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The Ordinary Limit for Varieties over $\mathbb{Z}[x_1, \ldots, x_r]$

Jan Stienstra
Mathematisch Instituut, Universiteit Utrecht
Postbus 80.010, 3508 TA Utrecht, the Netherlands
stien@math.uu.nl

Abstract. We investigate for families of smooth projective varieties over a localized polynomial ring $\mathbb{Z}[x_1, \ldots, x_r][D^{-1}]$ the conjugate filtration on De Rham cohomology $\otimes\mathbb{Z}/N\mathbb{Z}$. As N tends to ∞ this leads to the concept of the *ordinary limit*, which seems to be the *non-archimedean analogue* of the *large complex structure limit*.

Introduction

Many important families of Calabi-Yau threefolds appear in the following form: There is a ring $\mathbb{A} = \mathbb{Z}[x_1, \ldots, x_r][D^{-1}]$, with D some polynomial in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_r]$, and there is a smooth projective morphism $f : \mathbb{X} \to \mathbb{S} = \mathrm{Spec}\,\mathbb{A}$ of relative dimension 3 such that all De Rham cohomology groups $H^m_{DR}(\mathbb{X}/\mathbb{S})$ and all Hodge cohomology groups $H^j(\mathbb{X}, \Omega^i_{\mathbb{X}/\mathbb{S}})$ are free $\mathbb{A}$-modules and such that

$$\Omega^3_{\mathbb{X}/\mathbb{S}} \simeq \mathcal{O}_{\mathbb{X}}\,, \quad H^1(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) = H^2(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) = 0\,. \tag{0.1}$$

This happens, for instance, for complete intersections in a projective space $\mathbb{P}^d_{\mathbb{A}}$ given by homogeneous polynomials with coefficients in $\mathbb{A}$ and with degrees summing to $d+1$, like $\diamond$ quintic hypersurfaces in $\mathbb{P}^4$ $\diamond$ intersections of two cubics in $\mathbb{P}^5$ $\diamond$ intersections of four quadrics in $\mathbb{P}^7$. The conditions of smoothness of $\mathbb{X}/\mathbb{S}$ and freeness of cohomology then boil down to a condition that certain polynomial expressions in the coefficients should be invertible in $\mathbb{A}$.

In connection with Mirror Symmetry one is particularly interested in solutions of the Picard-Fuchs equations of the given family of Calabi-Yau threefolds. The Picard-Fuchs equations describe how a non-zero global differential 3-form varies in the family. For families like those in the first paragraph the Picard-Fuchs equations are completely defined over $\mathbb{A}$ and there are many analytical environments in which one may look for solutions. Traditionally one works in an *archimedean environment* represented by complex geometry and complex functions and one looks at Calabi-Yau threefolds *near the large complex structure limit*. In the present paper

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we want to work in a *non-archimedean environment* which is represented by projective systems of groups $\{M_N\}_{N\in\mathbb{N}}$, with M_N a module over $\mathbb{Z}/N\mathbb{Z}$, indexed by the positive integers with their divisibility relation, i.e. for every $N \in \mathbb{N}$ there is given a $\mathbb{Z}/N\mathbb{Z}$-module M_N and for every pair of positive integers (N, K) with K dividing N there is given a homomorphism $M_N \to M_K$.

In [12] it was pointed out that in the crystalline cohomology of *families of ordinary Calabi-Yau threefolds in the p-adic setting* the structure is analogous to that for Calabi-Yau threefolds near the large complex structure limit. The analogy comes from the fact that in both the complex and the p-adic situation there is a filtration on De Rham cohomology opposite to the Hodge filtration, stable under the Gauss-Manin connection and with associated graded module of Hodge-Tate type. For complex Calabi-Yau threefolds near the large complex structure limit one obtains the filtration from the *local monodromy around this limit point* [4, 11]. For families of ordinary Calabi-Yau threefolds over a base of positive characteristic one uses the conjugate filtration and the action of *Frobenius operators*. The present paper does not build on [12], but takes only the suggestion that by studying "ordinariness" and Frobenius operators one may discover interesting arithmetic-geometrical facts about Calabi-Yau threefolds. Ordinariness is a well defined condition for varieties in characteristic $p > 0$. The present paper focusses on the question: *What is the right notion of "ordinariness" for varieties over* $\mathbb{Z}[x_1, \ldots, x_r]$*?* For reasons of time and space we must postpone the discussion of topics which are specific for Calabi-Yau threefolds, like *canonical coordinates* and *Yukawa coupling* to another article. Without specific CY3 conditions the setting becomes:

Setting 0.1 *There is given a smooth projective morphism* $f : \mathbb{X} \to \mathbb{S} = \operatorname{Spec} \mathbb{A}$ *of relative dimension* d *with ring* $\mathbb{A}$ *étale over a polynomial ring* $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$ *and such that all De Rham cohomology groups* $H^m_{DR}(\mathbb{X}/\mathbb{S})$ *and all Hodge cohomology groups* $H^j(\mathbb{X}, \Omega^i_{\mathbb{X}/\mathbb{S}})$ *are free* $\mathbb{A}$*-modules. The* $\mathbb{A}$*-module* $H^{\mathsf{d}}(\mathbb{X}, \Omega^{\mathsf{d}}_{\mathbb{X}/\mathbb{S}})$ *should have rank* 1 *and be generated by an element* ϖ *from* $H^{\mathsf{d}}(\mathbb{X}, \Omega^{\mathsf{d}}_{\mathbb{X}/\mathbb{S},\log})$ *(cf. (6.15)). Finally for all* i, j *the pairing* $\langle\ ,\ \rangle :\ H^j(\mathbb{X}, \Omega^i_{\mathbb{X}/\mathbb{S}}) \times H^{\mathsf{d}-j}(\mathbb{X}, \Omega^{\mathsf{d}-i}_{\mathbb{X}/\mathbb{S}}) \to \mathbb{A}$*, defined by the rule* $\alpha \cdot \beta = \langle \alpha, \beta\rangle \varpi$*, should be non-degenerate.*

Note that this setting behaves well with respect to étale base change: if A is an étale algebra over $\mathbb{A}$, $S = \operatorname{Spec} A$ and $X = \mathbb{X} \times_{\mathbb{S}} S$, then $f : X \to S$ enjoys the same properties; namely $H^m_{DR}(X/S) = H^m_{DR}(\mathbb{X}/\mathbb{S}) \otimes_{\mathbb{A}} A$ for every m and $H^j(X, \Omega^i_{X/S}) = H^j(\mathbb{X}, \Omega^i_{\mathbb{X}/\mathbb{S}}) \otimes_{\mathbb{A}} A$ for all i, j and these are free A-modules; moreover, the induced pairing $H^j(X, \Omega^i_{X/S}) \times H^{\mathsf{d}-j}(X, \Omega^{\mathsf{d}-i}_{X/S}) \to A$ is non-degenerate.

The reason for requiring that 2 be invertible in $\mathbb{A}$ is some technical 2-torsion problem in the constructions of [13]; see loc. cit. 2.7-2.9. The reason for having $\mathbb{A}$ étale over $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$ instead of just of the form $\mathbb{Z}[x_1, \ldots, x_r][D^{-1}]$ is that for a continuation of this work one needs "vectors fixed by Frobenius". That requires solving certain systems of polynomial equations and thus more involved étale extensions than just Zariski localizations will be needed. Therefore we choose a formulation so that it is clear that the results remain valid after such an extension.

For varieties in characteristic $p > 0$ "ordinariness" can be described in various equivalent ways. In Section 1 we look at the condition of ordinariness which in characteristic p requires the *vanishing of all cohomology groups of all sheaves of* *exact differential forms*. Its straightforward generalization to our setting can be formulated as: *for fixed* $N \geq 2$:

$$H^j(X, Z_N\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \textit{is an isomorphism for all} \quad i, j, \tag{0.2}$$

where $Z_N\Omega^i_{X/S} := \{\omega \in \Omega^i_{X/S} \mid d\omega \in N \cdot \Omega^{i+1}_{X/S}\}$ is the sheaf of i-forms which are closed modulo N. Here we write X/S and not $\mathbb{X}/\mathbb{S}$ because the condition will usually only be satisfied after we restrict the original family to a (Zariski) open subset S of $\mathbb{S}$ which depends on N. Theorem 1.2 and Corollary 1.5 state that if Condition (0.2) is satisfied the *conjugate filtration* on $H^m_{DR}(X/S) \otimes \mathbb{Z}/N\mathbb{Z}$, for every m, is indeed opposite to the Hodge filtration and is stable for the Gauss-Manin connection. That is, however, not all we want. For one thing, as N varies, conditions and results should fit into projective systems. For the condition (0.2) this can be achieved (see Proposition 4.3) by replacing it by the (stronger) condition

$$H^j(X, Z_p\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \begin{array}{l}\textit{is an isomorphism for all } \ i, j \ \textit{ and} \\ \textit{all prime numbers } \ p \ \textit{ dividing } \ N.\end{array} \tag{0.3}$$

More importantly we also want *the associated graded module to be of Hodge-Tate type*. The associated graded module is the *Hodge cohomology* mod$N$:

$$\bigoplus_{i+j=m} H^j(X, \Omega^i_{X/S}) \otimes \mathbb{Z}/N\mathbb{Z}.$$

To formulate what "Hodge-Tate type" means we need Frobenius operators. The traditional point of view on Frobenius operators is that they arise in characteristic p, for only one prime number p at a time. Here we want to work at the level of algebraic geometry over $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$. In Sections 2–5 we describe the formalism of *generalized Witt vectors* and the *generalized De Rham-Witt complex* which yields a Frobenius operator and a Verschiebung operator for every positive integer. In Section 4 we relate $Z_N\Omega^i_{X/S}$ to Frobenius and Verschiebung operators. In Theorem 5.3 we relate the conjugate filtration, the Hodge filtration and the Hodge decomposition to what we call the *Hodge-Witt cohomology of* X/S. In Section 6 we investigate when the Hodge-Witt cohomology of X/S is of "Hodge-Tate type". It turns out that the conditions must be stengthened once more: instead of taking in (0.3) only primes p dividing N one should take all primes $p \leq N$. Thus we are led to propose

Definition 0.2 Let $f : X \to S = \operatorname{Spec} A$ satisfy the hypotheses in Setting 0.1. Let N be a positive integer. We say that X/S *is ordinary up to level* N if

$$H^j(X, Z_p\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \begin{array}{l}\textit{is an isomorphism for all } \ i, j \ \textit{ and} \\ \textit{all prime numbers } \ p \leq N.\end{array} \tag{0.4}$$

In Theorem 6.7 we show that if X/S is ordinary up to level N, then the Frobenius operators on Hodge-Witt cohomology induce for every prime number $p \leq N$ and for all i, j an isomorphism

$$\overline{F}_p : \overline{F}^*_p H^j(X, \Omega^i_{X/S}) \xrightarrow{\simeq} H^j(X, \Omega^i_{X/S}) \otimes \mathbb{Z}/p\mathbb{Z} \tag{0.5}$$

where

$$\overline{F}^*_p H^j(X, \Omega^i_{X/S}) = (A/pA) \otimes_A H^j(X, \Omega^i_{X/S})$$

with A/pA viewed as an A-module via the ring homomorphism $A \to A/pA\,, \quad a \mapsto a^p \bmod p$. Isomorphisms like (0.5) induced by Frobenius operators are the standard criterion for being of Hodge-Tate type. In the present situation it is probably better to say *of Hodge-Tate type up to level* N.

Setting 0.1, Definition 0.2, Theorems 1.2, 5.3 and 6.7 and the above description of "Hodge-Tate type up to level N" lead to the following conclusion.

Conclusion 0.3 *Let* $f : \mathbb{X} \to \mathbb{S} = \mathrm{Spec}\mathbb{A}$ *be as in Setting 0.1. Let* A *be an étale* $\mathbb{A}$*-algebra,* $S = \mathrm{Spec}A$ *and* $X = \mathbb{X} \times_{\mathbb{S}} S$*. Assume that* X/S *is ordinary up to level* N*, for some positive integer* N*. Then for every* m *and for every* $n \leq N$ *the conjugate filtration on* $H^m_{DR}(X/S) \otimes \mathbb{Z}/n\mathbb{Z}$ *is opposite to the Hodge filtration, is stable for the Gauss-Manin connection and the associated graded* $\bigoplus_{i+j=m} H^j(X, \Omega^i_{X/S}) \otimes \mathbb{Z}/n\mathbb{Z}$ *is of Hodge-Tate type up to level* N*.* □

This conclusion gives a somewhat weakened description of the actual structure. For a more precise description Propositions 5.2, 6.1 and Corollary 6.3 should also be taken into account. Moreover these suggest a reformulation in terms of *formal groups*, which might turn out to be quite attractive. A hint to this formal group structure is given in the Appendix.

The condition that X/S be ordinary up to level N will in general only be satisfied after restricting the original family $\mathbb{X}/\mathbb{S}$ to an open subset S of $\mathbb{S}$. This subset depends on N. As N moves up through $\mathbb{N}$ the corresponding open set will shrink. In the limit we have a subset $S_\infty \subset \mathbb{S}$.

- *We want to call* S_∞ *the ordinary limit set of* $\mathbb{X}/\mathbb{S}$.

S_∞ contains Spec $(\mathbb{A} \otimes \mathbb{Q})$, because there the conditions for ordinariness up to any level are trivially satisfied. But over Spec $(\mathbb{A} \otimes \mathbb{Q})$ the De Rham cohomology groups modulo N are just 0. So there there is no interesting conclusion. However S_∞ should be much larger than Spec $(\mathbb{A} \otimes \mathbb{Q})$ and should contain an affine set Spec (A_∞) on which no or just a few primes are invertible; see the example discussed below and also Remark 1.4. The geometry of the family $\mathbb{X}$ restricted to Spec (A_∞) is not yet clear (to me), but examples like the one below indicate that Spec (A_∞) contains a punctured formal neighborhood of the "large complex structure limit point" and that the aforementioned formal groups and their interaction via the Gauss-Manin connection extend over this limit point. I like to view this as analogous to the traditional complex situation where the limit fibre is a singular Calabi-Yau threefold and where one has a limit mixed Hodge structure.

To illustrate some of the above issues let us look at the pencil of elliptic curves

$$x(X^3 + Y^3 + Z^3) + XYZ = 0, \tag{0.6}$$

with the nine base points blown up. There are singular fibres for $x = 0$ and $x^3 = \frac{1}{27}$. Thus we work over the ring $\mathbb{A} = \mathbb{Z}[x, \frac{1}{2x(27x^3-1)}]$.

The formal group law (in an appropriate coordinatization) for these elliptic curves is

$$G(t_1, t_2) = \ell^{-1}(\ell(t_1) + \ell(t_2)). \tag{0.7}$$

with

$$\ell(t) \; = \; \sum_{m \geq 1} \frac{1}{m} \, a_m(x) t^m \; \in (\mathbb{A} \otimes \mathbb{Q})[[t]]$$

and

$$a_m(x) = \sum_{j\geq 0} \frac{(3j)!}{j!^3} \left(\begin{array}{c} m-1 \\ 3j \end{array} \right) x^{3j} \in \mathbb{Z}[x]. \tag{0.8}$$

For λ in some field of characteristic $p > 2$ it is well known that the elliptic curve $\lambda(X^3+Y^3+Z^3)+XYZ=0$ is ordinary if and only if its *Hasse-Witt invariant* $a_p(\lambda)$ is not zero; here we use one of many equivalent characterizations of ordinariness for elliptic curves in characteristic p. So the ordinary limit set S_∞ in this example is a union of affine sets Spec A_Π where Π runs over all (finite as well as infinite) subsets of the set $\mathcal{P}$ of all prime numbers and

$$A_\Pi := \mathbb{A}[a_p(x)^{-1},\, r^{-1} \mid p \in \Pi,\, r \in \mathcal{P} \setminus \Pi].$$

Thus, for $\Pi = \emptyset$, the empty set, $A_\emptyset = \mathbb{A} \otimes \mathbb{Q}$. On the other extreme, $\Pi = \mathcal{P}$ gives the ring we have in mind for the above mentioned A_∞; so,

$$A_\infty = A_\mathcal{P} = \mathbb{A}[a_p(x)^{-1} \mid p \text{ prime}].$$

Note that $a_p(0) = 1$ for all p and that, hence, A_∞ embeds into the Laurent series ring $\mathbb{Z}[\frac{1}{2}][[x]][x^{-1}]$. Thus we see that Spec A_∞ contains a punctured formal neighborhood of $x = 0$. The fibre over $x = 0$ is singular, consisting of three lines, and the monodromy around this fibre is maximally unipotent. In the physicists' language $x = 0$ is the large complex structure limit point. Note that $a_m(0) = 1$ for all $m \geq 1$ and that, hence, the formal group law (0.7) extends well over $x = 0$; in fact at $x = 0$ it is the standard multiplicative group law $t_1 + t_2 - t_1 t_2$.

For families of K3-surfaces, like $x(W^4 + X^4 + Y^4 + Z^4) + WXYZ = 0$, the story is the same (with 3 replaced by 4 in (0.8)). In particular the fibre at $x = \lambda$ is ordinary if its Hasse-Witt invariant is not zero. The relevant formal group is the formal Brauer group $H^2(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$; see the appendix.

For families of Calabi-Yau threefolds invertibility of the Hasse-Witt invariant associated with the Artin-Mazur formal group $H^3(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$ (see the appendix) is necessary, but may be not sufficient to ensure ordinariness. Yet I do expect that also in this CY3 case the ordinary limit set contains a punctured formal neighborhood of "the large complex structure limit point".

1 The conjugate filtration and ordinariness: Act 1

1.1 Definition and basic properties of the conjugate filtration. For a smooth projective morphism $X \to S$ of schemes the De Rham cohomology $H^m_{DR}(X/S)$ is by definition the hypercohomology $\mathbb{H}^m(X, \Omega^\bullet_{X/S})$ of the De Rham complex [1]

$$\Omega^\bullet_{X/S} \;:\; \ldots \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{i-1}_{X/S} \xrightarrow{d} \Omega^i_{X/S} \xrightarrow{d} \Omega^{i+1}_{X/S} \xrightarrow{d} \ldots$$

Every complex $\mathcal{C}^\bullet$ carries two natural filtrations by sub-complexes $\mathcal{C}^{\bullet \geq i}$ and $t_{\leq i}\mathcal{C}^\bullet$:

$$\left(\mathcal{C}^{\bullet \geq i}\right)^j = 0 \quad \text{if} \;\; j < i\,, \qquad \left(\mathcal{C}^{\bullet \geq i}\right)^j = \mathcal{C}^j \quad \text{if} \;\; j \geq i. \tag{1.1}$$

$$\left(t_{\leq i}\mathcal{C}^\bullet\right)^j = 0 \quad \text{if} \;\; j > i\,, \qquad \left(t_{\leq i}\mathcal{C}^\bullet\right)^j = \mathcal{C}^j \quad \text{if} \;\; j < i\,,$$

$$\left(t_{\leq i}\mathcal{C}^\bullet\right)^i = Z\mathcal{C}^i := \ker(d : \mathcal{C}^i \to \mathcal{C}^{i+1}). \tag{1.2}$$

[1] all sheaves in this paper are taken with respect to the Zariski topology

When applied to the De Rham complex $\Omega^{\bullet}_{X/S}$ these induce on $H^m_{DR}(X/S)$:
the *Hodge filtration*

$$\mathrm{Fil}^i_{\mathrm{Hodge}} H^m_{DR}(X/S) = \mathrm{image}(\mathbb{H}^m(X, \Omega^{\bullet \geq i}_{X/S}) \to \mathbb{H}^m(X, \Omega^{\bullet}_{X/S}))$$

and the *conjugate filtration*

$$\mathrm{Fil}^{\mathrm{con}}_i H^m_{DR}(X/S) = \mathrm{image}(\mathbb{H}^m(X, t_{\leq i}\Omega^{\bullet}_{X/S}) \to \mathbb{H}^m(X, \Omega^{\bullet}_{X/S})).$$

The Hodge filtration is decreasing and the conjugate filtration is increasing and moreover

$$\begin{aligned} \mathrm{Fil}^0_{\mathrm{Hodge}} H^m_{DR}(X/S) &= \mathrm{Fil}^{\mathrm{con}}_m H^m_{DR}(X/S) = H^m_{DR}(X/S)\,, \\ \mathrm{Fil}^{m+1}_{\mathrm{Hodge}} H^m_{DR}(X/S) &= \mathrm{Fil}^{\mathrm{con}}_{-1} H^m_{DR}(X/S) = 0\,. \end{aligned}$$

Let $B\Omega^i_{X/S} := \mathrm{image}(d : \Omega^{i-1}_{X/S} \to \Omega^i_{X/S})$ resp. $Z\Omega^i_{X/S} := \ker(d : \Omega^i_{X/S} \to \Omega^{i+1}_{X/S})$ denote the sheaves of exact resp. closed i-forms. There are short exact sequences of complexes

$$0 \to t_{\leq i}\Omega^{\bullet}_{X/S} \oplus \Omega^{\bullet \geq i+1}_{X/S} \to \Omega^{\bullet}_{X/S} \to B\Omega^{i+1}_{X/S}[-i] \to 0 \tag{1.3}$$

$$0 \to Z\Omega^i_{X/S}[-i] \to t_{\leq i}\Omega^{\bullet}_{X/S} \oplus \Omega^{\bullet \geq i}_{X/S} \to \Omega^{\bullet}_{X/S} \to 0 \tag{1.4}$$

$$0 \to Z\Omega^i_{X/S} \to \Omega^i_{X/S} \to B\Omega^{i+1}_{X/S} \to 0 \tag{1.5}$$

where $B\Omega^{i+1}_{X/S}[-i]$ (resp. $Z\Omega^i_{X/S}[-i]$) is the complex with $B\Omega^{i+1}_{X/S}$ (resp. $Z\Omega^i_{X/S}$) sitting in degree i and all other terms equal to 0. The corresponding exact sequences of hypercohomology groups show: *if the condition*

$$H^{m-i}(X, B\Omega^{i+1}_{X/S}) = H^{m-i-1}(X, B\Omega^{i+1}_{X/S}) = H^{m-i}(X, B\Omega^i_{X/S}) = 0 \tag{1.6}$$

is satisfied, then

$$H^m_{DR}(X/S) = \mathrm{Fil}^{\mathrm{con}}_i H^m_{DR}(X/S) \oplus \mathrm{Fil}^{i+1}_{\mathrm{Hodge}} H^m_{DR}(X/S) \tag{1.7}$$

$$H^{m-i}(X, \Omega^i_{X/S}) = \mathrm{Fil}^{\mathrm{con}}_i H^m_{DR}(X/S) \cap \mathrm{Fil}^i_{\mathrm{Hodge}} H^m_{DR}(X/S). \tag{1.8}$$

So if (1.6) holds for $i = 0, \ldots, m$, then the Hodge filtration and the conjugate filtration on $H^m_{DR}(X/S)$ are opposite and one has the *Hodge decomposition*:

$$H^m_{DR}(X/S) = \bigoplus_{i=0}^{m} H^{m-i}(X, \Omega^i_{X/S}). \tag{1.9}$$

Remark 1.1 This form of conjugate filtration plays no role in complex geometry, since for varieties over $\mathbb{C}$ with the complex topology the *Poincaré lemma* says that locally all closed differential forms are exact, and thus implies that the filtration $\{\mathrm{Fil}^{\mathrm{con}}_i\}$ jumps from null at $i = -1$ to all at $i = 0$. On the other hand, according to Hodge theory one obtains for complex Kähler manifolds a filtration opposite to the Hodge filtration by simply taking the complex conjugate of the Hodge filtration.

Condition (1.6) is a familiar condition for smooth projective varieties in positive characteristic [8, 9]: *When $f : X \to S$ is a proper smooth morphism between schemes of characteristic $p > 0$, one says that X is* ordinary over S *if for all i, j*

$$R^j f_* B\Omega^i_{X/S} = 0.$$

In case $S = \mathrm{Spec}\, k$ with k a perfect field of characteristic $p > 0$ this is one of many equivalent ways to define ordinariness (cf. [9] thm. IV 4.13). This one involves no concepts specific for characteristic p and it is tempting to use it also in other contexts as a quick way for splitting the Hodge filtration, with the conjugate filtration as its opposite. We want to work in the geometric setting of a smooth projective morphism $f : X \to S = \mathrm{Spec}\, A$ in which the ring A is étale over a polynomial ring $\mathbb{Z}[x_1,\dots,x_r]$. We have found that the following makes a good bridge between the various prime characteristics:

- *Work systematically with projective systems of groups $\{M_N\}_{N\in\mathbb{N}}$, with M_N a module over $\mathbb{Z}/N\mathbb{Z}$, indexed by the set $\mathbb{N}$ which is ordered by divisibility.*

Throughout this paper we use for an integer N and an abelian group G the **notation:**

$$\begin{array}{rcll} G\{N\} & := & G/NG & \text{in case of additive notation} \\ & := & G/G^N & \text{in case of multiplicative notation.} \end{array} \tag{1.10}$$

Thus instead of $\Omega^\bullet_{X/S}$ we take the complex $\Omega^\bullet_{X/S}\{N\}$. If the De Rham cohomology group $H^m_{DR}(X/S)$ and the Hodge cohomology group $H^{m-i}(X,\Omega^i_{X/S})$ are free A-modules, then

$$\begin{array}{rcl} \mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\}) & = & H^m_{DR}(X/S)\{N\}\,, \\ H^{m-i}(X,\Omega^i_{X/S}\{N\}) & = & H^{m-i}(X,\Omega^i_{X/S})\{N\}\,. \end{array} \tag{1.11}$$

Note that

$$\Omega^\bullet_{X/S}\{N\} = \Omega^\bullet_{X\times\mathrm{Spec}\,(\mathbb{Z}/N\mathbb{Z})\,/\,S\times\mathrm{Spec}\,(\mathbb{Z}/N\mathbb{Z})}\,.$$

The above arguments can be applied to $X\times\mathrm{Spec}\,(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{f} S\times\mathrm{Spec}\,(\mathbb{Z}/N\mathbb{Z})$. Before stating our conclusion we make a closer analysis of the condition

$$H^j(X, B(\Omega^i_{X/S}\{N\})) = 0 \qquad \textit{for all}\ \ i,j. \tag{1.12}$$

The cohomology sequences associated with the short exact sequences of complexes

$$0 \to Z(\Omega^i_{X/S}\{N\}) \to \Omega^i_{X/S}\{N\} \to B(\Omega^{i+1}_{X/S}\{N\}) \to 0$$

show that the condition (1.12) is equivalent with:

$$H^j(X, Z(\Omega^i_{X/S}\{N\})) \to H^j(X,\Omega^i_{X/S}\{N\}) \quad \textit{is an isomorphism for all} \quad i,j. \tag{1.13}$$

There is yet another equivalent form of this condition. For that we consider

$$Z_N\Omega^i_{X/S} := \{\omega\in\Omega^i_{X/S} \mid d\omega \in N\cdot\Omega^{i+1}_{X/S}\}. \tag{1.14}$$

It fits into the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Omega^i_{X/S} & \xrightarrow{\cdot N} & Z_N\Omega^i_{X/S} & \longrightarrow & Z(\Omega^i_{X/S}\{N\}) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^i_{X/S} & \xrightarrow{\cdot N} & \Omega^i_{X/S} & \longrightarrow & \Omega^i_{X/S}\{N\} & \to & 0 \end{array}$$

Under the assumptions on X/S made in Theorem 1.2 multiplication by N on $H^j(X,\Omega^i_{X/S})$ is injective. Therefore the ladder of cohomology groups for the above diagram splits into diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^j(X,\Omega^i_{X/S}) & \xrightarrow{\cdot N} & H^j(X,Z_N\Omega^i_{X/S}) & \to & H^j(X,Z(\Omega^i_{X/S}\{N\})) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & H^j(X,\Omega^i_{X/S}) & \xrightarrow{\cdot N} & H^j(X,\Omega^i_{X/S}) & \to & H^j(X,\Omega^i_{X/S}\{N\}) & \to & 0 \end{array}$$

This shows that condition (1.13), and hence also condition (1.12), is equivalent with condition (1.15) below.

Our conclusion from the above discussion is:

Theorem 1.2 *Let $f : X \to S = \operatorname{Spec} A$ be a smooth projective morphism in which the ring A is étale over a polynomial ring $\mathbb{Z}[x_1, \ldots, x_r]$. Assume that $H^m_{DR}(X/S)$ and $H^{m-i}(X, \Omega^i_{X/S})$ are free A-modules for all m, i. Fix $N \in \mathbb{N}$ and assume*

$$H^j(X, Z_N\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \text{is an isomorphism for all} \quad i, j. \tag{1.15}$$

Then one has for all m, i:

$$\begin{aligned} H^m_{DR}(X/S)\{N\} &= \mathrm{Fil}^{\mathrm{con}}_i \mathbb{H}^m(X, \Omega^\bullet_{X/S}\{N\}) \oplus \mathrm{Fil}^{i+1}_{\mathrm{Hodge}} H^m_{DR}(X/S)\{N\} \\ H^{m-i}(X, \Omega^i_{X/S})\{N\} &= \mathrm{Fil}^{\mathrm{con}}_i \mathbb{H}^m(X, \Omega^\bullet_{X/S}\{N\}) \cap \mathrm{Fil}^{i}_{\mathrm{Hodge}} H^m_{DR}(X/S)\{N\} \\ H^m_{DR}(X/S)\{N\} &= \bigoplus_{i=0}^{m} H^{m-i}(X, \Omega^i_{X/S})\{N\}. \end{aligned}$$

□

Remark 1.3 If K is a divisor of N there are canonical maps, induced by reducing mod$K$ on the De Rham complex $\Omega^\bullet_{X/S}\{N\}$,

$$\begin{aligned} H^m_{DR}(X/S)\{N\} &\to H^m_{DR}(X/S)\{K\} \\ \mathrm{Fil}^{\mathrm{con}}_i \mathbb{H}^m(X, \Omega^\bullet_{X/S}\{N\}) &\to \mathrm{Fil}^{\mathrm{con}}_i \mathbb{H}^m(X, \Omega^\bullet_{X/S}\{K\}) \\ \mathrm{Fil}^{i+1}_{\mathrm{Hodge}} H^m_{DR}(X/S)\{N\} &\to \mathrm{Fil}^{i+1}_{\mathrm{Hodge}} H^m_{DR}(X/S)\{K\} \end{aligned}$$

and thus the conclusion part of Theorem 1.2 fits well into a projective system indexed by the positive integers ordered by divisibility.

It is however not clear that if condition (1.15) holds for N, it also holds with K in place of N. So the condition part of Theorem 1.2 does not fit well into a projective system. Proposition 4.3 will show that this can be remedied by replacing (1.15) by the condition

$$H^j(X, Z_p\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \begin{array}{l}\text{is an isomorphism for all } i, j \text{ and} \\ \text{all prime numbers } p \text{ dividing } N.\end{array} \tag{1.16}$$

Remark 1.4 Theorem 1.2 is to be used as follows. Let us start with a smooth projective morphism $f : \mathbb{X} \to \mathbb{S} = \operatorname{Spec} \mathbb{A}$ in which the ring $\mathbb{A}$ is étale over a polynomial ring $\mathbb{Z}[x_1, \ldots, x_r]$ and assume that $H^m_{DR}(\mathbb{X}/\mathbb{S})$ and $H^{m-i}(\mathbb{X}, \Omega^i_{\mathbb{X}/\mathbb{S}})$ are free $\mathbb{A}$-modules for all m, i. Then, for a fixed N, condition (1.15) will in general not be satisfied for the whole $\mathbb{X}/\mathbb{S}$. Instead there will be a non-empty open (for the Zariski topology) $S \subset \mathbb{S}$ such that (1.15) does hold for $X = \mathbb{X} \times_{\mathbb{S}} S$. As such this is a trivial statement since over the open set on which N is invertible condition (1.15) is trivially satisfied; but over this set the conclusion of the theorem also trivializes: all groups involved are zero! The point is however that the open set on which (1.15) holds is much bigger: for $N = p$ a prime number and universal families $\mathbb{X} \to \mathbb{S}$ of complete intersections in projective space Illusie [8] shows that (1.12), and hence also (1.15), holds on a non-empty open part of the characteristic p locus in $\mathbb{S}$. The complement of this open subset in the characteristic p locus is the zero set of some ideal $\mathfrak{h}_p \subset \mathbb{A}$ with $p\mathbb{A} \subsetneq \mathfrak{h}_p$. Thus condition (1.15) for $N = p$ will be satisfied on the complement of the zero locus of the ideal $\mathfrak{h}_p$ in $\mathbb{S}$. Proposition 4.3 will show that for general N the open set on which (1.15) holds is at least as large as the intersection of the open sets for the prime divisors of N i.e. the complement of the zero set of the ideal $\prod_{p|N}\mathfrak{h}_p$.

1.2 Conjugate filtration and Gauss-Manin connection. Let us see how the conjugate filtration behaves with respect to the Gauss-Manin connection. First recall how Katz and Oda [10] constructed the Gauss-Manin connection from the filtration on $\Omega^\bullet_{X/\mathbb{Z}}$ formed by the powers of the graded ideal

$$\mathcal{J}^\bullet_{X/S} := d(f^*\mathcal{O}_S)\cdot\Omega^\bullet_{X/\mathbb{Z}}, \tag{1.17}$$

i.e.

$$\mathcal{J}^0_{X/S} := \mathcal{O}_X\,, \qquad \mathcal{J}^i_{X/S} := d(f^*\mathcal{O}_S)\cdot\Omega^{i-1}_{X/\mathbb{Z}} \quad \text{for } i\geq 1.$$

First one has to notice that, since X is smooth over S,

$$\Omega^\bullet_{X/\mathbb{Z}}/\mathcal{J}^\bullet_{X/S} = \Omega^\bullet_{X/S}\,, \qquad \mathcal{J}^\bullet_{X/S}/(\mathcal{J}^\bullet_{X/S})^2 = \Omega^1_{S/\mathbb{Z}}\otimes\Omega^{\bullet-1}_{X/S}.$$

Then one finds *the Gauss-Manin connection*

$$\nabla : \mathbb{H}^m(X,\Omega^\bullet_{X/S}) \to \Omega^1_{S/\mathbb{Z}}\otimes\mathbb{H}^m(X,\Omega^\bullet_{X/S}). \tag{1.18}$$

as the connecting map $\mathbb{H}^m(X,\Omega^\bullet_{X/S}) \to \mathbb{H}^{m+1}(X,\Omega^1_{S/\mathbb{Z}}\otimes\Omega^{\bullet-1}_{X/S})$ in the long exact cohomology sequence of

$$0\to\Omega^1_{S/\mathbb{Z}}\otimes\Omega^{\bullet-1}_{X/S}\to\Omega^\bullet_X/(\mathcal{J}^\bullet_{X/S})^2\to\Omega^\bullet_{X/S}\to 0.$$

This construction immediately shows on the one hand that the Hodge filtration need not be stable under the Gauss-Manin connection, but instead satisfies *Griffiths transversality*:

$$\nabla\mathrm{Fil}^i_{\mathrm{Hodge}}H^m_{DR}(X/S) \ \subset\ \Omega^1_{S/\mathbb{Z}}\otimes\mathrm{Fil}^{i-1}_{\mathrm{Hodge}}H^m_{DR}(X/S)\,, \tag{1.19}$$

and on the other hand that the conjugate filtration is stable under the Gauss-Manin connection:

$$\nabla\mathrm{Fil}_i^{\mathrm{con}}H^m_{DR}(X/S) \ \subset\ \Omega^1_{S/\mathbb{Z}}\otimes\mathrm{Fil}_i^{\mathrm{con}}H^m_{DR}(X/S). \tag{1.20}$$

All this remains valid when we go modulo some positive integer N, i.e. when we pull $X\to S\to\mathrm{Spec}\,\mathbb{Z}$ back to $X\times\mathrm{Spec}\,\mathbb{Z}/N\mathbb{Z}\to S\times\mathrm{Spec}\,\mathbb{Z}/N\mathbb{Z}\to\mathrm{Spec}\,\mathbb{Z}/N\mathbb{Z}$. As a result we find:

Corollary 1.5 *In the situation of Theorem 1.2 the conjugate filtration on $H^m_{DR}(X/S)\{N\}$ provides a filtration which is opposite to the Hodge filtration and which is stable under the Gauss-Manin connection.* □

2 Generalized Witt vectors

For a ring[2] A one defines its additive group of *generalized Witt vectors*[3] $\underline{\mathcal{W}A}$ to be the multiplicative group of one-variable formal power series with constant term 1 and coefficients in A:

$$\underline{\mathcal{W}A} := 1 + tA[[t]]. \tag{2.1}$$

[2]all rings in this paper are commutative with 1, unless explicitly stated otherwise

[3]We add the adjective *generalized* to emphasize that these are not the more common Witt vectors associated with a prime number p; the latter is the *p-typical part* of $\underline{\mathcal{W}A}$; cf. [3, 6, 2].

The group $\underline{\mathcal{W}A}$ naturally comes with the decreasing filtration:

$$\mathrm{Fil}_n\underline{\mathcal{W}A} := 1 + t^{n+1}A[[t]]\,, \qquad n \geq 0. \tag{2.2}$$

If a is an element of A we write $\underline{\underline{a}}$ for the power series of $(1-at)^{-1}$ viewed as an element of $\underline{\mathcal{W}A}$. Every element of $1+t^{n+1}A[[t]]$ can be written uniquely as a t-adically converging product $\prod_{i\geq n+1}(1-a_it^i)^{-1}$ with all $a_i \in A$. For $n \in \mathbb{N}$ the substitution $t \mapsto t^n$ induces an (additive) endomorphism V_n of $\underline{\mathcal{W}A}$. Thus the elements of $\mathrm{Fil}_n\underline{\mathcal{W}A}$ can be written uniquely as a sum

$$\sum_{i\geq n+1} V_i\underline{\underline{a_i}} \tag{2.3}$$

which converges with respect to the filtration (2.2). The operators V_n are called *Verschiebung operators.*

One can construct a continuous product on $\underline{\mathcal{W}A}$ so that $\underline{\mathcal{W}A}$ becomes a topological ring, and each $\mathrm{Fil}_n\underline{\mathcal{W}A}$ is an ideal. One can also construct continuous endomorphisms F_n for $n \in \mathbb{N}$ (see [3, 6]). These are called *Frobenius operators.* For computations in $\underline{\mathcal{W}A}$ the following relations plus continuity with respect to the filtration (2.2) suffice. For $k, m, n \in \mathbb{N}$, elements $\alpha, \beta \in \underline{\mathcal{W}A}$ and $a, b \in A$:

$$\begin{array}{l} \underline{\underline{a}}\cdot\underline{\underline{b}} = \underline{\underline{ab}}\,, \quad F_n\underline{\underline{a}} = \underline{\underline{a}}^n\,, \\ F_n(\alpha\beta) = (F_n\alpha)(F_n\beta)\,, \quad V_n(\alpha(F_n\beta)) = (V_n\alpha)\beta\,, \\ F_mV_m = m\,, \quad F_mF_n = F_{mn}\,, \quad V_mV_n = V_{mn}\,, \\ V_kF_m = F_mV_k \qquad \text{if} \quad (k,m) = 1. \end{array} \tag{2.4}$$

The third line of (2.4) shows $F_1 = V_1 =$ *identity operator*. The first line of (2.4) shows that the map

$$A \to \underline{\mathcal{W}A}\,, \qquad a \mapsto \underline{\underline{a}} \tag{2.5}$$

is a *homomorphism of multiplicative monoids*. One calls it the *Teichmüller lifting.*

As an exercise in computing with the relations one can check

$$V_m\mathrm{Fil}_n\underline{\mathcal{W}A} \subset \mathrm{Fil}_{mn+m-1}\underline{\mathcal{W}A}\,, \qquad F_m\mathrm{Fil}_{mn}\underline{\mathcal{W}A} \subset \mathrm{Fil}_n\underline{\mathcal{W}A}. \tag{2.6}$$

The ring of *generalized Witt vectors of length* n is, by definition,

$$\underline{\mathcal{W}}_n\underline{A} := \underline{\mathcal{W}A}/\mathrm{Fil}_n\underline{\mathcal{W}A}.$$

Note

$$\underline{\mathcal{W}}_1\underline{A} \simeq A\,, \qquad \underline{\underline{a}} \bmod \mathrm{Fil}_1\underline{\mathcal{W}A} \;\leftrightarrow a.$$

Since obviously $\mathrm{Fil}_n\underline{\mathcal{W}A} \subset \mathrm{Fil}_m\underline{\mathcal{W}A}$ if $n \geq m$ there are standard *truncation maps*

$$\mathbb{1}^n_m : \underline{\mathcal{W}}_n\underline{A} \to \underline{\mathcal{W}}_m\underline{A}\,. \tag{2.7}$$

When source and target of a truncation map are clear from the context we simply write $\mathbb{1}$ instead of $\mathbb{1}^n_m$.

3 The generalized De Rham-Witt complex

The constructions in Section 2 are functorial in A i.e. a ring homomorphism $A \to A'$ induces a ring homomorphism $\underline{\mathcal{W}A} \to \underline{\mathcal{W}A}'$ compatible with all the truncations, Frobenius and Verschiebung operators. So one can sheafify the constructions and thus obtain on every scheme X the *sheaves* $\underline{\mathcal{W}\mathcal{O}_X}$ *and* $\underline{\mathcal{W}}_n\underline{\mathcal{O}_X}$ *of generalized Witt vectors (of length n)*, together with all the truncations, Frobenius and Verschiebung operators.

In [13] we constructed for every scheme X on which 2 is invertible the *generalized De Rham-Witt complex*[4]. This is a sheaf $\underline{\mathcal{W}\Omega}^\bullet_X$ of anti-commutative differential graded algebras on X with the following properties (3.1)-(3.3).

Let $\underline{\mathcal{W}\Omega}^i_X$ denote the degree i component of $\underline{\mathcal{W}\Omega}^\bullet_X$. Then $\underline{\mathcal{W}\Omega}^i_X = 0$ for $i < 0$,

$$\underline{\mathcal{W}\Omega}^0_X = \underline{\mathcal{W}\mathcal{O}}_X. \tag{3.1}$$

For every integer $m \geq 1$ and every $i \geq 0$ there are homomorphisms of additive groups

$$F_m, V_m : \underline{\mathcal{W}\Omega}^i_X \to \underline{\mathcal{W}\Omega}^i_X \tag{3.2}$$

The operators F_m are called *Frobenius operators* and the operators V_m are called *Verschiebung operators*. On the sheaf of generalized Witt vectors $\underline{\mathcal{W}\mathcal{O}}_X$ they coincide with the earlier defined Frobenius and Verschiebung operators. The following relations hold for all m, n, for all sections α, β of $\underline{\mathcal{W}\Omega}^\bullet_X$ and all sections a of $\mathcal{O}_X$:

$$\begin{array}{l} F_n d\underline{\underline{a}} = \underline{\underline{a}}^{n-1} d\underline{\underline{a}}\,, \\ F_m V_m = m\,, \qquad F_m F_n = F_{mn}\,, \qquad V_m V_n = V_{mn}, \\ V_m d = m d V_m\,, \quad dF_m = m F_m d\,, \qquad F_m d V_m = d, \\ F_m(\alpha\beta) = (F_m\alpha)(F_m\beta)\,, \qquad V_m(\alpha(F_m\beta)) = (V_m\alpha)\beta, \\ V_n F_m = F_m V_n \qquad \text{if} \quad (n,m) = 1; \end{array} \tag{3.3}$$

here $d : \underline{\mathcal{W}\Omega}^i_X \to \underline{\mathcal{W}\Omega}^{i+1}_X$ is the differential of the differential graded algebra $\underline{\mathcal{W}\Omega}^\bullet_X$.

Remark 3.1 The relation $dF_m = mF_m d$ means on the one hand that F_m does not commute with d, but shows on the other hand that we get an operator $\mathbb{F}_m$ on $\underline{\mathcal{W}\Omega}^\bullet_X$ which does commute with d by defining

$$\mathbb{F}_m = m^i F_m \qquad \text{on } \underline{\mathcal{W}\Omega}^i_X. \tag{3.4}$$

For De Rham-Witt complex in characteristic p this is a standard construction, which is of great importance for the so-called *slope spectral sequence*; see [7, 9].

The filtration $\{\mathrm{Fil}_n \underline{\mathcal{W}\mathcal{O}}_X\}_{n\geq 0}$ can be extended to a decreasing filtration on $\underline{\mathcal{W}\Omega}^\bullet_X$ by defining:

$$\mathrm{Fil}_n\underline{\mathcal{W}\Omega}^\bullet_X := \textit{ ideal generated by } \mathrm{Fil}_n\underline{\mathcal{W}\mathcal{O}}_X \textit{ and } d\mathrm{Fil}_n\underline{\mathcal{W}\mathcal{O}}_X. \tag{3.5}$$

Each $\mathrm{Fil}_n\underline{\mathcal{W}\Omega}^\bullet_X$ is then a differential graded ideal in $\underline{\mathcal{W}\Omega}^\bullet_X$. The quotient

$$\underline{\mathcal{W}}_n\underline{\Omega}^\bullet_X := \underline{\mathcal{W}\Omega}^\bullet_X / \mathrm{Fil}_n\underline{\mathcal{W}\Omega}^\bullet_X \tag{3.6}$$

is called the *generalized De Rham-Witt complex of level n*. This is a differential graded algebra; $\underline{\mathcal{W}}_n\underline{\Omega}^i_X = 0$ if $i < 0$ and

$$\underline{\mathcal{W}}_n\underline{\Omega}^0_X = \underline{\mathcal{W}}_n\underline{\mathcal{O}}_X.$$

In particular $\underline{\mathcal{W}}_1\underline{\Omega}^\bullet_X$ is a differential graded algebra with degree 0 term equal to $\mathcal{O}_X$. Let $\Omega^\bullet_X = \Omega^\bullet_{X/\mathbb{Z}}$ be the absolute De Rham complex on X i.e. the complex of differential forms relative to $\mathbb{Z}$. The universal property of $\Omega^\bullet_X$ yields a homomorphism of complexes

$$\Omega^\bullet_X \to \underline{\mathcal{W}}_1\underline{\Omega}^\bullet_X$$

[4]Here the adjective *generalized* emphasizes that this is not the De Rham-Witt complex in characteristic p [7]; the latter can be recovered as the *p-typical part* of $\underline{\mathcal{W}\Omega}^\bullet_X$.

which happens to be surjective by [13] §2.16. On the other hand it has been shown in [13] §2.20 that there is a homomorphism of sheaves of differential graded algebras

$$\underline{\mathcal{W}\Omega}^{\bullet}_X \to \widetilde{\Omega}^{\bullet}_X;$$

where $\widetilde{\Omega}^i_X := \Omega^i_X/(i!\text{-torsion in } \Omega^i_X)$. The composite of these two maps is the obvious map $\Omega^{\bullet}_X \to \widetilde{\Omega}^{\bullet}_X$. *Very mild conditions on X will remove the $i!$-torsion and thus assure that*

$$\underline{\mathcal{W}}_1\underline{\Omega}^{\bullet}_X = \Omega^{\bullet}_X. \tag{3.7}$$

Since obviously $\mathrm{Fil}_n\underline{\mathcal{W}\Omega}^{\bullet}_X \subset \mathrm{Fil}_m\underline{\mathcal{W}\Omega}^{\bullet}_X$ if $n \geq m$ there are standard *truncation maps*

$$\mathbb{1}^n_m : \underline{\mathcal{W}}_n\underline{\Omega}_X \to \underline{\mathcal{W}}_m\underline{\Omega}_X\,. \tag{3.8}$$

When source and target of a truncation map are clear from the context we simply write $\mathbb{1}$ instead of $\mathbb{1}^n_m$.

Lemma 3.2 *For all m and n one has*

$$V_m\mathrm{Fil}_n\underline{\mathcal{W}\Omega}^{\bullet}_X \subset \mathrm{Fil}_{mn+m-1}\underline{\mathcal{W}\Omega}^{\bullet}_X\,, \qquad F_m\mathrm{Fil}_{mn}\underline{\mathcal{W}\Omega}^{\bullet}_X \subset \mathrm{Fil}_n\underline{\mathcal{W}\Omega}^{\bullet}_X. \tag{3.9}$$

Consequently V_m and F_m induce maps

$$V_m : \underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_X \to \underline{\mathcal{W}}_{nm+m-1}\underline{\Omega}^{\bullet}_X\,, \qquad F_m : \underline{\mathcal{W}}_{nm}\underline{\Omega}^{\bullet}_X \to \underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_X. \tag{3.10}$$

Proof Most of this follows easily from the relations (3.3) and (2.6). Only for $F_m d\mathrm{Fil}_{mn}\underline{\mathcal{W}\mathcal{O}}_X \subset \mathrm{Fil}_n\underline{\mathcal{W}\Omega}^1_X$ one needs the following slightly tricky calculation: take $i > mn$; let r be the greatest common divisor of i and m; set $i' = i/r$ and $m' = m/r$; take integers j, k such that $ji' + km' = 1$; then

$$\begin{aligned} F_m dV_i\underline{\underline{a}} &= F_{m'}dV_{i'}\underline{\underline{a}} = jF_{m'}V_{i'}d\underline{\underline{a}} + kdF_{m'}V_{i'}\underline{\underline{a}} \\ &= jV_{i'}(\underline{\underline{a}}^{m'-1}d\underline{\underline{a}}) + kdV_{i'}(\underline{\underline{a}}^{m'}) \\ &= jV_{i'}(\underline{\underline{a}}^{m'-1})d(V_{i'}\underline{\underline{a}}) + kdV_{i'}(\underline{\underline{a}}^{m'}). \end{aligned} \tag{3.11}$$

This lies in $\mathrm{Fil}_n\underline{\mathcal{W}\Omega}^1_X$ because $i' > n$. □

As indicated in [13] §3.1 one can define for a morphism $f : X \to S$ a relativized version of the De Rham-Witt complexes. Here are the details for the *relative generalized De Rham-Witt complex of level n.*

Since the constructions are functorial f induces a subsheaf $f^*\underline{\mathcal{W}}_n\underline{\mathcal{O}}_S \subset \underline{\mathcal{W}}_n\underline{\mathcal{O}}_X$. We let $\mathcal{J}^{\bullet}_{X/S,n}$ denote the graded ideal in $\underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_X$ generated by $d(f^*\underline{\mathcal{W}}_n\underline{\mathcal{O}}_S)$; i.e.

$$\mathcal{J}^0_{X/S,n} = 0\,, \qquad \mathcal{J}^i_{X/S,n} = d(f^*\underline{\mathcal{W}}_n\underline{\mathcal{O}}_S)\cdot\underline{\mathcal{W}}_n\underline{\Omega}^{i-1}_X \quad \text{for } i \geq 1, \tag{3.12}$$

and define

$$\underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_{X/S} := \underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_X/\mathcal{J}^{\bullet}_{X/S,n}. \tag{3.13}$$

Note in particular

$$\underline{\mathcal{W}}_n\underline{\Omega}^0_{X/S} = \underline{\mathcal{W}}_n\underline{\mathcal{O}}_X.$$

Clearly $\underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_{X/S}$ is a differential graded algebra over $\underline{\mathcal{W}}_n\underline{\mathcal{O}}_S$.

For sections α of $f^*\underline{\mathcal{W}}_n\underline{\mathcal{O}}_S$ and β of $\underline{\mathcal{W}}_n\underline{\Omega}^{i-1}_X$ one immediately computes

$$V_m((d\alpha)\beta) = V_m((F_m dV_m\alpha)\beta) = (dV_m\alpha)(V_m\beta),$$

while the more tricky calculation (3.11) shows

$$F_m(d(f^*\underline{\mathcal{W}}_{mn}\underline{\mathcal{O}}_S)\cdot\underline{\mathcal{W}}_{mn}\underline{\Omega}^{i-1}_X) \subset d(f^*\underline{\mathcal{W}}_n\underline{\mathcal{O}}_S)\cdot\underline{\mathcal{W}}_n\underline{\Omega}^{i-1}_X.$$

Thus V_m and F_m induce maps

$$\begin{aligned} V_m : \quad & \underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S} \to \underline{\mathcal{W}}_{nm+m-1} \underline{\Omega}^i_{X/S} \\ F_m : \quad & \underline{\mathcal{W}}_{mn} \underline{\Omega}^i_{X/S} \to \underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S} \end{aligned}$$

Exactly as in (3.7) very mild conditions on X will remove the $i!$-torsion from $\Omega^i_{X/S}$ and Ω^i_X, and imply that $\underline{\mathcal{W}}_1 \underline{\Omega}^\bullet_X = \Omega^\bullet_X$, $\mathcal{J}^\bullet_{X/S,1} = \mathcal{J}^\bullet_{X/S}$ (see (1.17)) and

$$\underline{\mathcal{W}}_1 \underline{\Omega}^\bullet_{X/S} = \Omega^\bullet_{X/S} . \tag{3.14}$$

4 Characterization of $Z_N \Omega^i_{X/S}$ by Frobenius and Verschiebung

Again we consider a morphism $f : X \to S$. For simplicity we assume $S = \mathrm{Spec} A$ for some ring A. Then each $\underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S}$ is a module over the ring $\underline{\mathcal{W}}_n \underline{A}$. For every N we have the ring homomorphism

$$F_N : \underline{\mathcal{W}}_{nN} \underline{A} \to \underline{\mathcal{W}}_n \underline{A}$$

making $\underline{\mathcal{W}}_n \underline{A}$ an algebra over $\underline{\mathcal{W}}_{nN} \underline{A}$. We now *define*

$$F_N^* \underline{\mathcal{W}}_{nN} \underline{\Omega}^i_{X/S} := \underline{\mathcal{W}}_n \underline{A} \otimes_{\underline{\mathcal{W}}_{nN} \underline{A}} \underline{\mathcal{W}}_{nN} \underline{\Omega}^i_{X/S}. \tag{4.1}$$

This is then a module over $\underline{\mathcal{W}}_n \underline{A}$. The map

$$F_N : \underline{\mathcal{W}}_{nN} \underline{\Omega}^i_{X/S} \to \underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S}$$

induces a *linear map* of modules over $\underline{\mathcal{W}}_n \underline{A}$

$$F_N : F_N^* \underline{\mathcal{W}}_{nN} \underline{\Omega}^i_{X/S} \to \underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S} , \qquad F_N(a \otimes \omega) = a \cdot F_N \omega. \tag{4.2}$$

The Cartier isomorphism is crucial for a good theory of the De Rham(-Witt) complex for schemes which are smooth over a scheme of characteristic $p > 0$; cf. [10, 9]. The following result (4.4), the proof of which uses in an essential way the Cartier isomorphism in characteristic p, is the closest we can get to this in our setting. For our purpose it gives one good characterization of $Z_N \Omega^i_{X/S}$ in terms of the generalized De Rham-Witt complex.

Proposition 4.1 *Let $f : X \to S = \mathrm{Spec}\, A$ be a smooth projective morphism and assume that the ring A is étale over a polynomial ring $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$. Then for all i*

$$\underline{\mathcal{W}}_1 \underline{\Omega}^i_{X/S} = \Omega^i_{X/S}. \tag{4.3}$$

Recall from (1.14) the definition of $Z_N \Omega^i_{X/S}$. Then for $N \geq 1$ and for all i

$$Z_N \Omega^i_{X/S} = \mathrm{image}(F_N : F_N^* \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S} \to \Omega^i_{X/S}). \tag{4.4}$$

Proof The smoothness of X over S and flatness of A over $\mathbb{Z}$ imply that there is no $i!$-torsion in $\Omega^i_{X/S}$ and hence that $\underline{\mathcal{W}}_1 \underline{\Omega}^i_{X/S} = \Omega^i_{X/S}$ (cf. (3.7)).

The inclusion $\subset$ in (4.4) follows from the basic relation $dF_N = NF_N d$. So let us concentrate on the other inclusion and first prove it in case N is a power of some prime number p. Write $S_p = S \times \mathrm{Spec}\, (\mathbb{Z}/p\mathbb{Z})$ and $X_p = X \times \mathrm{Spec}\, (\mathbb{Z}/p\mathbb{Z})$. Then X_p is smooth over S_p and

$$\Omega^\bullet_{X_p/S_p} = \Omega^\bullet_{X/S} \,/\, p \cdot \Omega^\bullet_{X/S}.$$

We are going make heavy use of [10] thm. 7.2. As for notation in loc. cit.:

$$\Omega^i_{X_p^{(p)}/S_p} := (A/pA) \otimes_A \Omega^i_{X/S}$$

where A/pA is considered as a module over A via the map $a \mapsto a^p \bmod p$.

Now take for some $\nu \geq 1$

$$\omega \in \Omega^i_{X/S} \quad \textit{such that} \quad d\omega \equiv 0 \bmod p^\nu \,. \tag{4.5}$$

This implies that $\omega \bmod p$ is a closed form in $\Omega^i_{X_p/S_p}$. The theorem on the Cartier isomorphism in characteristic p (see [10] thm. 7.2) shows therefore

$$\omega \bmod p = C^{-1}\alpha + d\beta \qquad \text{with} \quad \alpha \in \Omega^i_{X_p^{(p)}/S_p}\,,\ \beta \in \Omega^{i-1}_{X_p/S_p} \tag{4.6}$$

where C^{-1} is the *inverse Cartier operator*.
Choose $s_j \in A$, sections a_{kj} of $\mathcal{O}_X$ and section $\widetilde{\beta}$ of $\Omega^{i-1}_{X/S}$ such that

$$\beta = \widetilde{\beta} \bmod p\,, \qquad \alpha = \widetilde{\alpha} \bmod p \qquad \text{with} \quad \widetilde{\alpha} = \sum_j s_j \otimes a_{0j} da_{1j} \cdot \ldots \cdot da_{ij}.$$

Recall the following formulas from (3.3)

$$\begin{array}{l} F_p V_p = p\,, \qquad F_p d V_p = d\,, \\ F_p(\underline{\underline{a_0}} d\underline{\underline{a_1}} \cdot \ldots \cdot \underline{\underline{a_i}}) = a_0^p (a_1 \cdot \ldots \cdot a_i)^{p-1} da_1 \cdot \ldots \cdot da_i\,. \end{array} \tag{4.7}$$

Comparing these with the formulas for C^{-1} in [10] thm. 7.2 we see that (4.6) can be rewritten as

$$\begin{array}{l} \omega = F_p \omega_1 \qquad \text{with} \\ \omega_1 = \displaystyle\sum_j s_j \otimes \underline{\underline{a_{0j}}} d\underline{\underline{a_{1j}}} \cdot \ldots \cdot \underline{\underline{a_{ij}}} + 1 \otimes dV_p \widetilde{\beta} + 1 \otimes V_p \widetilde{\gamma} \ \in\ F_p^* \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}, \end{array}$$

some $\widetilde{\gamma} \in \Omega^i_{X/S}$. If $\nu = 1$ we are done. If $\nu > 1$ we note that (4.5) and $d\omega = pF_p d\omega_1$ imply

$$\sum_j s_j \otimes (a_{0j} a_{1j} \cdot \ldots \cdot a_{ij})^{p-1} da_{0j} da_{1j} \cdot \ldots \cdot da_{ij} \ +\ d\widetilde{\gamma} = F_p d\omega_1 \equiv 0 \bmod p.$$

Looking at the first term we see that this means

$$C^{-1} d\alpha \ +\ d\widetilde{\gamma} = 0.$$

From the theorem on the Cartier isomorphism ([10] thm. 7.2) we can now conclude $d\alpha = 0$ and thus as before

$$\widetilde{\alpha} = F_p \widetilde{\alpha_1} \qquad \text{with} \quad \widetilde{\alpha_1} \in F_p^* \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}$$

Lift $\widetilde{\alpha_1}$ to some $\widetilde{\widetilde{\alpha_1}} \in F_p^* \underline{\mathcal{W}}_{p^2} \underline{\Omega}^i_{X/S} = \underline{\mathcal{W}}_p \underline{A} \otimes_{\underline{\mathcal{W}}_{p^2} \underline{A}} \underline{\mathcal{W}}_{p^2} \underline{\Omega}^i_{X/S}$ such that $\widetilde{\alpha_1} = (\mathbb{1} \otimes \mathbb{1}) \widetilde{\widetilde{\alpha_1}}$. Then

$$\omega = F_{p^2} \left(\widetilde{\widetilde{\alpha_1}} + dV_{p^2} \widetilde{\beta} \right) + p\widetilde{\gamma}.$$

From this we see $d\widetilde{\gamma} = 0 \bmod p$ and thus

$$\widetilde{\gamma} = F_p \widetilde{\gamma}_1 \qquad \text{with} \quad \widetilde{\gamma}_1 \in F_p^* \underline{\mathcal{W}}_p \underline{\Omega}^{i-1}_{X/S}.$$

Take $\widetilde{\widetilde{\gamma}}_1 \in F_p^* \underline{\mathcal{W}}_{p^2} \underline{\Omega}^i_{X/S}$ such that $F_p \widetilde{\widetilde{\gamma}}_1 = p\widetilde{\gamma}_1$. Then

$$\omega = F_{p^2} \left(\widetilde{\widetilde{\alpha_1}} + \widetilde{\widetilde{\gamma}}_1 + dV_{p^2} \widetilde{\beta} \right).$$

If $\nu = 2$ we are done. If $\nu > 2$ we go on the same way untill finally

$$\omega = F_{p^\nu}\omega_{p^\nu} \qquad \text{with} \quad \omega_{p^\nu} \in F^*_{p^\nu}\underline{\mathcal{W}}_{p^\nu}\underline{\Omega}^i_{X/S}.$$

Now take an arbitrary positive integer N with prime decomposition $N = \prod_l p_l^{\nu_l}$. Consider

$$\omega \in \Omega^i_{X/S} \quad \textit{such that} \quad d\omega \equiv 0 \bmod N\,.$$

By the previous prime power case we know that for every l

$$\omega = F_{p_l^{\nu_l}}\eta_l \qquad \text{with} \quad \eta_l \in F^*_{p_l^{\nu_l}}\underline{\mathcal{W}}_{p_l^{\nu_l}}\underline{\Omega}^i_{X/S}$$

Take integers c_l such that $\sum_l c_l p_l^{-\nu_l}N = 1$. Then

$$\omega = \sum_l c_l\, F_{p_l^{\nu_l}}(p_l^{-\nu_l}N\,\eta_l) = F_N\left(\sum_l c_l\,\widetilde{\eta_l}\right)$$

with $\widetilde{\eta_l} \in F^*_N\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}$ such that $F_{p_l^{-\nu_l}N}\widetilde{\eta_l} = p_l^{-\nu_l}N\eta_l$. □

A second good characterization of $Z_N\Omega^i_{X/S}$ in terms of the generalized De Rham-Witt complex is provided by (4.9) below. This result is an analogue of the exact sequence [9] II(1.2.2):

$$0 \to W\Omega^{i-1} \stackrel{(F^n,-F^n d)}{\longrightarrow} W\Omega^{i-1} \oplus W\Omega^i \stackrel{(dV^n+V^n)}{\longrightarrow} W\Omega^i \to W_n\Omega^i \to 0$$

which is of fundamental importance for the theory of the De Rham-Witt complex in characteristic p.

Proposition 4.2 *Let $f : X \to S = \operatorname{Spec} A$ be a smooth projective morphism and assume that the ring A is étale over a polynomial ring $\mathbb{Z}[\frac{1}{2}][x_1,\dots,x_r]$. Take $N > 1$ and recall from (1.14) the definition of $Z_N\Omega^i_{X/S}$. Then one has the exact sequences*

$$0 \to \mathcal{O}_X \stackrel{V_N}{\longrightarrow} \underline{\mathcal{W}}_N\underline{\mathcal{O}}_X \longrightarrow \underline{\mathcal{W}}_{N-1}\underline{\mathcal{O}} \to 0 \tag{4.8}$$

and for all $i \geq 1$

$$0 \to Z_N\Omega^{i-1}_{X/S} \stackrel{(1,-\frac{1}{N}d)}{\longrightarrow} \Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S} \stackrel{dV_N+V_N}{\longrightarrow} \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S} \longrightarrow \underline{\mathcal{W}}_{N-1}\underline{\Omega}^i_{X/S} \to 0. \tag{4.9}$$

with a slight abuse of notation, in that the map V_N is actually $\mathbb{1} \circ V_N$ with $\mathbb{1} : \underline{\mathcal{W}}_{2N-1}\underline{\Omega}^\bullet_{X/S} \to \underline{\mathcal{W}}_N\underline{\Omega}^\bullet_{X/S}$ the standard truncation.

Proof It is immediately obvious from the definitions and constructions in Section 2 that the sequence (4.8) is exact. The only point where it is not immediately obvious from the definitions and constructions in Section 3 that the sequence (4.9) is exact, is at $\Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S}$.

The composite map $\Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S} \stackrel{dV_N+V_N}{\longrightarrow} \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S} \stackrel{F_N}{\longrightarrow} \Omega^i_{X/S}$ is in fact $d+N$. This shows

$$\ker\left(\Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S} \stackrel{dV_N+V_N}{\longrightarrow} \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}\right) \subset \operatorname{image}\left(Z_N\Omega^{i-1}_{X/S} \stackrel{(1,-\frac{1}{N}d)}{\longrightarrow} \Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S}\right)$$

For the $\supset$-inclusion we note that by (4.4) with $i-1$ in place of i it suffices to show that the composite map

$$F^*_N\underline{\mathcal{W}}_N\underline{\Omega}^{i-1}_{X/S} \stackrel{(F_N,-F_N d)}{\longrightarrow} \Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S} \stackrel{dV_N+V_N}{\longrightarrow} \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}$$

is zero. Since by definition $F_N^*\underline{\mathcal{W}}_N\underline{\Omega}^{i-1}_{X/S} = A \otimes_{\underline{\mathcal{W}}_N\underline{A}} \underline{\mathcal{W}}_N\underline{\Omega}^{i-1}_{X/S}$ this means that we must show for $a \in A$ and $\omega \in \underline{\mathcal{W}}_N\underline{\Omega}^{i-1}_{X/S}$

$$dV_N(aF_N\omega) = V_N(aF_Nd\omega).$$

With the relations in (3.3) the left hand side can be written as $d((V_Na)\omega)$, while the right hand equals $(V_Na)d\omega$. These two are equal since $V_Na \in \underline{\mathcal{W}}_N\underline{A}$. This completes the proof. □

With the techniques of the preceding proofs we can now also prove the claim made at the end of Remark 1.4.

Proposition 4.3 *Let $f : X \to S = \operatorname{Spec} A$ be a smooth projective morphism such that the ring A is étale over a polynomial ring $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$ and $H^m_{DR}(X/S)$ and $H^{m-i}(X, \Omega^i_{X/S})$ are free A-modules for all m, i. Fix $N \geq 2$ and assume that*

$$H^j(X, Z_p\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \textit{is an isomorphism for all } i,j \textit{ and all prime numbers } p \textit{ dividing } N. \tag{4.10}$$

Then

$$H^j(X, Z_N\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \quad \textit{is an isomorphism for all } i,j. \tag{4.11}$$

Proof Step 1. Consider the prime decomposition $N = \prod_l p_l^{\nu_l}$. Fix i, j. Assume that for every p_l with $\nu_l > 0$ the map $H^j(X, Z_{p_l^{\nu_l}}\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$ is an isomorphism and call this map ρ_l. Note that multiplication by $Np_l^{-\nu_l}$ induces a map

$$H^j(X, Z_{p_l^{\nu_l}}\Omega^i_{X/S}) \xrightarrow{\cdot Np_l^{-\nu_l}} H^j(X, Z_N\Omega^i_{X/S}).$$

Choose integers c_l such that $\sum_l c_l N p_l^{-\nu_l} = 1$. Then

$$\sum_l c_l N p_l^{-\nu_l} \rho_l^{-1} : H^j(X, \Omega^i_{X/S}) \to H^j(X, Z_N\Omega^i_{X/S})$$

is an inverse for the map in (4.11).

Step 2. The problem is thus reduced to showing that if p is a prime number for which (4.10) holds, then for every $\nu \geq 1$

$$H^j(X, Z_{p^\nu}\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \qquad \text{is an isomorphism for all } i,j. \tag{4.12}$$

The basic relationship between the Frobenius operator F_p and the inverse Cartier operator C^{-1} is expressed in the following commutative diagram

$$\begin{array}{ccc} F_p^*\underline{\mathcal{W}}_p\underline{\Omega}^i_{X/S} & \xrightarrow{F_p} & Z_p\Omega^i_{X/S} \\ \downarrow & & \downarrow \\ (A/pA) \otimes_A \Omega^i_{X/S} & \xrightarrow{C^{-1}} & (Z_p\Omega^i_{X/S})/(d\Omega^{i-1}_{X/S} + p\Omega^i_{X/S}) \end{array} \tag{4.13}$$

where the left hand vertical map is the one induced by the standard truncation $\underline{\mathcal{W}}_p\underline{\Omega}^i_{X/S} \to \Omega^i_{X/S}$ and the right hand vertical map is the obvious one. In the notation $(A/pA)\otimes_A$ the ring A/pA is an A-module via the ring homomorphism

$A \to A/pA\,,\ a \mapsto a^p \bmod p$. Composing with $F_{p^\nu} : F^*_{p^{\nu+1}}\underline{\mathcal{W}}_{p^{\nu+1}}\underline{\Omega} \to F^*_p\underline{\mathcal{W}}_p\underline{\Omega}^i_{X/S}$ in the upper left hand corner we obtain the commutative diagram

$$\begin{array}{ccc}
F^*_{p^{\nu+1}}\underline{\mathcal{W}}_{p^{\nu+1}}\underline{\Omega} & \xrightarrow{=} & F^*_{p^{\nu+1}}\underline{\mathcal{W}}_{p^{\nu+1}}\underline{\Omega} \\
F_{p^\nu}\downarrow & & \downarrow F_{p^{\nu+1}} \\
F^*_p\underline{\mathcal{W}}_p\underline{\Omega}^i_{X/S} & \xrightarrow{F_p} & Z_p\Omega^i_{X/S} \\
\downarrow & & \downarrow \\
(A/pA)\otimes_A \Omega^i_{X/S} & \xrightarrow{C^{-1}} & (Z_p\Omega^i_{X/S})/(d\Omega^{i-1}_{X/S} + p\Omega^i_{X/S})
\end{array}$$

According to [10] thm. 7.2 the inverse Cartier operator C^{-1} is an isomorphism. Taking cokernels of the vertical composite maps and using (4.4) one obtains an isomorphism

$$(A/pA)\otimes_A \frac{\Omega^i_{X/S}}{Z_{p^\nu}\Omega^i_{X/S} + p\Omega^i_{X/S}} \xrightarrow{\simeq} \frac{Z_p\Omega^i_{X/S}}{Z_{p^{\nu+1}}\Omega^i_{X/S} + p\Omega^i_{X/S}}\ .$$

On cohomology this gives an isomorphism

$$H^j\left(X, (A/pA)\otimes_A \frac{\Omega^i_{X/S}}{Z_{p^\nu}\Omega^i_{X/S} + p\Omega^i_{X/S}}\right) \xrightarrow{\simeq} H^j\left(X,\ \frac{Z_p\Omega^i_{X/S}}{Z_{p^{\nu+1}}\Omega^i_{X/S} + p\Omega^i_{X/S}}\right).$$

Since the Frobenius endomorphism $A/pA \to A/pA\,,\ a \mapsto a^p$, is a flat map the group on the left hand side is isomorphic to

$$(A/pA)\otimes_A H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^\nu}\Omega^i_{X/S} + p\Omega^i_{X/S}}\right).$$

Hypothesis (4.10) implies that the group on the right hand side is isomorphic to

$$H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^{\nu+1}}\Omega^i_{X/S} + p\Omega^i_{X/S}}\right).$$

These arguments show in particular:

$$\textit{if } H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^\nu}\Omega^i_{X/S} + p\Omega^i_{X/S}}\right) = 0 \ \textit{ then } H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^{\nu+1}}\Omega^i_{X/S} + p\Omega^i_{X/S}}\right) = 0.$$

This implication can be combined with implications resulting from the exact cohomology sequences of the short exact sequences of sheaves

$$0 \to \frac{\Omega^i_{X/S}}{Z_{p^r}\Omega^i_{X/S}} \xrightarrow{\cdot p} \frac{\Omega^i_{X/S}}{Z_{p^{r+1}}\Omega^i_{X/S}} \longrightarrow \frac{\Omega^i_{X/S}}{Z_{p^{r+1}}\Omega^i_{X/S} + p\Omega^i_{X/S}} \to 0$$

for every $r \geq 0$. This leads to the conclusion:

$$\textit{If} \qquad H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^{\nu-1}}\Omega^i_{X/S}}\right) = H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^\nu}\Omega^i_{X/S}}\right) = 0 \qquad \textit{for all}\ \ j,$$

$$\textit{then} \qquad H^j\left(X,\ \frac{\Omega^i_{X/S}}{Z_{p^{\nu+1}}\Omega^i_{X/S}}\right) = 0 \qquad \textit{for all}\ \ j.$$

Thus we can derive (4.12) from (4.10) by induction. □

5 The conjugate filtration and ordinariness: Act 2

5.1 The generalized Hodge-Witt complex. Let X be a scheme on which 2 is invertible. For $n \in \mathbb{N}$ we define the *generalized Hodge-Witt complex of level n on X* to be the graded algebra

$$\underline{\mathcal{W}}_n\underline{\Omega}_X^{\oplus} := \bigoplus_{i\geq 0} \underline{\mathcal{W}}_n\underline{\Omega}_X^{i}[-i] \tag{5.1}$$

with zero differential. As graded algebras $\underline{\mathcal{W}}_n\underline{\Omega}_X^{\oplus}$ and $\underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}$ are the same, but they carry different differentials. The fundamental relation

$$dF_N = NF_N d$$

implies that the maps $F_N : \underline{\mathcal{W}}_{nN}\underline{\Omega}_X^{i} \to \underline{\mathcal{W}}_n\underline{\Omega}_X^{i}$ together yield, for every n, a *homorphism of differential graded algebras*

$$\Phi_N : \underline{\mathcal{W}}_{nN}\underline{\Omega}_X^{\oplus} \longrightarrow \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{N\}. \tag{5.2}$$

Not only is Φ_N a homorphism of differential graded algebras, but it preserves several other important algebraic structures: see (5.3)–(5.10).
For $k > n$ one has the commutative square

$$\begin{array}{ccc} \underline{\mathcal{W}}_{kN}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_N} & \underline{\mathcal{W}}_k\underline{\Omega}_X^{\bullet}\{N\} \\ \mathbb{1}\downarrow & & \downarrow\mathbb{1} \\ \underline{\mathcal{W}}_{nN}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_N} & \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{N\} \end{array} \tag{5.3}$$

in which the vertical maps are the canonical truncation maps.
For $K|N$ one has the commutative square

$$\begin{array}{ccc} \underline{\mathcal{W}}_{nN}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_N} & \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{N\} \\ F_{N/K}\downarrow & & \downarrow\mathbb{1} \\ \underline{\mathcal{W}}_{nK}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_K} & \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{K\} \end{array} \tag{5.4}$$

For $K|N$ one has the commutative diagram

$$\begin{array}{ccc} \underline{\mathcal{W}}_{nK}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_K} & \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{K\} \\ \mathbb{1}\circ V_{N/K}\downarrow & & \downarrow \cdot N/K \\ \underline{\mathcal{W}}_{nN}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_N} & \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{N\} \end{array} \tag{5.5}$$

The fundamental relation

$$dF_m = mF_m d$$

also shows that there is a *homomorphism of differential graded algebras*

$$\mathbb{F}_m : \underline{\mathcal{W}}_{nm}\underline{\Omega}_X^{\bullet} \longrightarrow \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\,, \qquad \mathbb{F}_m\alpha = m^i F_m\alpha \quad \text{for} \quad \alpha \in \underline{\mathcal{W}}_{nm}\underline{\Omega}_X^{i} \tag{5.6}$$

and similarly for $\underline{\mathcal{W}}_{\cdot}\underline{\Omega}_X^{\oplus}$ in place of $\underline{\mathcal{W}}_{\cdot}\underline{\Omega}_X^{\bullet}$. The following square is commutative

$$\begin{array}{ccc} \underline{\mathcal{W}}_{mnN}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_N} & \underline{\mathcal{W}}_{mn}\underline{\Omega}_X^{\bullet}\{N\} \\ \mathbb{F}_m\downarrow & & \downarrow\mathbb{F}_m \\ \underline{\mathcal{W}}_{nN}\underline{\Omega}_X^{\oplus} & \xrightarrow{\Phi_N} & \underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}\{N\} \end{array} \tag{5.7}$$

Recall that we defined the ideal $\mathcal{J}_{X/S,n}^{\bullet}$ in $\underline{\mathcal{W}}_n\underline{\Omega}_X^{\bullet}$ by the formulas in (3.12). With the same formulas we define the ideal $\mathcal{J}^{\oplus}_{X/S,n}$ in $\underline{\mathcal{W}}_n\underline{\Omega}^{\oplus}_X$. So as ideals $\mathcal{J}^{\oplus}_{X/S,n}$ and $\mathcal{J}^{\bullet}_{X/S,n}$ are the same, but they carry different differentials. The map Φ_N restricts to a homomorphism of differential graded ideals

$$\Phi_N:\ \mathcal{J}^{\oplus}_{X/S,nN}\longrightarrow \mathcal{J}^{\bullet}_{X/S,n}\{N\}. \tag{5.8}$$

Modding out by these ideals we get the *relative generalized Hodge-Witt (resp. De Rham-Witt) complex of level* n:

$$\underline{\mathcal{W}}_n\underline{\Omega}^{\oplus}_{X/S}:=\underline{\mathcal{W}}_n\underline{\Omega}^{\oplus}_X/\mathcal{J}^{\oplus}_{X/S,n}\,,\qquad \underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_{X/S}:=\underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_X/\mathcal{J}^{\bullet}_{X/S,n}. \tag{5.9}$$

The map Φ induces a homomorphism of differential graded algebras

$$\Phi_N:\ \underline{\mathcal{W}}_{nN}\underline{\Omega}^{\oplus}_{X/S}\ \longrightarrow\ \underline{\mathcal{W}}_n\underline{\Omega}^{\bullet}_{X/S}\{N\}. \tag{5.10}$$

Fact: There are obvious analogues of (5.3)–(5.7) with X/S instead of X.

Remark 5.1 If X is a smooth scheme over a perfect field of characteristic $p>2$, if $n=N=p^j$ and if we restrict to the p-typical parts, then Φ_N induces on the cohomology sheaves the *higher Cartier isomorphism* C^{-j} of [9] p. 77. In the same situation, with $j=1$, the map Φ_p plays a decisive role in Deligne and Illusie's algebraic proof of the degeneration of the Hodge-to-De Rham spectral sequence; see [5] thm. 2.1.

5.2 From Hodge-Witt to De Rham-Witt. From now on we assume that $f:X\to S=\operatorname{Spec}A$ is a smooth projective morphism in which the ring A is étale over a polynomial ring $\mathbb{Z}[\frac{1}{2}][x_1,\ldots,x_r]$ and that $H^m_{DR}(X/S)$ and $H^{m-i}(X,\Omega^i_{X/S})$ are free A-modules for all m,i.

Note that the smoothness of X over S and flatness of A over $\mathbb{Z}$ imply that there is no $i!$-torsion in $\Omega^i_{X/S}$ and hence that

$$\underline{\mathcal{W}}_1\underline{\Omega}^{\bullet}_{X/S}\ =\ \Omega^{\bullet}_{X/S}.$$

Fix $N\in\mathbb{N}$. The homomorphism Φ_N from (5.10) induces on the m-th hypercohomology the map

$$\Phi_N:\bigoplus_j H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S})\longrightarrow \mathbb{H}^m(X,\Omega^{\bullet}_{X/S}\{N\}). \tag{5.11}$$

Since the De Rham cohomology groups are free A-modules and A is flat over $\mathbb{Z}$ we have

$$\mathbb{H}^m(X,\Omega^{\bullet}_{X/S}\{N\})=H^m_{DR}(X/S)\{N\}.$$

Proposition 5.2 *Φ_N relates as follows to the structures on $H^m_{DR}(X/S)\{N\}$ discussed in Section 1.*

• Hodge filtration: *For all N,m,i*

$$\Phi_N\left(\bigoplus_{j\geq i} H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S})\right)\subset \mathrm{Fil}^i_{\mathrm{Hodge}}H^m_{DR}(X/S)\{N\}. \tag{5.12}$$

• Conjugate filtration: *For all N,m,i*

$$\Phi_N\left(\bigoplus_{j\leq i} H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S})\right)\subset \mathrm{Fil}_i^{\mathrm{con}}\mathbb{H}^m(X,\Omega^{\bullet}_{X/S}\{N\}). \tag{5.13}$$

• Gauss-Manin connection: *For all* N, j

$$\nabla \circ \Phi_N \left(H^{m-j}(X, \underline{\mathcal{W}}_N \underline{\Omega}^j_{X/S}) \right) \subset \Omega^1_{S/\mathbb{Z}} \otimes \Phi_N \left(H^{m-j+1}(X, \underline{\mathcal{W}}_N \underline{\Omega}^{j-1}_{X/S}) \right) \quad (5.14)$$

Proof (5.12) and (5.13) are obvious. Let us prove (5.14). Recall from (5.8) that Φ_N maps $\mathcal{J}^{\oplus}_{X/S,N}$ into $\mathcal{J}^{\bullet}_{X/S,1}\{N\}$. So there is a commutative diagram with exact rows and vertical maps Φ_N:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{J}^{\oplus}_{X/S,N} & \to & \underline{\mathcal{W}}_N \underline{\Omega}^{\oplus}_X & \to & \underline{\mathcal{W}}_N \underline{\Omega}^{\oplus}_{X/S} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{J}^{\bullet}_{X/S,1}/(\mathcal{J}^{\bullet}_{X/S,1})^2\{N\} & \to & \Omega^{\bullet}_X/(\mathcal{J}^{\bullet}_{X/S,1})^2\{N\} & \to & \Omega^{\bullet}_{X/S}\{N\} & \to & 0 \end{array}$$

In the ladder of hypercohomology groups there is the commutative square

$$\begin{array}{ccc} \mathbb{H}^m(X, \underline{\mathcal{W}}_N \underline{\Omega}^{\oplus}_{X/S}) & \to & \mathbb{H}^{m+1}(X, \mathcal{J}^{\oplus}_{X/S,N}) \\ \downarrow & & \downarrow \\ \mathbb{H}^m(X, \Omega^{\bullet}_{X/S}\{N\}) & \to & \mathbb{H}^{m+1}(X, \mathcal{J}^{\bullet}_{X/S,1}/(\mathcal{J}^{\bullet}_{X/S,1})^2\{N\}) \end{array}$$

in which the bottom map is in fact the Gauss-Manin connection. To finish the proof we note that $\mathcal{J}^0_{X/S,N} = 0$ while for every $j \geq 1$

$$\begin{aligned} \Phi_N(\mathcal{J}^j_{X/S,N}) &= \Phi_N(d(f^* \underline{\mathcal{W}}_N \underline{\mathcal{O}}_S) \cdot \underline{\mathcal{W}}_N \underline{\Omega}^{j-1}_X) \\ &= \Phi_N(d(f^* \underline{\mathcal{W}}_N \underline{\mathcal{O}}_S)) \cdot \Phi_N(\underline{\mathcal{W}}_N \underline{\Omega}^{j-1}_X) \subset \Omega^1_S \cdot \Phi_N(\underline{\mathcal{W}}_N \underline{\Omega}^{j-1}_X). \end{aligned}$$

□

Each $H^j(X, \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S})$ is a module over the ring $\underline{\mathcal{W}}_N \underline{A}$. Viewing A as an algebra over $\underline{\mathcal{W}}_N \underline{A}$ via the ring homomorphism

$$F_N : \underline{\mathcal{W}}_N \underline{A} \to A$$

we define

$$F_N^* H^j(X, \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S}) := A \otimes_{\underline{\mathcal{W}}_N \underline{A}} H^j(X, \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S}). \quad (5.15)$$

This is a module over A. The map Φ_N in (5.11) induces a *A-linear map*

$$\Phi_N : \bigoplus_j F_N^* H^{m-j}(X, \underline{\mathcal{W}}_N \underline{\Omega}^j_{X/S}) \longrightarrow H^m_{DR}(X/S)\{N\} \quad (5.16)$$

$$\Phi_N(a \otimes \omega) = a \cdot \Phi_N \omega.$$

Theorem 5.3 *Let* $f : X \to S = \text{Spec } A$ *be a smooth projective morphism with ring A étale over a polynomial ring* $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$. *Assume that* $H^m_{DR}(X/S)$ *and* $H^{m-i}(X, \Omega^i_{X/S})$ *are free A-modules for all* m, i. *Fix* $N \in \mathbb{N}$ *and assume that* $H^j(X, Z_N \Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$ *is an isomorphism for all* i, j. *Fix* m. *Then the following three statements are equivalent*

(a) *For* $i = 0, \ldots, m$:

$$\Phi_N \left(\bigoplus_{j \geq i} F_N^* H^{m-j}(X, \underline{\mathcal{W}}_N \underline{\Omega}^j_{X/S}) \right) = \text{Fil}^i_{\text{Hodge}} H^m_{DR}(X/S)\{N\}. \quad (5.17)$$

(b) *For $i=0,\ldots,m$:*

$$\Phi_N\left(\bigoplus_{j\le i} F_N^* H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S})\right) = \mathrm{Fil}_i^{\mathrm{con}}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\}). \tag{5.18}$$

(c) *For $i=0,\ldots,m$ the map*

$$F_N : F_N^* H^{m-i}(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^{m-i}(X,\Omega^i_{X/S}) \qquad \textit{is surjective.}$$

Proof From the discussion preceding Theorem 1.2 we know that the hypotheses imply for $i=0,\ldots,m$

$$\begin{aligned}\mathrm{Fil}^i_{\mathrm{Hodge}}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\})/\mathrm{Fil}^{i+1}_{\mathrm{Hodge}}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\}) &= H^{m-i}(X,\Omega^i_{X/S})\{N\},\\ \mathrm{Fil}^{\mathrm{con}}_i\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\})/\mathrm{Fil}^{\mathrm{con}}_{i-1}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\}) &= H^{m-i}(X,\Omega^i_{X/S})\{N\}.\end{aligned}$$

On the other hand it is obvious that Φ_N induces maps

$$\begin{aligned}&F_N^*H^{m-i}(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S})\to\mathrm{Fil}^i_{\mathrm{Hodge}}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\})/\mathrm{Fil}^{i+1}_{\mathrm{Hodge}}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\}),\\ &F_N^*H^{m-i}(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S})\to\mathrm{Fil}^{\mathrm{con}}_i\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\})/\mathrm{Fil}^{\mathrm{con}}_{i-1}\mathbb{H}^m(X,\Omega^\bullet_{X/S}\{N\}).\end{aligned}$$

These are in fact equal to the map

$$F_N^*H^{m-i}(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^{m-i}(X,\Omega^i_{X/S})\{N\} \tag{5.19}$$

which is F_N followed by reduction mod$N$. Thus we see that statements (a) and (b) are both equivalent to the statement that the map in (5.19) is surjective for $i=0,\ldots,m$. The latter statement is clearly also implied by (c). Conversely, since

$$H^{m-i}(X,\Omega^i_{X/S}) \xrightarrow{V_N} H^{m-i}(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \xrightarrow{F_N} H^{m-i}(X,\Omega^i_{X/S})$$

is just multiplication by N, we see that $NH^{m-i}(X,\Omega^i_{X/S})$ is contained in the image of F_N and thus we can conclude that if the map in (5.19) is surjective, then

$$F_N:\ F_N^*H^{m-i}(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^{m-i}(X,\Omega^i_{X/S})$$

is surjective too, i.e. (c) holds. □

5.3 Compatibilities as N varies. From (5.3) we get for $K|N$ a commutative square

$$\begin{array}{ccc}\bigoplus_j H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S}) & \xrightarrow{\Phi_N} & H^m_{DR}(X/S)\{N\}\\ F_{N/K}\downarrow & & \downarrow \mathrm{mod}K\\ \bigoplus_j H^{m-j}(X,\underline{\mathcal{W}}_K\underline{\Omega}^j_{X/S}) & \xrightarrow{\Phi_K} & H^m_{DR}(X/S)\{K\}\end{array} \tag{5.20}$$

On one side we have the projective system of groups $\{H^m_{DR}(X/S)\{N\}\}_{N\in\mathbb{N}}$ with for $K|N$ the reduce- mod K-map. On the other side we must consider the projective system of groups $\{H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S})\}_{N\in\mathbb{N}}$ with for $K|N$ the map $F_{N/K}$.

From (5.5) we get for $K|N$ a commutative square

$$\begin{array}{ccc}\bigoplus_j H^{m-j}(X,\underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S}) & \xrightarrow{\Phi_N} & H^m_{DR}(X/S)\{N\}\\ V_{N/K}\uparrow & & \uparrow \cdot N/K\\ \bigoplus_j H^{m-j}(X,\underline{\mathcal{W}}_K\underline{\Omega}^j_{X/S}) & \xrightarrow{\Phi_K} & H^m_{DR}(X/S)\{K\}\end{array} \tag{5.21}$$

On one side we have the inductive system of groups $\{H^m_{DR}(X/S)\{N\}\}_{N\in\mathbb{N}}$ with for $K|N$ the map induced by multiplication by N/K. On the other side we have the inductive system of groups $\{H^{m-j}(X, \underline{\mathcal{W}}_N\underline{\Omega}^j_{X/S})\}_{N\in\mathbb{N}}$ with for $K|N$ the map $V_{N/K}$ (or rather $\mathbb{1}_N^{N+N/K-1}\circ V_{N/K}$).

In connection with the systems on the right it may be worthwhile to note that

$$\mathbb{Z}/N\mathbb{Z} \simeq \frac{1}{N}\mathbb{Z}/\mathbb{Z}, \qquad z \bmod N\mathbb{Z} \leftrightarrow \frac{z}{N} \bmod \mathbb{Z}$$

and that via this isomorphism the map $\mathrm{mod}K : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/K\mathbb{Z}$ corresponds with the multiplication by $N/K : \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \frac{1}{K}\mathbb{Z}/\mathbb{Z}$ and the map $\cdot N/K : \mathbb{Z}/K\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ corresponds with the inclusion $\frac{1}{K}\mathbb{Z}/\mathbb{Z} \subset \frac{1}{N}\mathbb{Z}/\mathbb{Z}$.

6 The conjugate filtration and ordinariness: Act 3

6.1.

The general hypotheses for this section are: $f : X \to S = \mathrm{Spec}\, A$ is a smooth projective morphism such that the ring A is étale over a polynomial ring $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$ and $H^m_{DR}(X/S)$ and $H^{m-i}(X, \Omega^i_{X/S})$ are free A-modules for all m, i.

We are going to analyse the projective systems of groups $\{H^j(X, \underline{\mathcal{W}}_n\underline{\Omega}^i_{X/S})\}_{n\in\mathbb{N}}$ with for $k < n$ the map induced by the truncation $\mathbb{1}^n_k : \underline{\mathcal{W}}_n\underline{\Omega}^i_{X/S} \to \underline{\mathcal{W}}_k\underline{\Omega}^i_{X/S}$. Later we look at the issue of surjectivity of the maps

$$F_N : F_N^* H^j(X, \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S}).$$

Proposition 6.1 *Under the general hypotheses of this section the sequence*

$$0 \to H^j(X, \mathcal{O}_X) \xrightarrow{V_N} H^j(X, \underline{\mathcal{W}}_N\underline{\mathcal{O}}_X) \xrightarrow{\mathbf{1}} H^j(X, \underline{\mathcal{W}}_{N-1}\underline{\mathcal{O}}_X) \to 0 \qquad (6.1)$$

is exact and the map, induced by truncation,

$$H^j(X, \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \qquad (6.2)$$

is surjective for all j and all $N \geq 2$.
Note: we make here a slight abuse of notation by writing V_N instead of $\mathbb{1}_N^{2N-1}\circ V_N$.

Proof Take $N \geq 2$. Consider the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{V_N} \underline{\mathcal{W}}_N\underline{\mathcal{O}}_X \xrightarrow{\mathbf{1}} \underline{\mathcal{W}}_{N-1}\underline{\mathcal{O}} \to 0$$

and the following piece of the associated sequence of cohomology groups

$$\to H^j(X, \mathcal{O}_X) \xrightarrow{V_N} H^j(X, \underline{\mathcal{W}}_N\underline{\mathcal{O}}_X) \longrightarrow H^j(X, \underline{\mathcal{W}}_{N-1}\underline{\mathcal{O}}_X) \to H^{j+1}(X, \mathcal{O}_X) \xrightarrow{V_N}$$

One has for every k the map $F_N : H^k(X, \underline{\mathcal{W}}_N\underline{\mathcal{O}}_X) \to H^k(X, \mathcal{O}_X)$ and $F_N V_N = N$. Since $H^k(X, \mathcal{O}_X)$ is a free A-module and A is flat over $\mathbb{Z}$, multiplication by N is injective. Hence in the exact sequence of cohomology groups all maps V_N are injective and the long exact cohomology sequence splits up into the short exact sequences (6.1). □

Proposition 6.2 *Fix $N \geq 2$. Assume in addition to the general hypotheses of this section that $H^j(X, Z_N\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$ is an isomorphism for all i, j. Then the sequence*

$$0 \to H^j(X, \Omega^i_{X/S}) \xrightarrow{V_N} H^j(X, \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^j(X, \underline{\mathcal{W}}_{N-1}\underline{\Omega}^i_{X/S}) \to 0 \qquad (6.3)$$

is exact for all i,j.

Proof Fix i. Set $\mathbb{K} := \ker(\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S} \to \underline{\mathcal{W}}_{N-1}\underline{\Omega}^i_{X/S})$. The we have according to Proposition 4.2 exact sequences

$$0 \to Z_N\Omega^{i-1}_{X/S} \xrightarrow{(1,-\frac{1}{N}d)} \Omega^{i-1}_{X/S} \oplus \Omega^i_{X/S} \xrightarrow{dV_N+V_N} \mathbb{K} \to 0 \tag{6.4}$$

$$0 \to \mathbb{K} \to \underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S} \longrightarrow \underline{\mathcal{W}}_{N-1}\underline{\Omega}^i_{X/S} \to 0 \tag{6.5}$$

The additional hypothesis and the exact cohomology sequence of (6.4) together show that the maps

$$H^j(X,\Omega^i_{X/S}) \xrightarrow{V_N} H^j(X,\mathbb{K}) \tag{6.6}$$

are surjective. Because of the relation $F_NV_N = N$ and the fact that multiplication by N on $H^j(X,\Omega^i_{X/S})$ is injective, the map $H^j(X,\Omega^i_{X/S}) \xrightarrow{V_N} H^j(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S})$ is injective. Therefore the maps (6.6) are in fact isomorphisms and the exact cohomology sequence of (6.5) breaks up into the short exact sequences (6.3). □

As a consequence of Propositions 6.2 and 4.3 we have:

Corollary 6.3 *Fix* $N \geq 2$. *Assume in addition to the general hypotheses of this section that for all prime numbers* $p \leq N$ *condition (4.10) holds, i.e.*

$$H^j(X,Z_p\Omega^i_{X/S}) \to H^j(X,\Omega^i_{X/S}) \qquad \text{is an isomorphism for all} \quad i,j.$$

Then for all i,j *and for every* $n \leq N$ *the sequence*

$$0 \to H^j(X,\Omega^i_{X/S}) \xrightarrow{V_n} H^j(X,\underline{\mathcal{W}}_n\underline{\Omega}^i_{X/S}) \xrightarrow{\mathbf{1}^n_{n-1}} H^j(X,\underline{\mathcal{W}}_{n-1}\underline{\Omega}^i_{X/S}) \to 0 \tag{6.7}$$

is exact and, hence, the map, induced by truncation,

$$H^j(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^j(X,\Omega^i_{X/S}) \tag{6.8}$$

is surjective. □

Fix N,i,j. Under the hypotheses of Proposition 6.1 (for $i=0$) and Corollary 6.3 (for $i \geq 1$) the map, induced by truncation,

$$\mathbb{1}^N_1 : H^j(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^j(X,\Omega^i_{X/S})$$

is surjective. So we can pick a basis $\omega_1,\ldots,\omega_h$ for $H^j(X,\Omega^i_{X/S})$ and lift each ω_k to an $\widetilde{\omega_k}$ in $H^j(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S})$. For every $n \leq N$ we denote the image of $\widetilde{\omega_k}$ under the standard truncation $\mathbb{1}^N_n : H^j(X,\underline{\mathcal{W}}_N\underline{\Omega}^i_{X/S}) \to H^j(X,\underline{\mathcal{W}}_n\underline{\Omega}^i_{X/S})$ also by $\widetilde{\omega_k}$.

For every $n \leq N$ we then have an $h \times h$-matrix $\mathrm{MAT}_{\widetilde{\omega}}(F_n)$ with entries in A in which the k-th column gives the coordinates of $F_n\widetilde{\omega_k}$ with respect to the basis $\omega_1,\ldots,\omega_h$:

$$(F_n\widetilde{\omega_1},\ldots,F_n\widetilde{\omega_h}) \;=\; (\omega_1,\ldots,\omega_h)\cdot \mathrm{MAT}_{\widetilde{\omega}}(F_n). \tag{6.9}$$

Lemma 6.4 *Fix* N,i,j. *Assume the hypotheses of Proposition 6.1 if* $i=0$ *and of Corollary 6.3 if* $i \geq 1$. *Let* $\omega_1,\ldots,\omega_h$ *and* $\widetilde{\omega_1},\ldots,\widetilde{\omega_h}$ *be as above. Then for every* $n \leq N$ *and* $\xi \in H^j(X,\underline{\mathcal{W}}_n\underline{\Omega}^i_{X/S})$ *there are unique elements* $a_{l,k} \in A$ *for* $l = 1,\ldots,n$ *and* $k = 1,\ldots,h$ *such that*

$$\xi = \sum_{k,l} V_l(\underline{\underline{a_{l,k}}}\widetilde{\omega_k}) \tag{6.10}$$

where we have simplified the notation by writing V_l instead of $\mathbb{1}_n^{nl+l-1} \circ V_l$.

Proof Fix $n \leq N$ and $\xi \in H^j(X, \underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S})$. Since $\omega_1, \ldots, \omega_h$ is an A-basis of $H^j(X, \Omega^i_{X/S})$ there are uniquely determined elements $a_{k,1} \in A$ such that:

$$\mathbb{1}_1^n(\xi) = \sum_{k=1}^{h} a_{1,k}\omega_k.$$

Then the truncation map $\mathbb{1}_2^n$ maps $\xi - \sum_{k=1}^h \underline{\underline{a_{1,k}}}\widetilde{\omega_k}$ into the kernel of the map $\mathbb{1}_1^2 : H^j(X, \underline{\mathcal{W}}_2 \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$. From the exact sequence (6.1) resp. (6.7) we see that there are $a_{2,1}, \ldots, a_{2,h} \in A$ such that

$$\mathbb{1}_2^n(\xi - \sum_{k=1}^{h} \underline{\underline{a_{1,k}}}\widetilde{\omega_k}) = V_2(\sum_{k=1}^{h} a_{2,k}\omega_k).$$

Then the truncation map $\mathbb{1}_3^n$ maps $\xi - \sum_{k=1}^h \underline{\underline{a_{1,k}}}\widetilde{\omega_k} - \sum_{k=1}^h V_2(\underline{\underline{a_{2,k}}}\widetilde{\omega_k})$ into the kernel of the map $\mathbb{1}_2^3 : H^j(X, \underline{\mathcal{W}}_3 \underline{\Omega}^i_{X/S}) \to H^j(X, \underline{\mathcal{W}}_2 \underline{\Omega}^i_{X/S})$. This kernel equals the image of $V_3 : H^j(X, \Omega^i_{X/S}) \to H^j(X, \underline{\mathcal{W}}_3 \underline{\Omega}^i_{X/S})$. It is clear how one can go on this way till one has reached (6.10). □

Lemma 6.5 *Fix N, i, j. Assume the hypotheses of Proposition 6.1 if $i = 0$ and of Corollary 6.3 if $i \geq 1$. Let p be a prime number, $p \leq N$. Assume that the map*

$$F_p : F_p^* H^j(X, \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \qquad \textit{is surjective.} \tag{6.11}$$

Then for every $\nu \in \mathbb{N}$ such that $p^\nu \leq N$ the map

$$F_{p^\nu} : F_{p^\nu}^* H^j(X, \underline{\mathcal{W}}_{p^\nu} \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \qquad \textit{is surjective.} \tag{6.12}$$

Proof Assume $\nu \geq 2$. Let $\omega_1, \ldots, \omega_h$ and $\widetilde{\omega_1}, \ldots, \widetilde{\omega_h}$ be as above. Recall that we also write $\widetilde{\omega_k}$ for the image of $\widetilde{\omega_k}$ in $H^j(X, \underline{\mathcal{W}}_{p^\nu} \underline{\Omega}^i_{X/S})$ resp. $H^j(X, \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S})$ under the truncation $\mathbb{1}_{p^\nu}^N$ resp. $\mathbb{1}_p^N$. Then there are, according to the previous lemma, unique elements $a_{l,k,m} \in A$ such that

$$F_{p^{\nu-1}}\widetilde{\omega_m} = \sum_{k=1}^{h}\sum_{l=1}^{p} V_l(\underline{\underline{a_{l,k,m}}}\widetilde{\omega_k}).$$

Applying F_p we see

$$F_{p^\nu}\widetilde{\omega_m} \;=\; \sum_{k=1}^{h}\sum_{l=1}^{p} F_p V_l(\underline{\underline{a_{l,k,m}}}\widetilde{\omega_k}) \;=\; \sum_{k=1}^{h}(a_{1,k,m}^p F_p\widetilde{\omega_k} \;+\; p a_{p,k,m}\omega_k).$$

This shows that with the notation from (6.9) we have

$$\mathrm{MAT}_{\widetilde{\omega}}(F_{p^\nu}) \equiv \mathrm{MAT}_{\widetilde{\omega}}(F_p) \cdot \mathrm{MAT}_{\widetilde{\omega}}(F_{p^{\nu-1}})^{(p)} \mod p \tag{6.13}$$

where for a matrix $M = (a_{m,k})$ we denote $M^{(p^s)} = (a_{m,k}^{p^s})$. Induction now shows

$$\mathrm{MAT}_{\widetilde{\omega}}(F_{p^\nu}) \equiv \mathrm{MAT}_{\widetilde{\omega}}(F_p) \cdot \mathrm{MAT}_{\widetilde{\omega}}(F_p)^{(p)} \cdot \ldots \cdot \mathrm{MAT}_{\widetilde{\omega}}(F_p)^{(p^{\nu-1})} \mod p.$$

Assumption (6.11) implies that $\mathrm{MAT}_{\widetilde{\omega}}(F_p)$ is invertible $\bmod\, p$. Thus $\mathrm{MAT}_{\widetilde{\omega}}(F_{p^\nu})$ is invertible $\bmod\, p$ for every ν such that $p^\nu \leq N$. This in turn implies (6.12). □

Lemma 6.6 *Fix N, i, j. Assume the hypotheses of Proposition 6.1 if $i = 0$ and of Corollary 6.3 if $i \geq 1$. Assume that for every prime number p which divides N the map*

$$F_p : F_p^* H^j(X, \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$$

is surjective. Then the map

$$F_N : F_N^* H^j(X, \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$$

is surjective.

Proof Consider the prime decomposition $N = \prod_l p_l^{\nu_l}$. Choose integers c_l such that $\sum_l c_l N p_l^{-\nu_l} = 1$. Take $\xi \in H^j(X, \Omega^i_{X/S})$. From Lemma 6.5 we know that for every l the map

$$F_{p_l^{\nu_l}} : F^*_{p_l^{\nu_l}} H^j(X, \underline{\mathcal{W}}_{p_l^{\nu_l}} \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$$

is surjective. So there are elements $a_{l,m} \in A$ and $\alpha_{l,m} \in H^j(X, \underline{\mathcal{W}}_{p_l^{\nu_l}} \underline{\Omega}^i_{X/S})$ such that

$$\xi = \sum_m a_{l,m} F_{p_l^{\nu_l}} \alpha_{l,m}.$$

Then

$$F_N \left(\sum_{l,m} c_l a_{l,m} V_{N p_l^{-\nu_l}} \alpha_{l,m} \right) = \sum_{l,m} c_l N p_l^{-\nu_l} a_{l,m} F_{p_l^{\nu_l}} \alpha_{l,m} = \xi.$$

So F_N is surjective. □

6.2.

Theorem 5.3 raised the issue of surjectivity of the map

$$F_N : F_N^* H^j(X, \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S}).$$

Lemma 6.6 reduced the problem to the surjectivity of

$$F_p : F_p^* H^j(X, \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$$

for prime numbers p dividing N. Recall that in the circumstances of Proposition 6.1 and Corollary 6.3 one has short exact sequences

$$0 \to H^j(X, \Omega^i_{X/S}) \xrightarrow{V_n} H^j(X, \underline{\mathcal{W}}_n \underline{\Omega}^i_{X/S}) \xrightarrow{\mathbf{1}^n_{n-1}} H^j(X, \underline{\mathcal{W}}_{n-1} \underline{\Omega}^i_{X/S}) \to 0$$

From these sequences and the relations

$$F_p V_p = p\,, \qquad F_p V_n = V_n F_p \quad \text{if } n < p$$

one deduces that there is a map

$$\overline{F}_p : H^j(X, \Omega^i_{X/S})\{p\} \to H^j(X, \Omega^i_{X/S})\{p\} \tag{6.14}$$

making the following square commutative

$$\begin{array}{ccc} H^j(X, \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}) & \xrightarrow{F_p} & H^j(X, \Omega^i_{X/S}) \\ (\mathrm{mod} p) \circ \mathbb{1}^p_1 \downarrow & & \downarrow \mathrm{mod} p \\ H^j(X, \Omega^i_{X/S})\{p\} & \xrightarrow{\overline{F}_p} & H^j(X, \Omega^i_{X/S})\{p\} \end{array}$$

Set

$$\overline{F}_p^* H^j(X, \Omega^i_{X/S}) = (A/pA) \otimes_A H^j(X, \Omega^i_{X/S})$$

where A/pA is an A-module via the ring homomorphism $A \to A/pA$, $a \mapsto a^p \bmod p$. Then $\overline{F}_p$ induces the A-linear map

$$\overline{F}_p : \overline{F}_p^* H^j(X, \Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}), \qquad \overline{F}_p(a \otimes \alpha) := a \cdot \overline{F}_p \alpha.$$

Now it is clear that

$$F_p : F_p^* H^j(X, \underline{\mathcal{W}}_p \underline{\Omega}^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \qquad \text{is surjective}$$

if and only if

$$\overline{F}_p : \overline{F}_p^* H^j(X, \Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S}) \qquad \text{is surjective.}$$

To prove surjectivity for the latter $\overline{F}_p$ we need all hypotheses from Setting 0.1.

So from now on $f : X \to S = \mathrm{Spec} A$ is a smooth projective morphism of relative dimension d, such that the ring A is étale over a polynomial ring $\mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_r]$ and $H^m_{DR}(X/S)$ and $H^{m-i}(X, \Omega^i_{X/S})$ are free A-modules for all m, i.

Let $\Omega^{\mathsf{d}}_{X/S,\log}$ be the sheaf of *logarithmic* d*-forms*. It is locally generated by sections of the form

$$\frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_{\mathsf{d}}}{u_{\mathsf{d}}}. \tag{6.15}$$

We assume that $H^{\mathsf{d}}(X, \Omega^{\mathsf{d}}_{X/S})$ is a free A-module of rank 1 and that we can choose $\varpi \in H^{\mathsf{d}}(X, \Omega^{\mathsf{d}}_{X/S,\log})$ so that its image ϖ_1 in $H^{\mathsf{d}}(X, \Omega^{\mathsf{d}}_{X/S})$ is an A-basis of the latter. Using the product map

$$H^j(X, \Omega^i_{X/S}) \times H^{\mathsf{d}-j}(X, \Omega^{\mathsf{d}-i}_{X/S}) \to H^{\mathsf{d}}(X, \Omega^{\mathsf{d}}_{X/S})$$

in the Hodge cohomology algebra and the chosen ϖ we define the bilinear pairing

$$\langle\ ,\ \rangle :\ H^j(X, \Omega^i_{X/S}) \times H^{\mathsf{d}-j}(X, \Omega^{\mathsf{d}-i}_{X/S}) \to A \qquad \text{s.t.} \qquad \alpha \cdot \beta = \langle \alpha, \beta \rangle \varpi_1. \tag{6.16}$$

We assume that this pairing is non-degenerate for all i, j.

Theorem 6.7 *Fix $N \geq 2$. Assume in addition to the above hypotheses that $H^j(X, Z_p\Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S})$ is an isomorphism for all i, j and for all prime numbers $p \leq N$. Then for all i, j and for every prime number $p \leq N$ the map*

$$\overline{F}_p : \overline{F}_p^* H^j(X, \Omega^i_{X/S}) \to H^j(X, \Omega^i_{X/S})\{p\}$$

is an isomorphism.

Proof A section of the form (6.15) can be lifted to a section

$$(\underline{\underline{u_1}} \cdot \ldots \cdot \underline{\underline{u_{\mathsf{d}}}})^{-1} d\underline{\underline{u_1}} \wedge \ldots \wedge d\underline{\underline{u_{\mathsf{d}}}}$$

of $\underline{\mathcal{W}}_n \underline{\Omega}^{\mathsf{d}}_{X/S}$ for every n. Thus we get a morphism of sheaves

$$\Omega^{\mathsf{d}}_{X/S,\log} \to \underline{\mathcal{W}}_n \underline{\Omega}^{\mathsf{d}}_{X/S}$$

which composes with the truncation $\mathbb{1}^n_1 : \underline{\mathcal{W}}_n \underline{\Omega}^{\mathsf{d}}_{X/S} \to \Omega^{\mathsf{d}}_{X/S}$ to the inclusion $\Omega^{\mathsf{d}}_{X/S,\log} \subset \Omega^{\mathsf{d}}_{X/S}$. Let $\varpi_n \in H^{\mathsf{d}}(X, \underline{\mathcal{W}}_n \underline{\Omega}^{\mathsf{d}}_{X/S})$ denote the image of ϖ under the map $H^{\mathsf{d}}(\Omega^{\mathsf{d}}_{X/S,\log}) \to H^{\mathsf{d}}(\underline{\mathcal{W}}_n \underline{\Omega}^{\mathsf{d}}_{X/S})$. Using the formulas (3.3), first on sections and then on cohomology, one checks

$$F_l \varpi_{nl} = \varpi_n \qquad \text{for all} \quad n, l. \tag{6.17}$$

Take a basis $\omega_1, \ldots, \omega_h$ for $H^j(X, \Omega^i_{X/S})$ and a basis $\eta_1, \ldots, \eta_h$ for $H^{\mathsf{d}-j}(X, \Omega^{\mathsf{d}-i}_{X/S})$ such that the matrix

$$(\langle \omega_k, \eta_m \rangle)_{m,k}$$

is invertible over A; for this we use the assumption that the pairing $\langle\, , \,\rangle$ is non-degenerate. Lift each ω_k to an $\widetilde{\omega_k}$ in $H^j(X, \underline{\mathcal{W}}_N \underline{\Omega}^i_{X/S})$ and lift each η_m to an $\widetilde{\eta_m}$ in $H^{\mathsf{d}-j}(X, \underline{\mathcal{W}}_N \underline{\Omega}^{\mathsf{d}-i}_{X/S})$. Then, as in Lemma 6.4, there are unique elements $a_{l,k,m} \in A$ such that in $H^{\mathsf{d}}(X, \Omega^{\mathsf{d}}_{X/S})$

$$\widetilde{\omega_k} \cdot \widetilde{\eta_m} = \sum_l V_l(\underline{\underline{a_{l,k,m}}} \varpi_N). \tag{6.18}$$

Note $a_{1,k,m} = \langle \omega_k, \eta_m \rangle$. Let p be a prime number, $p \leq N$. Apply the operator $F_p \circ \mathbb{1}^N_p$ to (6.18):

$$(F_p \circ \mathbb{1}^N_p \widetilde{\omega_k}) \cdot (F_p \circ \mathbb{1}^N_p \widetilde{\eta_m}) = (a^p_{1,k,m} + p a_{p,k,m}) \varpi_1.$$

Taken modulo p this relation reads

$$\langle \overline{F}_p(\omega_k \bmod p), \overline{F}_p(\eta_m \bmod p) \rangle = \langle \omega_k \bmod p\, , \, \eta_m \bmod p \rangle^p.$$

In matrix form this is

$$\mathrm{MAT}_\omega(\overline{F}_p)^{\mathrm{transpose}} \cdot (\langle \omega_k, \eta_m \rangle) \cdot \mathrm{MAT}_\eta(\overline{F}_p) \equiv (\langle \omega_k, \eta_m \rangle^p) \mod p.$$

So the matrix $\mathrm{MAT}_\omega(\overline{F}_p)$ is invertible over A/pA. □

Remark 6.8 For $(i, j) \neq (\frac{\mathsf{d}}{2}, \frac{\mathsf{d}}{2})$ one can, after an initial choice of bases $\{\omega_1, \ldots, \omega_h\}$ for $H^j(X, \Omega^i_{X/S})$ and $\{\eta_1, \ldots, \eta_h\}$ for $H^{\mathsf{d}-j}(X, \Omega^{\mathsf{d}-i}_{X/S})$, change the latter by means of the matrix $(\langle \omega_k, \eta_m \rangle)^{-1}$ and thus obtain a basis $\{\zeta_1, \ldots, \zeta_h\}$ for $H^{\mathsf{d}-j}(X, \Omega^{\mathsf{d}-i}_{X/S})$ such that

$$\langle \omega_k, \zeta_m \rangle = \delta_{k,m} \qquad \text{(Kronecker's delta)}$$

Then the above proof shows

$$\mathrm{MAT}_\zeta(\overline{F}_p) = \mathrm{MAT}_\omega(\overline{F}_p)^{\text{transpose inverse}} \tag{6.19}$$

APPENDIX: Formal groups in the background

Behind the structures in Proposition 6.1 and Corollary 6.3 one can see formal groups. In this appendix we discuss the formal groups for Proposition 6.1, leaving the ones for Corollary 6.3 for future investigations. The formal groups for Proposition 6.1 are well known and were introduced by Artin and Mazur [1] as a generalization of the classical construction/definition of the *formal Picard group*: the latter is based on deformation of $H^1(X, \mathcal{O}^*_X)$; the new ones are based on deformation of $H^j(X, \mathcal{O}^*_X)$ for any j. Despite the abstract appearance of their definition these formal groups are often very manageable objects which can be used for concretely computing the action of Frobenius operators on the cohomology of sheaves of generalized Witt vectors.

We place our discussion of the Artin-Mazur formal groups in the geometric setting of this paper, i.e. $\mathbb{X}$, $\mathbb{S}$ and $\mathbb{A}$ are as in Setting 0.1. Unlike "ordinariness" Artin-Mazur formal groups require no further restrictive conditions; everything works with the original $\mathbb{X}$, $\mathbb{S}$ and $\mathbb{A}$.

The Artin-Mazur formal group $H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$ is defined as follows (m in the notation $\hat{\mathbb{G}}_{m,\mathbb{X}}$ refers to the *multiplicative* formal group). Let $\mathfrak{Nilalgs}_{\mathbb{A}}$ denote the category of nil-$\mathbb{A}$-algebras, i.e. associative commutative $\mathbb{A}$-algebras without unit element in which every element is nilpotent. For a nil-$\mathbb{A}$-algebra $\mathcal{N}$ we have the sheaf of nilalgebras $\mathcal{O}_{\mathbb{X}} \otimes_{\mathbb{A}} \mathcal{N}$ on $\mathbb{X}$. For local sections x and y of $\mathcal{O}_{\mathbb{X}} \otimes_{\mathbb{A}} \mathcal{N}$ we set

$$x \star y = x + y - xy.$$

The sheaf $\mathcal{O}_{\mathbb{X}} \otimes_{\mathbb{A}} \mathcal{N}$ with the binary operation $\star$ on its local sections is a sheaf of abelian groups on $\mathbb{X}$, which we denote by $\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})$. One defines $H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$ to be the functor

$$\begin{array}{rcl} H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}) \ : \ \mathfrak{Nilalgs}_{\mathbb{A}} & \longrightarrow & \mathfrak{Abelian\ groups} \\ \mathcal{N} & \mapsto & H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})). \end{array} \tag{6.20}$$

In the category $\mathfrak{Nilalgs}_{\mathbb{A}}$ one has in particular the nilalgebras $t\mathbb{A}[t]/(t^{n+1})$ for $n \geq 1$. Noticing that

$$\hat{\mathbb{G}}_{m,\mathbb{X}}(t\mathbb{A}[t]/(t^{n+1})) = 1 + t\mathcal{O}_{\mathbb{X}}[t]/(t^{n+1}) = \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}}$$

we recover the Witt vector cohomology groups

$$H^j(X, \hat{\mathbb{G}}_{m,\mathbb{X}}(t\mathbb{A}[t]/(t^{n+1}))) = H^j(X, \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}}).$$

According to Proposition 6.1 the truncation maps

$$H^j(X, \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}}) \to H^j(X, \underline{\mathcal{W}}_{n-1}\underline{\mathcal{O}}_{\mathbb{X}})$$

are surjective. So the limit $\lim_{\leftarrow n} H^j(X, \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}})$ contains elements which project onto a basis of $H^j(X, \mathcal{O}_{\mathbb{X}})$ as $\mathbb{A}$-module. A convenient notation for an element of $\lim_{\leftarrow n} H^{\mathsf{d}}(X, \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}})$ is $\gamma(t)$. For

$$\gamma(t) \in \lim_{\leftarrow n} H^{\mathsf{d}}(X, \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}})$$

we construct, as follows, a functorial homomorphism

$$\gamma : \mathcal{N} \to H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}))$$

from on the one side the functor $\mathfrak{Nilalgs}_{\mathbb{A}} \to \mathfrak{Sets}$ which assigns to a nilalgebra $\mathcal{N}$ the underlying set $\mathcal{N}$ to on the other side the functor which assigns to $\mathcal{N}$ the set underlying the group $H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}))$. For the construction of γ take a nilalgebra $\mathcal{N}$ and an element $u \in \mathcal{N}$. There is then an $n \in \mathbb{N}$ and an algebra homomorphism

$$g : t\mathbb{A}[t]/(t^{n+1}) \to \mathcal{N}\,, \qquad g(t) = u.$$

This g induces a group homomorphism

$$g_* : H^j(X, \underline{\mathcal{W}}_n\underline{\mathcal{O}}_{\mathbb{X}}) \to H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})).$$

We define

$$\gamma(u) = g_*(\gamma(t)) \in H^{\mathsf{d}}(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})).$$

and obtain thus a map of sets

$$\gamma : \mathcal{N} \to H^j(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}))\,, \quad u \mapsto \gamma(u).$$

The functoriality of this construction is obvious.

Now fix elements $\gamma_{j,1}(t),\ldots,\gamma_{j,h^{0j}}(t)$ in $\lim_{\leftarrow n} H^j(X,\underline{\mathcal{W}}_n\underline{\mathcal{O}_{\mathbb{X}}})$ such that their images in $H^j(X,\mathcal{O}_{\mathbb{X}})$ constitute an $\mathbb{A}$-module basis. Then the above construction gives a functorial map

$$\begin{array}{rccc} (\gamma_{j,1},\ldots,\gamma_{j,h^{0j}}): & \mathcal{N}^{\times h^{0j}} & \to & H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})) \\ & (u_1,\ldots,u_{h^{0j}}) & \mapsto & \sum_{k=1}^{h^{0j}}\gamma_k(u_k) \end{array} \tag{6.21}$$

Proposition 6.9 *(6.21) is a functorial bijection for every j.*

Proof First note that by a simple inductive limit argument it suffices to show bijectivity only for nilalgebras which are finitely generated over $\mathbb{A}$. For the finitely generated nilalgebras one proceeds by induction along so-called *small extensions*: a small extension is a surjective algebra homomorphism $\mathcal{N}\twoheadrightarrow\mathcal{N}'$ with kernel generated by a single element ϵ such that $\epsilon\mathcal{N}=0$. For every finitely generated nilalgebra $\mathcal{N}$ the trivial map $\mathcal{N}\to 0$ is the composite of finitely many small extensions.

For the induction step ones notes that

$$\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A})) = 1+\epsilon\mathbb{A}\otimes\mathcal{O}_{\mathbb{X}} \overset{\log}{\simeq} \epsilon\mathbb{A}\otimes\mathcal{O}_{\mathbb{X}}$$

and hence

$$H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A})) \overset{\log}{\simeq} H^j(\mathbb{X},\mathcal{O}_{\mathbb{X}}\otimes\epsilon\mathbb{A}) = H^j(\mathbb{X},\mathcal{O}_{\mathbb{X}})\otimes\epsilon\mathbb{A} \overset{\text{basis}}{\simeq} (\epsilon\mathbb{A})^{\times h^{0j}}. \tag{6.22}$$

For a small extension $\epsilon\mathbb{A}\hookrightarrow\mathcal{N}\twoheadrightarrow\mathcal{N}'$ the sequence of sheaves of groups on $\mathbb{X}$

$$0\to\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A})\to\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}))\to\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}')\to 0 \tag{6.23}$$

is obviously exact. There is, by functoriality, a commutative diagram

$$\begin{array}{ccccc} (\epsilon\mathbb{A})^{\times h^{0j}} & \hookrightarrow & \mathcal{N}^{\times h^{0j}} & \twoheadrightarrow & (\mathcal{N}')^{\times h^{0j}} \\ \downarrow & & \downarrow & & \downarrow \\ H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A})) & \to & H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})) & \to & H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}')) \end{array} \tag{6.24}$$

in which the vertical maps are given by (6.21) and the bottom row is part of the exact sequence of cohomology groups associated with (6.23):

$$\begin{array}{lllll} 0\to & H^0(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A}))\to & H^0(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})) & \to H^0(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}')) & \to \\ \to & H^1(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A}))\to & \ldots\quad\ldots\quad\ldots & \to H^{\mathsf{d}-1}(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}')) & \to \\ \to & H^{\mathsf{d}}(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A}))\to & H^{\mathsf{d}}(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})) & \to H^{\mathsf{d}}(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}')) & \to 0 \end{array}$$

The proposition can be proved by a double induction with inside an induction for $j=0,\ldots,\mathsf{d}$ an induction along small extensions. The result for $j-1$ provides surjectivity of $H^{j-1}(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}))\to H^{j-1}(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}'))$ and thus injectivity of $H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\epsilon\mathbb{A}))\to H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N}))$. Knowing this injectivity and assuming that in (6.24) the left and right vertical maps are bijective one can conclude that the middle map is bijective too. □

A functorial bijection such as (6.21) is called a *coordinatization of the functor* $H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}})$. A functor which allows such a coordinatization (i.e. a functorial bijection to the set valued functor $\mathcal{N}\mapsto\mathcal{N}^{\times h}$ for some h) *is called a formal group.* So in the circumstances of Setting 0.1 the Artin-Mazur functors are indeed formal groups.

There is a converse to the above result in that every coordinatization of the functor $H^j(\mathbb{X},\hat{\mathbb{G}}_{m,\mathbb{X}})$ provides a set of elements in $\lim_{\leftarrow n} H^j(X,\underline{\mathcal{W}}_n\underline{\mathcal{O}_{\mathbb{X}}})$ such that their images in $H^j(X, \mathcal{O}_{\mathbb{X}})$ constitute an $\mathbb{A}$-module basis. To see this one just has to evaluate the coordinatization at the projective system of nilalgebras $\{t\mathbb{A}[t]/(t^{n+1})\}_{n\geq 1}$ and to observe that functoriality implies that the coordinatization

$$(t\mathbb{A}[t]/(t^2))^{h^{0j}} \longrightarrow H^j(X, \mathcal{O}_X)$$

is not only a bijective map of sets but even an $\mathbb{A}$-linear isomorphism.

We are mainly interested in applying the theory to families of Calabi-Yau varieties. Since we do not want to blur the picture with unnecessary technicalities from formal group theory we assume henceforth

$$H^{\mathsf{d}}(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) \simeq \mathbb{A} \tag{6.25}$$

and we look only at the Artin-Mazur formal group $H^{\mathsf{d}}(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$. A coordinatization of this 1-dimensional formal group, i.e. a functorial bijection

$$\gamma : \mathcal{N} \xrightarrow{1:1} H^{\mathsf{d}}(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})),$$

provides a *formal group law*, i.e. a two variable power series $G(t_1, t_2) \in \mathbb{A}[[t_1, t_2]]$: if y_1, y_2 are elements in some nilalgebra $\mathcal{N}$, then

$$\gamma(y_1) + \gamma(y_2) = \gamma(G(y_1, y_2)) \qquad \text{in the group } H^{\mathsf{d}}(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}}(\mathcal{N})).$$

Since the ring $\mathbb{A}$ is flat over $\mathbb{Z}$ (i.e. the canonical map $\mathbb{A} \to \mathbb{A} \otimes \mathbb{Q}$ is injective), there is a power series

$$\ell(t) \;=\; \sum_{m\geq 1} \frac{a_m}{m}\, t^m \;\in (\mathbb{A} \otimes \mathbb{Q})[[t]]$$

in one variable t with all $a_m \in \mathbb{A}$ and $a_1 = 1$, such that

$$G(t_1, t_2) = \ell^{-1}(\ell(t_1) + \ell(t_2)).$$

The fact that the coefficients of the power series $G(t_1, t_2)$ have no denominators, puts a very strong structure on the sequence $\{a_m\}_{m\geq 1}$.

In [14] Theorem 1 it is shown how a concrete logarithm for the Artin-Mazur formal group $H^{\mathsf{d}}(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$ can be obtained from the defining equations when $\mathbb{X}$ is a complete intersection in $\mathbb{P}^N_{\mathbb{A}}$. Proposition 6.10 gives the result for Calabi-Yau varieties of dimension d (conditions $d_1 + \ldots + d_r \;=\; N+1$ and $N - s - 1 = \mathsf{d}$).

Proposition 6.10 *Let $\mathbb{A}$ be as in Setting 0.1. Let $P_1, \ldots, P_s$ be a* regular sequence *of homogeneous polynomials in $\mathbb{A}[Z_0, \ldots, Z_N]$ of degrees $d_1, \ldots, d_s$ with $d_1 + \ldots + d_s \;=\; N+1$ and $N - s = \mathsf{d}$. Let $\mathbb{X}$ be the subscheme of $\mathbb{P}^N_{\mathbb{A}}$ defined by the ideal $(P_1, \ldots, P_s)$. Assume $\mathbb{X}$ is flat over $\mathbb{A}$. Then the power series*

$$\ell(t) \;=\; \sum_{m\geq 1} \frac{a_m}{m}\, t^m$$

with

$$a_m \;:=\; \text{coefficient of } (Z_0 \cdot \ldots \cdot Z_N)^{m-1} \quad \text{in} \quad (P_1 \cdot \ldots \cdot P_s)^{m-1}$$

is a logarithm for the Artin-Mazur formal group $H^{\mathsf{d}}(\mathbb{X}, \hat{\mathbb{G}}_{m,\mathbb{X}})$. □

Well-known examples in which the conditions of this Proposition 6.10 are satisfied, are: ⋄ quintic hypersurfaces in $\mathbb{P}^4$ ⋄ intersections of two cubic hypersurfaces in $\mathbb{P}^5$ ⋄ intersections of four quadrics in $\mathbb{P}^7$.

[14] Theorem 2 gives a similar result for branched double coverings of $\mathbb{P}^N_{\mathbb{A}}$ and applies for instance to double coverings of $\mathbb{P}^3_{\mathbb{A}}$ branched along a hypersurface of degree 8.

Because of the enormous number of variables it is not practical to write out the a_m's explicitly for the general families in these examples. There are, however, manageable subfamilies, like:

Example 6.11 *For the family of Calabi-Yau hypersurfaces in* $\mathbb{P}^4$ *with equation*

$$Z_0Z_1Z_2Z_3Z_4 - x(Z_0^5+Z_1^5+Z_2^5+Z_3^5+Z_4^5) = 0$$

one has

$$a_m = \sum_{j\geq 0} (-1)^j \frac{(5j)!}{(j!)^5} \begin{pmatrix} m-1 \\ 5j \end{pmatrix} x^{5j}.$$

Note that a_m *is a polynomial of degree* $\leq m-1$.

The group

$$\lim_{\leftarrow n} H^{\mathsf{d}}(X, \hat{\mathbb{G}}_{m,\mathbb{X}}(t\mathbb{A}[t]/(t^{n+1}))) = \lim_{\leftarrow n} H^{\mathsf{d}}(X, \underline{\mathcal{W}}_n \underline{\mathcal{O}}_{\mathbb{X}})$$

is called the *module of curves on* or the *Cartier-Dieudonné module* of the formal group $H^{\mathsf{d}}(X, \hat{\mathbb{G}}_{m,\mathbb{X}})$. It comes equipped with operators $\underline{\underline{a}}$, V_k and F_k for $a \in \mathbb{A}$, $k \in \mathbb{N}$, defined as acting on elements $\gamma(t) \in \lim_{\leftarrow n} H^{\mathsf{d}}(X, \hat{\mathbb{G}}_{m,\mathbb{X}}(t\mathbb{A}[t]/(t^{n+1})))$ by

$$\begin{array}{rcll} \underline{\underline{a}}\gamma(t) &=& \gamma(at) & \text{i.e. induced by substution} \quad t \mapsto at \\ V_k\gamma(t) &=& \gamma(t^k) & \text{i.e. induced by substution} \quad t \mapsto t^k \\ F_k\gamma(t) &=& \boxplus_{l=0}^{k-1}\gamma(\zeta^l t^{1/k}) & \text{with } \ \zeta \ \text{ primitive } \ k\text{-th root of unity} \end{array} \tag{6.26}$$

where $\boxplus$ refers to the addition in the formal group, which corresponds with the standard addition in $\lim_{\leftarrow n} H^{\mathsf{d}}(X, \underline{\mathcal{W}}_n \underline{\mathcal{O}}_{\mathbb{X}})$. The operators F_k and V_k correspond with the earlier defined Frobenius and Verschiebung operators on $\lim_{\leftarrow n} H^{\mathsf{d}}(X, \underline{\mathcal{W}}_n \underline{\mathcal{O}}_{\mathbb{X}})$; the operator $\underline{\underline{a}}$ corresponds with multiplication by the element $\underline{\underline{a}} \in \underline{\mathcal{W}\mathbb{A}}$.

If $\gamma(t)$ projects onto an $\mathbb{A}$-basis of $H^{\mathsf{d}}(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ it gives rise to a coordinatization, a formal group law and a logarithm

$$\ell(t) \ = \sum_{m\geq 1} \frac{a_m}{m}\, t^m$$

for this formal group law. Since the coordinatization maps t to $\gamma(t)$ we can now concretely compute

$$\begin{array}{rcl} F_k\gamma(t) &=& \boxplus_{l=0}^{k-1}\gamma(\zeta^l t^{1/k}) \ = \ \ell^{-1}\left(\displaystyle\sum_{l=0}^{k-1} \ell(\zeta^l t^{1/k})\right) \\ &=& \ell^{-1}\left(\displaystyle\sum_{l=0}^{k-1}\sum_{m\geq 1} \frac{a_m}{m}(\zeta^l t^{1/k})^m\right) \\ &=& \ell^{-1}\,(a_k t + \text{ higher order terms}) \\ &=& a_k t \mod t^2. \end{array}$$

Looking back at (6.9) and the discussion preceding it we see: we have taken a basis $\overline{\gamma}$ for $H^{\mathsf{d}}(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$, lifted it to an element $\gamma(t) \in \lim_{\leftarrow n} H^{\mathsf{d}}(X, \hat{\mathbb{G}}_{m,\mathbb{X}}(t\mathbb{A}[t]/(t^{n+1}))) =$

$\lim_{\leftarrow n} H^{\mathsf{d}}(X, \underline{\mathcal{W}_n \mathcal{O}_{\mathbb{X}}})$ and computed the image of $F_k\gamma(t)$ in $H^{\mathsf{d}}(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$. The conclusion is

$$\mathrm{MAT}_{\gamma(t)}(F_k) = a_k. \tag{6.27}$$

This makes a direct link between the Frobenius operators

$$F_k : H^{\mathsf{d}}(X, \underline{\mathcal{W}_k \mathcal{O}_{\mathbb{X}}}) \to H^{\mathsf{d}}(X, \mathcal{O}_{\mathbb{X}})$$

and the coefficients of the logarithm for a formal group law of the Artin-Mazur formal group $H^{\mathsf{d}}(X, \hat{\mathbb{G}}_{m,\mathbb{X}})$.

Recall the following special case of (5.14):

$$\nabla \Phi_k H^{\mathsf{d}}(\mathbb{X}, \underline{\mathcal{W}_k \mathcal{O}_{\mathbb{X}}}) \ = \ 0.$$

This means here concretely

$$\nabla(a_k \overline{\gamma}) \equiv 0 \mod k.$$

Let $\omega \in H^0(\mathbb{X}, \Omega^{\mathsf{d}}_{\mathbb{X}/\mathbb{S}})$ be such that

$$\langle \omega, \overline{\gamma} \rangle = 1.$$

The Gauss-Manin connection ∇ on $H^{\mathsf{d}}_{DR}(\mathbb{X}/\mathbb{S})$ induces an action of the ring of differential operators $\mathrm{Diff}(\mathbb{S}/\mathbb{Z})$ (or rather of the subring generated by the derivations). From the above discussion we derive the following result which is a special case of [13] thm. 4.6:

Proposition 6.12 *With the above hypotheses and notations one has: if the differential operator* $L \in \mathrm{Diff}(\mathbb{S}/\mathbb{Z})$ *is such that*

$$L\omega = 0,$$

then

$$La_k \equiv 0 \bmod k\mathbb{A}.$$

□

Example 6.13 (continuation of Example 6.11) *For an appropriate choice of the nowhere vanishing holomorphic* 3*-form* ω *the* Picard-Fuchs operator*, i.e. the differential operator annihilating* ω*, is*

$$L = \theta^4 - 5^5 x^5 (\theta+1)(\theta+2)(\theta+3)(\theta+4) \qquad \textit{with} \quad \theta = x\frac{d}{dx}.$$

One easily checks that for

$$a_k = \sum_{j \geq 0} (-1)^j \frac{(5j)!}{(j!)^5} \begin{pmatrix} k-1 \\ 5j \end{pmatrix} x^{5j} \tag{6.28}$$

as in Example 6.11

$$La_k \equiv 0 \bmod k\mathbb{Z}[x] \tag{6.29}$$

On the other hand one has the power series

$$f(x) = \sum_{j \geq 0} \frac{(5j)!}{(j!)^5} x^{5j} \tag{6.30}$$

which satisfies

$$Lf = 0. \tag{6.31}$$

The striking resemblance between (6.28) and (6.30) and between (6.29) and (6.31) holds in great generality and shows where and how the "holomorphic solution of the Picard-Fuchs equation near the large complex structure limit" shows up in our "ordinary limit".

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