A CONSTRAINED NEVANLINNA-PICK INTERPOLATION PROBLEM FOR MATRIX-VALUED FUNCTIONS

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Abstract. Recent results of Davidson-Paulsen-Raghupathi-Singh give necessary and sufficient conditions for the existence of a solution to the Nevanlinna-Pick interpolation problem on the unit disk with the additional restriction that the interpolant should have the value of its derivative at the origin equal to zero. This concrete mild generalization of the classical problem is prototypical of a number of other generalized Nevanlinna-Pick interpolation problems which have appeared in the literature (for example, on a finitely-connected planar domain or on the polydisk). We extend the results of Davidson-Paulsen-Raghupathi-Singh to the setting where the interpolant is allowed to be matrix-valued and elaborate further on the analogy with the theory of Nevanlinna-Pick interpolation on a finitely-connected planar domain.

1. Introduction

The classical Nevanlinna-Pick interpolation problem [19, 17] has a data set

$$\mathfrak{D} : (z_1, w_1), \dots, (z_n, w_n) \tag{1.1}$$

where $(z_1, \dots, z_n)$ is an n-tuple of distinct points in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C}\colon |\lambda| < 1\}$ and $(w_1, \dots, w_n)$ is a collection of complex numbers in $\mathbb{C}$, and asks for the existence of a *Schur-class function* s, i.e., a holomorphic function s mapping the open unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$, such that s satisfies the set of interpolation conditions associated with the data set $\mathfrak{D}$:

$$s(z_j) = w_j \quad \text{for} \quad j = 1, \dots, n. \tag{1.2}$$

The well-known existence criterion [19] is the following: *there exists a Schur-class function s satisfying the interpolation conditions* (1.2) *if and only if the associated Pick matrix given by*

$$\mathbb{P} = \left[\frac{1 - w_i\overline{w}_j}{1 - z_i\overline{z}_j}\right]_{i,j=1,\dots,n} \tag{1.3}$$

is positive-semidefinite. In the recent paper [7], Davidson-Paulsen-Raghupathi-Singh considered a variant of the classical problem, where the Schur class $\mathcal{S}$ is replaced by the constrained Schur-class $\mathcal{S}_1$ consisting of Schur-class functions s satisfying the additional constraint $s'(0) = 0$. It is readily checked that $\mathcal{S}_1$ is the unit ball of the Banach algebra $H_1^\infty = \{f \in H^\infty(\mathbb{D}) :\ f'(0) = 0\}$. The constrained Nevanlinna-Pick interpolation problem considered in [7] then is:

CNP: *Find a function $s \in \mathcal{S}_1$ satisfying interpolation conditions* (1.2).

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In case $z_j = 0$ for some j, one can treat the problem as a standard Carathéodory-Fejér interpolation problem with interpolation condition on the derivative at 0:

$$s(z_j) = w_j \quad \text{for} \quad j = 1, \dots, n, \quad s'(0) = 0 \tag{1.4}$$

and the existence criterion has the same form: solutions exist if and only if a slightly modified Pick matrix $\mathbb{P}$ is positive-semidefinite. If no z_j is equal to 0, the solvability criterion is given in [7] as follows. With a pair (α, β) of complex numbers we associate the positive kernel on $\mathbb{D} \times \mathbb{D}$

$$K^{\alpha,\beta}(z, w) = (\alpha + \beta z)\overline{(\alpha + \beta w)} + \frac{z^2 \overline{w^2}}{1 - z\overline{w}}. \tag{1.5}$$

Theorem 1.1. *The problem* **CNP** *has a solution if and only if the matrix*

$$\left[(1 - w_i\overline{w_j})K^{\alpha,\beta}(z_i, z_j)\right]_{i,j=1,\dots,n} \tag{1.6}$$

is positive-semidefinite for each choice of $(\alpha, \beta) \in \mathbb{C}^2$ *satisfying* $|\alpha|^2 + |\beta|^2 = 1$.

In practice in Theorem 1.1 it suffices to work with pairs for which $\alpha \neq 0$. The paper [7] gives a second solution criterion for the problem **CNP**.

Theorem 1.2. *The problem* **CNP** *has a solution if and only if there exists* $\lambda \in \mathbb{D}$ *such that the matrix*

$$\left[\frac{z_i^2\overline{z}_j^2 - \varphi_\lambda(w_i)\overline{\varphi_\lambda(w_j)}}{1 - z_i\overline{z_j}}\right]_{i,j=1,\dots,n} \tag{1.7}$$

is positive-semidefinite, where $\varphi_\lambda(z) = \dfrac{z - \lambda}{1 - \overline{\lambda}z}$.

As pointed out in [7], the criteria (1.6) and (1.7) are complementary in the following sense. Using (1.6) to check that a solution exists appears not to be practical since one must check the positive-semidefiniteness of infinitely many matrices; on the other hand, to check that a solution does not exists is a finite test (if one is lucky): exhibit one admissible parameter (α, β) such that the associated Pick matrix $(1 - w_i\overline{w_j})K^{\alpha,\beta}(z_i, z_j)$ is not positive-semidefinite. The situation with the criterion (1.7) is the reverse. To check that a solution exists using criterion (1.7) is a finite test: one need only be lucky enough to find a single λ for which the associated matrix (1.7) is positive-semidefinite. On the other hand, the check via (1.7) that a solution does not exist requires checking the lack of positive-semidefiniteness for an infinite family of Pick matrices.

The paper [7] also considers the matrix-valued version of the constrained Nevanlinna-Pick problem. Here one specifies a data set

$$\mathfrak{D} : (z_1, W_1), \dots, (z_n, W_n) \tag{1.8}$$

where $z_1, \dots, z_n$ are distinct nonzero points in $\mathbb{D}$ as before, but where now W_1, $\dots$, W_n are $k \times k$ complex matrices. We let $(\mathcal{S}_1)^{k \times k}$ be the $k \times k$ matrix-valued constrained Schur-class

$$(\mathcal{S}_1)^{k\times k} = \{S \in (H^\infty(\mathbb{D}))^{k\times k} \colon \|S\|_\infty \le 1 \text{ and } S'(0) = 0\}.$$

Then the $k \times k$ matrix-valued constrained Nevanlinna-Pick problem is:

MCNP: *Find* $S \in (\mathcal{S}_1)^{k\times k}$ *satisfying interpolation conditions*

$$S(z_j) = W_j \quad \textit{for} \quad j = 1, \dots, n. \tag{1.9}$$

It is not difficult to show that a *necessary* condition for the existence of a solution to the problem **MCNP** is that the matrix

$$\left[(I_k - W_i W_j^*) K^{\alpha,\beta}(z_i, z_j)\right]_{i,j=1,\dots,n} \tag{1.10}$$

be positive-semidefinite for all pairs of complex numbers (α, β) with $|\alpha|^2 + |\beta|^2 = 1$. The main result in [7] on the problem **MCNP** is that this condition in general is not sufficient: *there exist three distinct nonzero points z_1, z_2, z_3 in $\mathbb{D}$ together with three $k \times k$ matrices W_1, W_2, W_3 for some integer $k > 1$ so that the 3×3 block matrix $[(I_k - W_i W_j^*) K^{\alpha,\beta}(z_i, z_j)]$ is positive semidefinite for all choices of (α, β) with $|\alpha|^2 + |\beta|^2 = 1$ yet there is no function $S \in (\mathcal{S}_1)^{k\times k}$ so that $S(z_i) = W_i$ for $i = 1, 2, 3$.* One of the main results of the present paper is that nevertheless Theorem 1.1 can be recovered for the matrix-valued case if one enriches the collection $\left[(I - W_i W_j^*) K^{\alpha,\beta}(z_i, z_j)\right]$ of Pick matrices to be checked for positive-semidefiniteness.

To state our results, we introduce some notation. For ℓ, ℓ' a pair of integers satisfying $1 \le \ell \le \ell' \le k$, we let $\mathbb{G}(\ell' \times \ell)$ be the set of pairs (α, β) of $\ell' \times \ell$ matrices subject to the constraints that $\alpha\alpha^* + \beta\beta^* = I_{\ell'}$ and that α be injective. If two elements (α, β) and (α', β') of $\mathbb{G}(\ell' \times \ell)$ are related by $\alpha' = U\alpha$, $\beta' = U\beta$ for a $\ell' \times \ell'$ unitary matrix U, then the kernel functions $K^{\alpha,\beta}$ and $K^{\alpha',\beta'}$ (defined as in (1.11) below) are the same. Thus what is of interest are the *equivalence classes* of elements of $\mathbb{G}(\ell' \times \ell)$ induced by this equivalence relation $(\alpha, \beta) \equiv (\alpha', \beta')$. If we drop the restriction that α is injective, then such equivalence classes are in one-to-one correspondence with the *Grassmannian* (as suggested vaguely by our notation $\mathbb{G}(\ell' \times \ell)$) of ℓ'-dimensional subspaces of 2ℓ-dimensional space; with the injectivity restriction on α in place, $\mathbb{G}(\ell' \times \ell)$ still corresponds to a dense subset of this Grassmannian. This is the canonical generalization of the analysis in [7] where the special case $\ell' = \ell = 1$ appears in the same context.

Given any $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$, we define the $\ell \times \ell$-matrix kernel function $K^{\alpha,\beta}(z, w)$ on $\mathbb{D} \times \mathbb{D}$ with values in $\mathbb{C}^{\ell\times\ell}$ by

$$K^{\alpha,\beta}(z, w) = (\alpha^* + \overline{w}\beta^*)(\alpha + z\beta) + \frac{\overline{w^2} z^2}{1 - \overline{w} z} I_\ell. \tag{1.11}$$

Then we have the following result.

Theorem 1.3. *The problem* **MCNP** *has a solution if and only if for each $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$ and for all n-tuples $(X_1, \dots, X_n)$ of $k \times \ell$ matrices $X_1, \dots, X_n$ where $1 \le \ell \le \ell' \le k$, it is the case that*

$$\sum_{i,j=1}^{n} \operatorname{trace}\left[X_j K^{\alpha,\beta}(z_i, z_j) X_i^* - W_j^* X_j K^{\alpha,\beta}(z_i, z_j) X_i^* W_i\right] \ge 0. \tag{1.12}$$

In case $k = 1$, we have $\ell' = \ell = 1$ and then $\mathbb{G}(1 \times 1)$ consist of pairs of complex numbers (α, β) subject to $|\alpha|^2 + |\beta|^2 = 1$ and the associated kernel $K^{\alpha,\beta}$ is as in (1.5). Then the quantities $X_1, \dots, X_n$ are just complex numbers and it is straightforward to see that the condition in Theorem 1.3 collapses to the positive-semidefiniteness of the single matrix in (1.6) for each $(\alpha, \beta) \in \mathbb{G}(1 \times 1)$. In this way we see that Theorem 1.3 contains Theorem 1.1 as a corollary. For the general case of Theorem 1.3, we see that restriction of Theorem 1.3 to the case where α, β are scalar $k \times k$ matrices leads to the necessity of the positive-semidefiniteness of the matrix (1.10).

We also have an extension of Theorem 1.2 to the matrix-valued setting. Given a data set $\mathfrak{D}$ as in (1.8) with no z_i equal to 0, the Pick matrix associated with the standard matrix-valued Nevanlinna-Pick problem for this data set is

$$\mathbb{P} = \left[\frac{I_k - W_i W_j^*}{1 - z_i \overline{z}_j} \right]_{i,j=1,\dots n}. \tag{1.13}$$

We define auxiliary matrices

$$Z = \begin{bmatrix} z_1 I_k & & \\ & \ddots & \\ & & z_n I_k \end{bmatrix}, \quad E = \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix}, \quad W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix}. \tag{1.14}$$

For X a free-parameter $k \times k$ matrix, known results on matrix-valued Carathéodory-Fejér interpolation (see e.g. [4]) show that the Pick matrix for the Carathéodory-Fejér interpolation problem of finding $S \in (\mathcal{S})^{k \times k}$ satisfying the extended set of interpolation conditions

$$S(z_j) = W_j \quad \text{for} \quad j = 1, \dots, n \quad \text{and} \quad S(0) = X, \;\; S'(0) = 0$$

is given by

$$\mathbb{P}'_X = \begin{bmatrix} \mathbb{P} & E - WX^* & Z(E - WX^*) \\ E^* - XW^* & I_k - XX^* & 0 \\ (E^* - XW^*)Z^* & 0 & I_k - XX^* \end{bmatrix}. \tag{1.15}$$

A standard Schur-complement manipulation shows that the Pick matrix $\mathbb{P}'_X$ being positive-semidefinite is equivalent to the positive-semidefiniteness of the matrix

$$\mathbb{P}_X = \begin{bmatrix} \mathbb{P} & E - WX^* & Z(E - WX^*) & 0 & 0 \\ E^* - XW^* & I_k & 0 & X & 0 \\ (E^* - XW^*)Z^* & 0 & I_k & 0 & X \\ 0 & X^* & 0 & I_k & 0 \\ 0 & 0 & X^* & 0 & I_k \end{bmatrix}. \tag{1.16}$$

We are thus led to the following result.

Theorem 1.4. *The problem* **MCNP** *has a solution if and only if there exists a $k \times k$ matrix X so that the matrix $\mathbb{P}_X$ given by* (1.16) *is positive-semidefinite.*

For the case $k = 1$ the criterion in Theorem 1.4, while of the same flavor, is somewhat different from the criterion in Theorem 1.2 since the free parameter $X = [x]$ in the matrix $\mathbb{P}_x$ appears linearly in (1.16) while the free parameter λ in the matrix $\mathbb{P}_\lambda$ in (1.7) appears in a linear-fractional form in the matrix entries. There is an advantage in the linear representation (1.16) over the fractional representation (1.7) (or the quadratic in (1.15)); specifically, the search for a parameter X which makes the matrix $\mathbb{P}_X$ positive-semidefinite is a problem of a standard type called a Linear Matrix Inequality (LMI) for which efficient numerical algorithms are available for determining if a solution exists (see [16]). However in Section 4 we also give an analytic treatment concerning existence and parametrization of solutions for the LMIs occurring here; only in some special cases does one get a clean positivity test for existence of solutions and, when solutions exist, a complete linear-fractional description for the set of all solutions. The type of LMIs occurring here are similar to the *structured LMIs* arising in the robust control theory for control systems subject to structured balls of uncertainty with a so-called linear-fractional-transformation model—see [9].

It is not immediately clear how to convert the criterion in Theorem 1.3 to an LMI; what is true is that if one is lucky enough to find a particular $(\alpha, \beta) \in \mathbb{G}(1 \times 1)$

along with a tuple $X_1, \dots, X_n$ of $k \times \ell$ matrices for which the matrix (1.12) is not positive-semidefinite, then it necessarily follows that the matrix-valued constrained Nevanlinna-Pick interpolation problem does not have a solution.

We remark that Theorems 1.1 and 1.3 can be seen as parallel to the situation for Nevanlinna-Pick interpolation over a finitely-connected planar domain (see [1, 3]). For $\mathcal{R}$ a bounded domain in the complex plane bounded by $g+1$ disjoint analytic Jordan curves, Abrahamse [1] identified a family of reproducing kernel Hilbert spaces $H^2_{\mathbf{u}}(\mathcal{R})$ indexed by points $\mathbf{u}$ in the g-torus $\mathbb{T}^g$ with corresponding kernels $K_{\mathbf{u}}(z,w)$. His result is that, given n distinct points $z_1, \dots, z_n$ in $\mathcal{R}$ and n complex values $w_1, \dots, w_n$, there exists a holomorphic function f mapping $\mathcal{R}$ into the closed unit disk $\overline{\mathbb{D}}$ which also satisfies the interpolation conditions $f(z_i) = w_i$ for $1 \le i \le n$ if and only if the $n \times n$ matrix $[(1 - w_i \overline{w_j}) K_{\mathbf{u}}(z_i, z_j)]_{i,j=1,\dots,n}$ is positive semidefinite for all $\mathbf{u} \in \mathbb{T}^g$. The first author [3] obtained a commutant lifting theorem for the finitely-connected planar-domain setting which, when specified to a matrix-valued interpolation problem over $\mathcal{R}$, yields the following result.

Theorem 1.5. *For each $\ell = 1, 2, \dots$ there exists a family of $\ell \times \ell$ matrix-valued kernels $K_{\mathbf{U}}(z,w)$ on $\mathcal{R}$ indexed by g-tuples $\mathbf{U} = (U_1, \dots, U_g) \in \mathcal{U}(\ell)^g$ of $\ell \times \ell$ unitary matrices so that the following holds: given distinct points $z_1, \dots, z_n \in \mathcal{R}$ and $k \times k$ matrices $W_1, \dots, W_n$, there exists a holomorphic function F on $\mathcal{R}$ with values in the closed unit ball of $k \times k$ matrices which satisfies the interpolation conditions*

$$F(z_i) = W_i \quad \text{for} \quad i = 1, \dots, n \tag{1.17}$$

if and only if, for any choice of $\mathbf{U} \in \mathcal{U}(\ell)^g$ and n-tuple $X_1, \dots, X_n$ of $k \times \ell$ matrices for $1 \le \ell \le k$, it holds that

$$\sum_{i,j=1,\dots,n} \operatorname{trace}\left[X_j K_{\mathbf{U}}(z_i, z_j) X_i^* - W_j^* X_j K_{\mathbf{U}}(z_i, z_j) X_i^* W_i\right] \ge 0. \tag{1.18}$$

We conclude that Theorem 1.1 can be viewed as an analogue of Abrahamse's result [1] while Theorem 1.3 can be viewed as the parallel analogue of the matrix-valued extension of Abrahamse's result Theorem 1.5.

In the sequel we actually formulate and prove a more general version of Theorem 1.4, where the algebra H^∞_1 is replaced by an algebras of the form $H^\infty_B := \mathbb{C} + BH^\infty$ with B a finite Blaschke product. The algebra H^∞_1 is recovered as the special case where $B(z) = z^2$. Particular examples of the case where each zero of $B(z)$ has simple multiplicity were studied in [25] and other results in this direction are given in [22]. Additional motivation for the study of algebras like H^∞_B comes from the theory of algebraic curves: Agler-McCarthy [2] have shown that the algebra H^∞_1 is isometrically isomorphic to the algebra $H^\infty(V)$ of bounded analytic functions on the variety $\{(z^2, z^3) \colon z \in \mathbb{D}\}$; we expect that algebras of the type H^∞_B will play a similar role for more general varieties. We expect that Theorem 1.3 can also be extended to the setting where the algebra H^∞_1 is replaced by H^∞_B; we have chosen to restrict ourselves to the case $B(z) = z^2$ to keep the notation simple and explicit.

The paper is organized as follows. In Section 2 we prove Theorem 1.3. As a corollary we obtain a distance formula for the subspace $\mathcal{I}_{\mathfrak{D}}$ of elements in $(H^\infty_1)^{k \times k}$ satisfying the associated set of homogeneous interpolation conditions; this extends another scalar-valued result from [7]. We also obtain a Beurling-Lax theorem for this setting and provide some detail on how the lifting theorem from [3] leads to Theorem 1.5. In Section 3 we prove our extension of Theorem 1.4. In Section 4 we provide a further analysis of the LMIs which arise here. We use this analysis in Section 5 to discuss the geometry of the set of values $w = s(z_0)$ (where z_0 is a nonzero point in the disk not equal to one of the interpolation nodes $z_1, \dots, z_n$) associated with the set of all constrained Schur-class solutions of a constrained Nevanlinna-Pick interpolation problem.

2. Proof of Theorem 1.3

2.1. **Preliminaries.** We review some preliminaries and notation. $\mathcal{C}^{2,k\times\ell}$ denotes the space of $k \times \ell$ complex matrices considered as a Hilbert space with the inner product

$$\langle X, Y\rangle_{\mathcal{C}^{2,k\times\ell}} = \text{trace}[Y^* X].$$

Similarly, $(L^2)^{k\times\ell}$ is the space of $k\times\ell$ matrices with entries equal to L^2 functions on the circle $\mathbb{T} = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$ considered as a Hilbert space with the inner product

$$\langle F, G\rangle_{(L^2)^{k\times\ell}} = \frac{1}{2\pi}\int_{\mathbb{T}} \text{trace}[G(z)^* F(z)]\,|dz|.$$

Related spaces are the Banach space $(L^1)^{k\times\ell}$ of $k\times\ell$ matrices with entries equal to L^1 functions on $\mathbb{T}$ with norm given by

$$\|F\|_{(L^1)^{k\times\ell}} = \frac{1}{2\pi}\int_{\mathbb{T}} \text{trace}[(F(z)^* F(z))^{1/2}]\,|dz|,$$

and the Banach space $(L^\infty)^{k\times\ell}$ of $k\times\ell$ matrices with entries equal to L^∞ functions on $\mathbb{T}$ with the supremum norm $\|\;\cdot\|_\infty$. We will also use the Hilbert space $(H^2)^{k\times\ell}$ and the Banach spaces $(H^1)^{k\times\ell}$ and $(H^\infty)^{k\times\ell}$ which are the restrictions of $(L^2)^{k\times\ell}$, $(L^1)^{k\times\ell}$ and $(L^\infty)^{k\times\ell}$, respectively, to the matrices whose entries are analytic functions.

The Riesz representation theorem for this setting identifies the dual of $(L^1)^{k\times\ell}$ as the space $(L^\infty)^{\ell\times k}$ via the pairing

$$[F, G]_{(L^\infty)^{\ell\times k}\times(L^1)^{k\times\ell}} = \frac{1}{2\pi}\int_{\mathbb{T}} \text{trace}[F(z)G(z)]\,|dz|. \tag{2.1}$$

Moreover, the pre-annihilator of $(H^\infty)^{k\times\ell}$ under this pairing is given by

$$((H^\infty)^{k\times\ell})_\perp = z\cdot(H^1)^{\ell\times k}. \tag{2.2}$$

We shall make use of the following matrix-valued analogue of the Riesz factorization theorem.

Theorem 2.1. *Any $h \in (H^1)^{k\times k}$ can by factored as*

$$h = gf_0$$

where $g \in (H^2)^{k\times\ell}$ is such that its transpose $g^\top$ is outer, $f_0 \in (H^2)^{\ell\times k}$, and

$$\|f_0\|_2^{1/2} = \|g\|_2^{1/2} = \|h\|_1.$$

Proof. It is well known (see [15]) that $h^\top$ has an inner-outer factorization $h^\top = h_i^\top h_o^\top$ for a $k\times\ell$ inner function h_i and an outer function $h_o^\top$ in $(H^1)^{\ell\times k}$. We next factor h_o as

$$h_o = g\widetilde{f}_0$$

where $g \in (H^2)^{k\times\ell}$, $\widetilde{f}_0 \in (H^2)^{\ell\times k}$ and

$$\widetilde{f}_0^*\widetilde{f}_0 = (h_o^* h_o)^{1/2}, \quad g^* g = \widetilde{f}_0\widetilde{f}_0^* \text{ on } \mathbb{T}$$

(see [23, Theorem 4] where the operator-valued version is done). Then $h = gf_0$ with $f_0 = \widetilde{f}_0 h_i$ has the requisite properties. $\square$

2.2. **Representation spaces for $(H_1^\infty)^{k\times k}$ and proof of necessity.** We fix a positive integer k. Let $1 \le \ell \le \ell' \le k$. We define $\mathbb{G}(\ell' \times \ell)$ as in the introduction:

$$\mathbb{G}(\ell' \times \ell) = \{(\alpha, \beta)\colon \alpha, \beta \in \mathbb{C}^{\ell' \times \ell}, \ \ \alpha\alpha^* + \beta\beta^* = I_{\ell'} \text{ and } \ker \alpha = \{0\}\}. \tag{2.3}$$

For $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$ let $H^2_{\alpha,\beta}$ be the subspace

$$H^2_{\alpha,\beta} = \{F \in (H^2)^{k\times \ell}\colon \text{ for some } U \in \mathbb{C}^{k\times \ell'}, \, F(0) = U\alpha \text{ and } F'(0) = U\beta\} \tag{2.4}$$

of $(H^2)^{k\times\ell}$. Then $H^2_{\alpha,\beta}$ is a Hilbert space in its own right with norm taken to be the restriction of the norm of $(H^2)^{k\times\ell}$. These spaces turn out to be models for representations of the algebra $(H_1^\infty)^{k\times k}$.

Proposition 2.2. *Let $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$. If $S \in (H_1^\infty)^{k\times k}$ and $f \in H^2_{\alpha,\beta}$, then also $S \cdot f \in H^2_{\alpha,\beta}$. Moreover, the multiplication operator M_S of S on $H^2_{\alpha,\beta}$ satisfies*

$$\|M_S\|_{op} = \|S\|_\infty. \tag{2.5}$$

Proof. If $S(z) = S_0 + z^2\widetilde{S}(z) \in (H_1^\infty)^{k\times k}$ with $\widetilde{S} \in (H^\infty)^{k\times k}$, then

$$S_0 + z^2\widetilde{S}(z))(U\alpha + zU\beta + O(z^2)) = (S_0U)\alpha + z(S_0U)\beta + O(z^2) \in H^2_{\alpha,\beta}$$

for any $U\alpha + zU\beta + O(z^2) \in H^2_{\alpha,\beta}$.

We temporarily write $\widetilde{M_S}$ for the multiplication operator of S on $(H^2)^{k\times\ell}$. For $\ell = 1$ it is well known that $\widetilde{M_S}$ satisfies $\|\widetilde{M_S}\|_{op} = \|S\|_\infty$ (see e.g. [15]), and it is not difficult to see that this equality holds for the case $\ell \neq 1$ as well. We write M_S for the multiplication operator of S as an operator on $H^2_{\alpha,\beta}$. Since the norm of $H^2_{\alpha,\beta}$ is just the restriction of the norm of $(H^2)^{k\times\ell}$, the inequality $\|M_S\|_{op} \le \|S\|_\infty$ is immediate. Choose $f_n \in (H^2)^{k\times\ell}$ of norm 1 so that $\|\widetilde{M_S} f_n\| \to \|S\|_\infty$. Then the functions $g_n(z) := z^2 f_n(z)$ belong to $H^2_{\alpha,\beta}$, and $\|M_S g_n\|_{H^2_{\alpha,\beta}} = \|\widetilde{M_S} f_n\|_{(H^2)^{k\times\ell}} \to \|S\|_\infty$, which proves that $\|M_S\|_{op} = \|S\|_\infty$. $\square$

Remark 2.3. In case α is invertible, it can be shown that $(H_1^\infty)^{k\times k}$ is *exactly* the left multiplier space for $H^2_{\alpha,\beta}$ (see [7, Proposition 3.1] for the case $k = 1$). In the proof of Theorem 1.3 to follow, what comes up is the case α injective. As the reproducing kernels $K^{\alpha,\beta}$ in (1.11) can be approximated by reproducing kernels $K^{\alpha',\beta'}$ with α' injective, Theorem 1.3 remains valid if one restricts to (α, β) with α injective.

Proposition 2.4. *For $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$, the space $H^2_{\alpha,\beta}$ is a reproducing kernel Hilbert space with reproducing kernel $K^{\alpha,\beta}$ given by* (1.11) *and having the reproducing property*

$$\langle f(w), X\rangle_{\mathcal{C}^{2,k\times\ell}} = \langle f, XK^{\alpha,\beta}(\cdot, w)\rangle_{H^2_{\alpha,\beta}}. \tag{2.6}$$

Moreover, for $F \in (H_1^\infty)^{k\times k}$, the operator $M_F\colon f \mapsto F \cdot f$ of multiplication on the left by F takes $H^2_{\alpha,\beta}$ into itself and the action of M_F^* on kernel elements $XK^{\ell,(\alpha,\beta)}(w, \cdot)$ is given by

$$M_F^* XK^{\alpha,\beta}(\cdot, w) = F(w)^* XK^{\alpha,\beta}(\cdot, w). \tag{2.7}$$

Proof. Let $f(z) = U\alpha + zU\beta + z^2\widetilde{f}(z)$ be a generic element of $H^2_{\alpha,\beta}$. By direct computation we have, for $X \in \mathbb{C}^{k\times\ell}$,

$$\begin{aligned}\langle f, XK^{\alpha,\beta}(\cdot,w)\rangle_{H^2_{\alpha,\beta}} &= \langle U\alpha + zU\beta + z^2\widetilde{f}(z), XK^{\alpha,\beta}(\cdot,w)\rangle_{H^2_{\alpha,\beta}}\\ &= \langle U\alpha + zU\beta, X(\alpha^* + \overline{w}\beta^*)(\alpha + z\beta)\rangle_{H^2_{\alpha,\beta}} + \left\langle z^2\widetilde{f}(z), X\frac{\overline{w}^2 z^2}{1-\overline{w}z}\right\rangle_{H^2_{\alpha,\beta}}\\ &= \operatorname{trace}\left[(\alpha + w\beta)X^*U(\alpha\alpha^* + \beta\beta^*)\right] + \langle w^2\widetilde{f}(w), X\rangle_{\mathcal{C}^{2,k\times\ell}}\\ &= \langle U(\alpha + w\beta), X\rangle_{\mathcal{C}^{2,k\times\ell}} + \langle w^2\widetilde{f}(w), X\rangle_{\mathcal{C}^{2,k\times\ell}}\\ &= \langle f(w), X\rangle_{\mathcal{C}^{2,k\times\ell}}\end{aligned}$$

in agreement with (2.6) as wanted. We next compute, for $f \in H^2_{\alpha,\beta}$, $F \in (H^\infty_1)^{k\times k}$ and $X \in \mathbb{C}^{k\times\ell}$,

$$\begin{aligned}\langle f, M_F^* XK^{\alpha,\beta}(\cdot,w)\rangle_{H^2_{\alpha,\beta}} &= \langle M_F f, XK^{\alpha,\beta}(\cdot,w)\rangle_{H^2_{\alpha,\beta}}\\ &= \langle F(w)f(w), X\rangle_{\mathcal{C}^{2,k\times\ell}}\\ &= \langle f(w), F(w)^*X\rangle_{\mathcal{C}^{2,k\times\ell}}\\ &= \langle f, F(w)^*XK^{\alpha,\beta}(\cdot,w)\rangle_{H^2_{\alpha,\beta}}\end{aligned}$$

and thus (2.7) follows as well. □

Proof of necessity in Theorem 1.3. Suppose that there exists a function S in the constrained Schur class $(\mathcal{S}_1)^{k\times k}$ satisfying interpolation conditions (1.9) for given n distinct nonzero points $z_1,\dots,z_n$ in $\mathbb{D}$ and $k\times k$ matrices $W_1,\dots,W_n$. Given $(\alpha,\beta) \in \mathbb{G}(\ell'\times\ell)$ for some ℓ',ℓ $(1 \le \ell \le \ell' \le k)$, define a subspace $\mathcal{M}$ of $H^2_{\alpha,\beta}$ by

$$\mathcal{M} = \overline{\operatorname{span}}\left\{XK^{\alpha,\beta}(\cdot,z_j)\colon X \in \mathcal{C}^{2,k\times\ell},\ j = 1,\dots,n\right\}.$$

From (2.7) we see that $\mathcal{M}$ is invariant under M_S^*. From the assumption that $\|S\|_\infty \le 1$, we know by (2.5) that $\|M_F^*|_{\mathcal{M}}\| \le 1$. Hence, for a generic element

$$f = \sum_{j=1}^n X_j K^{\alpha,\beta}(\cdot,z_j) \in \mathcal{M}, \tag{2.8}$$

necessarily

$$\|f\|^2 - \|M_S^* f\|^2 \ge 0.$$

Substituting (2.8) for f into the latter inequality, expanding out inner products and using the reproducing property (2.6) and the equality (2.7) then leaves us with (1.12). □

2.3. **Proof of sufficiency.** The proof will follow the duality proof as in [23, 7] using the adaptations for the matrix-valued case as done in [3] in a different context. A key ideal in the algebra $(H^\infty_1)^{k\times k}$ is the set of all functions $F \in (H^\infty)^{k\times k}$ which satisfy the homogeneous interpolation conditions associated with the data set $\mathfrak{D}$:

$$\mathcal{I}_{\mathfrak{D}} := \{F \in (H^\infty_1)^{k\times k}\colon F(z_i) = 0 \ \text{ for } \ i = 1,\dots,n\}. \tag{2.9}$$

The first step is to compute the pre-annihilator of $\mathcal{I}_{\mathfrak{D}}$. To this end we introduce the dual version of $\mathbb{G}(\ell'\times\ell)$:

$$\mathbb{G}(\ell'\times\ell)^* = \{(a,b)\colon a,b \in \mathbb{C}^{\ell\times\ell'},\ a \text{ onto},\ a^*a + b^*b = I_{\ell'}\}.$$

For $(a,b) \in \mathbb{G}(\ell' \times \ell)^*$ we then define associated spaces

$$\begin{aligned} H^1_{r,(a,b)} &= \{h \in (H^1)^{\ell \times k} \colon \text{for some } U \in \mathbb{C}^{\ell \times k},\ h(0) = aU \text{ and } h'(0) = bU\}, \\ H^\infty_{r,(a,b)} &= \{F \in (H^\infty)^{\ell \times k} \colon \text{for some } U \in \mathbb{C}^{\ell \times k},\ F(0) = aU \text{ and } F'(0) = bU\}, \\ H^2_{r,(a,b)} &= \{f \in (H^2)^{\ell \times k} \colon \text{for some } U \in \mathbb{C}^{\ell \times k},\ f(0) = aU \text{ and } f'(0) = bU\}. \end{aligned}$$

Proposition 2.5. *There exists a pair $(a,b) \in \mathbb{G}(k \times k)^*$ so that the subspace $\mathcal{I}_{\mathfrak{D}}$ given by (2.9) can be expressed as*

$$\mathcal{I}_{\mathfrak{D}} = B_{\mathfrak{D}} H^\infty_{r,(a,b)} \quad \textit{where} \quad B_{\mathfrak{D}}(z) = \prod_{k=1}^n \frac{z - z_k}{1 - \overline{z_k} z} I_k$$

is the scalar Blaschke product with zeros equal to the interpolation nodes $z_1, \dots, z_n$ given by the data set (1.8).

Proof. By definition (2.9) we have $\mathcal{I}_{\mathfrak{D}} = (H^\infty_1)^{k \times k} \cap B_{\mathfrak{D}} (H^\infty)^{k \times k}$. Then $F \in \mathcal{I}_{\mathfrak{D}}$ means that F has the form $F = B_{\mathfrak{D}} G$ for a $G \in (H^\infty)^{k \times k}$ with the additional property that

$$F'(0) = B_{\mathfrak{D}}(0) G'(0) + B'_{\mathfrak{D}}(0) G(0) = 0.$$

By assumption no z_i is equal to 0 and hence $B_{\mathfrak{D}}(0)$ is invertible. Thus we can solve for $G'(0)$ in terms of $G(0)$:

$$G'(0) = -B_{\mathfrak{D}}(0)^{-1} B'_{\mathfrak{D}}(0) G(0)$$

or

$$G(0) = \widetilde{a} \widetilde{U}, \quad G'(0) = \widetilde{b} \widetilde{U}$$

where

$$\widetilde{a} = I_k, \quad \widetilde{b} = -B_{\mathfrak{D}}(0)^{-1} B'_{\mathfrak{D}}(0), \quad \widetilde{U} = G(0).$$

If we normalize $(\widetilde{a}, \widetilde{b})$ to (a,b) according to the formula

$$\begin{aligned} a &= \widetilde{a}(I + B'_{\mathfrak{D}}(0)^* B_{\mathfrak{D}}(0)^{*-1} B_{\mathfrak{D}}(0)^{-1} B'_{\mathfrak{D}}(0))^{-1/2}, \\ b &= \widetilde{b}(I + B'_{\mathfrak{D}}(0)^* B_{\mathfrak{D}}(0)^{*-1} B_{\mathfrak{D}}(0)^{-1} B'_{\mathfrak{D}}(0))^{-1/2}, \\ U &= (I + B'_{\mathfrak{D}}(0)^* B_{\mathfrak{D}}(0)^{*-1} B_{\mathfrak{D}}(0)^{-1} B'_{\mathfrak{D}}(0))^{1/2} \widetilde{U}, \end{aligned} \tag{2.10}$$

then $(a,b) \in \mathbb{G}(k \times k)^*$ and

$$G(0) = aU, \quad G'(0) = bU$$

shows that $G \in H^\infty_{r,(a,b)}$. Note that the a and b constructed in (2.10) are independent of the function $F \in \mathcal{I}_{\mathfrak{D}}$, so that this choice of a and b works for any $F \in \mathcal{I}_{\mathfrak{D}}$. □

Remark 2.6. For our setting here $B_{\mathfrak{D}}(z)$ is a scalar Blaschke product so the (a,b) produced by the recipe (2.10) are actually scalar $k \times k$ matrices. The proof of the lemma applies more generally to the setting where $B_{\mathfrak{D}}$ is not scalar, but we still assume that $B_{\mathfrak{D}}(0)$ is invertible. Due to this observation, the analysis here applies equally well should we wish to study left-tangential interpolation problems in the class $(H^\infty_1)^{k \times k}$ rather than just full-matrix-valued interpolation as in (1.9).

We next compute a pre-annihilator of $\mathcal{I}_{\mathfrak{D}}$. Here we use the notation $\ker_\ell X$ and $\operatorname{im}_\ell X$ for a matrix X to indicate the subspace of row vectors arising as the kernel (respectively image) of X when X is considered as an operator with row-vector argument acting on the left, i.e., for $X \in \mathbb{C}^{k\times k'}$,

$$\ker_\ell X = \{v \in \mathbb{C}^{1\times k} : vX = 0\},$$
$$\operatorname{im}_\ell X = \{u \in \mathbb{C}^{1\times k'} : u = vX \text{ for some } v \in \mathbb{C}^{1\times k}\}.$$

Proposition 2.7. *Suppose that $\mathcal{I}_{\mathfrak{D}} = (H_1^\infty)^{k\times k} \cap B_{\mathfrak{D}}(H^\infty)^{k\times k}$ is expressed as $B_{\mathfrak{D}} H^\infty_{r,(a,b)}$ as in Proposition 2.5. Then the pre-annihilator $(\mathcal{I}_{\mathfrak{D}})_\perp$ of $\mathcal{I}_{\mathfrak{D}}$ is given by*

$$(\mathcal{I}_{\mathfrak{D}})_\perp = z^{-1} H^1_{a_\perp, b_\perp} B_{\mathfrak{D}}^* \tag{2.11}$$

where $(a_\perp, b_\perp) \in \mathbb{G}(k\times k)$ is chosen so that

$$\operatorname{im}_\ell \begin{bmatrix} b_\perp & a_\perp \end{bmatrix} = \ker_\ell \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.12}$$

Proof. Let $F(z) = B_{\mathfrak{D}}(z)(aU + zbU + z^2\widetilde{F}(z))$ be a generic element of $\mathcal{I}_{\mathfrak{D}}$ and suppose that $h \in (L^1)^{k\times k}$ is in the pre-annihilator $(\mathcal{I}_{\mathfrak{D}})_\perp$. Then

$$[F,h]_{(L^\infty)^{k\times k}\times(L^1)^{k\times k}} = \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\Big[B_{\mathfrak{D}}(z)(aU + zbU + z^2\widetilde{F}(z))h(z)\Big]\,|dz| = 0$$

for all such F. In particular

$$\frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\Big[B_{\mathfrak{D}}(z)z^2\widetilde{F}(z)h(z)\Big]\,|dz| = 0$$

for all $\widetilde{F} \in (H^\infty)^{k\times k}$ from which we conclude that $z^2 h(z)B_{\mathfrak{D}}(z) \in \big((H^\infty)^{k\times k}\big)_\perp = z(H^1)^{k\times k}$. We may therefore write $h(z) = z^{-1}\widetilde{h}(z)B_{\mathfrak{D}}(z)^*$ where $\widetilde{h}(z) \in (H^1)^{k\times k}$. Write

$$\widetilde{h}(z) = \widetilde{h}(0) + z\widetilde{h}'(0) + z^2\widetilde{\widetilde{h}}(z) \quad \text{with} \quad \widetilde{\widetilde{h}} \in (H^1)^{k\times k}.$$

Then $h \in (\mathcal{I}_{\mathfrak{D}})_\perp$ forces in addition

$$\begin{aligned} 0 &= \Big[B_{\mathfrak{D}}(z)(aU + zbU),\, (z^{-1}\widetilde{h}(0) + \widetilde{h}'(0) + z\widetilde{\widetilde{h}}(z))B_{\mathfrak{D}}(z)^*\Big]_{(L^\infty)^{k\times k}\times(L^1)^{k\times k}} \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\Big[(aU + zbU)(z^{-1}\widetilde{h}(0) + \widetilde{h}'(0))\Big]\,|dz| \\ &= \operatorname{trace}\Big[aU\widetilde{h}'(0) + bU\widetilde{h}(0)\Big] = \operatorname{trace}\Big[U(\widetilde{h}'(0)a + \widetilde{h}(0)b)\Big] \end{aligned}$$

for all $k\times k$ matrices U. As the analysis is reversible, we conclude that $h \in (\mathcal{I}_{\mathfrak{D}})_\perp$ if and only if $h(z) = z^{-1}\widetilde{h}(z)B_{\mathfrak{D}}(z)^*$ where $\widetilde{h}(z) \in (H^1)^{k\times k}$ is such that

$$\begin{bmatrix} \widetilde{h}'(0) & \widetilde{h}(0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

From the definitions, this is just the assertion that $h \in z^{-1}H^1_{a_\perp,b_\perp}$ where $(a_\perp, b_\perp)$ is constructed as in (2.12). It is also not difficult to see that injectivity (and hence invertibility since a is square) of a is then equivalent to surjectivity (and hence also invertibility) of $a_\perp$, i.e., $(a,b) \in \mathbb{G}(k\times k)^*$ is equivalent to $(a_\perp, b_\perp) \in \mathbb{G}(k\times k)$. □

Proposition 2.8. *Suppose that $h \in (H^1)^{k\times k}$ is in $(\mathcal{I}_{\mathfrak{D}})_\perp$. Then there is an $(\alpha,\beta) \in \mathbb{G}(\ell',\ell)$ for some $1 \le \ell \le \ell' \le k$ so that h can be factored in the form*

$$h(z) = g(z)f(z)^* \tag{2.13}$$

where

$$g \in H^2_{\alpha,\beta}, \quad f \in (L^2)^{k\times\ell} \ominus \left(H^2_{\alpha,\beta} \cap B_{\mathfrak{D}}(H^2)^{k\times\ell}\right), \quad \|g\|_2^{1/2} = \|f\|_2^{1/2} = \|h\|_1. \tag{2.14}$$

Proof. By Proposition 2.7 we may write $h = z^{-1}\widetilde{h}B_{\mathfrak{D}}^*$ where $\widetilde{h} \in H^1_{a_\perp,b_\perp}$. As $\widetilde{h}$ is in $(H^1)^{k\times k}$, by Theorem 2.1 we may factor $\widetilde{h}$ as $\widetilde{h} = g \cdot f_0$, where $g \in (H^2)^{k\times\ell}$, $f \in (H^2)^{\ell\times k}$, $g^\top$ outer, and

$$\|g\|_2^{1/2} = \|f_0\|_2^{1/2} = \|h\|_1. \tag{2.15}$$

Combining factorizations for h and $\widetilde{h}$ gives $h = z^{-1}\widetilde{h}B_{\mathfrak{D}}^* = z^{-1}gf_0B_{\mathfrak{D}}^* = gf^*$ where we have set

$$f = zB_{\mathfrak{D}}f_0^*$$

and (2.13) holds with $g \in H^2_{\alpha,\beta}$ and f so constructed. Since $\|f\|_2 = \|f_0\|_2$, the norm equalities in the third part of (2.14) follow from (2.15). Since $g^\top$ is outer, $g(0)$ has trivial kernel. Let

$$\ell' := \operatorname{rank}(g(0)g(0)^* + g'(0)g'(0)^*). \tag{2.16}$$

Assuming that $h \neq 0$, we then have $1 \leq \ell \leq \ell' \leq k$. Then there exists an injective $k \times \ell'$ matrix X which solves the factorization problem

$$g(0)g(0)^* + g'(0)g'(0)^* = XX^*.$$

Define $\ell' \times \ell$ matrices α, β by

$$\alpha = (X^*X)^{-1}X^*g(0), \quad \beta = (X^*X)^{-1}X^*g'(0).$$

Then one can check that α is injective and $\alpha\alpha^* + \beta\beta^* = I_{\ell'}$, i.e., $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$. Moreover, (2.16) implies that $\operatorname{Im}\begin{bmatrix} g(0) & g'(0) \end{bmatrix} = \operatorname{Im} X$, and thus $\begin{bmatrix} g(0) & g'(0) \end{bmatrix} = X\begin{bmatrix} \alpha & \beta \end{bmatrix}$. Hence $g \in H^2_{\alpha,\beta}$.

It remains to verify that $f \in (L^2)^{k\times k} \ominus \left(H^2_{a,b} \cap B_{\mathfrak{D}}(H^2)^{k\times\ell}\right)$. To this end we observe first that $(H_1^\infty)^{k\times k} \cdot g$ is dense in $H^2_{a,b}$. Indeed, the L^2-closure of $(H_1^\infty)^{k\times k} \cdot g$ satisfies

$$[(H_1^\infty)^{k\times k}g]^- = \mathbb{C}^{k\times k}g + [z^2(H^\infty)^{k\times k}g]^- = \mathbb{C}^{k\times k}g + z^2(H^2)^{k\times k} = H^2_{a,b}.$$

The second identity follows because $g^\top$ is outer. Next we observe that

$$[\mathcal{I}_{\mathfrak{D}}g]^- = H^2_{a,b} \cap B_{\mathfrak{D}}(H^2)^{k\times\ell}, \tag{2.17}$$

again since $g^\top$ is outer. Indeed, the containment $\subset$ is clear as the left-hand side is in $H^2_{a,b}$ and vanishes at the points $z_1, \ldots, z_d$. Moreover, evidently both sides of (2.17) have codimension n in $H^2_{a,b}$ and (2.17) follows.

Hence, to check that f is orthogonal to $H^2_{a,b} \cap B_{\mathfrak{D}}(H^2)^{k\times k}$, it suffices to check that f is orthogonal to $\mathcal{I}_{\mathfrak{D}}g$. As $\mathcal{I}_{\mathfrak{D}} = B_{\mathfrak{D}}H^\infty_{r,(a,b)}$ by Proposition 2.5, we conclude that it suffices to check that f is orthogonal to elements of the form $\phi := B_{\mathfrak{D}}Fg$ with $F \in H^\infty_{r,(a,b)}$. We compute

$$\langle \phi, f\rangle = \frac{1}{2\pi}\int_{\mathbb{D}} \operatorname{trace}\left[B_{\mathfrak{D}}Fg \cdot z^{-1}f_0B_{\mathfrak{D}}^*\right] |dz| = \frac{1}{2\pi}\int_{\mathbb{D}} \operatorname{trace}\left[z^{-1}\widetilde{h}F\right] |dz|.$$

If we expand out F and $\widetilde{h}$ as

$$F(z) = F(0) + zF'(0) + z^2\widetilde{F}(z), \quad \widetilde{h}(z) = \widetilde{h}(0) + z\widetilde{h}'(0) + z^2\widetilde{\widetilde{h}}(z),$$

we see that the zeroth Fourier coefficient of $z^{-1}\widetilde{h}(z)F(z)$ is simply

$$[z^{-1}\widetilde{h}(z)F(z)]_0 = \widetilde{h}(0)F'(0) + \widetilde{h}'(0)F(0)$$

and hence

$$\frac{1}{2\pi}\int_{\mathbb{D}} \operatorname{trace}\left[z^{-1}\widetilde{h}F\right]\, |dz| = \operatorname{trace}\left[\widetilde{h}(0)F'(0) + \widetilde{h}'(0)F(0)\right]. \tag{2.18}$$

As $F \in H^\infty_{r,(a,b)}$ and $\widetilde{h} \in H^1_{a_\perp, b_\perp}$, we know that there are matrices U and V so that

$$\begin{bmatrix}\widetilde{h}'(0) & \widetilde{h}(0)\end{bmatrix} = U\begin{bmatrix} b_\perp & a_\perp\end{bmatrix}, \quad \begin{bmatrix} F(0) \\ F'(0)\end{bmatrix} = \begin{bmatrix} a \\ b\end{bmatrix} V.$$

Hence

$$\widetilde{h}(0)F'(0) + \widetilde{h}'(0)F(0) = \begin{bmatrix}\widetilde{h}'(0) & \widetilde{h}(0)\end{bmatrix}\begin{bmatrix} F(0) \\ F'(0)\end{bmatrix} = U\begin{bmatrix} b_\perp & a_\perp\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} V.$$

As the left hand side expression equals zero by (2.12), it follows that (2.18) vanishes as needed. ☐

Proof of sufficiency in Theorem 1.3. We assume that the data set $\mathfrak{D}$ as in (1.8) is such that condition (1.12) is satisfied for all $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$ and n-tuples $X = (X_1, \dots, X_n)$ of $k \times \ell$ matrices $(1 \le \ell \le \ell' \le k)$. By a coordinate-wise application of the theory of Lagrange interpolation, we can find a matrix polynomial $Q \in (H^\infty_1)^{k\times k}$ which satisfies the interpolation conditions (1.9) (but probably *not* the additional norm constraint $\|Q\|_\infty \le 1$). By the proof of the necessity in Theorem 1.3 we see that condition (1.12) can be translated to the operator-theoretic form

$$\left\| P_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} M_Q \Big|_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} \right\| \le 1 \ \text{ for all } \ (\alpha,\beta) \in \mathbb{G}(\ell' \times \ell) \ \text{ for } \ 1 \le \ell \le \ell' \le k \tag{2.19}$$

where we let $\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}$ be the subspace of $H^2_{\alpha,\beta}$ given by

$$\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta} = H^2_{\alpha,\beta} \ominus \left(H^2_{\alpha,\beta} \cap B_{\mathfrak{D}}(H^2)^{k\times\ell}\right).$$

We use Q to define a linear functional on $(\mathcal{I}_{\mathfrak{D}})_\perp$ by

$$\mathbb{L}_Q(h) = \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\left[Q(z)h(z)\right]\, |dz|. \tag{2.20}$$

By Proposition 2.8 we may factor h as $h = gf^*$ with g and f as in (2.14). Then we may convert the formula for $\mathbb{L}_Q(h)$ to an L^2-inner product as follows:

$$\begin{aligned} \mathbb{L}_Q(h) &= \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\left[Q(z)h(z)\right]\, |dz| \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\left[Q(z)g(z)f(z)^*\right]\, |dz| \\ &= \langle Qg, f\rangle_{(L^2)^{k\times\ell}} = \left\langle P_{H^2_{\alpha,\beta}} Qg,\, P_{\left(H^2_{\alpha,\beta}\cap B_{\mathfrak{D}}(H^2)^{k\times\ell}\right)^\perp} f\right\rangle_{(L^2)^{k\times\ell}} \end{aligned} \tag{2.21}$$

since $f \in \left(H^2_{\alpha,\beta} \cap B_{\mathfrak{D}}(H^2)^{k\times\ell}\right)^\perp$ by construction and since $H^2_{\alpha,\beta}$ is invariant under the multiplication operator M_Q as $Q \in (H^\infty_1)^{k\times k}$. As

$$\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta} = H^2_{\alpha,\beta} \cap \left(H^2_{\alpha,\beta} \cap B_{\mathfrak{D}}(H^2)^{k\times\ell}\right)^\perp,$$

we may continue the computation (2.21) as follows:

$$\mathbb{L}_Q(h) = \left\langle P_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} Qg, P_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} f \right\rangle_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}}. \tag{2.22}$$

We claim that the linear functional $\mathbb{L}_Q$ has linear-functional norm at most 1. To see this we note that

$$\begin{aligned} \|\mathbb{L}_Q\| &= \sup_{h \in (\mathcal{I}_{\mathfrak{D}})_\perp \colon \|h\|_1 \le 1} |\mathbb{L}_Q(h)| \\ &= \sup_{g, \widetilde{f} \in \mathcal{M}^{\mathfrak{D}}_{\alpha,\beta} \colon \|g\|_2, \|\widetilde{f}\|_2 \le 1, (\alpha,\beta) \in \mathbb{G}(\ell' \times \ell)} \left| \left\langle P_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} Qg, \widetilde{f} \right\rangle_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} \right| \le 1 \end{aligned}$$

where we use the norm preservation properties (2.14) in the factorization (2.13) and where we use the assumption (2.19) for the last step. This verifies that $\|\mathbb{L}_Q\| \le 1$.

By the Hahn-Banach Theorem we may extend $\mathbb{L}_Q$ to a linear functional of norm at most 1 defined on the whole space $(L^1)^{k \times k}$. By the Riesz representation theorem for this setting (see (2.1)), such a linear functional has the form $\mathbb{L}_S$ for an $S \in (L^\infty)^{k \times k}$ with implementation given by

$$\mathbb{L}_S(h) = \frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{trace}\left[S(z) h(z)\right] |dz|$$

where also $\|\mathbb{L}_S\| = \|S\|_\infty$. As $\|\mathbb{L}_S\| \le 1$, we have $\|S\|_\infty \le 1$. As $\mathbb{L}_S$ is an extension of $\mathbb{L}_Q$, we conclude that

$$\mathbb{L}_{S-Q}|_{(\mathcal{I}_{\mathfrak{D}})_\perp} = \mathbb{L}_S|_{(\mathcal{I}_{\mathfrak{D}})_\perp} - \mathbb{L}_Q = 0,$$

or $S - Q \in ((\mathcal{I}_{\mathfrak{D}})_\perp)^\perp = \mathcal{I}_{\mathfrak{D}}$. In concrete terms, this means that $S \in (H^\infty_1)^{k \times k}$ (since $Q \in (H^\infty_1)^{k \times k}$ and $\mathcal{I}_{\mathfrak{D}} \subset H^\infty_1$), and that S satisfies the interpolation conditions (1.9) (since Q satisfies (1.9) and elements of $\mathcal{I}_{\mathfrak{D}}$ satisfy the associated homogeneous interpolation conditions as in (2.9)). As we noted above that $\|S\|_\infty \le 1$, we conclude that S is in fact in $(\mathcal{S}_1)^{k \times k}$ and is a solution of the interpolation problem as described in Theorem 1.3. This concludes the proof of Theorem 1.3. □

As a corollary we have the following distance formula for $\mathcal{I}_{\mathfrak{D}} \subset (H^\infty_1)^{k \times k}$.

Corollary 2.9. *Let $\mathfrak{D}$ be a data set as in* (1.8)*. Then for any $F \in (H^\infty_1)^{k \times k}$,*

$$\operatorname{dist}(F, \mathcal{I}_{\mathfrak{D}}) = \sup_{(\alpha,\beta) \in \mathbb{G}(\ell' \times \ell), 1 \le \ell \le \ell' \le k} \left\| M_F^*|_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} \right\|. \tag{2.23}$$

Proof. Given $F \in (H^\infty_1)^{k \times k}$, the distance of F to $\mathcal{I}_{\mathfrak{D}}$ can be identified with

$$\operatorname{dist}(F, \mathcal{I}_{\mathfrak{D}}) = \inf\{\|G\| \colon G \in (H^\infty_1) \text{ and } G(z_j) = F(z_j) \ \text{ for } \ j = 1, \dots, n\}.$$

By rescaling the result in Theorem 1.3, we see that this infimum in turn can be computed as

$$\begin{aligned} \inf\{M \colon \sum_{i,j=1}^n \operatorname{trace}\left[M^2 \cdot X_j K^{\alpha,\beta}(z_i, z_j) X_i^* - W_j^* X_j K^{\alpha,\beta}(z_i, z_j) X_i^* W_i\right] \ge 0 \\ \text{for all } (\alpha, \beta) \in \mathbb{G}(\ell' \times \ell),\, X_1, \dots, X_n \in \mathbb{C}^{k \times \ell},\, 1 \le \ell \le \ell' \le k\}. \end{aligned}$$

By a routine rescaling of the equivalence of the kernel criterion (1.12) and the operator-norm criterion (2.19), this infimum in turn is the same as

$$\inf\left\{M \colon \left\| (M_F)^*|_{\mathcal{M}^{\mathfrak{D}}_{\alpha,\beta}} \right\| \le M \text{ for all } (\alpha, \beta) \in \mathbb{G}(\ell' \times \ell) \text{ for } 1 \le \ell \le \ell' \le k \right\}.$$

This last formula finally is clearly equal to the supremum given in (2.23). □

2.4. **A Beurling-Lax theorem for $(H_1^\infty)^{k\times k}$.** We begin with the following variant of the classical Beurling-Lax theorem.

Theorem 2.10. *Suppose that $\mathcal{M} \subset (H^2)^{k\times k}$ is invariant under the multiplication operator $M_F\colon h(z) \mapsto F(z)h(z)$ for each $F \in (H^\infty)^{k\times k}$. Then, for some $\ell \le k$ there is an $\ell \times k$ co-inner function J (so $J \in (H^\infty)^{\ell\times k}$ with $J(z)J(z)^* = I_\ell$ for almost all $z \in \mathbb{T}$) so that*

$$\mathcal{M} = (H^2)^{k\times \ell} \cdot J.$$

Proof. Let $\mathcal{E}$ be the subspace

$$\mathcal{E} = \mathcal{M} \ominus (z\mathbb{C}^{k\times k})\mathcal{M}.$$

Note that $\mathcal{E}$ is invariant under multiplication on the left by elements of $\mathbb{C}^{k\times k}$. It follows that there is a isometric right multiplication operator R_J which is an isometry from $\mathbb{C}^{k\times \ell}$ onto $\mathcal{E}$, i.e., so that $\mathcal{E} = \mathbb{C}^{k\times\ell}J$. As $\mathcal{E} \subset (H^2)^{k\times k}$, the entries of J are analytic. One can check that $\mathcal{E}$ is wandering for $z\mathbb{C}^{k\times k}$, i.e., $z^n\mathbb{C}^{k\times k}\mathcal{E} \perp z^m\mathbb{C}^{k\times k}\mathcal{E}$ for $n \ne m$. As $\cap_{n\ge 0} z^n\mathbb{C}^{k\times k}\mathcal{M} = \{0\}$, we conclude that $\mathcal{M}$ has the orthogonal decomposition

$$\mathcal{M} = \bigoplus_{n=0}^{\infty} z^n\mathbb{C}^{k\times k}J = (H^2)^{k\times\ell}J.$$

In particular, for all $X, Y \in \mathbb{C}^{k\times k}$ and for all $n \in \mathbb{Z}$ not equal to 0 we have

$$\begin{aligned} 0 = \langle z^n XJ(z), YJ(z)\rangle &= \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{trace}\left[z^n XJ(z)J(z)^*Y^*\right] \, |dz| \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} z^n \operatorname{trace}\left[J(z)J(z)^*Y^*X\right] \, |dz| \end{aligned}$$

which forces $J(z)J(z)^*$ to be a constant. As we arranged that right multiplication by J is an isometry, the constant must be I_ℓ and Theorem 2.10 follows. □

The Beurling-Lax theorem for the algebra $(H_1^\infty)^{k\times k}$ is as follows. For this theorem we introduce the notation $\overline{\mathbb{G}}(\ell'\times\ell)$ for the set of pairs (α,β) of $\ell'\times\ell$ matrices with the property that $\alpha\alpha^* + \beta\beta^* = I_{\ell'}$, i.e., $\overline{\mathbb{G}}(\ell'\times\ell)$ is defined as $\mathbb{G}(\ell'\times\ell)$ but without the injectivity condition on α.

Theorem 2.11. *Suppose that the nonzero subspace $\mathcal{M}$ of $(H^2)^{k\times k}$ is invariant under $M_F\colon h(z) \mapsto F(z)h(z)$ for all $F \in (H_1^\infty)^{k\times k}$. Then there is a co-inner function $J \in (H^\infty)^{\ell\times k}$ and an $(\alpha,\beta) \in \overline{\mathbb{G}}(\ell'\times\ell)$ for some $1 \le \ell \le \ell' \le k$ so that $\mathcal{M} = H^2_{\alpha,\beta}\cdot J$.*

Proof. Let $\widetilde{\mathcal{M}} := \left[(H^\infty)^{k\times k}\cdot\mathcal{M}\right]^-$. Then

$$\widetilde{\mathcal{M}} \supset \mathcal{M} \supset (H_1^\infty)^{k\times k}\cdot\mathcal{M} \supset \left[z^2(H^\infty)^{k\times k}\cdot\mathcal{M}\right]^- = z^2\mathcal{M}.$$

By Theorem 2.10, there is an $\ell\times k$ co-inner function J so that $\widetilde{\mathcal{M}} = (H^2)^{k\times\ell}\cdot J$. Then

$$(H^2)^{k\times\ell}J \supset \mathcal{M} \supset z^2(H^2)^{k\times\ell}J$$

from which we see that $\mathcal{M}$ has the form

$$\mathcal{M} = \mathcal{S}\cdot J + z^2(H^2)^{k\times\ell}J$$

for some subspace $\mathcal{S} \subset (H^2)^{k\times\ell} \ominus z^2(H^2)^{k\times\ell}$. As $\mathcal{S}$ is invariant under the operators M_X of multiplication by a constant $k \times k$ matrix on the left, we conclude that $\mathcal{S}$ must have the form

$$\mathcal{S} = \{X + zY \colon \begin{bmatrix} X & Y \end{bmatrix} = U \begin{bmatrix} \alpha & \beta \end{bmatrix} \text{ for some } (\alpha, \beta) \in \overline{\mathbb{G}}(\ell' \times \ell)\}$$

from which it follows that $\mathcal{M} = H^2_{\alpha,\beta} \cdot J$. $\square$

2.5. **An analogous problem: Nevanlinna-Pick interpolation on a finitely-connected planar domain.** The purpose of this section is to make explicit how the commutant lifting theorem from [3] leads to a proof of Theorem 1.5. Toward this goal we first need a few preliminaries.

Let $\mathcal{R}$ be a bounded finitely-connected planar domain with the boundary consisting of $g+1$ analytic closed simple curves $C_0, C_1, \dots, C_g$ where C_0 is the boundary of the unbounded component of $\mathbb{C} \setminus \mathcal{R}$. There is a standard procedure (see [1]) for introducing g disjoint simple curves $\gamma_1, \dots, \gamma_g$ so that $R \setminus \boldsymbol{\gamma}$ (where we set $\boldsymbol{\gamma}$ equal to the union $\boldsymbol{\gamma} = \gamma_1 \cup \dots \cup \gamma_g$) is simply connected. For each cut γ_i we assign an arbitrary orientation, so that points z not on γ_i but in a sufficiently small neighborhood of γ_i in $\mathcal{R}$ can be assigned a location of either "to the left" of γ_i or "to the right" of γ_i. For f a function on $\mathcal{R} \setminus \boldsymbol{\gamma}$ and $z \in \gamma_i$, we let $f(z_-)$ denote the limit of $f(\zeta)$ as ζ approaches z from the left of γ_i in $\mathcal{R}$, and similarly we let $f(z_+)$ denote the limit of $f(\zeta)$ as ζ approaches z from the right of γ_i in $\mathcal{R}$, whenever these limits exist. We also fix a point $t \in R$ and let $dm_t(z)$ be the harmonic measure on ∂R for the point $t \in R$; thus

$$u(t) = \int_{\partial R} u(\zeta)\, dm_t(\zeta)$$

whenever u is harmonic on $\mathcal{R}$ and continuous on the closure $\overline{R}$.

Suppose that we are given a g-tuple $\mathbf{U} = (U_1, \dots, U_g)$ of $\ell \times \ell$ unitary matrices, denoted as $\mathbf{U} \in \mathcal{U}(\ell)^g$. We also fix a positive integer k. We let $H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ denote the space of all $(k \times \ell)$-matrix-valued holomorphic functions F on $R \setminus \boldsymbol{\gamma}$ such that

$$F(z_-) = F(z_+) U_i \text{ for } z \in \gamma_i$$

(so $\operatorname{trace}(F(z)^* F(z))$ extends by continuity to a single-valued function on all of $\mathcal{R}$) and such that

$$\|F\|_2 := \int_{\partial R} \operatorname{trace}\left(F(z)^* F(z)\right)\, dm_t(z) < \infty.$$

Then $H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ is a Hilbert space. Given a bounded $k \times k$-matrix-valued function F on $\mathcal{R}$ (denoted as $F \in (H^\infty(\mathcal{R}))^{k\times k}$) and given any $\mathbf{U} \in \mathcal{U}(\ell)^g$ as above, we may consider the multiplication operator $M_F^{\mathbf{U}}$ on $H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ given by

$$M_F^{\mathbf{U}} \colon f(z) \mapsto F(z) f(z) \text{ for } f \in H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U}).$$

Then it can be shown that

$$\|F\|_\infty = \sup_{k \colon 1 \le k \le \ell} \sup_{\mathbf{U} \in \mathcal{U}(k)^g} \left\| M_F^{\mathbf{U}} \right\|.$$

For the particular case where $\mathbf{U} = (I_k, \dots, I_k)$ has all entries equal to the $k \times k$ identity matrix I_k, we write $\mathbf{I}_k$ for $\mathbf{U}$. In case $k = 1$ we write $\mathbf{1}$ for $(1, \dots, 1) \in \mathcal{U}(1)^g$.

Let us now assume that we are given the data set

$$z_1, \dots, z_n \in R, \quad W_1, \dots, W_n \in \mathbb{C}^{k\times k}$$

for an interpolation problem (1.17). For $\mathbf{U} \in \mathcal{U}(\ell)^g$ we define subspaces $\mathcal{M}^{\mathbf{U}}$ and $\mathcal{N}^{\mathbf{U}}$ of $H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ by

$$\mathcal{M}^{\mathbf{U}} := \{f \in H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})\colon f(z_i) = 0 \text{ for } i = 1, \dots, n\},$$
$$\mathcal{N}^{\mathbf{U}} := H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U}) \ominus \mathcal{M}^{\mathbf{U}}.$$

Let F_0 be any particular solution of the interpolation conditions (1.17); e.g., one can take such a solution to be a matrix polynomial constructed by solving a scalar Lagrange interpolation problem for each matrix entry. For each $\mathbf{U} \in \mathcal{U}(\ell)^g$ define an operator $\Gamma^{\mathbf{U}}$ on $\mathcal{N}^{\mathbf{U}}$ by

$$\Gamma^{\mathbf{U}} h = P_{\mathcal{N}^{\mathbf{U}}}(F_0 \cdot h) \text{ for } h \in \mathcal{N}^{\mathbf{U}}.$$

It is not difficult to see that $\Gamma^{\mathbf{U}}$ is independent of the choice of $(H^\infty(\mathcal{R}))^{k\times k}$-solution of the interpolation conditions (1.17). The following result is a more concrete expression of Theorem 4.3 and the Remark immediately following from [3].

Theorem 2.12. *There exists an $F \in (H^\infty(\mathcal{R}))^{k\times k}$ with $\|F\|_\infty \le 1$ so that*

$$P_{\mathcal{N}^{\mathbf{I}_k}} M_F^{\mathbf{I}_k}|_{\mathcal{N}^{\mathbf{I}_k}} = P_{\mathcal{N}^{\mathbf{I}_k}} M_{F_0}^{\mathbf{I}_k}|_{\mathcal{N}^{\mathbf{I}_k}},$$

or, equivalently, so that F satisfies the interpolation conditions (1.17)*, if and only if*

$$\sup_{\ell\colon 1\le \ell\le k} \ \sup_{\mathbf{U}\colon \mathbf{U}\in\mathcal{U}(\ell)^g} \left\|\Gamma^{\mathbf{U}}\right\| \le 1. \tag{2.24}$$

To convert the criterion (2.24) to a positivity test, we proceed as follows. For $\mathbf{U} \in \mathcal{U}(\ell)^g$ there is a $(\ell \times \ell)$-matrix-valued kernel function $K_{\mathbf{U}}(z, w)$ defined on $\mathcal{R} \times \mathcal{R}$ so that

$$xK_{\mathbf{U}}(\cdot, w) \in H^2_{\mathcal{C}^{2,1\times\ell}}(\mathbf{1}, \mathbf{U}) \text{ for each } x \in \mathbb{C}^{1\times\ell},$$
$$\langle f, xK_{\mathbf{U}}(\cdot, w)\rangle_{H^2_{\mathcal{C}^{2,1\times\ell}}(\mathbf{1},\mathbf{U})} = \langle f(w), x\rangle_{\mathcal{C}^{2,1\times\ell}} = f(w)x^*.$$

Then this same kernel $K_{\mathbf{U}}(z, w)$ serves as the kernel function for the space of $(k\times\ell)$-matrix-valued functions $H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ in the sense that for all $f \in H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ and $X \in \mathbb{C}^{k\times\ell}$ we have $XK_{\mathbf{U}}(\cdot, w) \in H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U})$ and

$$\langle f, XK_{\mathbf{U}}(\cdot, w)\rangle_{H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k,\mathbf{U})} = \langle f(w), X\rangle_{\mathcal{C}^{2,k\times\ell}} = \operatorname{trace}\left(f(w)X^*\right).$$

Then one can check that the following hold:

$$\overline{\operatorname{span}}\left\{XK_{\mathbf{U}}(\cdot, z_j)\colon X \in \mathbb{C}^{k\times\ell},\ j = 1, \dots, n\right\} = \mathcal{N}^{\mathbf{U}}, \tag{2.25}$$
$$\left(\Gamma^{\mathbf{U}}\right)^* : XK_{\mathbf{U}}(\cdot, z_j) \mapsto W_j^* XK_{\mathbf{U}}(\cdot, z_j). \tag{2.26}$$

With these preliminaries out of the way, Theorem 1.5 follows as a consequence of Theorem 2.12 by the following standard computations.

Proof of Theorem 1.5 via Theorem 2.12. By (2.25), a generic element f of $\mathcal{N}^{\mathbf{U}}$ is given by $f(z) = \sum_{j=1}^n X_j K_{\mathbf{U}}(\cdot, z_j)$ where $X_1, \dots, X_g \in \mathbb{C}^{k\times\ell}$. By (2.26),

$$\left(\Gamma^{\mathbf{U}}\right)^* : \sum_{j=1}^n X_j K_{\mathbf{U}}(\cdot, z_j) \mapsto \sum_{j=1}^n W_j^* X_j K_{\mathbf{U}}(\cdot, z_j).$$

Hence $\|\Gamma^{\mathbf{U}}\| = \left\|(\Gamma^{\mathbf{U}})^*\right\| \leq 1$ if and only if

$$\left\|\sum_{j=1}^{n} X_j K_{\mathbf{U}}(\cdot, z_j)\right\|^2 - \left\|\sum_{j=1}^{n} W_j^* X_j K_{\mathbf{U}}(\cdot, z_j)\right\|^2 \quad \text{for all } X_1, \dots, X_n \in \mathbb{C}^{k\times\ell}. \tag{2.27}$$

Use of the reproducing kernel property of $K_{\mathbf{U}}(z,w)$ converts (2.27) to (1.18). This holding true for all $\mathbf{U} \in \mathcal{U}(\ell)^g$ for each ℓ with $1 \leq \ell \leq k$ is equivalent to the existence of a solution of the interpolation problem of norm at most 1 by Theorem 2.12 □

Remark 2.13. If $\mathbf{U} = (U_1, \dots, U_g) \in \mathcal{U}(\ell)^g$ has a direct sum decomposition $\mathbf{U} = \mathbf{U}' \oplus \mathbf{U}''$ where

$$\mathbf{U}' = (U_1', \dots, U_g') \in \mathcal{U}(\ell')^g \text{ and } \mathbf{U}'' = (U_1'', \dots, U_g'') \in \mathcal{U}(\ell'')^g$$

where $\ell = \ell' + \ell''$, then it is easily seen that we have the orthogonal direct-sum decomposition

$$H^2_{\mathcal{C}^{2,k\times\ell}}(\mathbf{I}_k, \mathbf{U}) = H^2_{\mathcal{C}^{2,k\times\ell'}}(\mathbf{I}_k, \mathbf{U}') \oplus H^2_{\mathcal{C}^{2,k\times\ell''}}(\mathbf{I}_k, \mathbf{U}'')$$

which splits the action of $M_F^{\mathbf{U}}$:

$$M_F^{\mathbf{U}} = M_F^{\mathbf{U}'} \oplus M_F^{\mathbf{U}'''}.$$

It follows that in Theorem 1.5 we need only consider irreducible elements $\mathbf{U}$ of $\mathcal{U}(\ell)^g$ for each $\ell = 1, \dots, k$, or, alternatively, we need consider only $\mathcal{U}(\ell)^g$ with $\ell = k$.

In particular, for the case of $g = 1$ where each element $\mathbf{U} \in \mathcal{U}(k)^1$ reduces to a single unitary operator U, a consequence of the spectral theorem is that any such U is a direct sum of scalar U's ($k = 1$). Hence, for the case $g = 1$ in Theorem 1.5, it suffices to consider the condition (1.18) with $U = \lambda I_k$ where λ is on the unit circle $\mathbb{T}$. McCullough in [13] showed that in fact all these kernel functions (or more precisely, a dense set) are needed, even when the interpolation nodes $z_1, \dots, z_n$ are specified (see [14] for a refinement of this result). Remarkably, for the case of scalar interpolation ($k = 1$), if one specifies the interpolation nodes, Fedorov-Vinnikov [10] (see [13] for an alternate proof) showed that one can find two points on $\mathbb{T}$ for which positive-semidefiniteness of the associated Pick matrix guarantees solvability for the Nevanlinna-Pick interpolation problem.

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4 in the following more general setting. Let $\lambda_1, \dots, \lambda_m$ be distinct points in $\mathbb{D}$ and $r_1, \dots, r_m \in \mathbb{Z}_+$ so that $r_i \geq 1$ for $i = 1, \dots, m$. We write B for the finite Blaschke product

$$B(z) = \prod_{i=1}^{m} \left(\frac{z - \lambda_i}{1 - \overline{\lambda}_i z}\right)^{r_i}. \tag{3.1}$$

Let $(H_B^\infty)^{k\times k}$ be the subalgebra of $(H^\infty)^{k\times k}$ consisting of the matrix-functions in $(H^\infty)^{k\times k}$ whose entries are in the subalgebra $\mathbb{C} + BH^\infty$ of H^∞, and let $(\mathcal{S}_B)^{k\times k}$ be the constrained Schur class of functions in $(H_B^\infty)^{k\times k}$ of supremum norm at most one.

Then we consider the following matrix-valued constrained Nevanlinna-Pick interpolation problem: *given a data set* $\mathfrak{D}$ *as in (1.8), find a function* S *in* $(\mathcal{S}_B)^{k\times k}$ *satisfying the interpolation conditions*

$$S(z_i) = W_i \quad \textit{for} \quad i = 1, \dots, n. \tag{3.2}$$

We will assume that $z_i \neq \lambda_j$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, and return to the case where $z_i = \lambda_j$ for some i and j at the end of the section.

To state the solution criterion for the constrained Nevanlinna-Pick interpolation problem we introduce some more notations. Define $Z, \widetilde{W} \in \mathbb{C}^{kn\times kn}$ and $E \in \mathbb{C}^{kn\times k}$ by

$$Z = \begin{bmatrix} z_1 I_k & & \\ & \ddots & \\ & & z_n I_k \end{bmatrix}, \quad E = \begin{bmatrix} I_k \\ \vdots \\ I_k \end{bmatrix}, \quad \widetilde{W} = \begin{bmatrix} W_1 & & \\ & \ddots & \\ & & W_n \end{bmatrix}, \tag{3.3}$$

and define $J \in \mathbb{C}^{kd\times kd}$ and $\widetilde{E} \in \mathbb{C}^{kd\times k}$, where $d = r_1 + \cdots + r_m$, by

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}, \quad \widetilde{E} = \begin{bmatrix} \widetilde{E}_1 \\ \vdots \\ \widetilde{E}_m \end{bmatrix}, \tag{3.4}$$

where for $i = 1, \dots, m$ the matrices $J_i \in \mathbb{C}^{kr_i\times kr_i}$ and $\widetilde{E}_i \in \mathbb{C}^{kr_i\times k}$ are given by

$$J_i = \begin{bmatrix} \lambda_i I_k & 0 & \cdots & \cdots & 0 \\ I_k & \lambda_i I_k & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_k & \lambda_i I_k \end{bmatrix}, \quad \widetilde{E}_i = \begin{bmatrix} I_k \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{3.5}$$

Let $\mathbb{P}$ be the Pick matrix associated with the standard matrix-valued Nevanlinna-Pick problem, i.e., as in (1.13), and let $\mathbb{Q} \in \mathbb{C}^{kd\times kd}$ and $\widetilde{\mathbb{Q}} \in \mathbb{C}^{kd\times kn}$ be the unique solutions to the Stein equations

$$\mathbb{Q} - J\mathbb{Q}J^* = \widetilde{E}\widetilde{E}^*, \qquad \widetilde{\mathbb{Q}} - J\widetilde{\mathbb{Q}}Z^* = \widetilde{E}E^*, \tag{3.6}$$

that is, $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are given by the convergent infinite series

$$\mathbb{Q} = \sum_{i=0}^{\infty} J^i \widetilde{E}\widetilde{E}^* J^{*i} \quad \text{and} \quad \widetilde{\mathbb{Q}} = \sum_{i=0}^{\infty} J^i \widetilde{E}E^* Z^{*i}. \tag{3.7}$$

For the special case considered here there are more explicit formulas available (see e.g. Appendix A.2 in [4]) but we have no need for them. In particular one can see directly from the series expansion for $\mathbb{Q}$ in (3.7) or quote general theory (using the fact that the pair $(J, \widetilde{E})$ is controllable—see [4]) that $\mathbb{Q}$ is positive definite and hence invertible. In fact from the infinite series in (3.7) one can see that $Q \geq I$.

For $X \in \mathbb{C}^{k\times k}$ and $j \in \mathbb{Z}_+$ set $X_j \in \mathbb{C}^{jk\times jk}$,

$$X_j = \begin{bmatrix} X & & \\ & \ddots & \\ & & X \end{bmatrix},$$

and define $\mathbb{P}_X \in \mathbb{C}^{k(n+2d)\times k(n+2d)}$ by

$$\mathbb{P}_X = \begin{bmatrix} \mathbb{P} & (I - \widetilde{W}X_n^*)\widetilde{\mathbb{Q}}^* & 0 \\ \widetilde{\mathbb{Q}}(I - X_n\widetilde{W}^*) & \mathbb{Q} & X_d \\ 0 & X_d^* & \mathbb{Q}^{-1} \end{bmatrix}. \tag{3.8}$$

Then we have the following result.

Theorem 3.1. *Suppose we are given an interpolation data set of the form $\mathfrak{D}$ as in (1.8) with $W_1, \dots, W_n$ equal to $k \times k$ matrices along with a Blaschke product B as in (3.1). Assume that $z_i \neq \lambda_j$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. Then there exists a constrained Schur-class function S in $(\mathcal{S}_B)^{k\times k}$ satisfying the interpolation conditions (3.2) if and only if there exists a $k \times k$ matrix X so that the matrix $\mathbb{P}_X$ in (3.8) is positive-semidefinite.*

Proof. Notice that an (unconstrained) Schur class function S in $(H^\infty)^{k\times k}$, i.e., $\|S\|_\infty \leq 1$, is in the constrained Schur class $(\mathcal{S}_B)^{k\times k}$ if and only if

$$S^{(j)}(\lambda_i) = 0 \quad \text{for } i = 1, \dots, m, \text{ and } j = 1, \dots, r_j - 1, \tag{3.9}$$

$$S(\lambda_1) = \cdots = S(\lambda_m). \tag{3.10}$$

(If $r_j = 1$ for some j, we interpret (3.9) as vacuous.) So, alternatively we seek Schur class functions S satisfying the constraints (3.2), (3.9) and (3.10).

Now assume that S is a solution to the constrained Nevanlinna-Pick interpolation problem. Set $X = S(\lambda_1)$. Then X is contractive, and S is also a solution to the matrix-valued Carathéodory-Fejér interpolation problem: *find Schur class functions S in $(H^\infty)^{k\times k}$ satisfying (3.2), (3.9) and*

$$S(\lambda_i) = X \quad \textit{for} \quad i = 1, \dots, m. \tag{3.11}$$

Conversely, for any contractive $k \times k$ matrix X, a Schur class function in $(H^\infty)^{k\times k}$ satisfying (3.2), (3.9) and (3.11) is obviously also a solution to the constrained Nevanlinna-Pick problem.

From this we conclude that the solutions of the constrained Nevanlinna-Pick problem correspond to the solutions of the Carathéodory-Fejér problem defined by the interpolation conditions (3.2), (3.9) and (3.11), where X is a free-parameter in the set of (contractive) $k \times k$ matrices.

Given a $k \times k$ matrix X, the Carathéodory-Fejér problem described above is known to have a solution if and only if the Pick matrix $\widetilde{\mathbb{P}}_X$ given by

$$\widetilde{\mathbb{P}}_X = \begin{bmatrix} \mathbb{P} & (I - \widetilde{W}X_n^*)\widetilde{\mathbb{Q}}^* \\ \widetilde{\mathbb{Q}}(I - X_n\widetilde{W}^*) & \mathbb{Q} - X_d\mathbb{Q}X_d^* \end{bmatrix} \tag{3.12}$$

is positive-semidefinite; see [4]. Here we use the same notations as in the definition of the matrix $\mathbb{P}_X$ in (3.8). The theorem then follows from the fact that $\mathbb{Q}$ is invertible and $\widetilde{\mathbb{P}}_X$ can be written as

$$\widetilde{\mathbb{P}}_X = \begin{bmatrix} \mathbb{P} & (I - \widetilde{W}X_n^*)\widetilde{\mathbb{Q}}^* \\ \widetilde{\mathbb{Q}}(I - X_n\widetilde{W}^*) & \mathbb{Q} \end{bmatrix} - \begin{bmatrix} 0 \\ X_d \end{bmatrix} \mathbb{Q} \begin{bmatrix} 0 & X_d^* \end{bmatrix}$$

so that, with a standard Schur complement argument (see [5, Remark I.1.2]), we see that for each $X \in \mathbb{C}^{k\times k}$ the matrix $\widetilde{\mathbb{P}}_X$ is positive-semidefinite if and only if the matrix $\mathbb{P}_X$ in (3.8) is positive-semidefinite. $\square$

Proof of Theorem 1.4. The problem considered in Theorem 1.4 is the special case of the problem considered in this section where $m = 1$, $r_1 = 2$ and $\lambda_1 = 0$, i.e., $B(z) = z^2$. In that case the solutions $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ to the Stein equations (3.6) are given by

$$\mathbb{Q} = \begin{bmatrix} I_k & 0 \\ 0 & I_k \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbb{Q}} = \begin{bmatrix} I_k & \cdots & I_k \\ \overline{z}_1 I_k & \cdots & \overline{z}_n I_k \end{bmatrix},$$

so that

$$\widetilde{\mathbb{Q}}(I - X_n \widetilde{W}^*) = \begin{bmatrix} E^* - XW^* \\ (E^* - XW^*)Z^* \end{bmatrix}$$

with W as in (1.14). It thus follows that in this special case the matrix $\mathbb{P}_X$ in (3.8) reduces to (1.16). □

We conclude this section with some remarks concerning Theorem 3.1.

3.1. **Parametrization of the set of all solutions.** For each $X \in \mathbb{C}^{k\times k}$ so that $\mathbb{P}_X$ in (3.8) is positive-semidefinite, the solutions to the Carathéodory-Fejér problem defined by the interpolation conditions (3.2), (3.9) and (3.11) can be described explicitly as the image of a linear-fractional-transformation T_{Θ_X} acting on the unit ball of matrix-valued H^∞ of some size. Therefore, if one happens to be able to find all $X \in \mathbb{C}^{k\times k}$ for which $\mathbb{P}_X$ is positive-semidefinite, then in principle one can describe the set of all solutions to the constrained Nevanlinna-Pick problem as the union of the images of these linear-fractional-transformations T_{Θ_X}.

3.2. **The case where B and $B_{\mathfrak{D}}$ have overlapping zeros.** Theorem 3.1 (and also Theorem 1.4) only considers the case that $\{\lambda_1, \dots, \lambda_m\}$ and $\{z_1, \dots, z_n\}$ do not intersect. In case the intersection is not empty, say

$$\{\lambda_1, \dots, \lambda_m\} \cap \{z_1, \dots, z_n\} = \{z_{i_1}, \dots, z_{i_p}\},$$

then solutions exist if and only if $W_{i_1} = \cdots = W_{i_p}$ and $\mathbb{P}_{W_{i_1}}$ is positive-semidefinite. Indeed, this is the case since functions in the constrained Schur class $(\mathcal{S}_B)^{k\times k}$ have to satisfy (3.10) and for a solution S the matrix $\mathbb{P}_{S(\lambda_1)}$ is positive-semidefinite.

The Pick matrix obtained in this case includes some degeneracy since the interpolation condition at z_{i_j} is listed twice. There therefore exists a reduced Pick matrix whose size, for the case $k = 1$, is still larger than the number of interpolation points n. In [25] the size of this reduced Pick matrix is identified with the dimension of the C^*-envelope $C^*(\mathfrak{A})$ for the algebra $\mathfrak{A} = H^\infty_B / \mathcal{I}_{\mathfrak{D}}$, where $\mathcal{I}_{\mathfrak{D}}$ is the ideal

$$\mathcal{I}_{\mathfrak{D}} := \{f \in (H^\infty_B) \colon f(z_i) = 0 \text{ for } i = 1, \dots, n\}.$$

3.3. **The criterion in Theorem 1.2.** If $X \in \mathbb{C}^{k\times k}$ is a strict contraction, then $\mathbb{Q} - X_d \mathbb{Q} X_d^*$ is invertible (if X is not a strict contraction one can use a Moore-Penrose inverse), and, again with a Schur complement argument, it follows that $\widetilde{\mathbb{P}}_X$ in (3.12) is positive-semidefinite if and only if the matrix

$$\widehat{\mathbb{P}}_X := \mathbb{P} - (I - \widetilde{W} X_n^*) \widetilde{\mathbb{Q}}^* (\mathbb{Q} - X_d \mathbb{Q} X_d^*)^{-1} \widetilde{\mathbb{Q}} (I - X_n \widetilde{W}^*) \tag{3.13}$$

is positive-semidefinite. When specified for the setting considered in [7], i.e., as in Theorems 1.1 and 1.2, the matrix $\widehat{\mathbb{P}}_\lambda$ is conjugate to the matrix (1.6). More specifically, the matrix (1.6) is equal to $T\widehat{\mathbb{P}}_\lambda T^*$ with

$$T = \begin{bmatrix} \frac{1-\overline{\lambda}w_1}{\sqrt{1-|\lambda|^2}} & & \\ & \ddots & \\ & & \frac{1-\overline{\lambda}w_n}{\sqrt{1-|\lambda|^2}} \end{bmatrix}.$$

Thus the matrix (1.6) being positive-semidefinite corresponds to $\mathbb{P}_\lambda$ in (1.16) (for $k = 1$) positive-semidefinite.

3.4. **More general algebras.** The argument used in the proof of Theorem 3.1 extends to more general subalgebras of H^∞.

Let $B_1, \dots, B_l$ be finite Blaschke products with B_i having zeros $\lambda_1^{(i)}, \dots, \lambda_{m_i}^{(i)}$ for $i = 1, \dots l$. Then one can consider Nevanlinna-Pick interpolation for functions in the intersection of the algebras $H^\infty_{B_i}$. In case all zeros $\lambda_j^{(i)}$ are distinct this gives rise to a criterion where one has to check positive-semidefiniteness of a Pick matrix with l free parameters. Given a finite Blaschke product B and a finite number of polynomials $p_1, \dots, p_l$, let $\mathfrak{A}_{p_1,\dots,p_l}$ be the subalgebra of H^∞ generated by $p_1, \dots, p_l$. Nevanlinna-Pick interpolation for functions in the subalgebra $\mathfrak{A}_{p_1,\dots,p_l} + BH^\infty$ gives rise to coupling conditions more complicated than (3.10) and more intricate Pick matrices than (3.8) with possibly more then one free parameter.

For example, consider the case of three distinct points λ_1, λ_2 and λ_3 in $\mathbb{D}$ with associated the Blaschke product

$$B(z) = \left(\frac{z-\lambda_1}{1-\overline{\lambda}_1 z}\right)^3 \left(\frac{z-\lambda_2}{1-\overline{\lambda}_2 z}\right)^3 \left(\frac{z-\lambda_3}{1-\overline{\lambda}_3 z}\right)^2$$

and polynomials $p_1(z) = 1$ and $p_2(z) = z^2$. Those functions f which are in the algebra $\mathfrak{A}_{p_1,p_2} + BH^\infty$ can then be characterized as those f in H^∞ that satisfy $f'(\lambda_i) = 0$ for $i = 1, 2, 3$ along with

$$\begin{aligned} f''(\lambda_1) &= f''(\lambda_2), \\ 2f(\lambda_1) + \lambda_1^2 f''(\lambda_1) &= 2f(\lambda_2) + \lambda_2^2 f''(\lambda_2) = 2f(\lambda_3) + \lambda_3^2 f''(\lambda_3). \end{aligned} \tag{3.14}$$

By specifying both values in the coupling conditions (3.14) we return to a standard Carathéodory-Fejér problem; thus, for the general problem we obtain a Pick matrix with two free parameters.

3.5. **The non-square case.** We expect that the techniques used here can enable one to obtain a non-square version of Theorems 1.3 and 3.1 where one seeks $S \in (\mathcal{S}_1)^{k\times k'}$ (or, more generally, $S \in (\mathcal{S}_B)^{k\times k'}$) with $k \neq k'$ which satisfies some prescribed set of interpolation conditions. Note that $(\mathcal{S}_B)^{k\times k'}$ is no longer an algebra but rather an *operator space* when $k \neq k'$ (see e.g. [18]). There is an abstract notion of a non-square analogue of a C^*-algebra, namely the J^*-algebras introduced by Harris [11]. Perhaps looking for the J^*-algebra envelop of a quotient operator space will give insight into interpolation and dilation theory, as is the case for the algebra setting (see [7]).

4. Analysis of the LMI criterion of Theorem 1.4

Recall from the introduction that the LMI criterion of Theorem 1.4 is equivalent to the existence of a matrix X such that the Pick matrix $\mathbb{P}'_X$ given by (1.15) be positive semidefinite, i.e.,

$$\begin{bmatrix} \mathbb{P} & \widetilde{E} + \widetilde{W}\widetilde{X}^* \\ \widetilde{E}^* + \widetilde{X}\widetilde{W}^* & I - \widetilde{X}\widetilde{X}^* \end{bmatrix} \geq 0, \tag{4.1}$$

where $\mathbb{P}$ is the standard Pick matrix (1.13),

$$\widetilde{E} = \begin{bmatrix} E & ZE \end{bmatrix}, \quad \widetilde{W} = \begin{bmatrix} W & ZW \end{bmatrix} \quad \text{and} \quad \widetilde{X} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \tag{4.2}$$

with E, Z and W as in (1.14).

We forget for now about the structure in the matrices $\widetilde{E}$, $\widetilde{W}$ and $\widetilde{X}$, and just assume that $\widetilde{E}$, $\widetilde{W}$ are given $k \times nk$ matrices and $\widetilde{X}$ is a free-parameter $2k \times 2k$ matrix. In case $\mathbb{P}$ is positive definite we obtain, after taking the Schur complement with respect to $\mathbb{P}$, that (4.1) is equivalent to the positive-semidefiniteness condition

$$\begin{aligned} I - \widetilde{X}\widetilde{X}^* - (\widetilde{E}^* + \widetilde{X}\widetilde{W}^*)\mathbb{P}^{-1}(\widetilde{E} + \widetilde{W}\widetilde{X}^*) = \\ = \begin{bmatrix} I & \widetilde{X} \end{bmatrix} \begin{bmatrix} I - \widetilde{E}^*\mathbb{P}^{-1}\widetilde{E} & -\widetilde{E}^*\mathbb{P}^{-1}\widetilde{W} \\ -\widetilde{W}^*\mathbb{P}^{-1}\widetilde{E} & -(I + \widetilde{W}^*\mathbb{P}^{-1}\widetilde{W}) \end{bmatrix} \begin{bmatrix} I \\ \widetilde{X}^* \end{bmatrix} \succeq 0. \end{aligned} \tag{4.3}$$

Now assume that $\mathbb{P}$ is positive definite and that the matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} := \begin{bmatrix} I - \widetilde{E}^*\mathbb{P}^{-1}\widetilde{E} & -\widetilde{E}^*\mathbb{P}^{-1}\widetilde{W} \\ -\widetilde{W}^*\mathbb{P}^{-1}\widetilde{E} & -(I + \widetilde{W}^*\mathbb{P}^{-1}\widetilde{W}) \end{bmatrix} \tag{4.4}$$

is invertible. Then (4.3) can be interpreted as saying that the subspace

$$\mathcal{G}_{X^*} = \begin{bmatrix} I \\ \widetilde{X}^* \end{bmatrix} \mathbb{C}^{2k} \tag{4.5}$$

is a $2k$-dimensional positive subspace in the Kreĭn space $(\mathbb{C}^{4k}, M)$, i.e., $\mathbb{C}^{4k}$ endowed with the Kreĭn-space inner product

$$[x, y]_M = \langle Mx, y \rangle_{\mathbb{C}^{4k}}$$

(we refer the reader to [6] for elementary facts concerning Kreĭn spaces). Conversely, any $2k$-dimensional positive subspace $\mathcal{G}$ of $(\mathbb{C}^{4k}, M)$ has the form (4.5) as long as

$$\mathcal{G} \cap \begin{bmatrix} 0 \\ \mathbb{C}^{2k} \end{bmatrix} = \{0\}. \tag{4.6}$$

Since $M_{22} = -(I + \widetilde{W}^*\mathbb{P}^{-1}\widetilde{W})$ is strictly negative definite, the subspace $\left[\begin{smallmatrix} 0 \\ \mathbb{C}^{2k} \end{smallmatrix}\right]$ is uniformly negative in $(\mathbb{C}^{4k}, M)$ and hence condition (4.6) is automatic for any positive subspace $\mathcal{G} \subset (\mathbb{C}^{4k}, M)$. We conclude that *there exist* $(2k \times 2k)$*-matrix solutions* $\widetilde{X}$ *to* (4.3) *if and only if the invertible matrix* M *has at least* $2k$ *positive eigenvalues.* As already observed above, $M_{22} < 0$ so M must have at least $2k$ negative eigenvalues. We conclude that *there exist solutions to* (4.3) *if and only if*

M has exactly $2k$ positive eigenvalues, or, equivalently, again by a standard Schur-complement argument [5, Remark I.1.2], *if and only if the Schur complement*

$$\begin{aligned}\Lambda &= I - \widetilde{E}^*\mathbb{P}^{-1}\widetilde{E} + \widetilde{E}^*\mathbb{P}^{-1}\widetilde{W}(I + \widetilde{W}^*\mathbb{P}^{-1}\widetilde{W})^{-1}\widetilde{W}^*\mathbb{P}^{-1}\widetilde{E} \\ &= I - \widetilde{E}^*(\mathbb{P} + \widetilde{W}\widetilde{W}^*)^{-1}\widetilde{E} \end{aligned} \tag{4.7}$$

is positive definite. Given that this is the case, we can then factor M as $M = A^{-1*}JA^{-1}$ where we set $J = \left[\begin{smallmatrix} I_{2k} & 0 \\ 0 & -I_{2k}\end{smallmatrix}\right]$ and where $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right]$ is an invertible $4k \times 4k$ matrix with block entries a_{ij} taken to have size $2k \times 2k$ for $i, j = 1, 2$. Then *the set X^* for which* (4.3) *holds can be expressed as the set of all $2k \times 2k$ matrices X so that X^* has the form*

$$X^* = (a_{21}K + a_{22})(a_{11}K + a_{12})^{-1}$$

for some contractive $2k \times 2k$ matrix K. With a little more algebra, such an image of a linear-fractional map can be converted to the form of a matrix ball. Results of this type go back at least to [24, 12, 20, 21]. The following summary for our situation here can be seen as a direct result of the discussion here and Theorem 1.6.3 of [8].

Theorem 4.1. *Assume that $\mathbb{P}$ is positive definite and that M in* (4.4) *is invertible. Then there exists an $2k \times 2k$ matrix $\widetilde{X}$ such that* (4.1) *holds if and only if Λ in* (4.7) *is positive semidefinite. In that case Λ is positive definite and all $\widetilde{X}$ such that* (4.1) *holds are given by*

$$\widetilde{X} = C + L^{\frac{1}{2}}KR^{\frac{1}{2}}, \tag{4.8}$$

where K is a free-parameter contractive $2k \times 2k$ matrix. Here $R = \Lambda$, and C and L are the $2k \times 2k$ matrices given by

$$C = -\widetilde{E}^*(\mathbb{P} + \widetilde{W}\widetilde{W}^*)^{-1}\widetilde{W}, \quad L = I - \widetilde{W}^*(\mathbb{P} + \widetilde{W}\widetilde{W}^*)^{-1}\widetilde{W}.$$

Moreover, (4.1) *holds with strict inequality if and only if K is a strict contraction.*

Under the assumptions of Theorem 4.1 we thus get a full description of all matrices $\widetilde{X}$ so that (4.1) is satisfied. This however is not enough to solve the problem **MCNP**, since the matrix $\widetilde{X}$ is also required to be of the form (4.2). We thus get the following result.

Theorem 4.2. *Let $\mathfrak{D}$ be a data set as in* (1.8). *Assume that the Pick matrix $\mathbb{P}$ in* (1.13) *is positive definite, and that M in* (4.4) *is invertible. Here $\widetilde{E}$ and $\widetilde{W}$ are given by* (4.2) *with E, W and Z as in* (1.14). *Then there exists a solution to the problem* **MCNP** *if and only if (1) Λ in* (4.7) *is positive definite and (2) there exists a contractive $2k \times 2k$ matrix K such that $\widetilde{X}$ given by* (4.8) *is of the form in* (4.2).

We now consider the LMI criterion for the problem **CNP**, i.e., the Nevanlinna-Pick problem for functions in the scalar-valued constrained Schur class $\mathcal{S}_1$. Let $\mathfrak{D}$ be a data set as in (1.1). If $|w_i| = 1$ for some i, then a solution exists if and only if $w_i = w_j$ for all $i, j = 1, \dots, n$. In that case, the solution is unique and equal to the constant function with value w_i, and the Pick matrix $\mathbb{P}_x$ in (1.16) (with $k = 1$) is positive semidefinite only for $x = w_i$. On the other hand, if $|w_i| < 1$ for all i, then for $\mathbb{P}_x$ to be positive semidefinite it is necessary that $|x| < 1$. In the latter case we obtain the following result.

Theorem 4.3. *Let $\mathfrak{D}$ be a data set as in* (1.1) *with $|w_i| < 1$ for all i. Define Z, E and W by* (1.14) *with $k = 1$, and assume that $\Delta = \mathbb{P} + WW^* + ZWW^*Z$ is positive definite. Set $\widetilde{\Delta}$ equal to*

$$\widetilde{\Delta} = \mathbb{P} - EE^* - ZEE^*Z^* + (WE^* + ZWE^*Z^*)\Delta^{-1}(EW^* + ZEW^*Z^*).$$

Then there exists a solution to the problem **CNP** *if and only if (1) the matrix $\widetilde{\Delta}$ is positive semidefinite and (2) there exists a contractive $n \times n$ matrix K such that*

$$\widetilde{X} := \Delta^{-1}(EW^* + ZEW^*Z^*) + \Delta^{-\frac{1}{2}}K\widetilde{\Delta}^{\frac{1}{2}} \tag{4.9}$$

is a scalar multiple of the identity matrix I_n. Moreover, given that $\widetilde{\Delta}$ is positive semidefinite and given $x \in \mathbb{D}$, the Pick matrix $\mathbb{P}_x$ in (1.16) *is positive semidefinite if and only if $\widetilde{X} := \overline{x}I_n$ has the form* (4.9) *for some contractive $n \times n$ matrix K. Finally, for the existence of an $x \in \mathbb{D}$ with $\mathbb{P}_x$ positive definite it is necessary that $\widetilde{\Delta}$ be positive definite, and if this is the case, then $\mathbb{P}_x$ is positive definite if and only if $\widetilde{X} := \overline{x}I_n$ has the form* (4.9) *for some strictly contractive $n \times n$ matrix K.*

Proof. It follows from Theorem 1.4, specified to the case $k = 1$, that a solution to problem **CNP** exists if and only if there exists an $x \in \mathbb{C}$ such that the Pick matrix $\mathbb{P}_x$ in (1.16) is positive semidefinite, or equivalently, the Pick matrix $\mathbb{P}'_x$ in (1.15) is positive semidefinite. In this case, the term $I - XX^*$ is just $1 - |x|^2$. As explained above, without loss of generality we may assume that $|x| < 1$.

Now fix an $x \in \mathbb{D}$. We can then take the Schur complement of $\mathbb{P}'_x$ with respect to the block $\begin{bmatrix} 1-|x|^2 & 0 \\ 0 & 1-|x|^2 \end{bmatrix}$ to obtain that $\mathbb{P}'_x$ is positive semidefinite if and only if

$$\mathbb{P} - \frac{1}{1-|x|^2}(E - \overline{x}W)(E^* - xW^*) - \frac{1}{1-|x|^2}Z(E - \overline{x}W)(E^* - xW^*)Z^* \geq 0. \tag{4.10}$$

After multiplication with $1 - |x|^2$ and rearranging terms it follows that (4.10) is equivalent to

$$|x|^2\Delta - \overline{x}(WE^* + ZWE^*Z^*) - x(EW^* + ZEW^*Z^*) + \mathbb{P} - EE^* - ZEE^*Z^* \leq 0$$

and thus equivalent to

$$\begin{aligned}(x\Delta^{\frac{1}{2}} - (WE^* + ZWE^*Z^*)\Delta^{-\frac{1}{2}})(\overline{x}\Delta^{\frac{1}{2}} - \Delta^{-\frac{1}{2}}(EW^* + ZEW^*Z^*)) \leq \\ \leq \mathbb{P} - EE^* - ZEE^*Z^* + (WE^* + ZWE^*Z^*)\Delta^{-1}(EW^* + ZEW^*Z^*) = \tilde{\Delta}.\end{aligned} \tag{4.11}$$

It follows in particular that $\widetilde{\Delta}$ must be positive semidefinite for a solution to exist.

Assume that $x \in \mathbb{D}$ is such that $\mathbb{P}'_x$ is positive semidefinite, and thus also $\widetilde{\Delta}$ is positive semidefinite. Then by the previous computations we see that

$$\widetilde{K} = \overline{x}\Delta^{\frac{1}{2}} - \Delta^{-\frac{1}{2}}(EW^* + ZEW^*Z^*) \tag{4.12}$$

satisfies $\widetilde{K}^*\widetilde{K} \leq \widetilde{\Delta}$. Using Douglas factorization lemma we obtain a contractive matrix K such that $\widetilde{K} = K\widetilde{\Delta}^{\frac{1}{2}}$, and thus

$$\begin{aligned}\Delta^{-1}(EW^* + ZEW^*Z^*) + \Delta^{-\frac{1}{2}}K\widetilde{\Delta}^{\frac{1}{2}} &= \Delta^{-\frac{1}{2}}(\Delta^{-\frac{1}{2}}(EW^* + ZEW^*Z^*) + \widetilde{K}) \\ &= \Delta^{-\frac{1}{2}}(\overline{x}\Delta^{\frac{1}{2}}) = \overline{x}I_n\end{aligned}$$

is a scalar multiple of the identity I_n.

Conversely, assume that $\tilde{\Delta}$ is positive semidefinite and that there exists a contractive matrix K such that (4.9) is a scalar multiple of I_k; say (4.9) is equal to αI_k. Then take x equal to $\overline{\alpha}$. It follows that $\widetilde{K} = K\widetilde{\Delta}^{\frac{1}{2}}$ is given by (4.12) and satisfies $\widetilde{K}^*\widetilde{K} \leq \widetilde{\Delta}$. In other words, for this choice of x the inequality (4.11) holds. Hence the Pick matrix $\mathbb{P}'_x$ is positive semidefinite, and a solution exists.

To verify the last statement, note that $\mathbb{P}'_x$ is positive definite if and only if the Schur complement of $\mathbb{P}'_x$ with respect to the block $\begin{bmatrix} 1-|x|^2 & 0 \\ 0 & 1-|x|^2 \end{bmatrix}$ is positive definite, which is the same as having strict inequality in (4.11). Since the left hand side in (4.11) is positive semidefinite is follows right away that $\widetilde{\Delta}$ must be positive definite for $\mathbb{P}'_x$ to be positive definite. So assume $\widetilde{\Delta} > 0$. If $x \in \mathbb{D}$ with $\mathbb{P}'_x$ positive definite, then $\widetilde{K}$ in (4.12) satisfies $\widetilde{K}^*\widetilde{K} < \widetilde{\Delta}$, and thus $\widetilde{K} = K\widetilde{\Delta}^{\frac{1}{2}}$ for some strict contraction K. With the same argument as above it then follows that $\widetilde{X} = xI_n$ is given by (4.9). Conversely, if K is a strictly contractive matrix such that $\widetilde{X}$ in (4.9) is a scalar multiple of the identity, say $\widetilde{X} = \alpha I_n$, then as above it follows that (4.11) holds with $x = \overline{\alpha}$ but now with strict inequality. □

As in Theorem 4.1, the criterion of Theorem 4.3 is one of verifying whether there exists an element in a certain matrix ball that has a specified structure. In the special case of a constrained Nevanlinna-Pick problem with just one point ($n = 1$) the result of Theorem 4.3 provides a definitive answer to the problem **CNP**.

Corollary 4.4. *Let $\mathfrak{D}$ be a data set as in (1.1) with $n = 1$ and $|w_1| < 1$. Then a solution always exists, and set of values x for which the Pick matrix $\mathbb{P}_x$ is positive semidefinite is the closed disk $\overline{\mathbb{D}}(c; r) = \{x\colon |x - c| \leq r\}$ with center c and radius r given by*

$$c = \frac{w_1(1 - |z_1|^4)}{1 - |z_1|^4|w_1|^2} \quad \text{and} \quad r = \frac{|z_1|^2(1 - |w_1|^2)}{1 - |z_1|^4|w_1|^2}. \tag{4.13}$$

Moreover, $\mathbb{P}_x$ is positive definite if and only if x is in the open disk $\mathbb{D}(c; r) = \{x\colon |x - c| < r\}$.

Proof. The condition that (4.9) be a scalar multiple of I_n is trivially satisfied because $n = 1$. It follows, after some computations, that

$$\Delta = \frac{1 - |w_1|^2|z_1|^4}{1 - |z_1|^2} > 0 \quad \text{and} \quad \widetilde{\Delta} = \left(\frac{|z_1|^2(1 - |w_1|^2)}{1 - |w_1|^2|z_1|^4}\right)^2 \Delta > 0.$$

Thus, by Theorem 4.3, a solution exists. Moreover, the last part of Theorem 4.3 tells us that the set of x so that $\mathbb{P}_x$ is positive semidefinite is given by the complex conjugates of $\widetilde{X}$ in (4.9), where K is now an element of the closed disk $\overline{\mathbb{D}}$. Thus $\mathbb{P}_x$ is positive semidefinite for all x in the closed disk with center c and radius r given by

$$\begin{aligned} c &= \overline{\Delta^{-1}(\overline{w_1}(1 + |z_1|^2))} = \frac{w_1(1 - |z_1|^4)}{1 - |z_1|^4|w_1|^2}, \\ r &= \Delta^{-\frac{1}{2}}\widetilde{\Delta}^{\frac{1}{2}} = \left(\widetilde{\Delta}\Delta^{-1}\right)^{\frac{1}{2}} = \frac{|z_1|^2(1 - |w_1|^2)}{1 - |z_1|^4|w_1|^2}. \end{aligned}$$

□

Remark 4.5. We observed that for the problem **CNP** the points x so that the Pick matrix $\mathbb{P}_x$ in (1.16) is positive semidefinite must be in the open unit disk $\mathbb{D}$ whenever all values $w_1, \ldots, w_n$ from the data set are in $\mathbb{D}$. For the special case that $n = 1$, Corollary 4.4 tells us that the set of x for which $\mathbb{P}_x$ is positive semidefinite is given by a closed disk with center c and radius r given by (4.13). It should then be the case that this disk in contained in $\mathbb{D}$. This is in fact so, since

$$\begin{aligned} 1-|c|-r &= \frac{1-|z_1|^4|w_1|^2+|z_1|^4|w_1|-|w_1|-|z_1|^2+|z_1|^2|w_1|^2}{1-|z_1|^4|w_1|^2} \\ &= \frac{(1-|z_1|^2)(1-|w_1|)(1-|z_1|^2|w_1|)}{1-|z_1|^4|w_1|^2} > 0. \end{aligned}$$

5. Interpolation bodies associated with a set of interpolants

Let us consider the classical (unconstrained) matrix-valued interpolation problem:

MNP: *Given points $z_1,\dots,z_n \in \mathbb{D}$ and matrices $W_1,\dots,W_n \in \mathbb{C}^{k\times k}$, find $S \in \mathcal{S}^{k\times k}$ satisfying interpolation conditions*

$$S(z_j) = W_j \quad \textit{for} \quad j=1,\dots,n. \tag{5.1}$$

and assume that the associated Pick matrix

$$\mathbb{P} = \left[\frac{I-W_iW_j^*}{1-z_i\overline{z_j}}\right]_{i,j=1,\dots,n}$$

is invertible. Choose a point $z_0 \neq z_1,\dots,z_n$ and consider the problem of characterizing the associated interpolation body

$$\mathfrak{B} = \mathfrak{B}(\mathfrak{D},\mathcal{S}^{k\times k},z_0) := \{\widetilde{X} = S(z_0)\colon S\in\mathcal{S}^{k\times k} \text{ satisfies (5.1)}\}.$$

Then application of the classical Pick matrix condition to the $(n+1)$-point set $\{z_1,z_2,\dots,z_n,z_0\}$ leads to the characterization of the Pick body as the set of all $\widetilde{X}$ for which (4.1) is satisfied, where we take

$$\widetilde{E} = \begin{bmatrix} I_k \\ \vdots \\ I_k\end{bmatrix}(1-|z_0|^2)^{1/2}, \quad \widetilde{W} = \begin{bmatrix} W_1 \\ \vdots \\ W_n\end{bmatrix}(1-|z_0|^2)^{1/2}.$$

Hence as an application of Theorem 4.1 we arrive at the well known result (see [8, Section 5.5] for the case of the Schur problem) that interpolation bodies associated with Schur-class interpolants for a data set $\mathfrak{D}$ can be described as matrix balls.

We now consider the interpolation body for constrained Schur-class interpolants for a data set $\mathfrak{D}$:

$$\mathfrak{B} = \mathfrak{B}((\mathfrak{D},(\mathcal{S}_1)^{k\times k},z_0) = \{\widetilde{X} = S(z_0)\colon S\in(\mathcal{S}_1)^{k\times k} \text{ satisfies (5.1)}\}.$$

For simplicity we assume that $k=1$ and $n=1$. We are thus led to the following problem: *Given nonzero points z_0, z_1 in the unit disk $\mathbb{D}$, describe the set*

$$\mathcal{B}_{z_0} = \{w_0\colon s\in\mathcal{S}_1,\ s(z_1)=w_1, s(z_0)=w_0\}. \tag{5.2}$$

Given a data set $(z_1, w_1; z_0)$ as above along with a complex parameter x, we introduce auxiliary matrices as follows:

$$\mathbb{P}'_x = \begin{bmatrix} 1-|x|^2 & 0 & 1-\overline{w_1}x \\ 0 & 1-|x|^2 & \overline{z_1}(1-\overline{w_1}x) \\ 1-w_1\overline{x} & z_1(1-w_1\overline{x}) & \frac{1-|w_1|^2}{1-|z_1|^2} \end{bmatrix},$$

$$E = \begin{bmatrix} 1 \\ \overline{z_0} \\ \frac{1}{1-\overline{z_0}z_1} \end{bmatrix} \delta_0^{1/2}, \quad W_x = \begin{bmatrix} -x \\ -\overline{z_0}x \\ -\frac{w_1}{1-\overline{z_0}z_1} \end{bmatrix} \delta_0^{1/2} \text{ where } \delta_0 := 1-|z_0|^2. \tag{5.3}$$

We then define numbers (i.e., 1×1 matrices) c_x and R_x by

$$c_x = -\, E^*(\mathbb{P}'_x + W_xW_x^*)^{-1}W_x, \quad R_x = \ell_x^{1/2} r_x^{1/2} \text{ where}$$
$$\ell_x = 1 - W_x^* \left(\mathbb{P}'_x + W_xW_x^*\right)^{-1} W_x, \quad r_x = 1 - E^* \left(\mathbb{P}'_x + W_xW_x^*\right)^{-1} E. \tag{5.4}$$

The following result gives an indication of how the geometry of the interpolation body $\mathcal{B}_{z_0}$ is more complicated for the constrained case in comparison with the unconstrained case where it is simply a disk; specifically, we identify a union of a 1-parameter family of disks, where the parameter itself runs over a certain subset of a disk, as a subset of $\mathcal{B}_{z_0}$. In general we use the notation

$$\overline{\mathbb{D}}(C,R) = \{w \in \mathbb{C} \colon |w-C| \le R\}$$

for the closed disk in the complex plane with center $C \in \mathbb{C}$ and radius $R > 0$.

Proposition 5.1. *The union of disks*

$$\bigcup_{x \in \mathbb{D}(c,r), r_x > 0} \overline{\mathbb{D}}(c_x, R_x) \tag{5.5}$$

is a subset of the interpolation body $\mathcal{B}_{z_0}$ in (5.2)*. Here c, r are as in* (4.13) *and r_x, c_x, R_x are as in* (5.4)*.*

Proof. As an application of Theorem 1.4 we see that $w_0 \in \mathcal{B}_{z_0}$ if and only if there exists an $x \in \mathbb{D}$ so that the Pick matrix

$$\mathbb{P}_{x,w_0} = \begin{bmatrix} \frac{1-|w_1|^2}{1-|z_1|^2} & \frac{1-w_1\overline{w_0}}{1-z_1\overline{z_0}} & 1-w_1\overline{x} & z_1(1-w_1x) \\ \frac{1-\overline{w_1}w_0}{1-\overline{z_1}z_0} & \frac{1-|w_0|^2}{1-|z_0|^2} & 1-w_0\overline{x} & z_0(1-w_0x) \\ 1-\overline{w_1}x & 1-\overline{w_0}x & 1-|x|^2 & 0 \\ \overline{z_1}(1-\overline{w_1}x) & \overline{z_0}(1-\overline{w_0}x) & 0 & 1-|x|^2 \end{bmatrix}$$

is positive semidefinite. Interchanging the first two rows with the last two rows and similarly for the columns brings us to

$$\mathbb{P}'_{x,w_0} = \begin{bmatrix} 1-|x|^2 & 0 & 1-\overline{w_1}x & \delta_0^{1/2}(1-\overline{w_0}x) \\ 0 & 1-|x|^2 & \overline{z_1}(1-\overline{w_1}x) & \delta_0^{1/2}\overline{z_0}(1-\overline{w_0}x) \\ 1-w_1\overline{x} & z_1(1-w_1\overline{x}) & \frac{1-|w_1|^2}{1-|z_1|^2} & \delta_0^{1/2}\frac{1-w_1\overline{w_0}}{1-\overline{z_0}z_1} \\ \delta_0^{1/2}(1-w_0\overline{x}) & \delta_0^{1/2}z_0(1-w_0\overline{x}) & \delta_0^{1/2}\frac{1-\overline{w_1}w_0}{1-z_0\overline{z_1}} & 1-|w_0|^2 \end{bmatrix}$$

where we also multiplied the last row and last column by $\delta_0^{1/2} = (1-|z_0|^2)^{1/2}$. Next observe that we can write $\mathbb{P}_{x,w_0}$ as

$$\mathbb{P}'_{x,w_0} = \begin{bmatrix} \mathbb{P}'_x & E + W_x\overline{w_0} \\ E^* + \overline{w_0}W_x^* & 1-w_0\overline{w_0} \end{bmatrix}$$

where $\mathbb{P}'_x$, E and W_x are as in (5.3). We conclude that w_0 is in the interpolation body $\mathcal{B}_{z_0}$ if and only if there is an $x \in \mathbb{D}$ for which the matrix $\mathbb{P}'_{x,w_0}$ is positive-semidefinite. By Corollary 4.4 we see in particular that $\mathbb{P}'_x$ is positive definite if and only if $x \in \mathbb{D}(c,r)$ where c and r are as in (4.13). For a fixed such x we apply Theorem 4.1 (tailored to the case $k = 1$) to see that then $\mathbb{P}'_{x,w_0}$ is positive semidefinite in case $r_x > 0$ and $w_0 \in \overline{\mathbb{D}}(c_x, R_x)$. □

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