Motives from Diffraction.

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Dedicated to Jaap Murre and Spencer Bloch

Abstract

We look at geometrical and arithmetical patterns created from a finite subset of $\mathbb{Z}^n$ by diffracting waves and bipartite graphs. We hope that this can make a link between Motives and the Melting Crystals/Dimer models in String Theory.

Introduction.

Why is it that, occasionally, mathematicians studying Motives *and physicists searching for a* Theory of Everything *seem to be looking at the same examples, just from different angles? Should the Theory of Everything include properties of Numbers? Does Physics yield realizations of Motives which have not been considered before in the cohomological set-up of motivic theory?*

Calabi-Yau varieties of dimensions 1 and 2, being elliptic curves and K3-surfaces, have a long and rich history in number theory and geometry. Calabi-Yau varieties of dimension 3 have played an important role in many developments in String Theory. The discovery of *Mirror Symmetry* attracted the attention of physicists and mathematicians to Calabi-Yau's near the *large complex structure limit* [13, 20]. Some analogies between String Theory and Arithmetic Algebraic Geometry near this limit were discussed in [16, 17, 18]. Recently new models appeared, called *Melting Crystals* and *Dimers* [14, 11], which led to interesting new insights in String Theory, without going near the large complex structure limit. The present paper is an attempt to find motivic aspects of these new models. We look at geometrical and enumerative patterns associated with a finite subset $\mathfrak{A}$ of $\mathbb{Z}^n$. The geometry comes from waves diffracting on $\mathfrak{A}$ and from a periodic weighted bipartite graph generated by $\mathfrak{A}$. The latter is related to the dimers (although here we can not say more about this relation). Since the tori involved in these models are naturally dual to each other there seems to be some sort of mirror symmetry between the diffraction and the graph pictures. The enumerative patterns count lattice points on the diffraction pattern, points on varieties over finite fields and paths on the graph. They are expressed

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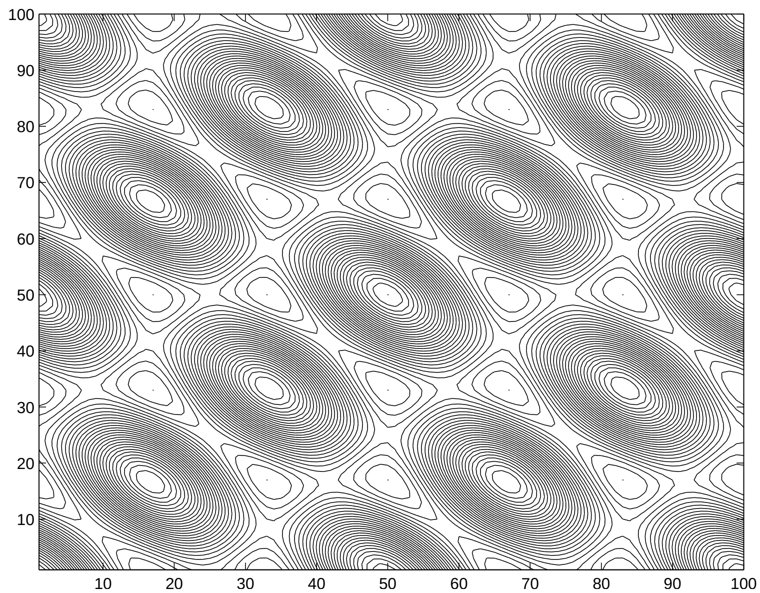

Figure 1: *Diffraction pattern for* $\mathfrak{A} = \{(1,0), (0,1), (-1,-1)\} \subset \mathbb{Z}^2$*, all* $c_{\mathsf{a}} = 1$.

through a sequence of polynomials $\mathsf{B}_N(z)$ with coefficients in $\mathbb{Z}$ and via a limit for $z \in \mathbb{C}$:

$$\mathsf{Q}(z) = \lim_{N\to\infty} |\mathsf{B}_N(z)|^{-N^{-n}} . \tag{1}$$

Limit formulas like (1) appear frequently and in very diverse contexts in the literature, e.g. for *entropy in algebraic dynamical systems* in [7] Theorem 4.9, for *partition function per fundamental domain in dimer models* in [11] Theorem 3.5, for *integrated density of states* in [9] p.206. Moreover, $\mathsf{Q}(z)$ appears as *Mahler measure* in [4], as the exponential of a *period in Deligne cohomology* in [6, 15], and in *instanton counts* in [18]; see the remark at the end of Section 4.

With some additional restrictions $\mathfrak{A}$ provides the toric data for a family of Calabi-Yau varieties and various well-known results about Calabi-Yau varieties near the large complex structure limit can be derived from the Taylor series expansion of $\log \mathsf{Q}(z)$ near $z = \infty$; see the Remark at the end of Section 5. In the present paper we are not so much interested in the large complex structure limit. Instead we focus on the polynomials $\mathsf{B}_N(z)$ and the limit formula (1). *This does not require conditions of 'Calabi-Yau type'.*

When waves are diffracted at some finite set $\mathfrak{A}$ of points in a plane, the diffraction pattern observed in a plane at large distance is, according to the Frauenhofer model, the absolute value squared of the Fourier transform of $\mathfrak{A}$. There is no mathematical reason to restrict this model to dimension 2. Also the points may have weights ≥ 1. So, we take a finite subset $\mathfrak{A}$ of $\mathbb{Z}^n$ and positive integers c_{a} ($\mathsf{a} \in \mathfrak{A}$). These data can be summarized as a distribution

$\mathcal{D} = \sum_{\mathsf{a}\in\mathfrak{A}} c_{\mathsf{a}}\delta_{\mathsf{a}}$, where δ_{a} denotes the Dirac delta distribution, evaluating test functions at the point a. The Fourier transform of $\mathcal{D}$ is the function $\widehat{\mathcal{D}}(\mathsf{t}) = \sum_{\mathsf{a}\in\mathfrak{A}} c_{\mathsf{a}} e^{-2\pi i\langle \mathsf{t},\mathsf{a}\rangle}$ on $\mathbb{R}^n$; here $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. The diffraction pattern consists of the level sets of the function

$$|\widehat{\mathcal{D}}(\mathsf{t})|^2 = \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}}c_{\mathsf{b}} e^{2\pi i\langle \mathsf{t},\mathsf{a}-\mathsf{b}\rangle} = \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}}c_{\mathsf{b}} \cos(2\pi\langle \mathsf{t},\mathsf{a}-\mathsf{b}\rangle)\,.$$

This function is periodic with period lattice $\Lambda^\vee$ dual to the lattice Λ spanned over $\mathbb{Z}$ by the differences $\mathsf{a}-\mathsf{b}$ with $\mathsf{a},\mathsf{b}\in\mathfrak{A}$. *Throughout this note we assume that Λ has rank n.* Looking at the intersections of the diffraction pattern with the lattices $\frac{1}{N}\Lambda^\vee$ we introduce the enumerative data

$$\mathrm{mult}_N(r) := \sharp\{\mathsf{t}\in \tfrac{1}{N}\Lambda^\vee/\Lambda^\vee \mid |\widehat{\mathcal{D}}(\mathsf{t})|^2 \;=\; r\,\} \qquad \text{for} \quad N\in\mathbb{N},\, r\in\mathbb{R}\,, \tag{2}$$

and use these to define polynomials $\mathsf{B}_N(z)$ as follows:

Definition 1

$$\mathsf{B}_N(z) := \prod_{r\in\mathbb{R}} (z-r)^{\mathrm{mult}_N(r)}\,. \tag{3}$$

One could also introduce the generating function $\mathsf{F}(z,T) := \sum_{N\in\mathbb{N}} \mathsf{B}_N(z)\, T^{N^n}$, but except for the classical number theory of the case $n=1$ (see Section 6.1), and the observation that Formula (1) gives $\mathsf{Q}(z)$ as the radius of convergence of $\mathsf{F}(z,T)$ as a complex power series in T, we do not yet have appealing results about $\mathsf{F}(z,T)$.

For the graph model we start from the same data: the finite set $\mathfrak{A}\subset\mathbb{Z}^n$, the weights c_{a} ($\mathsf{a}\in\mathfrak{A}$) and the lattice Λ spanned by the differences $\mathsf{a}-\mathsf{b}$ with $\mathsf{a},\,\mathsf{b}\in\mathfrak{A}$. We must now assume that

$$\mathfrak{A}\cap\Lambda \;=\; \emptyset.$$

One can then construct a weighted bipartite graph Γ as follows. Bipartite graphs have two kinds of vertices, often called black and white. The set of *black vertices* of Γ is Λ. The set of *white vertices* of Γ is $\mathfrak{A}+\Lambda$. Note that $\mathfrak{A}+\Lambda$ is just one single coset of Λ in $\mathbb{Z}^n$. In Γ there is an (oriented) edge from vertex v_1 to vertex v_2 if and only if v_1 is black, v_2 is white and $\mathsf{v}_2-\mathsf{v}_1\in\mathfrak{A}$. If $\mathsf{v}_2-\mathsf{v}_1=\mathsf{a}\in\mathfrak{A}$ the edge is said to be of *type* a and gets *weight* c_{a}.

The graph Γ is Λ-periodic and for every $N\in\mathbb{N}$ one has the finite graph $\Gamma_N := \Gamma/N\Lambda$, which is naturally embedded in the torus $\mathbb{R}^n/N\Lambda$. By a *closed path* of length $2k$ on Γ or Γ_N we mean a sequence of edges $(e_1, e_2, \ldots, e_{2k-1}, e_{2k})$ such that for $i=1,\ldots,k$ the intersection $e_{2i-1}\cap e_{2i}$ contains a white vertex and $e_{2i}\cap e_{2i+1}$ contains a black vertex; here $e_{2k+1}=e_1$. By the weight of such a path we mean the product of the weights of the edges $e_1, e_2, \ldots, e_{2k-1}, e_{2k}$. We denote the set of closed paths of length $2k$ on Γ_N by $\Gamma_N(2k)$. Enumerating the closed paths on Γ_N according to length and weight we prove in Section 3 that this leads to a new interpretation of the polynomials $B_N(z)$:

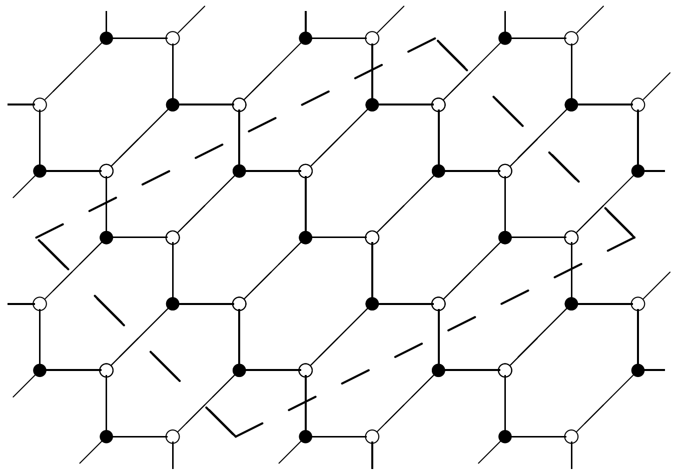

Figure 2: *A fundamental parallelogram of the lattice* 3Λ *and a piece of the bipartite graph* Γ *for* $\mathfrak{A} = \{(1,0), (0,1), (-1,-1)\} \subset \mathbb{Z}^2$*, and all* $c_{\mathsf{a}} = 1$*. By identifying opposite sides of the parallelogram one obtains the graph* Γ_3*.*

Theorem 1

$$B_N(z) \,=\, z^{N^n} \exp\left(- \sum_{k\geq 1} \sum_{\gamma \in \Gamma_N(2k)} \mathrm{weight}(\gamma) z^{-k} \right) . \tag{4}$$

■

Formulas (3) and (4) transfer the enumerative data between the two models.

We pass to algebraic geometry with the Laurent polynomial

$$W(x_1, \ldots, x_n) \,=\, \sum_{\mathsf{a},\mathsf{b} \in \mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} \mathsf{x}^{\mathsf{a}-\mathsf{b}} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \,, \tag{5}$$

which satisfies $|\widehat{\mathcal{D}}(\mathsf{t})|^2 \,=\, W(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$; here $\mathsf{x}^\lambda := \prod_{j=1}^n x_j^{\lambda_j}$ if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$. For $N \in \mathbb{N}$ let μ_N denote the group of N-th roots of unity and let $\mu_N^\Lambda := \mathrm{Hom}(\Lambda, \mu_N)$ be the group of homomorphisms from the lattice Λ to μ_N. Thus the defining formula (3) can be rewritten as:

$$\mathsf{B}_N(z) = \prod_{\mathsf{x} \in \mu_N^\Lambda} (z - W(\mathsf{x})) \,. \tag{6}$$

Written in the form (6) the polynomials $\mathsf{B}_N(z)$ appear as direct generalizations of quantities introduced by Lehmer [12] for a 1-variable (i.e. $n = 1$) polynomial $W(x)$. Using (6) one can easily show (Proposition 2) that the polynomials $\mathsf{B}_N(z)$ have integer coefficients and that $\mathsf{B}_{N'}(z)$ divides $\mathsf{B}_N(z)$ in $\mathbb{Z}[z]$ if N' divides N in $\mathbb{Z}$. Thus for $h \in \mathbb{Z}$ also $\mathsf{B}_N(h)$ is an integer. Lehmer was particularly interested in the prime factorization of these integers in case $n = 1$ [12, 7, 15]. Also for general $n \geq 1$ these prime factorizations must be interesting, for instance because they relate to counting points on varieties over finite fields; see Section 4 for details. *Thus prime factorization gives a third occurrence of* $\mathsf{B}_N(z)$ *in enumerative problems, related to counting points on varieties over finite fields.* $\mathsf{Q}(z)$ appears in [4, 6, 15] as *Mahler measure* with ties to special values of L-functions. It would be nice if the limit formula (1) together with the prime factorization of the numbers $\mathsf{B}_N(z)$ (with $z \in \mathbb{Z}$) could shed new light on these very intriguing ties.

In *Section 1* we study the density distribution of the level sets in the diffraction pattern. Passing from measures to complex functions with the Hilbert transform we find one interpretation of $\mathsf{Q}(z)$, $\mathsf{B}_N(z)$ and (1). In *Section 2* we briefly discuss another interpretation in connection with the spectrum of a discretized Laplace operator. In *Section 3* we prove Theorem 1. In *Section 4* we pass to toric geometry, where the diffraction pattern reappears as the intersection of a real torus with a family of hypersurfaces in a complex torus and where $\log \mathsf{Q}(z)$ becomes a period integral, while the prime factorization of $\mathsf{B}_N(z)$ for $z \in \mathbb{Z}$ somehow relates to counting points on those hypersurfaces over finite fields. In *Section 5* we discuss sequences of integers which appear as moments of measures, path counts on graphs and coefficients in Taylor expansions of solutions of Picard-Fuchs differential equations. Finally, in *Section 6* we present some concrete examples.

1 The diffraction pattern.

The function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$ is periodic with period lattice $\Lambda^\vee$ dual to the lattice Λ:

$$\begin{array}{lcl} \Lambda^\vee & := & \{\mathsf{t} \in \mathbb{R}^n \mid \langle \mathsf{t}, \mathsf{a} - \mathsf{b} \rangle \in \mathbb{Z}\,,\ \forall \mathsf{a}, \mathsf{b} \in \mathfrak{A}\}\,, \\ \Lambda & := & \mathbb{Z}\text{--}\mathrm{Span}\{\mathsf{a} - \mathsf{b} \mid \mathsf{a}, \mathsf{b} \in \mathfrak{A}\}. \end{array}$$

Throughout this note we assume that the lattices Λ *and* $\Lambda^\vee$ *have rank* n.

Because of this periodicity $|\widehat{\mathcal{D}}(\mathsf{t})|^2$ descends to a function on $\mathbb{R}^n/\Lambda^\vee$. Defining for $\mathsf{t} \in \mathbb{R}^n$ the function $e_\mathsf{t} : \mathbb{R}^n \to \mathbb{C}$ by $e_\mathsf{t}(\mathsf{v}) = e^{2\pi i \langle \mathsf{t}, \mathsf{v} \rangle}$ we obtain an isomorphism of real tori

$$\mathbb{R}^n/\Lambda^\vee \simeq \mathbb{U}^\Lambda\,, \qquad \mathsf{t} \mapsto e_\mathsf{t}\,, \tag{7}$$

where $\mathbb{U}^\Lambda := \mathrm{Hom}(\Lambda, \mathbb{U})$ is the torus of group homomorphisms from the lattice Λ to the unit circle $\mathbb{U} := \{x \in \mathbb{C} \mid |x| = 1\}$. Recall that a group homomorphism $\psi : \Lambda \to \mathbb{U}$ induces an algebra homomorphism ψ_* from the group algebra $\mathbb{C}[\Lambda]$ to $\mathbb{C}$. Thus $\mathbb{C}[\Lambda]$ is the natural algebra of functions on $\mathbb{U}^\Lambda$ and ψ_* evaluates functions at the point ψ of $\mathbb{U}^\Lambda$. The inclusion $\Lambda \subset \mathbb{Z}^n$ identifies $\mathbb{C}[\Lambda]$ with the subalgebra of the algebra of Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, which consists of $\mathbb{C}$-linear combinations of the monomials $\mathsf{x}^\lambda := \prod_{j=1}^n x_j^{\lambda_j}$ with $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$. Thus, via (7), the function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$ coincides with the Laurent polynomial $W(x_1, \ldots, x_n)$ defined in (5).

Positivity of the coefficients c_{a} implies that the function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$ attains its maximum exactly at the points $\mathsf{t} \in \Lambda^\vee$. In terms of the torus $\mathbb{U}^\Lambda$ and the function $W : \mathbb{U}^\Lambda \to \mathbb{R}$ this means that W attains its maximum exactly at the origin 1 of the torus group $\mathbb{U}^\Lambda$:

$$\forall \mathsf{x} \in \mathbb{U}^\Lambda \setminus \{1\} : \quad W(\mathsf{x}) \; < \; W(1) \; = \; C^2 \qquad \text{with} \quad C := \sum_{\mathsf{a} \in \mathfrak{A}} c_{\mathsf{a}} \, .$$

Some important aspects of the density distribution in the diffraction pattern are captured by the function

$$V : \mathbb{R} \to \mathbb{R}\,, \qquad V(r) := \frac{\operatorname{volume}\{\mathsf{t} \in \mathbb{R}^n / \Lambda^\vee \mid |\widehat{\mathcal{D}}(\mathsf{t})|^2 \le r\}}{\operatorname{volume}(\mathbb{R}^n / \Lambda^\vee)} \, . \tag{8}$$

We view the derivative $dV(r)$ of V as a measure on $\mathbb{R}$. In our analysis it will be important that the measure $dV(r)$ is also the push forward of the standard measure $dt_1\, dt_2\, \ldots dt_n$ on $\mathbb{R}^n$ by the function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$.

Another insight into the diffraction pattern comes from its intersection with the torsion subgroup of $\mathbb{U}^\Lambda$. For $N \in \mathbb{N}$ let $\mu_N \subset \mathbb{U}$ denote the group of N-th roots of unity. Then the group of N-torsion points in $\mathbb{U}^\Lambda$ is $\mu_N^\Lambda := \operatorname{Hom}(\Lambda, \mu_N)$ and (2) can be rewritten as

$$\operatorname{mult}_N(r) = \sharp(\mu_N^\Lambda \cap W^{-1}(r)) \qquad \text{for} \quad N \in \mathbb{N},\, r \in \mathbb{R} \, .$$

Moreover we set, in analogy with (8),

$$V_N(r) := \frac{1}{N^n} \sharp\{\mathsf{x} \in \mu_N^\Lambda \mid W(\mathsf{x}) \le r\} \qquad \text{for} \quad N \in \mathbb{N},\, r \in \mathbb{R} \, .$$

The derivative of the step function $V_N : \mathbb{R} \to \mathbb{R}$ is the distribution

$$dV_N(r) \; = \; N^{-n} \sum_r \operatorname{mult}_N(r)\, \delta_r \, , \tag{9}$$

which assigns to a continuous function f on $\mathbb{R}$ the value

$$\int_{\mathbb{R}} f(r) dV_N(r) := N^{-n} \sum_r \operatorname{mult}_N(r)\, f(r) = N^{-n} \sum_{\mathsf{x} \in \mu_N^\Lambda} f(W(\mathsf{x})) \, . \tag{10}$$

One thus finds a limit of distributions

$$\lim_{N \to \infty} dV_N(r) \; = \; dV(r) \, ; \tag{11}$$

by definition, this means that for every continuous function f on $\mathbb{R}$

$$\lim_{N \to \infty} \int_{\mathbb{R}} f(r) dV_N(r) \; = \; \int_{\mathbb{R}} f(r) dV(r) \, . \tag{12}$$

Measure theory is connected with complex function theory by the Hilbert transform. The Hilbert transform of the measure $dV(r)$ is the function $-\frac{1}{\pi}\mathsf{H}(z)$ defined by

$$\mathsf{H}(z) := \int_{\mathbb{R}} \frac{1}{z-r} dV(r) \qquad \text{for} \quad z \in \mathbb{C} \setminus \mathcal{I}\,; \tag{13}$$

here $\mathcal{I} := \overline{\{r \in \mathbb{R} \mid 0 < V(r) < 1\}} \subset [0, C^2]$ is the support of the measure $dV(r)$. The measure can be recovered from its Hilbert transform because for $r_0 \in \mathbb{R}$

$$\frac{dV}{dr}(r_0) = \frac{1}{2\pi i} \lim_{\epsilon \in \mathbb{R}, \epsilon \downarrow 0} \left(\mathsf{H}(r_0 + i\epsilon) \,-\, \mathsf{H}(r_0 - i\epsilon) \right) .$$

Another way of writing the connection between $dV(r)$ and $\mathsf{H}(z)$ is

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) \mathsf{H}(z) dz \;=\; \int_{\mathbb{R}} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-r} dz \right) dV(r) \;=\; \int_{\mathbb{R}} f(r) dV(r) \tag{14}$$

for holomorphic functions f defined on some open neighborhood U of the interval $\mathcal{I}$ in $\mathbb{C}$ and closed paths γ in $U \setminus \mathcal{I}$ encircling $\mathcal{I}$ once counter clockwise.

Next we consider the function

$$\mathsf{Q} : \mathbb{C} \setminus \mathcal{I} \longrightarrow \mathbb{R}_{>0}\,, \qquad \mathsf{Q}(z) := \exp\left(- \int_{\mathbb{R}} \log|z-r| dV(r) \right) . \tag{15}$$

This function satisfies $\frac{d}{dz} \log \mathsf{Q}(z) = -\mathsf{H}(z)$ and thus (14) can be rewritten as

$$\frac{-1}{2\pi i} \oint_{\gamma} f(z) d \log \mathsf{Q}(z) \;=\; \int_{\mathbb{R}} f(r) dV(r)\,.$$

This means that, at least intuitively, the functions $\mathsf{Q}(z)$ and $e^{-V(r)}$ correspond to each other via some kind of comparison isomorphism.

In order to find the analogue of (11) in terms of functions on $\mathbb{C} \setminus \mathcal{I}$ we apply (10) to the function $f(r) = \log|z-r|$ on $\mathbb{R}$ with fixed $z \in \mathbb{C} \setminus \mathcal{I}$:

$$\int_{\mathbb{R}} \log|z-r| dV_N(r) = N^{-n} \log|\mathsf{B}_N(z)| \tag{16}$$

where $\mathsf{B}_N(z) = \prod_{r \in \mathcal{I}} (z-r)^{\mathrm{mult}_N(r)} = \prod_{\mathsf{x} \in \mu_N^\Lambda} (z - W(\mathsf{x}))$ as in (3) and (6).

Combining (12), (15) and (16) we find the limit announced in (1):

Proposition 1 $\quad \mathsf{Q}(z) = \lim_{N \to \infty} |\mathsf{B}_N(z)|^{-N^{-n}}$ *for every* $z \in \mathbb{C} \setminus \mathcal{I}$. ■

2 The Laplacian perspective.

Convolution with the distribution $\mathcal{D}$ gives the operator $\mathsf{D}f(\mathsf{v}) := \sum_{\mathsf{a}\in\mathfrak{A}} c_{\mathsf{a}} f(\mathsf{v}-\mathsf{a})$ on the space of $\mathbb{C}$-valued functions on $\mathbb{R}^n$. Let $\overline{\mathsf{D}}f(\mathsf{v}) := \sum_{\mathsf{a}\in\mathfrak{A}} c_{\mathsf{a}} f(\mathsf{v}+\mathsf{a})$ and

$$\Delta := \mathsf{D}\overline{\mathsf{D}}\,, \qquad \Delta f(\mathsf{v}) = \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} f(\mathsf{v}+\mathsf{a}-\mathsf{b})\,.$$

For a sufficiently differentiable function f on $\mathbb{R}^n$ the Taylor expansion

$$\Delta f(\mathsf{v}) = C^2 f(\mathsf{v}) \;+\; \tfrac{1}{2} \sum_{i,j=1}^{n} \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} (a_i - b_i)(a_j - b_j) \frac{\partial^2 f}{\partial v_i \partial v_j}(\mathsf{v}) + \ldots$$

shows that the difference operator $\Delta - C^2$ is a *discrete approximation of the Laplace operator* corresponding to the Hessian of the function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$ at its maximum.

Remark. In [9] Gieseker, Knörrer and Trubowitz investigate Schrödinger equations in solid state physics via a discrete approximation of the Laplacian. In their situation the Schrödinger operator is the discretized Laplacian *plus a periodic potential function*. So from the perspective of [9] the present note deals only with the (simple) case of zero potential. On the other hand we consider more general discretization schemes and possibly higher dimensions.

We now turn to the spectrum of Δ. For $\mathsf{t} \in \mathbb{R}^n$ the function $e_{\mathsf{t}} : \mathbb{R}^n \to \mathbb{C}$ given by $e_{\mathsf{t}}(\mathsf{v}) = e^{2\pi i \langle \mathsf{t},\mathsf{v}\rangle}$ is an eigenfunction for Δ with eigenvalue $|\widehat{\mathcal{D}}(\mathsf{t})|^2$:

$$\Delta e_{\mathsf{t}}(\mathsf{v}) \;=\; \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} e^{2\pi i \langle \mathsf{t},\mathsf{v}+\mathsf{a}-\mathsf{b}\rangle} \;=\; |\widehat{\mathcal{D}}(\mathsf{t})|^2\, e_{\mathsf{t}}(\mathsf{v})\,.$$

Take a positive integer N. The space of $\mathbb{C}$-valued C^∞-functions on $\mathbb{R}^n$ which are periodic for the sublattice $N\Lambda$ of Λ is spanned by the functions e_{t} with t in the dual lattice $\frac{1}{N}\Lambda^\vee$. The characteristic polynomial of the restriction of Δ to this space is therefore (see (3) and (6))

$$\prod_{\mathsf{t}\,\in\,\frac{1}{N}\Lambda^\vee/\Lambda^\vee} (z - |\widehat{\mathcal{D}}(\mathsf{t})|^2) \;=\; \prod_{\mathsf{x}\in\mu_N^\Lambda} (z - W(\mathsf{x})) \;=\; \prod_{r\in\mathbb{R}} (z-r)^{\mathrm{mult}_N(r)} \;=\; \mathsf{B}_N(z)\,.$$

With (9) and (11) the measure $dV(r)$ can now be interpreted as the density of the eigenvalues of Δ on the space of $\mathbb{C}$-valued C^∞-functions on $\mathbb{R}^n$ which are periodic for some sublattice $N\Lambda$ of Λ.

3 Enumeration of paths on a periodic weighted bipartite graph; proof of Theorem 1.

In this section we prove Theorem 1. Recall from the Introduction just before Theorem 1 the various ingredients: the finite set $\mathfrak{A} \subset \mathbb{Z}^n$, the weights c_{a}, the lattice Λ and the graphs Γ and Γ_N. Recall also the closed paths on Γ_N, their lengths and weights, and the set $\Gamma_N(2k)$ of closed paths of length $2k$ on Γ_N.

Consider a path $(e_1, e_2, \ldots, e_{2k-1}, e_{2k})$ on Γ with edge e_i going from black to white if i is odd, respectively from white to black if i is even. Let s denote the starting point of the path (i.e. the black vertex of edge e_1). Let for $j = 1, \ldots, k$ edge e_{2j-1} be of type a_j and edge e_{2j} of type b_j. Then the end point of the path (i.e. the black vertex of e_{2k}) is $\mathsf{s} + \sum_{j=1}^{k}(\mathsf{a}_j - \mathsf{b}_j)$. The weight of the path is $\prod_{j=1}^{k} c_{\mathsf{a}_j} c_{\mathsf{b}_j}$. The path closes on Γ_N if and only if $\sum_{j=1}^{k}(\mathsf{a}_j - \mathsf{b}_j) \in N\Lambda$.

Next recall from (5) that $W(\mathsf{x}) = \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} \mathsf{x}^{\mathsf{a}-\mathsf{b}}$ and set

$$\mathsf{m}_k^{(N)} := N^{-n} \sum_{\mathsf{x}\in\mu_N^{\Lambda}} W(\mathsf{x})^k \,. \tag{17}$$

So $\mathsf{m}_k^{(N)}$ is the sum of the coefficients of those monomials in $W(\mathsf{x})^k$ with exponent in $N\Lambda$. In view of the above considerations $\mathsf{m}_k^{(N)}$ is therefore equal to the sum of the weights of the paths on Γ which start at s, have length $2k$ and close in Γ_N. Since on Γ_N there are N^n black vertices and on a path of length $2k$ there are k black vertices we conclude

$$\frac{N^n}{k}\mathsf{m}_k^{(N)} = \sum_{\gamma\in\Gamma_N(2k)} \mathrm{weight}(\gamma)\,. \tag{18}$$

From (6) one sees for $|z| > C^2$

$$N^{-n}\log B_N(z) = \log z + N^{-n}\sum_{\mathsf{x}\in\mu_N^{\Lambda}} \log(1 - W(\mathsf{x})z^{-1}) = \log z - \sum_{k\geq 1}\frac{\mathsf{m}_k^{(N)}}{k} z^{-k}\,. \tag{19}$$

Combining (18) and (19) we find

$$B_N(z) = z^{N^n}\exp\left(-\sum_{k\geq 1}\sum_{\gamma\in\Gamma_N(2k)} \mathrm{weight}(\gamma)\, z^{-k}\right).$$

This finishes the proof of Theorem 1. ■

Remark. In the Laplacian perspective $N^n\mathsf{m}_k^{(N)}$ is the trace of the operator Δ^k on the space of $\mathbb{C}$-valued C^∞-functions on $\mathbb{R}^n$ which are periodic for the sublattice $N\Lambda$ of Λ. The polynomial $B_N(z)$ is the characteristic polynomial of Δ on this space. Formula (19) gives the well-known relation between the characteristic polynomial of an operator and the traces of its powers.

Remark. One may refine the above enumerations by keeping track of the homology class to which the closed path belongs. That means that instead of (17) one extracts from the polynomial $W(\mathsf{x})^k$ the subpolynomial consisting of terms with exponent in $N\Lambda$. Such a refinement of the enumerations with homology data appears also in the theory of dimer models (cf. [11]), but its meaning for the diffraction pattern is not clear.

4 Algebraic geometry.

The polynomial $\mathsf{B}_N(z) = \prod_{\mathsf{x}\in\mu_N^\Lambda}(z - W(\mathsf{x}))$ has coefficients in the ring of integers of the cyclotomic field $\mathbb{Q}(\mu_N)$ and is clearly invariant under the Galois group of $\mathbb{Q}(\mu_N)$ over $\mathbb{Q}$. Consequently, the coefficients of $\mathsf{B}_N(z)$ lie in $\mathbb{Z}$. The same argument applies to the polynomial $\mathsf{B}_N(z)\mathsf{B}_{N'}(z)^{-1} = \prod_{\mathsf{x}\in\mu_N^\Lambda\setminus\mu_{N'}^\Lambda}(z - W(\mathsf{x}))$ if N' divides N. Thus we have proved

Proposition 2 *For every $N \in \mathbb{N}$ the coefficients of $\mathsf{B}_N(z)$ lie in $\mathbb{Z}$. If N' divides N in $\mathbb{Z}$, then $\mathsf{B}_{N'}(z)$ divides $\mathsf{B}_N(z)$ in $\mathbb{Z}[z]$.* ■

Fix a prime number p and a positive integer $\nu \in \mathbb{Z}_{>0}$. Let $\mathbb{W}(\mathbb{F}_{p^\nu})$ denote the ring of Witt vectors of the finite field $\mathbb{F}_{p^\nu}$ (see e.g. [3]). So, $\mathbb{W}(\mathbb{F}_{p^\nu})$ is a complete discrete valuation ring with maximal ideal $p\mathbb{W}(\mathbb{F}_{p^\nu})$ and residue field $\mathbb{F}_{p^\nu}$. The Teichmüller lifting is a map $\tau : \mathbb{F}_{p^\nu} \longrightarrow \mathbb{W}(\mathbb{F}_{p^\nu})$ such that

$$x \equiv \tau(x) \bmod p\,, \qquad \tau(xy) = \tau(x)\tau(y) \qquad \forall x, y \in \mathbb{F}_{p^\nu}\,.$$

Every non-zero $x \in \mathbb{F}_{p^\nu}$ satisfies

$$x^{p^\nu - 1} = 1\,.$$

Thus there is an isomorphism $\mu_{p^\nu-1} \simeq \mathbb{F}_{p^\nu}^*$. Such an isomorphism composed with the Teichmüller lifting gives an embedding $j : \mu_{p^\nu-1} \hookrightarrow \mathbb{W}(\mathbb{F}_{p^\nu})$. Thus for $\mathsf{x} \in \mu_{p^\nu-1}^\Lambda$ we get

$$W(j(\mathsf{x})) \in \mathbb{W}(\mathbb{F}_{p^\nu})\,.$$

Recall the p-adic valuation on $\mathbb{Z}$: for $k \in \mathbb{Z}$, $k \neq 0$:

$$v_p(k) \;:=\; \max\{v \in \mathbb{Z} \mid p^v \text{ divides } k\}\,.$$

Proposition 3 *For p, ν as above and for $z \in \mathbb{Z}$ the p-adic valuation of the integer $\mathsf{B}_{p^\nu-1}(z)$ satisfies*

$$v_p(\mathsf{B}_{p^\nu-1}(z)) \geq \sharp\{\xi \in (\mathbb{F}_{p^\nu}^*)^n \mid W(\xi) = z \text{ in } \mathbb{F}_{p^\nu}\}\,. \tag{20}$$

Proof: From (6) we obtain the product decomposition, with factors in $\mathbb{W}(\mathbb{F}_{p^\nu})$,

$$\mathsf{B}_N(z) = \prod_{\xi \in (\mathbb{F}_{p^\nu}^*)^n} (z - W(\tau(\xi)))\,.$$

The result of the proposition now follows because

$$z - W(\tau(\xi)) \in p\mathbb{W}(\mathbb{F}_{p^\nu}) \qquad \Leftrightarrow \qquad W(\xi) = z \text{ in } \mathbb{F}_{p^\nu}$$

∎

Remark. In (30) we give an example showing that in (20) we may have a strict inequality.

Remark about the relation with Mahler measure and L-functions. The *logarithmic Mahler measure* $\mathsf{m}(F)$ and the *Mahler measure* $\mathsf{M}(F)$ of a Laurent polynomial $F(x_1, \ldots, x_n)$ with complex coefficients are:

$$\begin{aligned} \mathsf{m}(F) &:= \frac{1}{(2\pi i)^n} \oint\!\!\oint_{|x_1|=\ldots=|x_n|=1} \log|F(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdot \ldots \cdot \frac{dx_n}{x_n}\,, \\ \mathsf{M}(F) &:= \exp(\mathsf{m}(F))\,. \end{aligned}$$

Boyd [4] gives a survey of many (two-variable) Laurent polynomials for which $\mathsf{m}(F)$ equals (numerically to many decimal places) a 'simple' non-zero rational number times the derivative at 0 of the L-function of the projective plane curve Z_F defined by the vanishing of F:

$$\mathsf{m}(F) \cdot \mathbb{Q}^* \;=\; \mathrm{L}'(Z_F, 0) \cdot \mathbb{Q}^*\,. \tag{21}$$

Deninger [6] and Rodriguez Villegas [15] showed that the experimentally observed relations (21) agree with predictions from the Bloch-Beilinson conjectures. Rodriguez Villegas [15] provided actual proofs for a few special examples.

Since the measure $dV(r)$ is the push forward of the measure $dt_1\, dt_2 \ldots dt_n$ on $\mathbb{R}^n$ by the function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$, one can rewrite Formula (15) as:

$$-\log \mathsf{Q}(z) \;=\; \frac{1}{(2\pi i)^n} \int_{\mathbb{U}^n} \log|z - W(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \frac{dx_2}{x_2} \ldots \frac{dx_n}{x_n}\,. \tag{22}$$

On the right hand side of (22) we now recognize the logarithmic Mahler measure of the Laurent polynomial $z - W(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

For fixed $z \in \mathbb{Z}$ Formulas (1) and (20) provide a link between $\mathsf{Q}(z)$ and counting points over finite fields on the variety with equation $W(x_1, \ldots, x_n) = z$. It may be an interesting challenge to further extend these ideas to a proof of a result like (21).

5 Moments.

Important invariants of the measure $dV(r)$ are its *moments* m_k $(k \in \mathbb{Z}_{\geq 0})$:

$$\begin{aligned} \mathsf{m}_k &:= \int_{\mathbb{R}} r^k dV(r) \;=\; \int_0^1 \cdots \int_0^1 |\widehat{\mathcal{D}}(t_1,\ldots,t_n)|^{2k} dt_1 \ldots dt_n \\ &= \text{constant term of Fourier series } \; |\widehat{\mathcal{D}}(t_1,\ldots,t_n)|^{2k} \\ &= \text{constant term of Laurent polynomial } \; W(x_1,\ldots,x_n)^k\,. \end{aligned} \tag{23}$$

The relation between the moments and the functions $\mathsf{H}(z)$, $\mathsf{Q}(z)$ defined in (13) and (15) is: for $z \in \mathbb{R}$, $z > C^2$,

$$\mathsf{H}(z) \;=\; \sum_{k\geq 0} \mathsf{m}_k z^{-k-1}\,, \qquad \mathsf{Q}(z) = z^{-1} \exp\left(\sum_{k\geq 1} \frac{\mathsf{m}_k}{k} z^{-k} \right)\,. \tag{24}$$

It is clear that the moments m_k of $dV(r)$ are non-negative integers. They satisfy all kinds of arithmetical relations. There are, for instance, recurrences like (28) and congruences like the following

Lemma 1 $\mathsf{m}_{kp^{\alpha+1}} \equiv \mathsf{m}_{kp^{\alpha}} \bmod p^{\alpha+1}$ *for every prime number* p *and* $k, \alpha \in \mathbb{Z}_{\geq 0}$.

Proof: The Laurent polynomial $W(x_1,\ldots,x_n)$ has coefficients in $\mathbb{Z}$. Therefore

$$W(x_1,\ldots,x_n)^{kp^{\alpha+1}} \equiv W(x_1^p,\ldots,x_n^p)^{kp^{\alpha}} \bmod p^{\alpha+1}\mathbb{Z}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]\,.$$

The lemma follows by taking constant terms. ■

Theorems 1.1, 1.2, 1.3 in [1] together with the above lemma immediately yield the following integrality result for series and product expansions:

Corollary 1 *For* $z \in \mathbb{R}$, $z > C^2$

$$z^{-1} \exp\left(\sum_{k\geq 1} \frac{\mathsf{m}_k}{k} z^{-k} \right) \;=\; z^{-1} + \sum_{k\geq 1} A_k z^{-k-1} \;=\; z^{-1} \prod_{k\geq 1} (1 - z^{-k})^{-b_k} \tag{25}$$

with $A_k, b_k \in \mathbb{Z}$ *for all* $k \geq 1$. ■

Remark. In [1, 8] the result of Corollary 1 is used to interpret $z\mathsf{Q}(z)$ as the Artin-Mazur zeta function of a dynamical system, provided the integers b_k are not negative. We have not yet found such a dynamical system within the present framework.

For $N \in \mathbb{N}$ the moments of the measure $dV_N(r)$ are, by definition,

$$\mathsf{m}_k^{(N)} := \int_{\mathbb{R}} r^k dV_N(r) = N^{-n} \sum_{\mathsf{x} \in \mu_N} W(\mathsf{x})^k\,.$$

These are the same numbers as in (17).

Proposition 4 *With the above notations we have*

$$\begin{array}{rcll} \mathsf{m}_k^{(N)} & \geq & \mathsf{m}_k \ \geq \ 0 & \textit{for all} \qquad N,\, k\,, \\ \mathsf{m}_k^{(N)} & = & \mathsf{m}_k \qquad \textit{if} & N > k \max_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} \max_{1\leq j\leq n} |a_j - b_j|\,. \end{array}$$

Proof: $N^n\,(\mathsf{m}_k^{(N)} - \mathsf{m}_k)$ is the sum of the coefficients of all non-constant monomials in the Laurent polynomial $W(x_1,\ldots,x_n)^k$ with exponents divisible by N. Since all coefficients of $W(x_1,\ldots,x_n)$ are positive, this shows $\mathsf{m}_k^{(N)} \geq \mathsf{m}_k \geq 0$.

Assume $N > k \max_{\mathsf{a},\mathsf{b}\in\mathfrak{A},\, 1\leq j\leq n} |a_j - b_j|$. Then all exponents in the monomials of the Laurent polynomial $W(x_1,\ldots,x_n)^k$ are $> -N$ and $< N$. So only the exponent of the constant term is divisible by N. Therefore $\mathsf{m}_k^{(N)} = \mathsf{m}_k$. ∎

Note the natural interpretation (and proof) of this proposition in terms of closed paths on the graph Γ_N: closed paths on Γ_N which are too short are in fact projections of closed paths on Γ.

Corollary 2 *For* $N > \ell \max\limits_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} \max\limits_{1\leq j\leq n} |a_j - b_j|$ *and* $|z| > C^2$*:*

$$\mathsf{Q}(z)\cdot|B_N(z)|^{N^{-n}} = \left|\exp\left(\sum_{k>\ell} \frac{\mathsf{m}_k - \mathsf{m}_k^{(N)}}{k} z^{-k}\right)\right|\,.$$

This not only gives an estimate for the rate of convergence of (1) with respect to the usual absolute value on $\mathbb{C}$*, but it also yields the following congruence of power series in* z^{-1}*:*

$$B_N(z)^{-N^{-n}} \equiv z^{-1} + \sum_{k\geq 1} A_k z^{-k-1} \mod z^{-\ell-1}$$

with A_k *as in (25)* ∎

Remark about the relation with the large complex structure limit. Since the measure $dV(r)$ is the push forward of the measure $dt_1\,dt_2\ldots dt_n$ on $\mathbb{R}^n$ by the function $|\widehat{\mathcal{D}}(\mathsf{t})|^2$, one can rewrite (13) as

$$\mathsf{H}(z) \;=\; \frac{1}{(2\pi i)^n}\int_{\mathbb{U}^n} \frac{1}{z - W(x_1,\ldots,x_n)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\cdots\frac{dx_n}{x_n}$$

for $z \in \mathbb{C}\setminus\mathcal{I}$. From this (and the residue theorem) one sees that $\mathsf{H}(z)$ is *a period of some differential form of degree* $n-1$ *along some* $(n-1)$*-cycle on the hypersurface in* $(\mathbb{C}^*)^n$ *given by the equation* $W(x_1,\ldots,x_n) = z$. As z varies we get a 1-parameter family of hypersurfaces. The function $\mathsf{H}(z)$ is a solution of the Picard-Fuchs differential equation associated with (that $(n-1)$-form on) this family of hypersurfaces. The Picard-Fuchs equation is equivalent with a recurrence relation for the coefficients m_k in the power series expansion (24) of

$\mathsf{H}(z)$ near $z = \infty$. All this is standard knowledge about Calabi-Yau varieties near the large complex structure limit and there is an equally standard algorithm to derive from the Picard-Fuchs differential equation enumerative information about numbers of instantons (or rational curves); see for instance [20, 13, 18].

On the other hand, we have the enumerative data $\mathsf{m}_k^{(N)}$ of the present paper. In the limit for $N \to \infty$ these yield the moments m_k and, hence, the Picard-Fuchs differential equation and eventually the instanton numbers.

6 Examples

6.1 $n = 1$

Mahler measures of one variable polynomials have a long history with many interesting results; see the introductory sections of [4, 7, 15]. We limit our discussion to one example, without a claim of new results. This simple, yet non-trivial, example has $n = 1$, $\mathfrak{A} = \{-1, 1\} \subset \mathbb{Z}$, $c_{-1} = c_1 = 1$ and hence

$$|\widehat{\mathcal{D}}(t)|^2 = 2 + 2\cos(4\pi t)\,, \qquad W(x) = (x + x^{-1})^2\,.$$

The moments are

$$\mathsf{m}_k \;=\; \text{constant term of } (x + x^{-1})^{2k} \;=\; \binom{2k}{k}$$

and hence by (24): for $z \in \mathbb{R}$, $z > 4$,

$$\begin{aligned}
\mathsf{H}(z) \;&=\; \sum_{k \geq 0} \binom{2k}{k} z^{-k-1} \;=\; \frac{1}{\sqrt{z(z-4)}}\,, \\
\mathsf{Q}(z) \;&=\; \exp\left(-\int \frac{dz}{\sqrt{z(z-4)}}\right) \;=\; \frac{1}{2}\left(z - 2 - \sqrt{z(z-4)}\right)\,.
\end{aligned}$$

Applying Formula (17) to the present example we find

$$\mathsf{m}_k^{(N)} = \sum_{j \equiv k \bmod N} \binom{2k}{j}\,,$$

which nicely illustrates Proposition 4.

Setting $z = 2 + u + u^{-1}$ one finds for the polynomials $\mathsf{B}_N(z)$ defined in (3):

$$\begin{aligned}
\mathsf{B}_N(z) \;&=\; \prod_{x^2 \in \mu_N} (z - 2 - x^2 - x^{-2}) \;=\; u^{-N} \prod_{x^2 \in \mu_N} (u - x^2)(u - x^{-2}) \\
&=\; u^N + u^{-N} - 2 \\
&=\; \left(\frac{1}{2}\left(z - 2 - \sqrt{z(z-4)}\right)\right)^N + \left(\frac{1}{2}\left(z - 2 + \sqrt{z(z-4)}\right)\right)^N - 2 \\
&=\; -2 + 2^{1-N} \sum_j \binom{N}{2j} z^j (z-4)^j (z-2)^{N-2j}\,.
\end{aligned}$$

So, $\mathsf{B}_N(z)$ *is up to some shift and normalization the* N*-th Čebyšev polynomial.*

The above computation also shows $\mathsf{B}_N(z) = \mathsf{Q}(z)^N + \mathsf{Q}(z)^{-N} - 2$ and thus, in agreement with (1),

$$\lim_{N\to\infty} \mathsf{B}_N(z)^{-N^{-1}} = \mathsf{Q}(z)\,.$$

For actual computation of $\mathsf{B}_N(z)$ in case $z \in \mathbb{Z}$ one can use the generating series identity:

$$\sum_{N\geq 1} \mathsf{B}_N(z)\,\frac{T^N}{N} = -\log\left(1-(z-4)\frac{T}{(1-T)^2}\right).$$

For $z = 6$ one finds (using PARI)

$$\begin{aligned}\sum B_N(6)T^N \;=\;& 2T + 12T^2 + 50T^3 + 192T^4 + 722T^5 + 2700T^6 + 10082T^7 \\ & +37632T^8 + 140450T^9 + 524172T^{10} + 1956242T^{11} \\ & +7300800T^{12} + 27246962T^{13} + 101687052T^{14} \\ & +379501250T^{15} + 1416317952T^{16} + 5285770562T^{17} + \ldots\end{aligned}$$

For primes p in the displayed range the number $B_{p-1}(6)$ is divisible by p^2 for $p \equiv \pm 1 \bmod 12$ and is not divisible by p for $p \equiv \pm 5 \bmod 12$ and is exactly divisible by p if $p = 2, 3$. We also checked $5^2 \mid B_{24}(6)$ and $7^2 \mid B_{48}(6)$. This agrees with the number of solutions of the equation $u + u^{-1} = 4$ in $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$.

If $z \in \mathbb{Z}, z > 4$, then $\mathsf{Q}(z) = \frac{1}{2}(\,z - 2 - \sqrt{z(z-4)})$ is a unit in the real quadratic field $\mathbb{Q}(\sqrt{z(z-4)})$. According to Dirichlet's class number formula it relates to the L-function of this real quadratic field:

$$\log(\mathsf{Q}(z)) = \frac{\sqrt{D}}{2h} L(1,\chi)$$

where D, h, χ are the discriminant, class number, character, respectively, of the real quadratic field $\mathbb{Q}(\sqrt{z(z-4)})$ (see e.g. [5]). The relations between Mahler measures and values of L-functions, which have been observed for some curves, are perfect analogues of the above class number formula (see [15]).

6.2 The honeycomb pattern.

For a nice two-dimensional example we take $\mathfrak{A} = \{(1,0),\,(0,1),\,(-1,-1)\} \subset \mathbb{Z}^2$, $c_{(1,0)} = c_{(0,1)} = c_{(-1,-1)} = 1$ and hence

$$\begin{aligned}|\widehat{\mathcal{D}}(t_1,t_2)|^2 &= 3 + 2\cos(2\pi(t_1-t_2)) + 2\cos(2\pi(2t_1+t_2)) + 2\cos(2\pi(t_1+2t_2)),\\ W(x_1,x_2) &= (x_1 + x_2 + x_1^{-1}x_2^{-1})(x_1^{-1} + x_2^{-1} + x_1x_2)\\ &= x_1x_2^{-1} + x_1^2x_2 + x_1^{-1}x_2 + x_1x_2^2 + x_1^{-2}x_2^{-1} + x_1^{-1}x_2^{-2} + 3\,.\end{aligned}$$

As a basis for the lattice Λ we take $(2,1)$ and $(-1,-2)$. This leads to coordinates $u_1 = x_1^2 x_2$ and $u_2 = x_1^{-1} x_2^{-2}$ on the torus $\mathbb{U}^\Lambda$. In these coordinates the function W reads

$$\begin{aligned} W(u_1,u_2) &= u_1 + u_1^{-1} + u_2 + u_2^{-1} + u_1^{-1}u_2 + u_1 u_2^{-1} + 3 \\ &= (u_1+u_2+1)(u_1^{-1}+u_2^{-1}+1)\,. \end{aligned} \tag{26}$$

Figure 2 shows a piece of the graph Γ. Figure 1 shows some level sets of the function $|\widehat{\mathcal{D}}(t_1,t_2)|^2$. The dual lattice $\Lambda^\vee$ is spanned by $(1,0)$ and $(\frac{1}{3},\frac{1}{3})$. The maximum of the function $|\widehat{\mathcal{D}}(t_1,t_2)|^2$ equals 9 and is attained at the points of $\Lambda^\vee$. The minimum of the function $|\widehat{\mathcal{D}}(t_1,t_2)|^2$ equals 0 and is attained at the points of $(0,-\frac{1}{3})+\Lambda^\vee$ and $(\frac{1}{3},0)+\Lambda^\vee$. There are saddle points with critical value 1 at $(-\frac{1}{6},\frac{1}{3})+\Lambda^\vee$, $(\frac{1}{3},-\frac{1}{6})+\Lambda^\vee$ and $(\frac{1}{6},\frac{1}{6})+\Lambda^\vee$. In terms of the coordinates u_1, u_2 the maximum lies at $(u_1,\,u_2)=(1,1)$, the minima at $(e^{2\pi i/3},e^{4\pi i/3})$, $(e^{4\pi i/3},e^{2\pi i/3})$ and the saddle points at $(1,-1),(-1,1),(-1,-1)$. The algebraic geometry of this example concerns the 1-parameter family of elliptic curves with equation $z - W(u_1,u_2) = 0$. In homogeneous coordinates $(U_0:U_1:U_2)$ on the projective plane $\mathbb{P}^2$, with $u_1 = U_1U_0^{-1}$, $u_2 = U_2U_0^{-1}$, this becomes a homogeneous equation of degree 3:

$$(U_0U_1 + U_0U_2 + U_1U_2)(U_0+U_1+U_2) - zU_0U_1U_2 \;=\; 0\,. \tag{27}$$

Beauville [2] showed that there are exactly six semi-stable families of elliptic curves over $\mathbb{P}^1$ with four singular fibres. The pencil (27) is one of these six. It has singular fibres at $z = 0,1,9,\infty$ with Kodaira types I_2, I_3, I_1, I_6, respectively. Note that the first three match the critical points and levels in the diffraction pattern. After blowing up the points $(1,0,0),(0,1,0),(0,0,1)$ of $\mathbb{P}^2$ one gets the DelPezzo surface dP_3. The elliptic pencil (27) naturally lives on dP_3. It has six base points, corresponding to six sections of the pencil. Since the base points have a zero coordinate, these sections do not intersect the real torus $\mathbb{U}^\Lambda$. Equations (26) and (27) also appear in the literature in connection with the string theory of dP_3.

Formula (23) and some manipulations of binomials give the moments:

$$\mathsf{m}_k \;=\; \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}\,.$$

These numbers satisfy the recurrence relation (see [19] Table 7)

$$(k+1)^2\mathsf{m}_{k+1} = (10k^2+10k+3)\mathsf{m}_k - 9k^2\mathsf{m}_{k-1}\,. \tag{28}$$

We refer to [19] Example $\mathcal{C}$ and to [18] Example ♯6 for relations of these numbers to modular forms and instanton counts. Golyshev [10] can derive the recurrence (28) from the quantum cohomology of dP_3. The numerical evidence for the relation (21) between Mahler measure and L-function in this example is given in [4] Table 2.

With Formulas (17) and (26) one easily calculates

$$\mathsf{m}_k^{(N)} = \sum_{i_1 \equiv i_2 \bmod N,\, j_1 \equiv j_2 \bmod N} \begin{pmatrix} k \\ i_1 \end{pmatrix} \begin{pmatrix} k - i_1 \\ j_1 \end{pmatrix} \begin{pmatrix} k \\ i_2 \end{pmatrix} \begin{pmatrix} k - i_2 \\ j_2 \end{pmatrix} .$$

Note that these formulas confirm $\mathsf{m}_k^{(N)} = \mathsf{m}_k$ for $k < N$.

Equation (27) is clearly invariant under permutations of U_0, U_1, U_2. Therefore the diffraction pattern has this S_3-symmetry too. Since only the critical points have a non-trivial stabilizer in S_3 the multiplicities $\mathrm{mult}_N(r)$ in this example satisfy

$$\begin{array}{lcll} \mathrm{mult}_N(9) & = & 1 & \forall N \\ \mathrm{mult}_N(0) & = & 2 & \text{if} \quad 3|N \\ \mathrm{mult}_N(1) & \equiv & 3 \bmod 6 & \text{if} \quad 2|N \\ \mathrm{mult}_N(r) & \equiv & 0 \bmod 6 & \text{if} \quad r \neq 0,\, 1,\, 9, \quad \forall N. \end{array} \tag{29}$$

We have computed the numbers $\mathrm{mult}_N(r)$ for some values of N. We found for instance

$$\mathsf{B}_6(z) = z^2\,(z-1)^{15}\,(z-3)^6\,(z-4)^6\,(z-7)^6\,(z-9)\,.$$

We computed $W(u_1, u_2) = (u_1 + u_2 + 1)(u_1^{-1} + u_2^{-1} + 1)$ for $u_1, u_2 \in \mathbb{F}_7^*$: the (i,j)-entry of the following 6×6-matrix is $W(i,j) \bmod 7$:

$$\begin{bmatrix} 2 & 3 & 0 & 3 & 0 & 1 \\ 3 & 3 & 4 & 0 & 1 & 1 \\ 0 & 4 & 0 & 1 & 4 & 1 \\ 3 & 0 & 1 & 3 & 4 & 1 \\ 0 & 1 & 4 & 4 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

This yields the following count of points over $\mathbb{F}_7$:

$z \bmod 7$	:	0	1	2	3	4	5	6
$\sharp\{\xi \in (\mathbb{F}_7^*)^2 \mid W(\xi) = z \text{ in } \mathbb{F}_7\}$	:	8	15	1	6	6	0	0

Thus we see that the inequality in (20) can be strict:

$$v_7(\mathsf{B}_6(53)) \;=\; 12 \;>\; 6 \;=\; \sharp\{\xi \in (\mathbb{F}_7^*)^2 \mid W(\xi) = 53 \text{ in } \mathbb{F}_7\}\,. \tag{30}$$

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