x_*(\omega) = 2.5\omega$  if  $\omega \in (0.81, 1]$  is an optimal solution of  $(IC_p)$  and  $(EC_p)$  for any  $p = 0$  or  $p \geq 1$  (observe that  $x_* \in \mathcal{L}_Z^p$  for any  $p \geq 1$ ). Moreover, our derivation shows  $x_*$  to be the essentially unique optimal solution of  $(IC_p)$  and  $(EC_p)$ .

Next, we turn to the existence results in [10].

**Corollary 4.10** ([10, Proposition 4.2]) *Suppose that  $u(z, \omega)$  is upper semicontinuous and non-decreasing in  $z$  for a.e.  $\omega$  in  $\Omega$  and concave in  $z$  for a.e.  $\omega$  in  $\Omega^{p^a}$ . Suppose also that  $u$  has growth property  $(\delta'_1)$  and that  $(\xi_{i,j})$  has diagonal structure. Then problem  $(IC_0)$  has an optimal solution  $x_*$ ,  $\int_\Omega \hat{\xi} |x_*| d\mu < +\infty$ , that is also an optimal solution of  $(EC_0)$ .*

PROOF. Condition  $(\gamma_1)$  holds by Proposition 4.3, since  $u(z, \omega)$  is nondecreasing in  $z$ . The conditions of Proposition 2.6 are thus fulfilled. This gives the existence result. QED

**Corollary 4.11** ([10, Theorems 4.1, 4.2]) *Suppose that  $u(z, \omega)$  is upper semicontinuous and non-decreasing in  $z$  for a.e.  $\omega$  in  $\Omega$ , concave in  $z$  for a.e.  $\omega$  in  $\Omega^{p^a}$  and increasing for a.e.  $\omega$  in some non-null subset of  $\Omega$ . Suppose also that  $u$  has growth property  $(\delta'_p)$  and that  $(\xi_{i,j})$  has diagonal structure and is order-equivalent to  $\hat{\xi}$  with  $\hat{\xi}^{-1} \in \mathcal{L}^p$ . Suppose also that there exists some  $\tilde{x} \in \mathcal{L}^p$  for which  $\omega \mapsto u(\tilde{x}(\omega), \omega)$  is essentially bounded. Then problem  $(EC_p)$  has an optimal solution that is also an optimal solution of  $(IC_p)$ .*

PROOF. Again, by Proposition 4.3  $u$  has property  $(\gamma_p)$  in view of the given monotonicity of  $u(z, \omega)$  in  $z$ . Since  $\hat{\xi}^{-1} \in \mathcal{L}^p$ , it is evident that  $\omega \mapsto u(\tilde{x}(\omega), \omega)/\hat{\xi}(\omega)$  is  $p$ -integrable. So all the conditions of Theorem 2.8 are valid and the result follows. QED

Observe that, by Example 4.2, the upper bounds for  $u$  in Theorems 4.1, 4.2 of [10] both imply the validity of  $(\delta'_p)$ , as used in the above corollary. Other improvements over the conditions used for the utility  $u$  in [10] are also quite evident; for instance, our concavity and monotonicity conditions are considerably weaker. We conclude this section by giving a very historical application of Theorem 2.8:

**Example 4.12** Let  $\Omega$  be the unit interval, equipped with Lebesgue measure  $\mu$ . The following formulation can be given of Newton's classical problem of least resistance [1, p. 17].

$$\inf_{y \in \mathcal{Y}^p} \left\{ \int_0^1 \frac{\omega}{1 + \dot{y}^2(\omega)} \mu(d\omega) : y(0) = 0, y(1) = \alpha_1, \dot{y} \geq 0 \right\}.$$

Here  $\alpha_1 > 0$  and  $\mathcal{Y}^p$  stands for the class of all  $p$ -absolutely continuous functions, i.e., the set of all functions  $y : [0, 1] \rightarrow \mathbb{R}$  for which there exists  $\dot{y} \in \mathcal{L}^p$  such that  $y(\omega) = y(0) + \int_0^\omega \dot{y} d\mu$  for every  $\omega \in \Omega$ . In [1] this problem is only studied for  $p = 1$ , but we wish to consider it also for  $p \geq 1$ . By substitution of  $x := \dot{y}$ , Newton's problem is seen to be precisely of the form  $(EC_p)$ , with  $m = d = 1$ ,  $u(z, \omega) := -\omega/(1 + z^2)$ ,  $\hat{\xi} = \xi_{1,1} \equiv 1$  (observe that  $\int_0^1 x = \int_0^1 \dot{y} = y(1) - y(0) = \alpha_1$ ). It is easy to check that all conditions of Theorem 2.8 hold in this example for any  $p \geq 1$  (use Remark 2.4). Thus, for any  $p \geq 1$  the above problem has an optimal solution. See [1, p. 60 ff.] for a complete description of this optimal solution. Just as in Examples 4.8 and 4.9, it could also be derived via Theorem 3.2.

## 5 Extensions

### 5.1 State-contingent consumption sets

The fact that  $u(z, \omega)$  is allowed to be  $-\infty$  can be exploited to absorb pointwise constraints on consumption of the type

$$x(\omega) \in X(\omega) \text{ for a.e. } \omega \text{ in } \Omega$$

in a very simple and direct way into the model. Here  $X : \Omega \rightarrow 2^{\mathbb{R}^d_+}$  denotes a multifunction with a  $\mathcal{F} \times \mathcal{B}(\mathbb{R}^d_+)$ -measurable graph. Such absorption comes about very simply by introducing

$$\tilde{u}(z, \omega) := \begin{cases} u(z, \omega) & \text{if } z \in X(\omega) \\ -\infty & \text{if } z \notin X(\omega) \end{cases}$$

Of course now the conditions for  $X$  must be such that  $\tilde{u}$  can be substituted for  $u$  in the various conditions. Observe that for  $\tilde{u}(z, \omega)$  to be upper semicontinuous [concave] in the variable  $z$ , it is sufficient to have  $X(\omega)$  closed [convex]. The reformulation of  $(\gamma_p)$  for  $\tilde{u}$  obviously yields a version that is easier to satisfy than the one used previously, and in Definition 2.3 one must simply replace the maximization domain  $\mathbb{R}^d_+$  by  $X(\omega)$ .

### 5.2 Optimal consumption over time

Other extensions and applications are to a time-dependent situation. First of all, one can specialize  $(IC_p)$  and  $(EC_p)$  to deterministic variational problems by setting  $\Omega := [0, T]$  and taking  $\mathcal{F}$  equal to the Lebesgue  $\sigma$ -algebra and  $\mu$  equal to the Lebesgue measure on  $[0, T]$ . This is the situation of optimal consumption or resource allocation over time, as considered by Aumann and Perles [4] and several others (e.g., see [17]).

Secondly, as in [10], one can *automatically* extend the main results of this paper to a stochastic time-dependent situation, simply by a suitable choice of the underlying measure space. In addition to the space  $\Omega$  of states of nature, whose distribution is given by the (probability) measure  $\mu$ ,

there is now also a time interval  $[0, T]$  and a filtration  $\{\mathcal{F}_t : t \in [0, T]\}$  of information  $\sigma$ -algebras (e.g., this could be the natural filtration with respect to some stochastic process of signals). Equip  $\tilde{\Omega} := [0, T] \times \Omega$  with the  $\sigma$ -algebra  $\tilde{\mathcal{F}}$  of progressively measurable sets (i.e.,  $A \in \tilde{\mathcal{F}}$  if and only if the section of  $A$  at  $t$  belongs to  $\mathcal{F}_t$  for each  $t$ ). If, moreover, a final wealth term is added to the objective function, then problem  $(IC_p)$  gets the following form (of course, the same can be done for  $(EC_p)$ ):

$$(\tilde{IC}_p) \quad \sup_{x \in \tilde{\mathcal{L}}_Z^p} \{\tilde{U}(x) : \int_{\Omega} \int_0^T x_t(\omega) \cdot \xi_{i,t}(\omega) dt \mu(d\omega) \leq \alpha_i, i = 1, \dots, m\}.$$

Here

$$\tilde{U}(x) := \int_{\Omega} \int_0^T u_t(x_t(\omega), \omega) dt \mu(d\omega) + \int_{\Omega} u_T(x_T(\omega), \omega) \mu(d\omega)$$

and  $\tilde{\mathcal{L}}_Z^p$  stands for  $(\mathcal{L}_+^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu}))^d$ , where  $\tilde{\mu} := \tilde{\mu}_1 + \tilde{\mu}_2$ , with  $\tilde{\mu}_1$  the product of the Lebesgue measure on  $[0, T]$  and  $\mu$ , and  $\tilde{\mu}_2$  the measure on  $[0, T] \times \Omega$  that is entirely concentrated on the subset  $\{T\} \times \Omega$  and coincides there with  $\mu$  (i.e.,  $\tilde{\mu}_2(A \times B) := 1_A(T)\mu(B)$ ). Observe that the strip  $\{T\} \times \Omega$  has  $\tilde{\mu}_1$ -measure zero, which makes it possible to treat the restrictions  $x|_{[0, T] \times \Omega}$  and  $x|_{\{T\} \times \Omega}$  as separate functions. The reformulated problem (10) of Cox and Huang [10], an optimal consumption-portfolio problem in static form, is a special case of  $(\tilde{IC}_p)$ .

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