On the Connes-Kreimer construction of Hopf Algebras

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Abstract: We give a universal construction of families of Hopf $\mathbb{P}$-algebras for any Hopf operad $\mathbb{P}$. As a special case, we recover the Connes-Kreimer Hopf algebra of rooted trees.

Keywords: Hopf operad, Hopf algebra, Hochschild cohomology.

In [K], [CK] a Hopf algebra H of rooted trees is discussed. This algebra originates in problems of renormalisation [K] and is closely related to the Hopf algebra introduced in [CM] in the context of cyclic homology and foliations. The algebra H is the polynomial algebra on countably many indeterminates T, one for each finite rooted tree T. Its comultiplication is given by the formula

$$\Delta(T) = 1 \otimes T + T \otimes 1 + \sum_c F_c \otimes R_c,$$

see [CK]. Here c ranges over all "cuts" (prunings) of the tree T. Such cuts are assumed non-empty, and to contain at most one edge on each branch. R_c is the part of the tree which remains after having performed the pruning, and F_c is the product of subtrees which have fallen on the ground. In [CK] it is proved that this comultiplication indeed makes H into a Hopf algebra. Furthermore, H is equipped with a linear endomorphism λ, which is a universal cocycle for a suitably defined Hochschild cohomology of Hopf algebras.

The first aim of this note is to show that all these properties can in fact be deduced from a more basic universal property of H. Namely, H is the initial object in the category of (commutative unitary) algebras equipped with a linear endomorphism. Having realized that this is the case, it becomes clear that H is in fact equipped with a large family of Hopf algebra structures, all making the endomorphism λ into a universal cocycle for the corresponding Hochschild cohomology. For example, for any two complex numbers q_1 and q_2, there is a coproduct on H, uniquely determined by the identity

$$\Delta(\lambda(T)) = \sum q_1^{|T_{(1)}|} \cdot T_{(1)} \otimes \lambda(T_{(2)}) + \lambda(T_{(1)}) \otimes q_2^{|T_{(2)}|} \cdot T_{(2)},$$

where $|T|$ denotes the number of nodes in the tree T. For $q_1 = 1$ and $q_2 = 0$ one recovers the Hopf algebra structure of [CK].

The second aim is to describe how this construction applies more generally to "algebras" for any operad $\mathbb{P}$ on an additive category, as soon as one has a well-behaved tensor product of algebras. More precisely, we will show that if $\mathbb{P}$ is a "Hopf operad" on a symmetric monoidal additive category, then the initial object in the category of $\mathbb{P}$-algebras equipped with a "linear" endomorphism is naturally equipped with a family of natural Hopf $\mathbb{P}$-algebra structures. The algebra of rooted trees then becomes the extreme instance of this construction where the operad $\mathbb{P}$ is the unit object in each degree.

Acknowledgements. My attention was first drawn to the algebra H by A. Connes at the "Karoubi Fest" in Paris (November 1998). I would like to thank Ezra Getzler and André Joyal for helpful discussion. I am indebted to the Dutch Science Foundation (NWO) for financial support. The main results of this paper were first presented at the Newton Institute, in February 1999.

1 Operads and algebras.

1.1 The underlying category. In this preliminary section we will consider operads on a category $\mathcal{C}$. We will assume that $\mathcal{C}$ is a symmetric monoidal additive category, with countable sums and quotients of actions by finite groups on objects of $\mathcal{C}$. (In most cases, $\mathcal{C}$ will be closed under all small colimits.) As an example, the reader may wish to keep the category of vector spaces over a field k in mind in what follows. We will write k for the unit object of $\mathcal{C}$, and a, l, r for the associativity and unit isomorphisms. The symmetry will be denoted by c, with components $c_{X,Y} : X \otimes Y \to Y \otimes X$. We will assume that $\otimes$ is an additive functor in each variable separately. Often, the isomorphisms a, l, r will be suppressed from the notation, and we identify $k \otimes X$ with X, and $X \otimes (Y \otimes Z)$ with $(X \otimes Y) \otimes Z$, etc. This is justified, on the basis of Mac Lane's coherence theorem. See [CWM] for details.

1.2 Operads. ([M], [KM], [GK], ...) We will consider operads $\mathbb{P}$ on such a category $\mathcal{C}$, and write $\mathbb{P}(n)$ for the object (of $\mathcal{C}$) of n-ary operations. We will always assume that our operads have a distinguished "unit element" $u : k \to \mathbb{P}(0)$. We will *not* assume that this map is an isomorphism, i.e. that $\mathbb{P}$ is unitary in the sense of [KM]. Many operads are unitary, but the constructions of 1.3 lead us out of unitary operads. Note that the unit $u : k \to \mathbb{P}(0)$ provides us with a unit $u_A : k \to A$ in any $\mathbb{P}$-algebra A.

The functor underlying the monad on $\mathcal{C}$ whose algebras are $\mathbb{P}$-algebras will be denoted by $F_{\mathbb{P}} : \mathcal{C} \to \mathcal{C}$; so for any object V in $\mathcal{C}$,

$$F_{\mathbb{P}}(V) = \coprod_{n \geq 0} \mathbb{P}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

This object $F_{\mathbb{P}}(V)$ is the free $\mathbb{P}$-algebra generated by V.

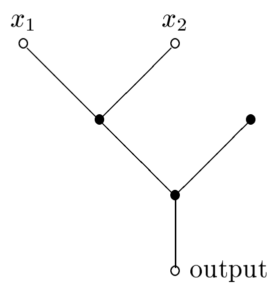
1.3 Two constructions. (i) If $\mathbb{P}$ is an operad on $\mathcal{C}$ and G is an object of $\mathcal{C}$, there is an operad $\mathbb{P}_G$ whose algebras are $\mathbb{P}$-algebras equipped with a map from G. Thus, $\mathbb{P}_G$ is obtained from $\mathbb{P}$ by adding G to the space $\mathbb{P}(0)$ of "constants" (nullary operations). Explicitly,

$$\mathbb{P}_G(n) = \coprod_{p \geq 0} \mathbb{P}(n+p) \otimes_{\Sigma_p} G^{\otimes p}.$$

Note that the initial $\mathbb{P}_G$-algebra $\mathbb{P}_G(0)$ is the free $\mathbb{P}$-algebra $F_{\mathbb{P}}(G)$ on G.

(ii) Let $\mathbb{P}$ be an operad on $\mathcal{C}$. A $\mathbb{P}[t]$-*algebra* is a pair (A, α) where A is a $\mathbb{P}$-algebra and $\alpha : A \to A$ is a map in $\mathcal{C}$. (We will often refer to maps in $\mathcal{C}$ as "*linear maps*", to contrast them with $\mathbb{P}$-algebra homomorphisms.) A map between $\mathbb{P}[t]$-algebras $(A, \alpha) \to (B, \beta)$ is a map of $\mathbb{P}$-algebras $f : A \to B$ such that $\beta f = f\alpha$. This defines a category of $\mathbb{P}[t]$-algebras. This category is the category of algebras for an operad, again denoted $\mathbb{P}[t]$. It is the operad obtained by freely adjoining a unary operation "t" to $\mathbb{P}$. It is not difficult to give an explicit description of $\mathbb{P}[t]$ in terms of trees, analogous to constructions in [GK]. We will not need such an explicit description.

1.4 Example. Let $\mathcal{C}$ be the category of vector spaces over a field k, and let $\mathbb{P}$ be the operad $\mathbb{P}(n) = k$. Its algebras are commutative unitary k-algebras, and the monad $F_{\mathbb{P}}$ associated to $\mathbb{P}$ is the symmetric algebra functor. The associated operad $\mathbb{P}[t]$ can be described as follows. The space $\mathbb{P}[t](n)$ is the vector space on rooted finite trees T, with one "*output node*", the root, and n "*input nodes*", labelled by $x_1, \dots, x_n$. The *inner nodes* represent application of the new unary operation t. For example, the tree



represents the binary operation $t(t(x_1 \cdot x_2) \cdot t(1))$. The tree $\circ$ consisting of just the output vertex represents the element (nullary operation) 1. We will refer to the algebra $\mathbb{P}[t](0)$ as the algebra of *finite rooted trees.* It can be identified with the Connes-Kreimer algebra H mentioned in the introduction. (There is a slight difference in notation, in that we have merged a product of trees into one tree with a new output node added to it.)

2 Hopf operads.

2.1 Coalgebras. Let $\mathcal{C}$ be a category as in 1.1. A *coalgebra* $\underline{X} = (X, \varepsilon, \Delta)$ is an object X of $\mathcal{C}$ equipped with a coassociative comultiplication $\Delta : X \to X \otimes X$, and a counit $\varepsilon : X \to k$ for this comultiplication. The associated category $\mathrm{Coalg}(\mathcal{C})$ is again a (symmetric) monoidal category, with the usual tensor product ($\underline{X} \otimes \underline{Y}$ is $X \otimes Y$ with as comultiplication the composition of $\Delta_X \otimes \Delta_Y : X \otimes Y \to (X \otimes X) \otimes (Y \otimes Y)$ and the symmetry $X \otimes c \otimes Y : (X \otimes X) \otimes (Y \otimes Y) \to (X \otimes Y) \otimes (X \otimes Y)$).

2.2 Hopf operads. A *Hopf operad* on $\mathcal{C}$ is an operad $\mathbb{P}$ on $\mathcal{C}$ equipped with additional structure making it an operad on $\mathrm{Coalg}(\mathcal{C})$. Thus, each $\mathbb{P}(n)$ has the structure of a coalgebra,

$$k \xleftarrow{\varepsilon} \mathbb{P}(n) \xrightarrow{\Delta} \mathbb{P}(n) \otimes \mathbb{P}(n), \tag{1}$$

this structure is Σ_n-invariant, and the structure maps of the operad $\mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \cdots \otimes \mathbb{P}(k_n) \to \mathbb{P}(k_1 + \cdots + k_n)$ are coalgebra maps. The notion of a Hopf operad has been introduced in [GJ]. (But beware that their coalgebras are not necessarily counital.) I will sometimes write $\underline{\mathbb{P}}$ for this operad on $\mathrm{Coalg}(\mathcal{C})$, as opposed to the operad $\mathbb{P}$ on $\mathcal{C}$. The Hopf operad $\mathbb{P}$ is *cocommutative* if each of the coalgebras $\mathbb{P}(n)$ is.

If $\mathbb{P}$ is a Hopf operad, then the tensor product $A \otimes B$ of two $\mathbb{P}$-algebras A and B is again a $\mathbb{P}$-algebra, by the maps

$$\mathbb{P}(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{\Delta \otimes \mathrm{id}} \mathbb{P}(n) \otimes \mathbb{P}(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{c}$$
$$(\mathbb{P}(n) \otimes A^{\otimes n}) \otimes (\mathbb{P}(n) \otimes B^{\otimes n}) \longrightarrow A \otimes B.$$

Moreover, the counits $\varepsilon : \mathbb{P}(n) \to k$ in (1) make k into a $\mathbb{P}$-algebra, which is a unit for this tensor product of k-algebras. Thus, the category of $\mathbb{P}$-algebras is again a monoidal category (symmetric if $\mathbb{P}$ is cocommutative). A coalgebra in this category of $\mathbb{P}$-algebras is the same thing as a $\underline{\mathbb{P}}$-algebra in the category $\mathrm{Coalg}(\mathcal{C})$ of coalgebras, and (as in [GJ]) will be referred to as a *Hopf* $\mathbb{P}$*-algebra*.

2.3 Example. The free $\mathbb{P}$-algebra $F_{\mathbb{P}}(G)$ on an object G has a canonical Hopf $\mathbb{P}$-algebra structure, cocommutative if $\mathbb{P}$ is. Indeed, since $F_{\mathbb{P}}(G)$ is *free*, the maps $0 : G \to k$ and $\mathrm{id} \otimes 1 + 1 \otimes \mathrm{id} : G \to F_{\mathbb{P}}(G) \otimes F_{\mathbb{P}}(G)$ into $\mathbb{P}$-algebras extend uniquely to $\mathbb{P}$-algebra maps

$$k \xleftarrow{\varepsilon} F_{\mathbb{P}}(G) \xrightarrow{\Delta} F_{\mathbb{P}}(G) \otimes F_{\mathbb{P}}(G),$$

and one easily checks that this provides the claimed structure.

3 The Connes-Kreimer construction.

Let $\mathbb{P}$ be a Hopf operad on a category $\mathcal{C}$ as before, and let $\mathbb{P}[t]$ be the associated operad whose algebras are $\mathbb{P}$-algebras equipped with a "linear" endomorphism. We now present a general construction of Hopf $\mathbb{P}$-algebras, of which the Connes-Kreimer Hopf algebra is a special case.

3.1 The initial $\mathbb{P}[t]$-algebra. Let (H, λ) denote the initial $\mathbb{P}[t]$-algebra, i.e. $(H, \lambda) = \mathbb{P}[t](0)$. Thus H is a $\mathbb{P}$-algebra, $\lambda : H \to H$ is a linear map (i.e. just an arrow in $\mathcal{C}$), and these have the following universal property: For any $\mathbb{P}$-algebra A and any linear map $\alpha : A \to A$, there is a *unique* $\mathbb{P}$-algebra map $\varphi : H \to A$ such that $\alpha\varphi = \varphi\lambda$.

3.2 Lemma. *There is a unique augmentation $\varepsilon : H \to k$ with $\lambda\varepsilon = 0$.*

Proof: Apply the universal property to the $\mathbb{P}$-algebra k with the zero endomorphism. $\square$

Next, let $\sigma_1, \sigma_2 : H \to H$ be two linear maps. Let

$$(\sigma_1, \sigma_2) = \sigma_1 \otimes \lambda + \lambda \otimes \sigma_2 : H \otimes H \to H \otimes H.$$

This gives $H \otimes H$ the structure of a $\mathbb{P}[t]$-algebra. So there is a unique $\mathbb{P}$-algebra map

$$\Delta = \Delta_{\sigma_1, \sigma_2} : H \to H \otimes H$$

such that $(\sigma_1, \sigma_2) \circ \Delta = \Delta \circ \lambda$.

3.3 Lemma. (i) *If $\varepsilon\sigma_i = \varepsilon$ for $i = 1, 2$ then $\varepsilon : H \to k$ is a counit for Δ.*
(ii) If, in addition, $\Delta\sigma_i = (\sigma_i \otimes \sigma_i)\Delta$ for $i = 1, 2$ then Δ is coassociative.

Proof: (i) Consider the maps

$$(H, \lambda) \xrightarrow{\Delta} (H \otimes H, (\sigma_1, \sigma_2)) \overset{\mathrm{id}\otimes\varepsilon}{\underset{\varepsilon\otimes\mathrm{id}}{\rightrightarrows}} (H, \lambda),$$

where on the right the isomorphisms $H \otimes k = H = k \otimes H$ have been suppressed. By initiality of H, it is enough to prove that $\mathrm{id}\otimes\varepsilon$ and $\varepsilon \otimes \mathrm{id}$ are $\mathbb{P}[t]$-homomorphisms. This is indeed the case, since

$$\begin{aligned}
(\mathrm{id}\otimes\varepsilon)(\sigma_1, \sigma_2) &= (\mathrm{id}\otimes\varepsilon)(\sigma_1 \otimes \lambda + \lambda \otimes \sigma_2) \qquad \text{(definition)} \\
&= \sigma_1 \otimes \varepsilon\lambda + \lambda \otimes \varepsilon\sigma_2 \\
&= \lambda \otimes \varepsilon\sigma_2 \qquad (\varepsilon\lambda = 0) \\
&= \lambda \otimes \varepsilon \qquad \text{(assumption)} \\
&= \lambda \circ (\mathrm{id}\otimes\varepsilon),
\end{aligned}$$

and similarly $(\varepsilon \otimes \mathrm{id})(\sigma_1, \sigma_2) = \lambda \circ (\varepsilon \otimes \mathrm{id})$.

(ii) Consider the map $\nu : H \otimes H \otimes H \to H \otimes H \otimes H$,

$$\nu = \lambda \otimes \sigma_2 \otimes \sigma_2 + \sigma_1 \otimes \lambda \otimes \sigma_2 + \sigma_1 \otimes \sigma_1 \otimes \lambda.$$

This makes $H^{\otimes 3}$ into a $\mathbb{P}[t]$-algebra, so there is a unique $\mathbb{P}[t]$-homomorphism $(H, \lambda) \to (H^{\otimes 3}, \nu)$. It thus suffices to show that $(\mathrm{id}\otimes\Delta)\Delta$ and $(\Delta \otimes \mathrm{id})\Delta$ both are. For the first,

$$\begin{aligned}
(\mathrm{id}\otimes\Delta)\Delta\lambda &= (\mathrm{id}\otimes\Delta)(\sigma_1 \otimes \lambda + \lambda \otimes \sigma_2)\Delta \\
&= (\sigma_1 \otimes \Delta\lambda + \lambda \otimes \Delta\sigma_2)\Delta \\
&= (\sigma_1 \otimes \sigma_1 \otimes \lambda + \sigma_1 \otimes \lambda \otimes \sigma_2 + \lambda \otimes \sigma_2 \otimes \sigma_2)(\mathrm{id}\otimes\Delta)\Delta \\
&= \nu(\mathrm{id}\otimes\Delta)\Delta.
\end{aligned}$$

The calculation for $(\Delta \otimes \mathrm{id})\Delta$ is similar. $\square$

The preceding lemmas prove:

3.4 Theorem. *The initial $\mathbb{P}[t]$-algebra (H, λ) has a natural family of Hopf $\mathbb{P}$-algebra structures, parametrized by pairs $\sigma_1, \sigma_2 : H \to H$ satisfying the conditions of Lemma 3.3.*

3.5 Example. The conditions of Lemma 3.3 are always satisfied if one takes σ_i to be the identity $H \to H$ or the composition of the counit $\varepsilon : H \to k$ and the unit $u : k \to H$, or any convex combination $\alpha \cdot \mathrm{id} + \beta \cdot u\varepsilon : C \to C$ of these two (for $\alpha, \beta : k \to k$ with $\alpha + \beta = \mathrm{id}$). This provides many different Hopf $\mathbb{P}$-algebra structures on H.

3.6 Example. Consider again the case of the commutative unitary algebra operad of 1.4. Then H is the algebra of finite rooted trees T. Note that $\varepsilon(T) = 0$ as soon as T has at least one inner node. Write $|T|$ for the number of inner nodes of T. Now let $q_1, q_2 \in k$ be any two numbers, and let

$$\sigma_i = q_i^{|T|} \cdot T, \quad \text{for } i = 1, 2$$

Then σ_1 and σ_2 satisfy the condition of Lemma 3.3. Thus for any two $q_1, q_2 \in k$, the algebra H has a Hopf algebra structure, with the usual counit, and with comultiplication completely determined by the identity

$$\Delta\lambda(T) = \sum q_1^{|T_{(1)}|} T_{(1)} \otimes \lambda(T_{(2)}) + \lambda(T_{(1)}) \otimes q_2^{|T_{(2)}|} \cdot T_{(2)}$$

where we write $\Delta(T) = \sum T_{(1)} \otimes T_{(2)}$ as usual [S]. For the values $q_1 = 1$ and $q_2 = 0$ one finds $\sigma_1 = \mathrm{id}$ and $\sigma_2 = \varepsilon$, and one recovers the Hopf algebra structure of [CK].

3.7 Remark. The results and examples in this section have been stated for the initial $\mathbb{P}[t]$-algebra $(H, \lambda) = \mathbb{P}[t](0)$. Similar facts hold for the free $\mathbb{P}[t]$-algebra generated by any object G of $\mathcal{C}$. Writing $(H[G], \lambda)$ for this algebra and $j : G \to H[G]$ for the universal map from G, one defines $\Delta : H[G] \to H[G] \otimes H[G]$ from σ_1 and σ_2 as the unique map of $\mathbb{P}[t]$-algebras satisfying $\Delta\lambda = (\sigma_1 \otimes \lambda + \lambda \otimes \sigma_2)\Delta$ as before and extending the map $u \otimes j + j \otimes u : G \to H[G] \otimes H[G]$ (where $u : k \to H[G]$ is the unit). However, rather than doing the calculation again, this can be seen as a formal consequence of the statements made for the initial algebra, because the free $\mathbb{P}[t]$-algebra on G is the initial $\mathbb{P}_G[t]$-algebra (cf. 1.3.(i)), and $\mathbb{P}_G$ is a Hopf operad whenever $\mathbb{P}$ is.

4 Hochschild cohomology.

In [CK] it is proved that for the Connes-Kreimer algebra (H, λ) (cf. Example 3.6), the map λ is a universal 1-cocycle for Hochschild cohomology. In this section, we show that this result extends to our more general construction.

Recall the definition of the Hochschild cohomology groups $H^*(A, M)$ for any algebra A and any bimodule M, from the complex with maps $A^{\otimes n} \to M$ as cochains (see e.g. [L, formula (1.5.1.1)]). Turning around all the arrows in a diagrammatic form of this definition, one obtains a cohomology $H^*(E, C)$ of a coalgebra C with coefficients in a bicomodule E, as the cohomology of the complex $C^n(E, C) = \mathrm{Hom}_{\mathcal{C}}(E, C^{\otimes n})$. Explicitly, this is the cohomology of the simplicial abelian group with the face maps $d_i : C^{n-1}(E, C) \to C^n(E, C)$ defined for $\varphi : E \to C^{\otimes(n-1)}$ by

$$d_i(\varphi) = \begin{cases} E \xrightarrow{l} C \otimes E \xrightarrow{C \otimes \varphi} C \otimes C^{\otimes n-1} = C^{\otimes n} & (i = 0) \\ E \xrightarrow{\varphi} C^{\otimes n-1} \xrightarrow{\Delta^{(i)}} C^{\otimes n} & (0 < i < n) \\ E \xrightarrow{r} E \otimes C \xrightarrow{\varphi \otimes C} C^{\otimes n} & (i = n). \end{cases}$$

Here l and r are the left and right coactions, and $\Delta^{(i)} = C^{\otimes(i-1)} \otimes \Delta \otimes C^{(n-i-1)}$. Note that this cohomology $H^*(E, C)$ is *contravariant* in E and *covariant* in C.

In particular, given "linear" maps $\sigma_1, \sigma_2 : C \to C$, we can view C itself as a C-bimodule ${}_{\sigma_1}C_{\sigma_2}$, with left action $C \xrightarrow{\Delta} C \otimes C \xrightarrow{\sigma_1 \otimes C} C \otimes C$ and right action $C \xrightarrow{\Delta} C \otimes C \xrightarrow{C \otimes \sigma_2} C \otimes C$. We denote the corresponding cohomology by

$$HH^*_{\sigma_1, \sigma_2}(C). \tag{2}$$

A map $\varphi : C \to C$ is a 1-cocycle for this cohomology precisely when

$$\Delta \circ \varphi = (\sigma_1 \otimes \varphi + \varphi \otimes \sigma_2)\Delta. \tag{3}$$

Now let us go back to the context of a Hopf operad $\mathbb{P}$ on our underlying category $\mathcal{C}$.

4.1 Natural twisting functions. Call σ a natural twisting function if σ assigns to each Hopf $\mathbb{P}$-algebra C a linear endomorphism $\sigma = \sigma^{(C)} : C \to C$, which is natural for morphisms of augmented $\mathbb{P}$-algebras (i.e. if $f : C \to D$ is such a morphism then $f \circ \sigma^{(C)} = \sigma^{(D)} \circ f$), and has the property that $\sigma^{(k)}$ is the identity. Note that this implies that $\varepsilon \circ \sigma^{(C)} = \varepsilon$. For example, the identity $C \to C$ and the composition $C \xrightarrow{\varepsilon} k \xrightarrow{u} C$ of the augmentation and the unit are natural twisting functions, as is any convex combination $\alpha \cdot \mathrm{id} + \beta \cdot u\varepsilon : C \to C$ of these two (for $\alpha, \beta : k \to k$ with $\alpha + \beta = \mathrm{id}$).

Now let (H, λ) be the initial $\mathbb{P}[t]$-algebra, and let $\sigma_1 = \sigma_1^{(H)}, \sigma_2 = \sigma_2^{(H)} : H \to H$ be the components of two natural twisting functions. Suppose that σ_1 and σ_2 define a Hopf $\mathbb{P}$-algebra structure (H, Δ, ε) on H, by Theorem 3.4. Observe that the defining equation $(\sigma_1, \sigma_2)\Delta = \Delta\lambda$ for the coproduct states precisely that λ is a 1-cocycle for $HH^*_{\sigma_1,\sigma_2}(H)$. The following theorem is now a consequence of the universal property (3.1) of (H, λ).

4.2 Theorem. *The map λ is the universal 1-cocycle. More explicitly, if B is a Hopf $\mathbb{P}$-algebra and γ is a 1-cocycle in the complex defining $HH^*_{\sigma_1,\sigma_2}(B)$, there is a unique Hopf $\mathbb{P}$-algebra map $c_\gamma : H \to B$ such that $c_\gamma \circ \lambda = \gamma \circ c_\gamma$.*

Proof: By the universal property of H and λ, there is a unique $\mathbb{P}$-algebra map $c = c_\gamma : H \to B$ such that $\gamma c = c\lambda$. It suffices to show that c is a coalgebra map. First, we show that c is a map of augmented algebras, i.e. $\varepsilon \circ c = \varepsilon$. By initiality of (H, λ), it suffices to show that the composite $(H, \lambda) \xrightarrow{c} (B, \gamma) \xrightarrow{\varepsilon} (k, 0)$ is a map of $\mathbb{P}[t]$-algebras; in other words, that $\varepsilon\gamma = 0$. To prove this, apply $\varepsilon \otimes \varepsilon$ to the cocycle condition $\Delta\gamma = (\sigma_1 \otimes \gamma + \gamma \otimes \sigma_2)\Delta$. Using that $(\varepsilon \otimes \varepsilon)\Delta = \varepsilon$, and $\varepsilon\sigma_i = \varepsilon$ (as observed above), this yields $\varepsilon\gamma = (\varepsilon \otimes \varepsilon\gamma + \varepsilon\gamma \otimes \varepsilon)\Delta = \varepsilon\gamma + \varepsilon\gamma$. Thus $\varepsilon\gamma = 0$, as desired.

Next, we show that the map c preserves coproducts. Observe that, by initiality of (H, λ), the square

$$
\begin{tikzcd}
(H, \lambda) \arrow[r, "\Delta"] \arrow[d, "c"'] & (H \otimes H, \sigma_1^{(H)} \otimes \lambda + \lambda \otimes \sigma_2^{(H)}) \arrow[d, "c \otimes c"] \\
(B, \gamma) \arrow[r, "\Delta"] & (B \otimes B, \sigma_1^{(B)} \otimes \gamma + \gamma \otimes \sigma_2^{(B)})
\end{tikzcd}
$$

necesarily commutes as soon as all four maps are $\mathbb{P}[t]$-algebra homomorphisms. The map $c \otimes c$ is the only one for which this still has to be shown. But, we have just proved that c is a map of augmented $\mathbb{P}$-algebras, so $c \circ \sigma_i^{(H)} = \sigma_i^{(B)} \circ c$ by naturality. Since alse $c\lambda = \gamma c$, the map $c \otimes c$ is indeed a map of $\mathbb{P}[t]$-algebras. This completes the proof of the theorem. $\square$

5 Remarks on functoriality.

We continue to work in the context of Hopf operads on a category $\mathcal{C}$ as in 1.1.

5.1 Adjoint functors. Let $\varphi : \mathbb{Q} \to \mathbb{P}$ be a map of Hopf operads. Then φ induces functors $\varphi^* : (\mathbb{P}\text{-algebras}) \to (\mathbb{Q}\text{-algebras})$ and $\overline{\varphi}^* : (\text{Hopf } \mathbb{P}\text{-algebras}) \to (\text{Hopf } \mathbb{Q}\text{-algebras})$. Also, φ gives a functor $\varphi^* : (\mathbb{P}[t]\text{-algebras}) \to (\mathbb{Q}[t]\text{-algebras})$, by $\varphi^*(B, \beta) = (\varphi^*(B), \beta)$. If the relevant coequalizers exists in $\mathcal{C}$ then the first functor φ^* has a left adjoint $\varphi_! : (\mathbb{Q}\text{-algebras}) \to (\mathbb{P}\text{-algebras})$, see e.g. [GJ]. Note that $\varphi^*(k) = k$ and that the (first) functor φ^* commutes with tensor products of algebras. Hence by adjointness, there are canonical maps of $\mathbb{P}$-algebras $\varphi_!(k) \to k$ and $\varphi_!(A \otimes B) \to \varphi_!(A) \otimes \varphi_!(B)$. Using these maps, one obtains a lifting of $\varphi_!$ to a left adjoint $\overline{\varphi}_! : (\text{Hopf-}\mathbb{P}\text{-algebras}) \to (\text{Hopf-}\mathbb{Q}\text{-algebras})$ for $\overline{\varphi}^*$.

Now let (H, λ) be the initial $\mathbb{P}[t]$-algebra and (K, μ) the one for $\mathbb{Q}$. Let $j_0 : (K, \mu) \to (\varphi^*(H), \lambda)$ be the unique map of $\mathbb{Q}[t]$-algebras, and note that this is a map of augmented $\mathbb{Q}$-algebras. Let $j : \varphi_!(K) \to H$ be the adjoint map; this is a map of augmented $\mathbb{P}$-algebras. Next, consider natural twisting functions σ_1, σ_2 on $\mathbb{Q}$-algebras. These also induce $\sigma_i : H \to H$ on any $\mathbb{P}$-algebra H, by $\sigma_i = \sigma_i^{(\varphi^*(H))}$.

5.2 Proposition. *Suppose σ_1 and σ_2 satisfy the conditions of Theorem 3.4 so as to make H and K into Hopf $\mathbb{P}$-(respectively $\mathbb{Q}$-)algebras. Then $j_0 : K \to \varphi^*(H)$ and $j : \varphi_!(K) \to H$ are maps of Hopf $\mathbb{P}$-(resp. $\mathbb{Q}$-)algebras.*

Proof: The second assertion for j follows from the first for j_0 by adjointness. To see that the map j_0 preserves the coproduct, simply apply initiality of (K, μ) to the square

$$\begin{tikzcd}
(K, \mu) \arrow[r, "\Delta"] \arrow[d, "j_0"'] & (K \otimes K, \sigma_1^{(K)} \otimes \mu + \mu \otimes \sigma_2^{(K)}) \arrow[d, "j_0 \otimes j_0"] \\
(\varphi^*(H), \lambda) \arrow[r, "\Delta"] & (\varphi^*(H) \otimes \varphi^*(H), \sigma_1^{(H)} \otimes \lambda + \lambda \otimes \sigma_2^{(H)}),
\end{tikzcd}$$

exactly as in the proof of Theorem 4.2. $\square$

5.3 The operad $\mathbb{B}$. A *pointed object* is an object X of $\mathcal{C}$ equipped with a "basepoint" $u : k \to X$. We call X *well-pointed* if X is equipped with an augmentation $\varepsilon : X \to k$ with $\varepsilon u = \mathrm{id}$. Such an object splits as $X = k \oplus \tilde{X}$ where $\tilde{X} = \mathrm{Ker}(\varepsilon)$. Let $\mathbb{B}$ be the operad whose algebras are pointed objects. If $\mathbb{P}$ is any (Hopf) operad then the unit of $\mathbb{P}$ gives a map of operads $u : \mathbb{B} \to \mathbb{P}$. We consider the left adjoint $u_!$ of the induced functor $u^* : (\mathbb{P}\text{-algebras}) \to (\mathbb{B}\text{-algebras})$.

5.4 Lemma. *If X is well-pointed then $u_!(X) = F_{\mathbb{P}}(\tilde{X})$, the free $\mathbb{P}$-algebra on $\tilde{X}$.*

Proof: Let $k \xrightarrow{u} X \xrightarrow{\varepsilon} k$ be a well-pointed object. Let $w : X \to F_{\mathbb{P}}(\tilde{X}) = F(\tilde{X})$ be the map $k \oplus \tilde{X} \to F(\tilde{X})$ obtained from the unit $u_{F(\tilde{X})} : k \to F(\tilde{X})$ of this free algebra together with the canonical map $\mu : \tilde{X} \to F(\tilde{X})$. We claim that w is the universal base-point preserving map from X into a $\mathbb{P}$-algebra. Indeed, suppose $f : X \to A$ is any map into the underlying object A of a $\mathbb{P}$-algebra $\underline{A}$, with $f \circ u = u_{\underline{A}}$. Since $F(\tilde{X})$ is the free algebra, the restriction $f \restriction \tilde{X} : \tilde{X} \to A$

extends uniquely to a $\mathbb{P}$-algebra map $\underline{f} : F(\tilde{X}) \to \underline{A}$. It is easy to check that $\underline{f} \circ w = f$ for this map $\underline{f}$. $\square$

Now let (A, α) be the initial $\mathbb{B}[t]$-algebra, and (H, λ) the initial $\mathbb{P}[t]$-algebra as before. Let σ_1, σ_2 be natural twisting functions on $\mathbb{B}$-algebras. Suppose $\sigma_1^{(A)}, \sigma_2^{(A)} : A \to A$ define a Hopf algebra structure on A, and $\sigma_1^{(H)}, \sigma_2^{(H)} : H \to H$ one on H, by Theorem 3.4.

5.5 Proposition. *There is a canonical retraction*

$$u_!(A) \underset{r}{\overset{j}{\rightleftarrows}} H, \quad r \circ j = \mathrm{id},$$

where j is a map of Hopf $\mathbb{P}$-algebras and r one of augmented $\mathbb{P}$-algebras.

Proof: The map $j : u_!(A) \to H$ is the one of Proposition 5.2. The map $r : H \to u_!(A)$ is the unique map $(H, \lambda) \to (u_!(A), \overline{\alpha})$ of $\mathbb{P}[t]$-algebras, for the map $\overline{\alpha}$ defined as follows. Since A has an augmentation ε with $\varepsilon\alpha = 0$ (Lemma 3.2), we can write $A = k \oplus \tilde{A}$ where α maps A into $\tilde{A}$. Also, the free $\mathbb{P}$-algebra $u_!(A) = F_{\mathbb{P}}(\tilde{A})$, briefly $F(\tilde{A})$, is augmented, hence splits as $u_!(A) = k \oplus F(\tilde{A})\tilde{}$. Now define $\overline{\alpha}$ on these two summands separately: on k it is the composition

$$k \xrightarrow{u} A \xrightarrow{\alpha} \tilde{A} \to F(\tilde{A})$$

and on the other summand it is the map

$$F(\tilde{A})\tilde{} \subseteq F(\tilde{A}) \xrightarrow{F(\tilde{\alpha})} F(\tilde{A})$$

where $\tilde{\alpha} : \tilde{A} \to \tilde{A}$ is the restriction of α. Note that the map $\overline{\alpha}$ thus defined satisfies the identities

$$\overline{\alpha} w = w\alpha, \ \varepsilon\overline{\alpha} = 0,$$

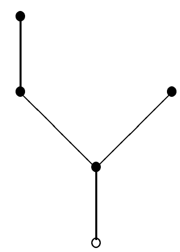
where $w : A \to u_!(A)$ is the universal map as in the proof of the previous lemma.

We claim that $r \circ j = \mathrm{id}$. By adjointness, it suffices to show $rjw = w$ as maps of pointed objects. Now $w\alpha = \overline{\alpha} w$ as we have seen. Also, $j : u_!(A) \to H$ is obtained from $j_0 : A \to u^*(H)$ by adjointness, hence $jw = j_0$. Thus $(rjw)\alpha = rj_0\alpha = r\lambda j_0 = \overline{\alpha} r j_0 = \overline{\alpha}(rjw)$. This shows that w and rjw are both maps of $\mathbb{B}[t]$-algebras on (A, α), hence equal by initiality.

It remains to observe that r respects the augmentation. Since $r : (H, \lambda) \to (u_!(A), \overline{\alpha})$ and $\varepsilon : (u_!(A), \overline{\alpha}) \to (k, 0)$ are both maps of $\mathbb{P}[t]$-algebras, so is the composite εr. So $\varepsilon r = \varepsilon$ by initiality of (H, λ). This shows that r preserves the augmentation, and completes the proof. $\square$

5.6 Example. Let (H, λ) be the Connes-Kreimer Hopf algebra of Example 3.6. For the same twisting functions $\sigma_1 = \mathrm{id}$ and $\sigma_2 = u\varepsilon$, the initial $\mathbb{B}[t]$-algebra (A, α) is the vector space with basis $x_0, x_1, x_2, \ldots$, where x_0 is the base point and $\alpha(x_n) = x_{n+1}$. Thus $u_!(A)$ is the algebra $k[x_1, x_2, \ldots]$, where we identify x_0 with $1 \in u_!(A)$. The Hopf algebra structure is given by $\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}$. The embedding j identifies $u_!(A)$ with the subalgebra of "linear trees" of H

(considered also in [CK]), and x_n with $\lambda^n(1) \in H$. The retraction $r : H \to u_!(A)$ sends a tree T to the product of all the maximal branches through T. For example, the tree



representing $\lambda(\lambda^2(1) \cdot \lambda(1))$ is sent to $x_3 \cdot x_1$. Note that r does not commute with coproducts.

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Utrecht, April 1999.