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Multivariable Operator-valued Nevanlinna-Pick Interpolation: a Survey

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Abstract. The theory of Nevanlinna-Pick and Carathéodory-Fejér interpolation for matrix- and operator-valued Schur class functions on the unit disk is now well established. Recent work has produced extensions of the theory to a variety of multivariable settings, including the ball and the polydisk (both commutative and noncommutative versions), as well as a time-varying analogue. Largely independent of this is the recent Nevanlinna-Pick interpolation theorem by P.S. Muhly and B. Solel for an abstract Hardy algebra set in the context of a Fock space built from a W^*-correspondence E over a W^*-algebra $\mathcal{A}$ and a $*$-representation σ of $\mathcal{A}$. In this review we provide an exposition of the Muhly-Solel interpolation theory accessible to operator theorists, and explain more fully the connections with the already existing interpolation literature. The abstract point evaluation first introduced by Muhly-Solel leads to a tensor-product type functional calculus in the main examples. A second kind of point-evaluation for the W^*-correspondence Hardy algebra, also introduced by Muhly and Solel, is here further investigated, and a Nevanlinna-Pick theorem in this setting is proved. It turns out that, when specified for examples, this alternative point-evaluation leads to an operator-argument functional calculus and corresponding Nevanlinna-Pick interpolation. We also discuss briefly several Nevanlinna-Pick interpolation results for Schur classes that do not fit into the Muhly-Solel W^*-correspondence formalism.

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1. Introduction

The classical interpolation theorems of Nevanlinna [74] and Pick [79], now approaching the age of one hundred years, can be stated as follows: *Given N distinct points $\lambda_1, \dots, \lambda_N$ in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$ together with N complex numbers $w_1, \dots, w_N$, there exists a Schur class function s* (i.e., s holomorphic from the unit disk $\mathbb{D}$ to the closed disk $\overline{\mathbb{D}}$) *such that*

$$s(\lambda_i) = w_i \text{ for } i = 1, \dots, N$$

if and only if the so-called Pick matrix

$$\mathbb{P} := \left[\frac{1 - w_i \overline{w_j}}{1 - \lambda_i \overline{\lambda_j}} \right]_{i,j=1}^{N}$$

is positive semidefinite.

There is a parallel result usually attributed to Carathéodory and Fejér as well as to Schur (see [33, 34, 50, 99, 100]) for the case where one prescribes an initial segment of Taylor coefficients at the origin rather than functional values at distinct points: *Given $N+1$ complex numbers $s_0, \dots, s_N$, there exists a Schur class function s such that*

$$\frac{1}{i!} \frac{d^i s}{d\lambda^i}(0) = s_i \text{ for } i = 0, \dots, N$$

if and only if the $(N+1) \times (N+1)$ lower-triangular Toeplitz matrix

$$\begin{bmatrix} s_0 & 0 & \cdots & 0 \\ s_1 & s_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ s_N & s_{N-1} & \cdots & s_0 \end{bmatrix}$$

is contractive.

Over the years these results have been an inspiration for new mathematics, often interacting symbiotically with assorted engineering applications. We mention in particular that the operator-theoretic formulation of the Nevanlinna-Pick/Carathéodory-Fejér interpolation problem due to Sarason [97] led to the advance in operator theory known as commutant lifting theory introduced by Sz.-Nagy-Foias [72]; see also [51, 53]. Commutant lifting in turn provides a unifying framework for the handling of a variety of interpolation problems of Nevanlinna-Pick/Carathéodory-Fejér type for matrix- and operator-valued functions on the disk, and, more generally, for left- and right-tangential interpolation with operator-argument (**LTOA/RTOA**); cf., [53]. The operator-argument approach to interpolation emerged as one of the most popular ways of handling Nevanlinna-Pick and Carathéodory-Fejér interpolation conditions in a unified way and can be seen to be equivalent to the Sarason formulation (see [24, 21]). We mention that it is the Sarason formulation which plays a prominent role in connection with the H^∞-control theory (see, e.g., [24]) where it takes the form of a model matching problem.

Our goal here is to survey recent generalizations of the theory of Nevanlinna-Pick interpolation to several-variable contexts. In addition there has been a lot of recent work on noncommutative function theory, where one plugs in a tuple of noncommuting operators (or matrices) as the function arguments and outputs a matrix or operator. In this regard, we mention the papers [2, 3, 4, 5, 11, 17, 18, 19, 21, 28, 29, 32, 40, 48, 65, 93, 96] for the commutative setting and [20, 37, 55, 58, 83, 85, 86, 87, 88] for the noncommutative setting. We mention yet another direction having an analogue of the Schur class and of Nevanlinna-Pick interpolation, namely: the unit ball of the algebra of lower triangular matrices acting as operators on $\ell^2(\mathbb{Z})$; in the 1990s there was a lot of activity on Beurling-Lax representation and an analogue of **LTOA/RTOA** interpolation theory having connections with robust control and model reduction for time-varying systems (see [8, 23, 41, 42, 98]).

Going beyond these types of results is the recent generalized Nevanlinna-Pick interpolation theorem [69] and the generalized Schur class [70] of Muhly-Solel, where the concept of *completely positive kernel* or *completely positive map* enters into various characterizations. In particular, the Muhly-Solel result, when specialized to various specific settings, is different from the standard **LTOA** or **RTOA** formulation, in that the point-evaluation is actually with a tensor-type functional calculus. The simplest instance of this is what we call *Riesz-Dunford* interpolation, where one is given operators Z and W on a Hilbert space $\mathcal{Z}$ and one seeks a scalar Schur class function s so that $s(Z) = W$. We discuss how a solution of this problem can be had by using existing theory for **LTOA**, for both the unit disk setting and the right half-plane setting. In particular, the theorem for the right half-plane setting gives a solution which appears quite different from that obtained by Cohen-Lewkowicz [36]. We also show how the Muhly-Solel solution criterion (involving complete positivity of a kernel or of a map between C^*-algebras) can be seen to be equivalent to the criterion (involving positivity of a single block matrix) obtained from the **LTOA** theory. A similar story holds in the setting of the unit ball (commutative and noncommutative). There is an analogue of the Riesz-Dunford Nevanlinna-Pick interpolation problem in the setting of the unit ball which is handled by the Muhly-Solel theory; to the uninitiated the solution criterion looks somewhat strange since it involves *complete positivity* (in the sense of [31]) rather than merely *positivity* of a kernel, or, in another formulation, *complete positivity* of a map between C^*-algebras rather than merely positive semidefiniteness of a single block operator matrix. We resolve this situation by giving a direct proof that the two solution criteria are equivalent. In the recent paper [71] Muhly and Solel introduced a second kind of point-evaluation in the context of a generalized Poisson kernel. We develop here some further properties of this point-evaluation and derive the corresponding Nevanlinna-Pick interpolation theorem. For instance, it is shown that the Muhly-Solel Hardy algebra with respect to this point-evaluation is in general not multiplicative; a multiplicative law reminiscent of that in the operator-argument functional calculus and time-varying system theory literature is proved. This connection with time-varying systems was already hinted upon in [71]. In fact, when specified for the standard one-variable Schur class, the Nevanlinna-Pick interpolation theorem for this alternative point-evaluation gives us precisely the corresponding left-tangential operator-argument result. The precise connections with the time-varying-system interpolation theory remain to be worked out.

There is yet another generalized theory of Schur class (or, more precisely, Schur-Agler class) and Nevanlinna-Pick interpolation where one defines a Schur class starting from a family of test functions (see [44, 46, 64] and see also [6, Chapter 13] for an introduction to this approach). It is well appreciated that the Agler theory for the polydisk is an example for this theory (indeed, this is the motivating example); precise specification of what other examples can be covered, such as the higher rank graph algebras of [62] and the Hardy algebras associated with product decompositions along semigroups more general than $\mathbb{Z}$ (see [102]), is an ongoing area of investigation [15].

In this survey of interpolation problems of Nevanlinna-Pick type in a variety of settings, we discuss only criteria for existence of solutions; we do not discuss characterizations of the various Schur classes via realization as the transfer function of a conservative linear system or of the construction of solutions or parametrization of solutions of interpolation problems via linear-fractional formulas, although in the various cases often such topics are worked out in the literature. In general we do not discuss the techniques used for establishing these solution criteria; let us only mention here that, just as in the classical case, there are a variety of techniques for analyzing these types of multivariable interpolation problems. We mention specifically commutant lifting theory [80, 81, 28, 29], the "lurking isometry" method [4, 5, 28, 29, 17, 18, 19], the Fundamental-Matrix-Inequality method of Potapov [32, 21] as well as the Ball-Helton Grassmannian Kreĭn-space method [49].

The paper [21] also surveys the various types of operator-valued interpolation problems but with a focus on the Drury-Arveson Schur-multiplier class. In addition to **LTOA**, the paper [21] treats a more general version of **LTOA** (where the joint spectra of $Z^{(1)}, \dots, Z^{(N)}$ are no longer required to be in the open ball $\mathbb{B}^d$), and a still more general Abstract Interpolation Problem (see [59] for the single-variable version). These more general formalisms handle more general interpolation problems (e.g., boundary interpolation) which go beyond the original results based on the commutant lifting approach. We do not discuss these more general problems here.

We focus on interpolation theory on the unit disk and multivariable generalizations of the unit disk; this means we ignore all the activity that has been going on of late on interpolation theory for the Nevanlinna class (holomorphic functions taking the right half-plane into itself) and its multivariable generalizations. We do however give the complex disk analogue of results from [35, 36] which are set out in a real right half-plane setting.

The paper is organized as follows. In Section 2 we review Nevanlinna-Pick interpolation for the single-variable case. Perhaps new is the derivation of the result for the tangential Riesz-Dunford interpolation problem due to Rosenblum-Rovnyak [96, Section 2.3] and for the full Riesz-Dunford interpolation problem as an application of known results for **LTOA**. In the third section various extensions of the one-variable theory for functions of several variables are discussed. We consider the unit ball case (both commutative and noncommutative) in Subsections 3.1 and 3.2, and the generalization to so-called free semigroupoid algebras [66, 68, 60, 61] in Subsection 3.3 where the starting point is a directed graph (also called a quiver). New in the free semigroupoid algebra setting is a more explicit formula for the point-evaluation and of the criterion for existence of solutions of the associated Nevanlinna-Pick interpolation problem. We also work out the Nevanlinna-Pick result for a specific quiver. This example corresponds to Nevanlinna-Pick interpolation for functions in a certain subalgebra of a matrix-valued H^∞-space. In Subsection 3.4 we develop parallel results for the Schur-Agler class in the polydisk setting for both the commutative and noncommutative settings. Most of the material in Sections 2 and 3 should be well known to the experts in the interpolation theory community; the focus here is on the connections among the various results. Section 4 contains the relatively new results on a generalized Schur class and Nevanlinna-Pick interpolation obtained by P.S. Muhly and B. Solel [69, 70]. The material is presented for the expert in operator theory not familiar with such notions as "W^*-correspondence" and "Hilbert module" which are more common in the operator algebra literature. The novel part of the exposition here is to define the point-evaluation in a direct way without making explicit use of the dual correspondence. Moreover, we prove a Nevanlinna-Pick theorem for the second kind of point-evaluation for the W^*-correspondence Schur class that was recently introduced in [71]. We recover the results for the single-variable case (Section 2) and the free semigroupoid algebra setting (Subsection 3.3) as an application of the general Muhly-Solel theory; a key ingredient here is careful understanding of the connections between completely positive kernels and completely positive maps versus merely positive semidefinite operator matrices, including an application of Choi's theorem [38]. A final subsection gives some perspective on connections of the Muhly-Solel theory with the time-varying interpolation theory of the 1990s. The last section discusses the test-function approach and directions for future work.

The notation is mostly standard but we mention here a few conventions for reference. For Ω any index set and B a Banach space with norm $\| \, \|_B$, the symbol $\ell^2_B(\Omega)$ denotes the space of B-valued norm-squared summable sequences indexed by Ω:

$$\ell^2_B(\Omega) = \Big\{ \xi\colon \Omega \to B\colon \sum_{\omega \in \Omega} \|\xi(\omega)\|^2_B < \infty \Big\}.$$

Most often the choice $\Omega = \mathbb{Z}$ (the integers) or $\Omega = \mathbb{Z}_+$ (the nonnegative integers) appears. For Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ the symbol $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ stands for the space of bounded linear operators from $\mathcal{U}$ into $\mathcal{Y}$. We write $H^2_{\mathcal{U}}(\mathbb{D})$ for the Hardy space of analytic functions $f : \mathbb{D} \to \mathcal{U}$ that can be extended to square integrable functions on the unit circle $\mathbb{T} := \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$. With $H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(\mathbb{D})$ we denote the Banach space of uniformly bounded analytic functions on the unit disc $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$.

2. Operator-valued Nevanlinna-Pick interpolation: the one-variable case

Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces. With $\mathcal{S}(\mathcal{U},\mathcal{Y})$ we denote the $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued Schur class, i.e., the set of all holomorphic functions S on the unit disc $\mathbb{D}$ whose values are contractive operators in $\mathcal{L}(\mathcal{U},\mathcal{Y})$:

$$\|S(\lambda)\| \leq 1 \text{ for all } \lambda \in \mathbb{D}. \tag{2.1}$$

In this section we consider variations on the classical Nevanlinna-Pick interpolation problem for functions in the Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$.

2.1. Standard functional calculus Nevanlinna-Pick interpolation problems

The standard operator-valued Nevanlinna-Pick interpolation problem and the left- and right-tangential versions are the following problems:

(1) The Full Operator-Valued Nevanlinna-Pick (**FOV-NP**) interpolation problem: *Given $\lambda_1,\dots,\lambda_N$ in $\mathbb{D}$ and operators $W_1,\dots,W_N$ in $\mathcal{L}(\mathcal{U},\mathcal{Y})$, determine when there exists a Schur class function $S \in \mathcal{S}(\mathcal{U},\mathcal{Y})$ such that $S(\lambda_i) = W_i$ for $i = 1,\dots,N$.*

(2) The Left-Tangential Nevanlinna-Pick (**LT-NP**) interpolation problem: *Given $\lambda_1,\dots,\lambda_N$ in $\mathbb{D}$, an auxiliary Hilbert space $\mathcal{C}$ and operators $X_1,\dots,X_N$ in $\mathcal{L}(\mathcal{Y},\mathcal{C})$ and $Y_1,\dots,Y_N$ in $\mathcal{L}(\mathcal{U},\mathcal{C})$, determine when there exists a Schur class function $S \in \mathcal{S}(\mathcal{U},\mathcal{Y})$ such that $X_iS(\lambda_i) = Y_i$ for $i = 1,\dots,N$.*

(3) The Right-Tangential Nevanlinna-Pick (**RT-NP**) interpolation problem: *Given $\lambda_1,\dots,\lambda_N$ in $\mathbb{D}$, an auxiliary Hilbert space $\mathcal{C}$ and operators U_1, …, U_N in $\mathcal{L}(\mathcal{C},\mathcal{U})$ and V_1, …, V_N in $\mathcal{L}(\mathcal{C},\mathcal{Y})$, determine when there exists a Schur class function $S \in \mathcal{S}(\mathcal{U},\mathcal{Y})$ such that $S(\lambda_i)U_i = V_i$ for $i = 1,\dots,N$.*

Note that the **FOV-NP** interpolation problem is a particular case of the **LT-NP** interpolation problem, namely with $\mathcal{C} = \mathcal{Y}$ and $X_i = I_{\mathcal{Y}}$ for $i = 1,\dots,N$. Moreover, **LT-NP** reduces to **RT-NP**, and vice versa, as follows. A function $S : \mathbb{D} \to \mathcal{L}(\mathcal{U},\mathcal{Y})$ is in the Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ if and only if the function $S^\sharp : \mathbb{D} \to \mathcal{L}(\mathcal{Y},\mathcal{U})$ defined by

$$S^\sharp(\lambda) = S(\overline{\lambda})^* \quad (\lambda \in \mathbb{D}) \tag{2.2}$$

is in $\mathcal{S}(\mathcal{Y},\mathcal{U})$. In particular, S is a solution to the **LT-NP** interpolation problem, with data as above, if and only if $S^\sharp : \mathbb{D} \to \mathcal{L}(\mathcal{Y},\mathcal{U})$ is a solution to the **RT-NP** with data $\overline{\lambda}_1,\dots,\overline{\lambda}_N \in \mathbb{D}$, $X_1^*,\dots,X_N^* \in \mathcal{L}(\mathcal{C},\mathcal{Y})$ and $Y_1^*,\dots,Y_N^* \in \mathcal{L}(\mathcal{C},\mathcal{U})$.

The relations among these variations on the classical Nevanlinna-Pick interpolation problem are also exhibited in their solutions; see, e.g., [24, 51].

Theorem 2.1. *Let the data for the* **FOV-NP**, **LT-NP** *and the* **RT-NP** *interpolation problem be as given in* (1), (2) *and* (3) *above.*

1. *A solution to the* **FOV-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm FOV} := \left[\frac{I_{\mathcal{Y}} - W_i W_j^*}{1 - \lambda_i \overline{\lambda}_j} \right]_{i,j=1}^N \tag{2.3}$$

is positive semidefinite.

2. *A solution to the* **LT-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm LT} := \left[\frac{X_i X_j^* - Y_i Y_j^*}{1 - \lambda_i \overline{\lambda}_j} \right]_{i,j=1}^N \tag{2.4}$$

is positive semidefinite.

3. *A solution to the* **RT-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm RT} := \left[\frac{U_i^* U_j - V_i^* V_j}{1 - \overline{\lambda}_i \lambda_j} \right]_{i,j=1}^N \tag{2.5}$$

is positive semidefinite.

2.2. Operator-argument functional calculus Nevanlinna-Pick interpolation problems

Let S be a Schur class function in $\mathcal{S}(\mathcal{U},\mathcal{Y})$ with Taylor coefficients $S_0, S_1, \dots$ in $\mathcal{L}(\mathcal{U},\mathcal{Y})$, i.e.,

$$S(\lambda) = \sum_{n=0}^{\infty} \lambda^n S_n \quad (\lambda \in \mathbb{D}).$$

For an auxiliary Hilbert space $\mathcal{C}$ and operators X in $\mathcal{L}(\mathcal{Y},\mathcal{C})$ and T in $\mathcal{L}(\mathcal{C})$ with $r_{\rm spec}(T) < 1$ the left-tangential operator-argument point-evaluation $(XS)^{\wedge L}(T)$ is given by

$$(XS)^{\wedge L}(T) = \sum_{n=0}^{\infty} T^n X S_n. \tag{2.6}$$

Similarly, for an auxiliary Hilbert space $\mathcal{C}$ and operators U in $\mathcal{L}(\mathcal{C},\mathcal{U})$ and A in $\mathcal{L}(\mathcal{C})$ with $r_{\rm spec}(A) < 1$ the right-tangential operator-argument point-evaluation $(SU)^{\wedge R}(A)$ is given by

$$(SU)^{\wedge R}(A) = \sum_{n=0}^{\infty} S_n U A^n. \tag{2.7}$$

With respect to these operator-argument functional calculi we consider the following tangential Nevanlinna-Pick interpolation problems.

(1) The Left-Tangential Nevanlinna-Pick interpolation problem with Operator-Argument (**LTOA-NP**): *Given an auxiliary Hilbert space $\mathcal{C}$ together with operators $T_1, \dots, T_N$ in $\mathcal{L}(\mathcal{C})$ with $r_{\rm spec}(T_i) < 1$ for $i = 1, \dots, N$ and operators $X_1, \dots, X_N$ in $\mathcal{L}(\mathcal{Y},\mathcal{C})$ and $Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{U},\mathcal{C})$, determine when there exists a Schur class function S in $\mathcal{S}(\mathcal{U},\mathcal{Y})$ so that $(X_i S)^{\wedge L}(T_i) = Y_i$ for $i = 1, \dots, N$.*

(2) The Right-Tangential Nevanlinna-Pick interpolation problem with Operator-Argument (**RTOA-NP**): *Given an auxiliary Hilbert space $\mathcal{C}$ together with operators $A_1, \dots, A_N$ in $\mathcal{L}(\mathcal{C})$ with $r_{\rm spec}(A_i) < 1$ for $i = 1, \dots, N$ and operators $U_1, \dots, U_N$ in $\mathcal{L}(\mathcal{C},\mathcal{U})$ and $V_1, \dots, V_N$ in $\mathcal{L}(\mathcal{C}, \mathcal{Y})$, determine when there exists a Schur class function S in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ so that $(SU_i)^{\wedge R}(A_i) = V_i$ for $i = 1, \dots, N$.*

The **LT-NP** (**RT-NP**) interpolation problem is the special case of the **LTOA-NP** (**RTOA-NP**) interpolation problem with $T_i = \lambda_i I_{\mathcal{U}}$ ($A_i = \lambda_i I_{\mathcal{U}}$). As in the case of the standard functional calculus left- and right-tangential interpolation problems, here also **LTOA-NP** reduces to **RTOA-NP** and vice versa; again via the transformation $S \mapsto S^\sharp$ in (2.2) and a similar transformation of the data. Moreover, it suffices to consider the **LTOA-NP** interpolation problem for the case $N = 1$, since the general case, with data as above, is covered by the **LTOA-NP** interpolation problem with data

$$T = \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_N \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix};$$

see [53, Section I.3] for more details. In particular, a multipoint classical interpolation problem of Nevanlinna-Pick Carathéodory-Fejér type can be conveniently encoded as a single-point **LTOA-NP** or **RTOA-NP** problem; we refer to [24, Sections 16.8-9] for the single-variable case and [19] for multivariable examples. However, we write out the results without this reduction for better comparison with the classical case.

The solutions to the **LTOA-NP** and **RTOA-NP** interpolation problems are given in the following theorem.

Theorem 2.2. *Let the data for the* **LTOA-NP** *and* **RTOA-NP** *interpolation problems be as given in* (1) *and* (2) *above.*

1. *A solution to the* **LTOA-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm LTOA} := \left[\ \textstyle\sum_{n=0}^\infty T_i^n (X_i X_j^* - Y_i Y_j^*) T_j^{*n} \ \right]_{i,j=1}^N \tag{2.8}$$

   is positive semidefinite.
2. *A solution to the* **RTOA-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm RTOA} := \left[\ \textstyle\sum_{n=0}^\infty A_i^{*n} (U_i^* U_j - V_i^* V_j) A_j^n \ \right]_{i,j=1}^N \tag{2.9}$$

   is positive semidefinite.

2.3. Riesz-Dunford functional calculus Nevanlinna-Pick interpolation problems

As a third variant we consider Nevanlinna-Pick interpolation problems for Schur class functions with a Riesz-Dunford functional calculus. In this case the Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ are both equal to $\mathbb{C}$, i.e., the Schur class functions are scalar-valued, but the unit disc is replaced by strict contractions on some fixed Hilbert space $\mathcal{Z}$. Given a Schur class function s in $\mathcal{S}(\mathbb{C},\mathbb{C})$ and an operator Z in $\mathcal{L}(\mathcal{Z})$ with spectral radius less than 1, we define $s(Z)$ via the Riesz-Dunford functional calculus:

$$s(Z) = \frac{1}{2\pi i}\int_{\rho\mathbb{T}} (\zeta I_{\mathcal{Z}} - Z)^{-1} s(\zeta)\, d\zeta = \sum_{n=0}^{\infty} s_n \cdot Z^n \quad \text{if} \quad s(\lambda) = \sum_{n=0}^{\infty} s_n \lambda^n. \tag{2.10}$$

Here the multiplication $s_n \cdot Z^n$ is scalar multiplication and $\rho < 1$ is chosen large enough so that the circle $\rho\mathbb{T}$ encloses the spectrum of Z.

With respect to this functional calculus we consider the following Nevanlinna-Pick interpolation problems.

(1) The Full Riesz-Dunford Nevanlinna-Pick (**FRD-NP**) interpolation problem: *Given a Hilbert space $\mathcal{Z}$, strict contractions $Z_1, \dots, Z_N$ in $\mathcal{L}(\mathcal{Z})$ and operators $W_1, \dots, W_N$ in $\mathcal{L}(\mathcal{Z})$, determine when there exists a Schur class function s in $\mathcal{S}(\mathbb{C},\mathbb{C})$ such that $s(Z_i) = W_i$ for $i = 1, \dots, N$.*

(2) The Left-Tangential Riesz-Dunford Nevanlinna-Pick (**LTRD-NP**) interpolation problem: *Given Hilbert spaces $\mathcal{Z}$ and $\mathcal{C}$, strict contractions $Z_1, \dots, Z_N$ in $\mathcal{L}(\mathcal{Z})$ and operators $X_1, \dots, X_N, Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{Z},\mathcal{C})$, determine when there exists a Schur class function s in $\mathcal{S}(\mathbb{C},\mathbb{C})$ such that $X_i s(Z_i) = Y_i$ for $i = 1, \dots, N$.*

(3) The Right-Tangential Riesz-Dunford Nevanlinna-Pick (**RTRD-NP**) interpolation problem: *Given Hilbert spaces $\mathcal{Z}$ and $\mathcal{C}$, strict contractions $Z_1, \dots, Z_N$ in $\mathcal{L}(\mathcal{Z})$ and operators $U_1, \dots, U_N, V_1, \dots, V_N$ in $\mathcal{L}(\mathcal{C},\mathcal{Z})$, determine when there exists a Schur class function s in $\mathcal{S}(\mathbb{C},\mathbb{C})$ such that $s(Z_i)U_i = V_i$ for $i = 1, \dots, N$.*

The **FRD-NP** interpolation problem is the special case of the **RTRD-NP** interpolation problem with $\mathcal{C} = \mathcal{Z}$ and $U_1 = \cdots = U_N = I_{\mathcal{Z}}$. Let us first consider the **RTRD-NP** interpolation problem with $\mathcal{C} = \mathbb{C}$, that is, the operators $U_1, \dots, U_N, V_1, \dots, V_N$ are vectors $u_1, \dots, u_N, v_1, \dots, v_N$ in $\mathcal{Z}$. This type of Nevanlinna-Pick interpolation problem is studied in the book [96] (see also [94, 95]), in a slightly more general form where only a local weak version of the Riesz-Dunford functional calculus is required; in particular, there it is shown how any classical Nevanlinna-Pick Carathéodory-Fejér interpolation problem can be encoded as a (even single-point) **RTRD-NP** problem. In this case $s(Z_i)u_i$ is equal to $(u_i s)^{\wedge L}(Z_i)$ as defined in (2.6), and thus, the **RTRD-NP** interpolation problem is a **LTOA-NP** interpolation problem with the same data set. If $\mathcal{C} \neq \mathbb{C}$ is finite dimensional with orthonormal basis $\{e_1, \dots, e_\kappa\}$, then the **RTRD-NP** interpolation problem can still be seen as a **LTOA-NP** interpolation problem. In this case the identity $s(Z_i)U_i = V_i$ holds if and only if

$$s(Z_i)U_ie_j = V_ie_j \text{ for } j = 1, \dots, \kappa.$$

Thus the **RTRD-NP** interpolation problem becomes a **LTOA-NP** interpolation problem with tangential interpolation conditions indexed by the Cartesian product set

$$\{1, \dots, N\} \times \{1, \dots, \kappa\}$$

and with interpolation data

$$Z_{i,j} := Z_i, \quad x_{i,j} := U_ie_j, \quad y_{i,j} := V_ie_j \text{ for } i = 1, \dots, N \text{ and } j = 1, \dots, \kappa.$$

The **LTRD-NP** interpolation problem reduces to a **RTOA-NP** interpolation problem in a similar way.

In conclusion, we have the following result.

Theorem 2.3. *Let the data for the* **FRD-NP**, **LTRD-NP** *and* **RTRD-NP** *interpolation problems be as given in (1) and (2) above with, for the* **LTRD-NP** *and* **RTRD-NP** *interpolation problems* $\dim \mathcal{C} = \kappa$ *and* $\{e_1, \dots, e_\kappa\}$ *an orthonormal basis for* $\mathcal{C}$, *and for the* **FRD-NP** *interpolation problem* $\dim \mathcal{Z} = \kappa$ *and* $\{e_1, \dots, e_\kappa\}$ *an orthonormal basis for* $\mathcal{Z}$.

1. *A solution to the* **FRD-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm FRD} := \left[\; \sum_{n=0}^\infty Z_i^n(e_{i'}e_{j'}^* - W_ie_{i'}e_{j'}^*W_j^*)Z_j^{*n} \; \right]_{(i,i'),(j,j')\in\{1,\dots,N\}\times\{1,\dots,\kappa\}} \tag{2.11}$$

is positive semidefinite.

2. *A solution to the* **LTRD-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm LTRD} := \left[\; \sum_{n=0}^\infty Z_i^{*n}(X_i^*e_{i'}e_{j'}^*X_j - Y_i^*e_{i'}e_{j'}^*Y_j)Z_j^n \; \right]_{(i,i'),(j,j')\in\{1,\dots,N\}\times\{1,\dots,\kappa\}} \tag{2.12}$$

is positive semidefinite.

3. *A solution to the* **RTRD-NP** *interpolation problem exists if and only if the associated Pick matrix*

$$\mathbb{P}_{\rm RTRD} := \left[\; \sum_{n=0}^\infty Z_i^n(U_ie_{i'}e_{j'}^*U_j^* - V_ie_{i'}e_{j'}^*V_j^*)Z_j^{*n} \; \right]_{(i,i'),(j,j')\in\{1,\dots,N\}\times\{1,\dots,\kappa\}} \tag{2.13}$$

is positive semidefinite.

The statements in Theorem 2.3 remain true for the case that $\mathcal{C}$ is a separable Hilbert space (i.e., $\kappa = \infty$). The Pick matrices $\mathbb{P}_{\rm FRD}$, $\mathbb{P}_{\rm LTRD}$ and $\mathbb{P}_{\rm RTRD}$ in this case are infinite operator matrices; positivity is then to be interpreted as positivity of all $M \times M$-finite sections for each $M \in \mathbb{Z}_+$.

We wish to point out that Cohen and Lewkowicz [35, 36] have studied the Full Riesz-Dunford Nevanlinna-Pick interpolation problem in a somewhat different setting (where the unit disk is replaced by the right half-plane and where the Schur class is replaced by positive-real-odd functions – holomorphic functions f on the right half-plane such that (i) $\overline{f(\lambda)} = f(-\overline{\lambda})$ and $f(\lambda) + \overline{f(\lambda)} \geq 0$ for $\lambda + \overline{\lambda} > 0$

and (ii) f has meromorphic pseudocontinuation to the left half-plane such that $\overline{f(-\overline{\lambda})} = f(\lambda)$). Adaption of these results to our setting leads to the following refinement of part (1) of Theorem 2.3. For simplicity we consider only the case where $N = 1$; just as is the case for **LTOA-NP**, this reduction is without loss of generality.

Theorem 2.4. *Suppose that we are given the data set*

$$Z \in \mathcal{L}(\mathcal{Z}), \quad W \in \mathcal{L}(\mathcal{Z})$$

*for a Full Riesz-Dunford Nevanlinna-Pick (***FRD-NP***) interpolation problem with* $N = 1$. *Assume that* $\dim \mathcal{Z} = \kappa < \infty$ *with* $\{e_1, \dots, e_\kappa\}$ *equal to an orthonormal basis for* $\mathcal{Z}$. *Then the following are equivalent:*

1. *A solution to the* **FRD-NP** *interpolation problem exists, i.e., the Pick matrix* $\mathbb{P}_{\mathrm{FRD}}$ *given by* (2.11) (*with* $N = 1$, $Z = Z_1$, $W = W_1$)
$$\mathbb{P}_{\mathrm{FRD}} := \left[\; \sum_{n=0}^{\infty} Z^n (e_i e_j^* - W e_i e_j^* W^*) Z^{*n} \; \right]_{i,j=1,\dots,\kappa} \tag{2.14}$$
is positive semidefinite.
2. *For each vector* $u \in \mathcal{Z}$, *the* **RTRD-NP** *problem*
   find a scalar Schur-class function s so that $s(Z)u = Wu$ (*where a priori, the solution* $s = s_u$ *depends on the vector* u)
has a solution, or, equivalently, for each vector $u \in \mathcal{Z}$ *the Pick matrix* $\mathbb{P}_{\mathrm{FRD},u}$ *given by*
$$\mathbb{P}_{\mathrm{FRD},u} = \sum_{n=0}^{\infty} Z^n (uu^* - Wuu^*W^*) Z^{*n} \tag{2.15}$$
is positive-semidefinite.
3. *The Pick matrix* $\mathbb{P}_{\mathrm{FRD},u}$ *is positive-semidefinite for each vector* $u \in \mathcal{Z}$ *satisfying the additional constraint*
$$u = \arg\max_{u'} \operatorname{rank} \begin{bmatrix} u' & Zu' & \cdots & Z^{n-1}u' \end{bmatrix}, \tag{2.16}$$
i.e., for any maximally Z-controllable vector u.
4. *The matrix* W *is in the double commutant* $\{Z\}''$ *of* Z *and the Pick matrix* $\mathbb{P}_{\mathrm{FRD},u}$ (2.15) *is positive-semidefinite for some maximally* Z*-controllable vector* u (*i.e., a vector* u *satisfying* (2.16)).

Proof. Note first that the statements in condition (1) are equivalent as an application of part (1) of Theorem 2.3 and the statements in condition (2) are equivalent as an application of part (3) of Theorem 2.3. We show (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) and (2) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2): Note that positive-semidefiniteness of $\mathbb{P}_{FRD}$ in (2.14) can equivalently be expressed as

$$\sum_{i,j=1}^{\kappa} u_i^* \left[\sum_{n=0}^{\infty} Z^n (e_i e_j^* - W e_i e_j^* W^*) Z^{*n} \right] u_j \geq 0 \text{ for all } u_1, \dots, u_\kappa \in \mathcal{Z}. \tag{2.17}$$

In particular, if we take u_i to be of the form $u_i = c_i u_0$ where $c_1, \dots, c_\kappa$ are scalars and $u_0 \in \mathbb{C}^\kappa$ is a fixed vector, condition (2.17) becomes

$$u_0^* \mathbb{P}_{FRD,u} u_0 \geq 0$$

where we have set $u = \sum_{i=1}^{\kappa} c_i e_i$. As u and u_0 are arbitrary, we see that we have verified condition (2).

(2) $\iff$ (3): Note that (2) $\Longrightarrow$ (3) is trivial. For the converse, one can use the fact that the set of maximal Z-controllability-rank vectors u form a dense set in the set of all vectors in $\mathcal{Z}$ – for details we refer to [36].

(2) $\Longrightarrow$ (4): Note that the second part of (4) follows from (2) trivially. It remains to show that (2) implies that $W \in \{Z\}''$. If we assume condition (1) of the theorem so that $W = s(Z)$ for some scalar Schur-class s, it is then trivial that $W \in \{Z\}''$; the issue is to prove that $W \in \{Z\}''$ under the seemingly weaker hypothesis (2). The following proof is adapted from a similar argument in [35] – see Remark 2.6 below for additional discussion.

We first observe that positive-semidefiniteness of the matrix $\mathbb{P}_{\mathrm{FRD},u}$ for all u is invariant under similarity transformations

$$(Z, W) \mapsto (T^{-1}ZT, T^{-1}WT).$$

More precisely, if we use the notation $\mathbb{P}_{\mathrm{FRD},u}(Z, W)$ to indicate the dependence of the matrix in (2.15) on Z, W, it is easily verified that

$$\mathbb{P}_{\mathrm{FRD},u}(T^{-1}ZT, T^{-1}WT) = T^{-1}\left(\mathbb{P}_{\mathrm{FRD},T^{-1}u}(Z, W)\right) T^{-1*} \tag{2.18}$$

from which the assertion follows.

We show next that, under hypothesis (2),

$$\textit{any } Z\textit{-invariant subspace of } \mathcal{Z} \textit{ is also } W\textit{-invariant.} \tag{2.19}$$

Indeed, by an appropriate choice of basis in $\mathcal{Z}$, we may assume that Z has the upper triangular form $Z = \left[\begin{smallmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{smallmatrix}\right]$ and W is given in block 2×2 form $W = \left[\begin{smallmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{smallmatrix}\right]$ with respect to the same basis. Our goal is to show that $W_{21} = 0$. Toward this goal, introduce the similarity transformation $T_\epsilon = \left[\begin{smallmatrix} I & 0 \\ 0 & \epsilon I \end{smallmatrix}\right]$. By the claim in the preceding paragraph, we may replace (Z, W) by (Z_ϵ, W_ϵ) where

$$Z_\epsilon := T_\epsilon^{-1} Z T_\epsilon = \left[\begin{smallmatrix} Z_{11} & \epsilon Z_{12} \\ 0 & Z_{22} \end{smallmatrix}\right], \quad W_\epsilon := \left[\begin{smallmatrix} W_{11} & \epsilon W_{12} \\ \frac{1}{\epsilon} W_{21} & W_{22} \end{smallmatrix}\right].$$

The hypothesis (2) tells us that the matrix

$$\sum_{n=0}^{\infty} Z_\epsilon^n \left(uu^* - W_\epsilon uu^* W_\epsilon^*\right) Z_\epsilon^{*n} \geq 0 \text{ for all } \epsilon > 0. \tag{2.20}$$

As we let ϵ tend to zero, the expression (2.20) has the same asymptotics as

$$\sum_{n=0}^{\infty} Z^n (uu^* - W_\epsilon uu^* W_\epsilon^*) Z^{*n}. \tag{2.21}$$

If we take the vector u to have the form $u = \left[\begin{smallmatrix} u_1 \\ 0 \end{smallmatrix}\right]$, then the $(2,2)$-block of the second term in the $n = 0$ term in (2.21) in turn has the form

$$-\begin{bmatrix} \frac{1}{\epsilon} W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \begin{bmatrix} u_1^* & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon} W_{21}^* \\ W_{22}^* \end{bmatrix} = -\frac{1}{\epsilon^2} W_{21} u_1 u_1^* W_{21}^*. \tag{2.22}$$

Since the vector u_1 is arbitrary, it is clear from (2.22) that it is not possible for the quantity in (2.20) to be positive-semidefinite for all $\epsilon > 0$ if $W_{21} \neq 0$. We therefore conclude that we must have $W_{21} = 0$ and (2.19) follows.

As a consequence of (2.19), if we choose a basis for $\mathcal{Z}$ to put Z in upper triangular Jordan form, say

$$Z = \begin{bmatrix} J_{\lambda_1}^{(n_1)} & & \\ & \ddots & \\ & & J_{\lambda_k}^{(n_k)} \end{bmatrix} \text{ where } J_{\lambda_i}^{(n_i)} = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \text{ (of size } n_i \times n_i),$$

then it follows that $W = \begin{bmatrix} W_{11} & & \\ & \ddots & \\ & & W_{kk} \end{bmatrix}$ is block diagonal with each block-diagonal entry at least in upper triangular form when expressed as a matrix in this same basis. To show that W is in the double commutant of Z, we must show that each such diagonal entry W_{ii} is in fact Toeplitz with repeated diagonal partial blocks corresponding to repeated partial Jordan blocks in Z. As a first step we argue that each diagonal block W_{ii} is upper triangular Toeplitz.

From the fact that the pair (Z, W) satisfies condition (2) in the statement of the theorem, it follows that the pair of diagonal blocks $(J_{\lambda_i}^{(n_i)}, W_{ii})$ does as well; indeed, this can be seen simply by restricting the vectors u to be of the form $u = \begin{bmatrix} 0 & \dots & u_i^\top & \dots & 0 \end{bmatrix}^\top$. It therefore suffices to assume that the original pair (Z, W) satisfies (2) with Z given in Jordan form $Z = J_\lambda^{(n)}$. Write $Z = \lambda I + N$ where N is the nilpotent Jordan cell. Set $T_t = I - tN$ where t is a real parameter, so $T_t \in \{Z\}'$. If we define $(Z_t, W_t) = (T_t^{-1} Z T_t, T_t^{-1} W T_t)$. Then $Z_t = Z$ for all t and a simple computation gives

$$W_t = W + tC$$

where

$$C = (N + \cdots + t^{n-2} N^{n-1})W - WN - (tN + \cdots + t^{n-1} N^{n-1})WN.$$

Note that $C = 0$ exactly when W is Toeplitz. By our assumption (2) combined with (2.18), we know that $\mathbb{P}_{\mathrm{RTRD},u}(Z_t, W_t) \geq 0$ for all $t \in \mathbb{R}$, i.e., the quantity

$$\sum_{n=0}^\infty Z^n \left[uu^* - (W + tC) uu^* (W + tC)^* \right] Z^{*n} \tag{2.23}$$

$$= \sum_{n=0}^\infty Z^n (uu^* - Wuu^*W^*) Z^{*n} - \sum_{n=0}^\infty Z^n \left[tCuu^*W + tWuu^*C + t^2 Cuu^*C^* \right] Z^{*n}$$

is positive semidefinite for all real t and all vectors u. If C is not zero, it is clear from the last term that this is not possible. It follows that we must have $C = 0$ and consequently W is upper triangular Toeplitz as wanted.

It remains to handle the case where the eigenvalues in some of the Jordan blocks may repeat. We therefore suppose that

$$Z = \begin{bmatrix} J_\lambda^{(n_1)} & 0 \\ 0 & J_\lambda^{(n_2)} \end{bmatrix} = \begin{bmatrix} \lambda I_{n_1} + N_1 & 0 \\ 0 & \lambda I_{n_2} + N_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

where say $n_1 \geq n_2$ and N_i is the upper triangular $n_i \times n_i$ nilpotent cell for $i = 1, 2$. We let X be the $n_2 \times n_1$ matrix given by

$$X = \begin{bmatrix} I_{n_2} & 0_{n_2 \times (n_1 - n_2)} \end{bmatrix}$$

and note that X has the intertwining property

$$X J_\lambda^{(n_1)} = J_\lambda^{(n_2)} X. \tag{2.24}$$

We introduce the similarity

$$S_\epsilon = \begin{bmatrix} I_{n_1} & 0 \\ X & \epsilon I_{n_2} \end{bmatrix}$$

depending on the real nonzero parameter ϵ. From (2.24) it follows that S_ϵ commutes with Z, so $Z_\epsilon := S_\epsilon^{-1} Z S_\epsilon = Z$ for each t. To show that $W \in \{Z\}''$, it suffices to show that X commutes with W as well. In general we have

$$W_\epsilon := S_\epsilon^{-1} W S_\epsilon = \begin{bmatrix} W_1 & 0 \\ \epsilon^{-1}(W_2 X - X W_1) & W_2 \end{bmatrix}.$$

By our assumption (2) in the statement of the theorem combined with the observation (2.18), we see that we know that the matrix

$$\begin{aligned} &\sum_{n=0}^{\infty} Z^n \left(uu^* - \begin{bmatrix} W_1 & 0 \\ \epsilon^{-1}(W_2 X - X W_1) & W_2 \end{bmatrix} uu^* \begin{bmatrix} W_1^* & \epsilon^{-1}(W_2 X - X W_1)^* \\ 0 & W_2^* \end{bmatrix} \right) Z^{*n} \\ &= \sum_{n=0}^{\infty} Z^n (uu^* - W uu^* W^*) Z^{*n} - \epsilon^{-1} \sum_{n=0}^{\infty} Z^n \begin{bmatrix} 0 & 0 \\ W_2 X - X W_1 & 0 \end{bmatrix} uu^* W^* Z^{*n} \\ &\quad - \epsilon^{-1} \sum_{n=0}^{\infty} Z^n W uu^* \begin{bmatrix} 0 & (W_2 X - X W_1)^* \\ 0 & 0 \end{bmatrix} Z^{*n} \\ &\quad - \epsilon^{-2} \sum_{n=0}^{\infty} Z^n \begin{bmatrix} 0 & 0 \\ W_2 X - X W_1 & 0 \end{bmatrix} uu^* \begin{bmatrix} 0 & (W_2 X - X W_1)^* \\ 0 & 0 \end{bmatrix} Z^{*n} \end{aligned} \tag{2.25}$$

is positive-semidefinite for all nonzero real numbers ϵ and all vectors u. As the vector u sweeps all possible vectors in $\mathbb{C}^\kappa$, it is clear that (2.25) cannot hold unless the commutator $W_2 X - X W_1$ is zero, as was wanted

(4) $\Longrightarrow$ (1): We consider first the case where Z is non-derogatory, i.e., in the Jordan form for Z, there is associated only one Jordan cell with each eigenvalue λ of Z. It is known that Z is non-derogatory if and only if there exists a vector u such that (Z, u) is controllable, i.e., if and only if the controllability matrix appearing in (2.16) has rank equal to the dimension κ of the whole state space for any maximally Z-controllable vector u. We therefore suppose that we are given a vector u such that the pair (Z, u) is controllable and the Pick matrix $\mathbb{P}_{FRD,u}$ is positive semidefinite. Note that $\mathbb{P}_{\mathrm{FRD},u}$ is the same as the matrix $\mathbb{P}_{\mathrm{LTOA}}$ in (2.8) if we take

$$N = 1, \quad T_1 = Z, \quad X_1 = u, \quad Y_1 = Wu.$$

Hence by part (2) of Theorem 2.3, the positive-semidefiniteness of $\mathbb{P}_{FRD,u}$ implies that we can find a scalar Schur-class function s so that $s(Z)u = Wu$. As we are also assuming that W is in the second commutant of Z, we know that W and Z commute (in fact, in the non-derogatory case, the commutant and double commutant of Z are the same). We therefore have

$$s(Z)Z^k u = Z^k s(Z)u = Z^k W u = W Z^k u$$

from which we get

$$s(Z)\begin{bmatrix} u & Zu & \cdots & Z^{\kappa-1}u \end{bmatrix} = W\begin{bmatrix} u & Zu & \cdots & Z^{\kappa-1}u \end{bmatrix}. \tag{2.26}$$

As the controllability matrix $\begin{bmatrix} u & Zu & \cdots & Z^{\kappa-1}u \end{bmatrix}$ is surjective (and in fact invertible since it is square), from (2.26) we see that s solves the FRD-NP problem $s(Z) = W$, i.e., condition (1) of the theorem holds.

Under the assumption that W is in the double commutant of Z, the derogatory case can be reduced to the non-derogatory case by working with the Jordan canonical form for Z; we refer to [36] for details. □

Remark 2.5. There is also a left-tangential version of Theorem 2.4 where conditions (2), (3) and (4) are replaced by their left versions:

2′. *For each linear functional $v \in \mathcal{L}(\mathcal{Z}, \mathbb{C})$, the* **LTRD-NP** *problem*

find a scalar Schur-class function s so that $vs(Z) = vW$ (*where a priori, the solution $s = s_v$ depends on the linear functional v*)

has a solution, or, equivalently, for each linear functional $v \in \mathcal{L}(\mathcal{Z}, \mathbb{C})$ the Pick matrix $\mathbb{P}_{v,\mathrm{FRD}}$ given by

$$\mathbb{P}_{v,\mathrm{FRD}} = \sum_{n=0}^{\infty} Z^{*n}(v^*v - W^*v^*vW)Z \tag{2.27}$$

is positive-semidefinite.

3′. *The Pick matrix $\mathbb{P}_{v,\mathrm{FRD}}$ is positive-semidefinite for each linear functional $v \in \mathcal{L}(\mathcal{Z}, \mathbb{C})$ satisfying the additional constraint*

$$v = \text{arg max}_{v'} \text{ rank} \begin{bmatrix} v' \\ v'Z \\ \vdots \\ v'Z^{n-1} \end{bmatrix}, \tag{2.28}$$

i.e., for any maximally Z-controllable linear functional v.

4′. *The operator W is in the double commutant $\{Z\}''$ of Z and the Pick matrix $\mathbb{P}_{v,\mathrm{FRD}}$ (2.27) is positive-semidefinite for some maximally Z-controllable linear functional v (i.e., a linear functional v satisfying (2.28)).*

Note that condition (1) in Theorem 2.4 implies condition (2′) trivially. The remaining implications (2′) $\iff$ (3′) and (2′) $\implies$ (4′) $\implies$ (1) can be proved by paralleling the proof of Theorem 2.4 (with the role of vectors u acting as operators from $\mathbb{C}$ into $\mathcal{Z}$ acting on the right replaced by linear functionals v acting on the left). In this way we see that each of (2′), (3′) and (4′) is also equivalent to any of (1), (2), (3) and (4) in Theorem 2.4.

Remark 2.6. For purposes of this remark we identify $\mathcal{Z}$ with $\mathbb{C}^\kappa$, so operators Z on $\mathcal{Z}$ can be viewed concretely as $\kappa \times \kappa$ complex matrices. The paper [35] is concerned with a "Lyapunov ordering" for real symmetric matrices based on solving Lyapunov inequalities: $ZS + SZ^* \geq 0$. Theorem 2.4 is closely related to the analogue of these results with two adjustments: we work with complex Hermitian rather than real symmetric matrices and we work with Stein inequalities $S - ZSZ^* \geq 0$ rather than Lyapunov inequalities. Following [35], given a matrix Z (say of size $\kappa \times \kappa$) we give a Stein ordering on a pair of complex Hermitian $\kappa \times \kappa$ matrices Z, W as follows: $Z \leq_{\mathfrak{S}} W$ *if* $S - WSW^* \geq 0$ *whenever* $S - ZSZ^* \geq 0$. As we are assuming that Z has spectral radius less than 1, the operator $\mathcal{S}_Z \colon X \mapsto X - ZXZ^*$ is invertible as an operator on complex Hermitian $\kappa \times \kappa$ matrices with inverse $\mathcal{S}_Z^{-1}$ given explicitly as

$$\mathcal{S}_Z^{-1}(X) = \sum_{n=0}^{\infty} Z^n X Z^n.$$

Then the condition $Z \leq_{\mathfrak{S}} W$ can be expressed in operator-theoretic form as

$$\mathcal{S}_W \circ \mathcal{S}_Z^{-1}(\mathcal{P}_+) \subset \mathcal{P}_+$$

where we use the notation $\mathcal{P}_+$ to denote the positive semidefinite $\kappa \times \kappa$ matrices, or, more explicitly,

$$X \in \mathbb{C}^{\kappa\times\kappa} \text{ with } X \geq 0 \Longrightarrow \mathbb{P}_X := \sum_{n=0}^{\infty} Z^n X Z^{*n} - W \left(\sum_{n=0}^{\infty} Z^n X Z^{*n} \right) W^* \geq 0. \tag{2.29}$$

An equivalent formulation of (2.29) is that, for all positive semidefinite $\kappa \times \kappa$ matrices X and Y we have

$$\begin{aligned} &\operatorname{tr}\left(\left[\sum_{n=0}^{\infty} Z^n X Z^{*n} - W \left(\sum_{n=0}^{\infty} Z^n X Z^{*n} \right) W^* \right] Y \right) \\ &= \operatorname{tr}\left(\sum_{n=0}^{\infty} [Z^{*n}(Y - W^*YW)Z^n]\, X \right) \geq 0. \end{aligned}$$

or, equivalently,

$$\sum_{n=0}^{\infty} [Z^{*n}(Y - W^*YW)Z^n] \geq 0 \text{ for all } Y \geq 0. \tag{2.30}$$

If we restrict to matrices Y which generate extreme rays in the cone of positive semidefinite matrices, i.e., Y of the form $Y = v^*v$ with v a row vector of size $1 \times \kappa$, then we may rewrite (2.30) as

$$\sum_{n=0}^{\infty} [Z^{*n}(v^*v - W^*v^*vW)Z^n] \geq 0 \text{ for all } v \in \mathbb{C}^{1\times\kappa}$$

which we recognize as condition (2′) in Remark (2.5). Thus the Stein ordering $Z \leq_{\mathfrak{S}} W$ can be interpreted as being able to solve the **LTRD-NP** interpolation problem $vs(Z) = vW$ along every row vector v and thus is equivalent to any of the other conditions listed in Theorem 2.4 and Remark 2.5.

Alternatively, by an argument parallel to the proof of (2) $\Longrightarrow$ (4) in Theorem 2.4, one can see that $Z \leq_{\mathfrak{S}} W$ implies that $W \in \{Z\}''$, so W commutes with Z; in fact this is proved in [35] for the real symmetric case (in which case the result holds with some modifications which can be ignored in the complex case) and is the inspiration for our proof of (2) $\Longrightarrow$ (4) in Theorem 2.4. We note that to check whether $Z \leq_{\mathfrak{S}} W$ it suffices to check whether the matrix $\mathbb{P}_X$ is positive semidefinite for matrices X which generate extreme rays in $\mathcal{P}_+$, i.e., for X of the form uu^* where now we take u to be a column vector in $\mathbb{C}^\kappa$. We use the fact that Z and W commute to get

$$\begin{aligned}
\mathbb{P}_{uu^*} - Z\mathbb{P}_{uu^*}Z^* &= \sum_{n=0}^{\infty} Z^n uu^* Z^{*n} - W\left(\sum_{n=0}^{\infty} Z^n uu^* Z^{*n}\right)W^* \\
&\quad - Z\left[\sum_{n=0}^{\infty} Z^n uu^* Z^{*n} - W\left(\sum_{n=0}^{\infty} Z^n uu^* Z^{*n}\right)W^*\right]Z^* \\
&= \sum_{n=0}^{\infty} Z^n uu^* Z^{*n} - \sum_{n=0}^{\infty} Z^n W uu^* W^* Z^{*n} \\
&\quad - \sum_{n=1}^{\infty} Z^n uu^* Z^{*n} - \sum_{n=1}^{\infty} Z^n W uu^* W^* Z^{*n} \\
&= uu^* - Wuu^*W^*.
\end{aligned}$$

from which we see that $\mathbb{P}_{uu^*} = \mathbb{P}_{\mathrm{FRD},u}$ as given by (2.15). Thus positive semidefiniteness of $\mathbb{P}_{uu^*}$ for all vectors u is equivalent to the positive semidefiniteness of the matrix $\mathbb{P}_{\mathrm{FRD},u}$ for all vectors $u \in \mathbb{C}^\kappa$, i.e., to condition (2) in Theorem 2.4. Hence, with use of the result that $Z \leq_{\mathfrak{S}} W \Longrightarrow W \in \{Z\}''$, we have a direct verification of the equivalence of $Z \leq_{\mathfrak{S}} W$ with being able to solve the **RTRD-NP** problem $s(Z)u = Wu$ along every column vector u.

Via either of these approaches we see that we may add a fifth statement to the list of equivalent statements in Theorem 2.4 and Remark 2.5:

5. $Z \leq_{\mathfrak{S}} W$ *in the sense of Cohen-Lewkowicz* (as adapted to complex Stein inequalities as done here).

Remark 2.7. As observed in the proof of (1) $\Longrightarrow$ (2), the positive semidefiniteness of the block matrix (2.14) is equivalent to condition (2.17) while (2.15) is equivalent to (2.17) holding for the vectors $u_1, \dots, u_\kappa$ of the special form

$$u_i = c_i u_0 \text{ for some scalars } c_1, \dots, c_\kappa \text{ and some fixed but arbitrary vector } u_0. \tag{2.31}$$

Since (1) and (2) in Theorem 2.4 are equivalent, we see that, conversely, the validity of (2.17) for all $u_1, \dots, u_\kappa$ of the special form (2.31) implies the validity of (2.17) for all $u_1, \dots, u_\kappa$. The following simple example shows that the reverse implication fails in general for block matrices $\mathbb{P}$ not coming from a Riesz-Dunford interpolation problem. Indeed, take $\kappa = 2$, $\mathcal{Z} = \mathbb{C}^2$ and $\mathbb{P} = \left[\begin{smallmatrix} \mathbb{P}_{11} & \mathbb{P}_{12} \\ \mathbb{P}_{21} & \mathbb{P}_{22} \end{smallmatrix}\right]$ where

$$\mathbb{P}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbb{P}_{12} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \mathbb{P}_{21}, \quad \mathbb{P}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

If we choose $u_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $u_2 = \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$, we get $\left[\begin{smallmatrix} u_1^* & u_2^* \end{smallmatrix}\right] \mathbb{P} \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] = -1$ and hence $\mathbb{P}$ is not positive definite. Nevertheless, if we restrict to u_1, u_2 of the special form (2.31), we have

$$\begin{aligned} \begin{bmatrix} \overline{c_1} u_0^* & \overline{c_2} u_0^* \end{bmatrix} \mathbb{P} \begin{bmatrix} c_1 u_0 \\ c_2 u_0 \end{bmatrix} &= u_0^* \begin{bmatrix} |c_1|^2 & \text{Re } c_1 \overline{c_2} \\ \text{Re } c_1 \overline{c_2} & |c_2|^2 \end{bmatrix} u_0 \\ &= \frac{1}{2} \left(\| \begin{bmatrix} \overline{c_1} & \overline{c_2} \end{bmatrix} u_0 \|^2 + \| \begin{bmatrix} c_1 & c_2 \end{bmatrix} u_0 \|^2 \right) \geq 0. \end{aligned}$$

It is a remarkable fact that (2.17) holding for $u_1, \dots, u_\kappa$ of the special form (2.31) implies that (2.17) holds for arbitrary $u_1, \dots, u_\kappa$ if $\mathbb{P}$ has the special form $\mathbb{P}_{\text{FRD}}$ for some **FRD-NP** problem.

Besides the standard definition of the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ used in this section there are various other ways to define $\mathcal{S}(\mathcal{U}, \mathcal{Y})$. One of these definitions is: *A function $S : \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ in case S defines a contractive multiplication operator M_S from the Hardy space $H^2_{\mathcal{U}}(\mathbb{D})$ into the Hardy space $H^2_{\mathcal{Y}}(\mathbb{D})$ via*

$$M_S : f(\lambda) = S(\lambda) f(\lambda) \quad (\lambda \in \mathbb{D}).$$

We also remark that the Hardy space $H^2_{\mathcal{U}}(\mathbb{D})$ is a reproducing kernel Hilbert space, whose reproducing kernel is the classical Szegő kernel

$$k(\lambda, \zeta) = \frac{1}{1 - \lambda \overline{\zeta}} \quad (\lambda, \zeta \in \mathbb{D}).$$

Another characterization of Schur class functions in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is based on an associated kernel function: *A function $S : \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ in case the function $k_S : \mathbb{D} \times \mathbb{D} \to \mathcal{L}(\mathcal{Y})$ given by*

$$k_S(\lambda, \zeta) = \frac{I_{\mathcal{Y}} - S(\lambda) S(\zeta)^*}{1 - \lambda \overline{\zeta}} \quad (\lambda, \zeta \in \mathbb{D})$$

is a positive kernel in the sense of Aronszajn [12] *adapted to the operator-valued setting: For any finite collection of points* $\omega_1, \dots, \omega_M \in \mathbb{D}$ *and vectors* $y_1, \dots, y_M \in \mathcal{Y}$ *it holds that*

$$\sum_{i,j=1}^{M} \langle k_S(\omega_i, \omega_j) y_j, y_i \rangle \geq 0.$$

The extensions of Nevanlinna-Pick interpolation theory for functions of several variables that we consider in the following sections are based on Schur-class functions defined as "contractive multipliers" or via "associated positive kernels", rather than as functions having contractive values as in (2.1).

3. Operator-valued Nevanlinna-Pick interpolation: the multivariable case

In this section we consider Nevanlinna-Pick interpolation problems for Schur-class function of several variables. The notion of Schur class function in these cases generalizes the definition via "contractive multipliers" or "positive kernels" mentioned at the end of Section 2, rather that the standard definition via "contractive values" (2.1) as given in Section 2.

3.1. The commutative unit ball setting

A much studied multivariable analogue of the classical Szegő kernel is the Drury-Arveson kernel k_d on the unit ball $\mathbb{B}^d = \{\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \colon \sum_{i=1}^{d} |\lambda_i|^2 < 1\}$ given by

$$k_d(\lambda, \zeta) = \frac{1}{1 - \langle \lambda, \zeta \rangle} = \frac{1}{1 - \lambda_1 \overline{\zeta}_1 - \cdots - \lambda_d \overline{\zeta}_d} \quad (\lambda, \zeta \in \mathbb{B}^d).$$

The associated reproducing kernel space $\mathcal{H}(k_d)$ is called the *Drury-Arveson space* which is the prototype for a reproducing kernel space with a *complete Pick kernel*; some of the seminal references on this topic are [47, 93, 13, 14, 5, 11, 29, 40, 48, 54, 65, 63, 84].

For a Hilbert space $\mathcal{U}$ we let $\mathcal{H}_{\mathcal{U}}(k_d)$ be the space $\mathcal{H}(k_d) \otimes \mathcal{U}$ of Drury-Arveson space functions with values in $\mathcal{U}$. It can be shown that a holomorphic function $h \colon \mathbb{B}^d \to \mathcal{U}$ with power series representation

$$h(\lambda) = \sum_{n \in \mathbb{Z}_+^d} h_n \lambda^n \quad \text{where } h_n \in \mathcal{U} \text{ for } n \in \mathbb{Z}_+^d$$

is in the vector-valued Drury-Arveson space $\mathcal{H}_{\mathcal{U}}(k_d)$ if and only if

$$\|h\|^2_{\mathcal{H}_{\mathcal{U}}(k_d)} = \sum_{n \in \mathbb{Z}^d} \frac{n!}{|n|!} \|h_n\|^2_{\mathcal{U}} < \infty.$$

Here we use standard multivariable notation: For

$$n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d \quad \text{and} \quad \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{B}^d \subset \mathbb{C}^d,$$

we set

$$\lambda^n = \lambda_1^{n_1} \cdots \lambda_d^{n_d}, \quad n! = n_1! \cdots n_d! \quad \text{and} \quad |n| = n_1 + \cdots + n_d.$$

For coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ we define the operator-valued Drury-Arveson Schur-multiplier class $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ to be the space of holomorphic functions $S\colon \mathbb{B}^d \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that the multiplication operator

$$M_S\colon f(\lambda) \mapsto S(\lambda) f(\lambda) \quad (\lambda \in \mathbb{B})$$

maps $\mathcal{H}_{\mathcal{U}}(k_d)$ contractively into $\mathcal{H}_{\mathcal{Y}}(k_d)$. We can then pose the Drury-Arveson space versions of the problems formulated in Section 2 by simply replacing the unit disk $\mathbb{D}$ by the unit ball $\mathbb{B}^d$ and the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ by the Drury-Arveson Schur-multiplier class $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$.

All results except the "right-tangential" versions extend in a natural way. The trick to reduce a right-tangential problem to a left-tangential problem via (2.2) fails to generalize to the Drury-Arveson setting; if $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and we set $S^\sharp(\lambda) := S(\overline{\lambda})^* = S(\overline{\lambda_1}, \ldots, \overline{\lambda_d})^*$ for each $\lambda \in \mathbb{B}^d$, then it is usually not the case that $S^\sharp$ is in $\mathcal{S}_d(\mathcal{Y}, \mathcal{U})$. Results on right-tangential Nevanlinna-Pick interpolation in the Drury-Arveson space exist (see [18, 19] for a general setting), but they are of the flavor of the results for Nevanlinna-Pick interpolation for the Schur-Agler class to be discussed in Subsection 3.4 below: rather than having a test with a single Pick matrix, the criterion involves being able to solve certain equations for a family of Pick matrices. This makes the theory for $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ completely asymmetric with respect to left versus right. We summarize the results in the following theorem, with the more complicated statement for right-tangential interpolation problems omitted.

Theorem 3.1.
(1) Drury-Arveson Full Operator-Valued Nevanlinna-Pick interpolation: *Suppose that we are given points $\lambda^{(1)}, \ldots, \lambda^{(N)}$ in $\mathbb{B}^d$ together with operators $W_1, \ldots, W_N$ in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then there exists an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ with $S(\lambda^{(i)}) = W_i$ for $i = 1, \ldots, N$ if and only if the Pick matrix*

$$\mathbb{P}_{\text{FOV}} := \left[\frac{I_{\mathcal{Y}} - W_i W_j^*}{1 - \langle \lambda^{(i)}, \lambda^{(j)} \rangle} \right]_{i,j=1}^N$$

is positive semidefinite.

(2) Drury-Arveson Left-Tangential Nevanlinna-Pick interpolation: *Suppose that we are given an auxiliary Hilbert space $\mathcal{C}$, points $\lambda^{(1)}, \ldots, \lambda^{(N)}$ in $\mathbb{B}^d$, operators $X_1, \ldots, X_N$ in $\mathcal{L}(\mathcal{Y}, \mathcal{C})$ and operators $Y_1, \ldots, Y_N$ in $\mathcal{L}(\mathcal{U}, \mathcal{C})$. Then there exists an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $X_i S(\lambda^{(i)}) = Y_i$ for $i = 1, \ldots, N$ if and only if the Pick matrix*

$$\mathbb{P}_{\text{LT}} := \left[\frac{X_i X_j^* - Y_i Y_j^*}{1 - \langle \lambda^{(i)}, \lambda^{(j)} \rangle} \right]_{i,j=1}^N$$

is positive semidefinite.

(3) Drury-Arveson Left-Tangential Nevanlinna-Pick interpolation with Operator-Argument: *Suppose that we are given an auxiliary Hilbert space $\mathcal{C}$ together with commutative d-tuples*

$$Z^{(1)} = (Z_1^{(1)}, \dots, Z_d^{(1)}), \dots, Z^{(N)} = (Z_1^{(N)}, \dots, Z_d^{(N)}) \in \mathcal{L}(\mathcal{C})^d,$$

i.e., $Z_k^{(i)} \in \mathcal{L}(\mathcal{C})$ for $i = 1, \dots, N$ and $k = 1, \dots, d$ and for each fixed i, the operators $Z_1^{(i)}, \dots, Z_d^{(i)}$ commute pairwise, with the property that each d-tuple $Z^{(i)}$ has joint spectrum contained in $\mathbb{B}^d$. Assume in addition that we are given operators $X_1, \dots, X_N$ in $\mathcal{L}(\mathcal{Y}, \mathcal{C})$ and operators $Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{U}, \mathcal{C})$. Then there is an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ so that

$$(X_i S)^{\wedge L}(Z^{(i)}) := \sum_{n \in \mathbb{Z}_+^d} (Z^{(i)})^n X_i S_n = Y_i \quad \textit{for} \quad i = 1, \dots, N$$

if and only if the associated Pick matrix

$$\mathbb{P}_{\text{LTOA}} := \left[\sum_{n \in \mathbb{Z}_+^d} (Z^{(i)})^n (X_i X_j^* - Y_i Y_j^*) (Z^{(j)})^{n*} \right]_{i,j=1}^N$$

is positive semidefinite. Here we use the multivariable notation: $Z^n = Z_1^{n_1} \cdots Z_d^{n_d}$ if $Z = (Z_1, \dots, Z_d) \in \mathcal{L}(\mathcal{C})^d$ and $n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$.

(4) Full Drury-Arveson Riesz-Dunford Nevanlinna-Pick interpolation: *Suppose that we are given commutative d-tuples $Z^{(1)}, \dots, Z^{(N)}$ as in statement* (3) *above, acting on a separable auxiliary Hilbert space $\mathcal{Z}$ with orthonormal basis $\{e_1, \dots, e_\kappa\}$ (with possibly $\kappa = \infty$). We also assume that we are given operators $W_1, \dots, W_N$ in $\mathcal{L}(\mathcal{Z})$. Then there exists a scalar Drury-Arveson Schur class function $s \in \mathcal{S}_d(\mathbb{C}, \mathbb{C})$ so that*

$$s(Z^{(i)}) := \sum_{n \in \mathbb{Z}_+^d} s_n (Z^{(i)})^n = W_i \quad \textit{for} \quad i = 1, \dots, N$$

if and only if the Pick matrix

$$\mathbb{P}_{\text{FRD}} := \left[\sum_{N \in \mathbb{Z}_+^d} (Z^{(i)})^n (e_{i'} e_{j'}^* - W_i e_{i'} e_{j'}^* W_j^*) (Z^{(j)})^{n*} \right]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}}$$

is positive semidefinite.

Proof. Statement (1) **FOV**-interpolation) was obtained in [11] and [40] via the method of descending from the corresponding result for the noncommutative case (see Theorem 3.2 below). Similarly the second and third statements (**LT** and LTOA interpolation) can be obtained as a consequence of the corresponding noncommutative result in [83] and [85]. Alternatively, one can obtain the results directly without reference to the noncommutative theory, as is done in [5, 29, 17]. The fourth statement (**RD**-interpolation) follows from the result on the **LTOA**-problem in the same way as was done for the single-variable case in Section 2. We note also that **FOV** is a special case of **LT** and that **LT** is a special case of **LTOA** just as in the single-variable case. We mention that an analogue of the Rosenblum-Rovnyak theory for the tangential Riesz-Dunford interpolation for this setting appears in [83], again via the connection with the noncommutative theory. □

3.2. The noncommutative unit ball setting

There is also a noncommutative version of the Drury-Arveson Schur class; see [85, 30]. To describe this Schur class, let $\{1,\dots,d\}$ be an alphabet consisting of d letters and let $\mathcal{F}_d$ be the associated free semigroup generated by the letters $1,\dots,d$ consisting of all words γ of the form $\gamma = i_N \cdots i_1$, where each $i_k \in \{1,\dots,d\}$ and where $N = 1,2,\dots$. For $\gamma = i_N \cdots i_1 \in \mathcal{F}_d$ we set $|\gamma| := N$ to be the *length* of the word γ. Multiplication of two words $\gamma = i_N \cdots i_1$ and $\gamma' = j_{N'} \cdots j_1$ is defined via concatenation:

$$\gamma\gamma' = i_N \cdots i_1 j_{N'} \cdots j_1.$$

The empty word $\emptyset$ is included in $\mathcal{F}_d$ and acts as the unit element for this multiplication; by definition $|\emptyset| = 0$. We set

$$\gamma\gamma'^{-1} = \begin{cases} \gamma'' & \text{if there is a } \gamma'' \in \mathcal{F}_d \text{ so that } \gamma = \gamma''\gamma', \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For a Hilbert space $\mathcal{U}$, the associated *Fock space* $\ell^2_{\mathcal{U}}(\mathcal{F}_d)$ is the $\mathcal{U}$-valued ℓ^2 space indexed by the free semigroup $\mathcal{F}_d$:

$$\ell^2_{\mathcal{U}}(\mathcal{F}_d) = \Big\{ u\colon \mathcal{F}_d \to \mathcal{U}\colon \sum_{\gamma \in \mathcal{F}_d} \|u(\gamma)\|^2_{\mathcal{U}} < \infty \Big\}.$$

Given two coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ we define the *noncommutative d-variable Schur class* $\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ to be the set consisting of all formal power series $s(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma$ in noncommuting indeterminates $z = (z_1,\dots,z_d)$ (where we think of $z^\gamma = z_{i_N} \cdots z_{i_1}$ if $\gamma = i_N \cdots i_1$) with coefficients $S_\gamma \in \mathcal{L}(\mathcal{U},\mathcal{Y})$ such that the associated Toeplitz matrix

$$R = \big[S_{\gamma\gamma'^{-1}} \big]_{\gamma,\gamma' \in \mathcal{F}_d}, \text{ where we set } S_{\text{undefined}} = 0,$$

defines a contraction operator from $\ell^2_{\mathcal{U}}(\mathcal{F}_d)$ into $\ell^2_{\mathcal{Y}}(\mathcal{F}_d)$.

There has been a variety of interpolation results for this setting, for instance Sarason type, Rosenblum-Rovnyak or tangential Riesz-Dunford type and left-tangential operator-argument type; we refer to [83, 40, 11, 85, 86]. As already mentioned, from results for the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ one can arrive at interpolation results for the Drury-Arveson space as in Theorem 3.1 by abelianizing the variables (see [13, 40, 11, 22]). We state here only the analogues of left-tangential operator-argument Nevanlinna-Pick and full Riesz-Dunford Nevanlinna-Pick interpolation for this noncommutative setting. The right-tangential versions again do not reduce to the left-tangential problems due to the same phenomenon discussed for the commutative case. In the noncommutative setting we distinguish between the *Riesz-Dunford functional calculus* which uses

$$f(Z) := \sum_{\gamma \in \mathcal{F}_d} f_\gamma Z^\gamma$$

where we set $Z^\gamma = Z_{i_N} \cdots Z_{i_1}$ if $\gamma = i_N \cdots i_1$, and the *transposed Riesz-Dunford functional calculus* which uses

$$f^\top(Z) := \sum_{\gamma \in \mathcal{F}_d} f_\gamma Z^{\gamma^\top}$$

where $\gamma^\top = i_1, \dots, i_N$ if $\gamma = i_N \cdots i_1$ and $Z^{\gamma^\top} = Z_{i_1} \cdots Z_{i_N}$ if $Z = (Z_1, \dots, Z_d)$.

Theorem 3.2.
(1) Free-semigroup algebra Left-Tangential Nevanlinna-Pick interpolation with Operator-Argument: *Suppose that we are given a coefficient Hilbert space $\mathcal{C}$ and a collection*

$$Z^{(1)} = (Z_1^{(1)}, \dots, Z_d^{(1)}), \dots, Z^{(N)} = (Z_1^{(N)}, \dots, Z_d^{(N)})$$

of (not necessarily commutative) d-tuples of operators on $\mathcal{C}$ such that, for each fixed $i = 1, \dots, N$, the block row matrix $\left[\begin{smallmatrix} Z_1^{(i)} & \cdots & Z_d^{(i)} \end{smallmatrix}\right]$ defines a strict contraction operator from $\mathcal{C}^d = \oplus_{k=1}^d \mathcal{C}$ into $\mathcal{C}$. In addition, suppose that we are given operators $X_1, \dots, X_N$ in $\mathcal{L}(\mathcal{Y}, \mathcal{C})$ and operators $Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{U}, \mathcal{C})$. Then there exists a formal power series $S(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma$ in the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ such that

$$(X_i S)^{\wedge L}(Z^{(i)}) := \sum_{\gamma \in \mathcal{F}_d} (Z^{(i)})^{\gamma^\top} X_i S_\gamma = W_i \quad \textit{for} \quad i = 1, \dots, N$$

if and only if the associated Pick matrix

$$\mathbb{P}_{\mathrm{LTOA}} := \left[\sum_{\gamma \in \mathcal{F}_d} (Z^{(i)})^\gamma (X_i X_j^* - Y_i Y_j^*) (Z^{(j)})^{\gamma *} \right]_{i,j=1}^N$$

is positive semidefinite.

(2) Free-semigroup algebra full transposed Riesz-Dunford Nevanlinna-Pick interpolation: *Suppose that we are given d-tuples of operators $Z^{(1)}, \dots, Z^{(N)}$ as in statement (1) above, acting on a separable auxiliary Hilbert space $\mathcal{Z}$ with orthonormal basis $\{e_1, \dots, e_\kappa\}$ (with possibly $\kappa = \infty$). Suppose also that we are given operators $W_1, \dots, W_N$ in $\mathcal{L}(\mathcal{Z})$. Then there exists a formal power series $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ in the scalar noncommutative Schur class $\mathcal{S}_{nc,d}(\mathbb{C}, \mathbb{C})$ satisfying the transposed Riesz-Dunford interpolation conditions*

$$s^\top(Z^{(i)}) := \sum_{\gamma \in \mathcal{F}_d} s_\gamma (Z^{(i)})^{\gamma^\top} = W_i \quad \textit{for} \quad i = 1, \dots, N \tag{3.1}$$

if and only if the associated Pick matrix

$$\mathbb{P}_{\mathrm{FRD}} := \left[\sum_{\gamma \in \mathcal{F}_d} (Z^{(i)})^{\gamma} (e_{i'} e_{j'}^* - W_i e_{i'} e_{j'}^* W_j^*) (Z^{(j)})^{\gamma *} \right]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}} \tag{3.2}$$

is positive semidefinite.

(3) Free-semigroup algebra full Riesz-Dunford Nevanlinna-Pick interpolation: *Suppose that we are given d-tuples of operators $Z^{(1)}, \dots, Z^{(N)}$ as in statement* (2) *above acting on a finite-dimensional auxiliary Hilbert space $\mathcal{Z}$ with orthonormal basis $\{e_1, \dots, e_\kappa\}$ along with operators $W_1, \dots, W_N$ in $\mathcal{L}(\mathcal{Z})$. Then there exists a formal power series $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ in the scalar noncommutative Schur class $\mathcal{S}_{nc,d}(\mathbb{C}, \mathbb{C})$ satisfying the Riesz-Dunford-type interpolation conditions*

$$s(Z^{(i)}) := \sum_{\gamma \in \mathcal{F}_d} s_\gamma (Z^{(i)})^\gamma = W_i \quad \textit{for} \quad i = 1, \dots, N \tag{3.3}$$

if and only if the associated Pick matrix

$$\mathbb{P}_{\mathrm{FRD}^*} := \left[\sum_{\gamma \in \mathcal{F}_d} (Z^{(i)})^{\gamma *} (e_{i'} e_{j'}^* - W_i^* e_{i'} e_{j'}^* W_j) (Z^{(j)})^{\gamma} \right]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}} \tag{3.4}$$

is positive semidefinite.

Proof. The first statement can be found in [85] as well as in [20]. A tangential version of the second statement (i.e., tangential Riesz-Dunford interpolation) analogous to the Rosenblum-Rovnyak theory for the single-variable case is given in [83]. The full transpose Riesz-Dunford interpolation result in the second statement follows from the first statement on **LTOA** in the same way as in Section 2 for the single-variable case.

We reduce the third statement to the second as follows. By taking adjoints, we may rewrite (3.1) as

$$\sum_{\gamma \in \mathcal{F}_d} \overline{s}_\gamma (Z^{(i)})^{\gamma^*} = \sum_{\gamma \in \mathcal{F}_d} \overline{s}_\gamma (Z^{(i)*})^{\gamma^\top} = W_i^* \tag{3.5}$$

where we set

$$Z^{(i)*} := (Z_1^{(i)*}, \dots, Z_d^{(i)*}) \text{ for } i = 1, \dots, N.$$

Note that

$$\tau \colon \{h_\gamma\}_{\gamma \in \mathcal{F}_d} \mapsto \{\overline{h}_\gamma\}_{\gamma \in \mathcal{F}_d}$$

is a conjugation (conjugate-linear norm-preserving involution) on $\ell^2(\mathcal{F}_d)$ and, if $R = \left[s_{\gamma\gamma'^{-1}} \right]_{\gamma,\gamma' \in \mathcal{F}_d}$ is the Toeplitz operator associated with the formal power series $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ acting on $\ell^2(\mathcal{F}_d)$, then $\tau \circ R \circ \tau = \overline{R}$ where $\overline{R} = \left[\overline{s}_{\gamma\gamma'^{-1}} \right]_{\gamma,\gamma' \in \mathcal{F}_d}$. We conclude that $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ is in the noncommutative Schur-multiplier class $\mathcal{S}_{nc,d}$ if and only if $\overline{s}(z) = \sum_{\gamma \in \mathcal{F}_d} \overline{s}_\gamma z^\gamma$ is in $\mathcal{S}_{nc,d}$. Thus there is an $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ in $\mathcal{S}_{nc,d}$ satisfying (3.5) if and only if there is another $s \in \mathcal{S}_{nc,d}$ (namely, $\overline{s}$) satisfying

$$s^\top(Z^{(i)*}) := \sum_{\gamma \in \mathcal{F}_d} s_\gamma (Z^{(i)*})^{\gamma^\top} = W_i^* \text{ for } i = 1, \dots, N. \tag{3.6}$$

By Part (2) of Theorem 3.2, this last condition is equivalent to the positive-semidefiniteness of the matrix $\mathbb{P}_{\mathrm{FRD}^*}$ in Part (3). This completes the proof of Theorem 3.2. □

3.3. Nevanlinna-Pick interpolation for Toeplitz algebras associated with directed graphs

This example is discussed in [69, Example 4.3 and pages 52–53] and [70, Section 5]. Here we work out the example, in particular, the nature of the point-evaluation and the Nevanlinna-Pick theorem, in more detail. The papers [66, 68, 101] introduced this class of algebras and studied the uniform closure of the algebra (called the quiver algebra) generated by creation operators. The associated weak-$*$ closed Toeplitz algebra obtained here is also known as a *free semigroupoid algebra*; the papers [60, 61] give concrete representations (some as explicit matrix-function algebras) arising from particular choices of the quiver (i.e., directed graph).

Quivers. Formally a *quiver* G is a quadruple $\{Q_0, Q_1, s, r\}$ that consists of two finite sets Q_0 and Q_1 and two maps s and r from Q_1 to Q_0. We think of the elements of Q_0 as vertices and those of Q_1 as arrows; for any $\alpha \in Q_1$ we think of α as an arrow from $s(\alpha)$ to $r(\alpha)$. It is possible to work with infinite sets Q_0 and Q_1 in which case they need to be equipped with a topology [60, 61], but we shall not consider that case here. With the quiver G we associate the *transposed quiver* $\widetilde{G} = \{Q_0, Q_1, r, s\}$ (with respect to G), i.e., where the source and range maps are interchanged. As a particular example, the reader is invited to take $Q_0 = \{v_0\}$ (i.e., the set of vertices is a singleton) and $Q_1 = \{\alpha_1, \dots, \alpha_d\}$ with $s(\alpha_k) = r(\alpha_k) = v_0$ for $k = 1, \dots, d$. The present example then collapses to the free semigroup algebra example $\mathcal{L}_d$ discussed in Subsection 3.2. The difference between the standard and the transposed Riesz-Dunford functional calculus depends on whether one starts with the quiver G or its reverse $\widetilde{G}$.

Paths. For each nonnegative integer n we write Q_n for the paths of length n and Γ for the collection $\cup_{n=0}^\infty Q_n$ of all finite paths of whatever length. Thus a $\gamma \in Q_n$ is an n-tuple $(\alpha_n, \dots, \alpha_1)$ consisting of arrows $\alpha_k \in Q_1$ with the property that $r(\alpha_k) = s(\alpha_{k+1})$ for $k = 1, \dots, n-1$. In that case we write $s_n(\gamma)$ for $s(\alpha_1)$ and $r_n(\gamma)$ for $r(\alpha_n)$. Note that for $n = 1$ this definition is consistent with that of Q_1 if the elements of Q_1 are seen as paths of length 1. For $n = 0$ we can view the elements of Q_0 as paths of length 0, with for any $v \in Q_0$, $r_0(v) = s_0(v) = v$. If the length n of γ in Γ is not specified, we write simply $s(\gamma)$ and $r(\gamma)$ rather than $s_n(\gamma)$ and $r_n(\gamma)$. The set of paths Γ associated with the quiver G forms a semigroupoid when multiplication is defined via concatenation: For

$$\gamma = (\alpha_n, \dots, \alpha_1), \gamma' = (\alpha'_m, \dots, \alpha'_1) \in \Gamma \text{ with } s(\gamma) = r(\gamma')$$

we set

$$\gamma \cdot \gamma' := (\alpha_n, \dots, \alpha_1, \alpha'_m, \dots, \alpha'_1).$$

Inversion is then given by

$$\gamma \cdot \gamma'^{-1} = \begin{cases} \gamma'' & \text{if } \gamma'' \in \Gamma \text{ so that } \gamma = \gamma'' \cdot \gamma', \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{3.7}$$

The Fock space and Toeplitz algebra associated with the quiver G. For a Hilbert space $\mathcal{U}$ with direct sum decomposition $\mathcal{U} = \oplus_{v \in Q_0} \mathcal{U}_v$ we write, with some abuse of notation, $\ell^2_{\mathcal{U}}(\Gamma)$ for the space $\oplus_{\gamma \in \Gamma} \mathcal{U}_{r(\gamma)}$; the $\mathcal{U}$-valued *Fock space defined by the quiver G*.

Given two coefficient Hilbert spaces $\mathcal{U} = \oplus_{v \in Q_0} \mathcal{U}_v$ and $\mathcal{Y} = \oplus_{v \in Q_0} \mathcal{Y}_v$, the associated Banach space of Toeplitz operators $\mathfrak{L}_\Gamma(\mathcal{U}, \mathcal{Y})$ consists of operators R from $\ell^2_{\mathcal{U}}(\Gamma)$ to $\ell^2_{\mathcal{Y}}(\Gamma)$ and hence can be given in terms of an infinite operator-matrix with rows and columns indexed by Γ:

$$R = [R_{\gamma,\gamma'}]_{\gamma,\gamma' \in \Gamma} \quad \text{with} \quad R_{\gamma,\gamma'} \in \mathcal{L}(\mathcal{U}_{r(\gamma')}, \mathcal{Y}_{r(\gamma)}).$$

The Toeplitz structure for this setting means that the matrix entries $R_{\gamma,\gamma'}$ are completely determined from the particular entries $R_\gamma := R_{\gamma, s(\gamma)}$ according to the rule

$$R_{\gamma,\gamma'} = R_{\gamma\gamma'^{-1}, r(\gamma')}$$

where we take $R_{\text{undefined},v} = 0$ for each $v \in Q_0$.

If $\mathcal{U} = \mathcal{Y}$, then we write $\mathfrak{L}_\Gamma(\mathcal{U})$ instead of $\mathfrak{L}_\Gamma(\mathcal{U}, \mathcal{U})$, and $\mathfrak{L}_\Gamma(\mathcal{U})$ is an algebra that we call the *Toeplitz algebra* associated with G and $\mathcal{U}$. In the special case that $\mathcal{U}_v = \mathcal{Y}_v = \mathbb{C}$ for all $v \in Q_0$, this Toeplitz algebra $\mathfrak{L}_\Gamma$ is otherwise known as the (weak-$*$ closed) *path algebra* corresponding to the quiver G. The algebra $\mathfrak{L}_\Gamma$ also appears as the weak-$*$ closed unital subalgebra of $\mathcal{L}(\ell^2_{\mathbb{C}}(\Gamma))$ generated by the creation operators C_α, for $\alpha \in Q_1$, given by

$$C_\alpha(\oplus_{\gamma \in \Gamma} f_\gamma) = \oplus_{\gamma' \in \Gamma} \tilde{f}_{\gamma'} \text{ where } \tilde{f}_{\gamma'} = \begin{cases} f_\gamma & \text{if } \gamma \in \Gamma \text{ so that } \alpha = \gamma' \cdot \gamma^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

or, more succinctly, where

$$\tilde{f}_{\gamma'} = f_{\alpha^{-1}\gamma'}$$

and we use the analogue of the convention (3.7) for the case where the inverse path is on the left.

The Schur class associated with the quiver G. Given a quiver G and coefficient spaces $\mathcal{U}$ and $\mathcal{Y}$ we define the noncommutative Schur class $\mathcal{S}_G(\mathcal{U}, \mathcal{Y})$ to be the set of formal power series $S(z) = \sum_{\gamma \in \Gamma} S_\gamma z^\gamma$ in noncommutative indeterminates $z = (z_\alpha \colon \alpha \in Q_1)$ (where $z^\gamma = z_{\alpha_n} \cdots z_{\alpha_1}$ in case $\gamma = (\alpha_n, \dots, \alpha_1) \in \Gamma$) such that the sequence of Taylor coefficients $S_\gamma \in \mathcal{L}(\mathcal{U}_{s(\gamma)}, \mathcal{Y}_{r(\gamma)})$ defines an element $S = \left[\ S_{\gamma,\gamma'}\ \right]_{\gamma,\gamma' \in \Gamma}$ (with $S_{\gamma,\gamma'} = S_{\gamma\gamma'^{-1}}$ and $S_{\text{undefined}} = 0$) in $\mathfrak{L}_\Gamma(\mathcal{U}, \mathcal{Y})$ of norm at most one. One can also define a commutative analog of the Schur class $\mathcal{S}_G(\mathcal{U}, \mathcal{Y})$, but we will not develop this here.

Now assume we are also given an auxiliary Hilbert space $\mathcal{Z}$ with direct sum decomposition $\mathcal{Z} = \oplus_{v \in Q_0} \mathcal{Z}_v$. We then define the generalized unit disc $\mathbb{D}_{G,\mathcal{Z}}$ associated with $\mathcal{Z}$ and the quiver G to be the set of tuples of operators of the form $(Z_\alpha \in \mathcal{L}(\mathcal{Z}_{s(\alpha)}, \mathcal{Z}_{r(\alpha)}) \colon \alpha \in Q_1)$ such that for each $v \in Q_0$ the row matrix formed by all Z_α with $r(\alpha) = v$ is a strict contraction:

$$Z_v := \left[\begin{array}{c} Z_\alpha \end{array} \right]_{\substack{\alpha \in Q_1 \\ r(\alpha)=v}} : \oplus_{\substack{\alpha \in Q_1 \\ r(\alpha)=v}} \mathcal{Z}_{s(\alpha)} \to \mathcal{Z}_v \text{ satisfies } \|Z_v\| < 1.$$

In other words, the operator matrix

$$\left[\begin{array}{c} Z_{v,\alpha} \end{array} \right]_{\substack{v \in Q_0 \\ \alpha \in Q_1}} : \oplus_{\alpha \in Q_1} \mathcal{Z}_{s(\alpha)} \to \mathcal{Z} \text{ with } Z_{v,\alpha} = \begin{cases} Z_\alpha & \text{if } v = r(\alpha), \\ 0 & \text{otherwise,} \end{cases}$$

is a strict contraction. For $Z \in \mathbb{D}_{G,\mathcal{Z}}$ given by the tuple $(Z_\alpha \colon \alpha \in Q_1)$ and $\gamma = (\alpha_n, \dots, \alpha_1) \in \Gamma$ we introduce the notation Z^γ for the operator

$$Z^\gamma = Z_{\alpha_n} \cdots Z_{\alpha_1} : \mathcal{Z}_{s_n(\gamma)} \to \mathcal{Z}_{r_n(\gamma)}. \tag{3.8}$$

In the sequel we shall use the abbreviations: $\mathcal{R}_v = \mathcal{U}_v \otimes \mathcal{Z}_v$ and $\mathcal{Q}_v = \mathcal{Y}_v \otimes \mathcal{Z}_v$ for each $v \in Q_0$, and $\mathcal{R} = \oplus_{v \in Q_0} \mathcal{R}_v$ and $\mathcal{Q} = \oplus_{v \in Q_0} \mathcal{Q}_v$.

Given a Schur class function S with Taylor coefficients $\{S_\gamma \colon \gamma \in \Gamma\}$ and $Z \in \mathbb{D}_{G,\mathcal{Z}}$ we define the value of S at Z to be given by the tensor functional-calculus:

$$S(Z) = \sum_{\gamma \in \Gamma} i_{\mathcal{Q}_{r(\gamma)}} (S_\gamma \otimes Z^\gamma) i^*_{\mathcal{R}_{s(\gamma)}} \in \mathcal{L}(\mathcal{R}, \mathcal{Q}). \tag{3.9}$$

Here we use the standard notation that for a subspace $\mathcal{V}$ of a Hilbert space $\mathcal{W}$ we write $i_\mathcal{V}$ for the canonical embedding of $\mathcal{V}$ into $\mathcal{W}$.

With respect to this tensor-product point-evaluation we consider what we will call the *Quiver Left-Tangential Tensor functional calculus Nevanlinna-Pick interpolation problem* (**QLTT-NP**): *Given a data set*

$$\mathfrak{D} : \; Z^{(1)}, \dots, Z^{(N)} \in \mathbb{D}_{G,\mathcal{Z}}, \; X_1, \dots, X_N \in \mathcal{L}(\mathcal{Q}, \mathcal{C}), \; Y_1, \dots, Y_N \in \mathcal{L}(\mathcal{R}, \mathcal{C}),$$

where $\mathcal{C}$ is an auxiliary Hilbert space, determine when there exists a Schur class function S in $\mathcal{S}_G(\mathcal{U}, \mathcal{Y})$ that satisfies

$$X_i S(Z^{(i)}) = Y_i \quad \textit{for} \quad i = 1, \dots, N.$$

The Riesz-Dunford functional calculus for this setting is then just the tensor functional calculus for the special case that the coefficient spaces $\mathcal{U}$ and $\mathcal{Y}$ are both equal to $\oplus_{v \in Q_0} \mathbb{C}$. Hence $\mathcal{R} = \mathcal{Q} = \mathcal{Z}$ and for $Z \in \mathbb{D}_{G,\mathcal{Z}}$ and a Schur class function s in $\mathcal{S}_G$ with Taylor coefficients $\{s_\gamma \colon \gamma \in \Gamma\}$ the value of s at Z is given by

$$s(Z) = \sum_{\gamma \in \Gamma} s_\gamma \cdot i_{\mathcal{Z}_{r(\gamma)}} Z^\gamma i^*_{\mathcal{Z}_{s(\gamma)}} \in \mathcal{L}(\mathcal{Z}).$$

The corresponding Nevanlinna-Pick problem is referred to as the *Quiver Left-Tangential Riesz-Dunford Nevanlinna-Pick interpolation problem* (**QLTRD-NP**).

We can also define an operator-argument functional-calculus for the Schur class $\mathcal{S}_G(\mathcal{U},\mathcal{Y})$. For this purpose, assume we have another Hilbert space $\mathcal{X}$, again admitting an orthogonal direct sum decomposition of the form $\mathcal{X}=\oplus_{v\in Q_0}\mathcal{X}_v$. The points in this case come from the generalized disk $\mathbb{D}_{\widetilde{G},\mathcal{X}}$, where $\widetilde{G}$ is the transposed quiver of G. Thus a $T\in\mathbb{D}_{\widetilde{G},\mathcal{X}}$ corresponds to a tuple $(T_\alpha\in\mathcal{L}(\mathcal{X}_{r(\alpha)},\mathcal{X}_{s(\alpha)})\colon \alpha\in Q_1)$ with the property that the operator matrix

$$\left[\ T_{v,\alpha}\ \right]_{\substack{v\in Q_0\\ \alpha\in Q_1}}:\oplus_{\alpha\in Q_1}\mathcal{X}_{r(\alpha)}\to\mathcal{X}\text{ with }T_{v,\alpha}=\begin{cases}T_\alpha & \text{if } v=s(\alpha),\\ 0 & \text{otherwise}\end{cases}$$

is a strict contraction. Given a $T=(T_\alpha\colon\alpha\in Q_1)\in\mathbb{D}_{\widetilde{G},\mathcal{X}}$ and $\gamma=(\alpha_n,\dots,\alpha_1)\in\Gamma$, then $\gamma^\top:=(\alpha_1,\dots,\alpha_n)\in\widetilde{\Gamma}$ and we set

$$T^{\gamma^\top}=T_{\alpha_1}\cdots T_{\alpha_n}:\mathcal{X}_{r(\gamma)}\to\mathcal{X}_{s(\gamma)}. \tag{3.10}$$

Then for a Schur class function $S\in\mathcal{S}_G(\mathcal{U},\mathcal{Y})$, a $T\in\mathbb{D}_{\widetilde{G},\mathcal{X}}$ and a block diagonal operator $X=\operatorname{diag}_{v\in Q_0}(X_v)$, with $X_v\in\mathcal{L}(\mathcal{Y}_v,\mathcal{X}_v)$, we define the left-tangential operator-argument point-evaluation $(XS)^{\wedge L}(T)$ by

$$(XS)^{\wedge L}(T)=\sum_{\gamma\in\Gamma} i_{\mathcal{X}_{s(\gamma)}}T^{\gamma^\top}X_{r(\gamma)}S_\gamma i^*_{\mathcal{U}_{s(\gamma)}}=\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}} i_{\mathcal{X}_{r(\widetilde{\gamma})}}T^{\widetilde{\gamma}}X_{s(\widetilde{\gamma})}S_{\widetilde{\gamma}^\top} i^*_{\mathcal{U}_{r(\widetilde{\gamma})}}\in\mathcal{L}(\mathcal{U},\mathcal{X}).$$

Notice that $(XS)^{\wedge L}(T)$ is a block diagonal operator in $\mathcal{L}(\mathcal{U},\mathcal{X})$, that is, the operator $(XS)^{\wedge L}(T)$ maps $\mathcal{U}_v$ into $\mathcal{X}_v$ for each $v\in Q_0$.

We then consider the *quiver Left-Tangential Nevanlinna-Pick interpolation problem with Operator Argument* (**QLTOA-NP**): *Given a data set*

$$T^{(i)}\in\mathbb{D}_{\widetilde{G},\mathcal{X}},\ X^{(i)}=\operatorname{diag}_{v\in Q_0}(X_v^{(i)}),\ Y^{(i)}=\operatorname{diag}_{v\in Q_0}(Y_v^{(i)}),\ \textit{for } i=1,\dots,N,$$

where $X_v^{(i)}\in\mathcal{L}(\mathcal{Y}_v,\mathcal{X}_v)$ *and* $Y_v^{(i)}\in\mathcal{L}(\mathcal{U}_v,\mathcal{X}_v)$, *determine when there exists a Schur class function* S *in* $\mathcal{S}_G(\mathcal{U},\mathcal{Y})$ *that satisfies*

$$(X^{(i)}S)^{\wedge L}(T^{(i)})=Y^{(i)}\quad\textit{for}\quad i=1,\dots,N.$$

The solutions to these interpolation problems are given in the following theorem. The proofs of these statements are given in Subsection 4.9 below.

Theorem 3.3. *Let the data for the* **QLTT-NP**, **QLTRD-NP** *and* **QLTOA-NP** *problems be as given above.*

1. *Assume that* $\mathcal{Z}$ *is separable, and that* $\{e_1^{(v)},\dots,e_{\kappa_v}^{(v)}\}$ *is an orthonormal basis for* $\mathcal{Z}_v$ *for each* $v\in Q_0$ *(with possibly* $\kappa_v=\infty$*). Then a solution to the* **QLTT-NP** *interpolation problem exists if and only if for each* $v\in Q_0$ *the associated Pick matrix* $\mathbb{P}^{(v)}_{QLTT}\in\mathcal{L}(\mathcal{C})^{\kappa_vN\times\kappa_vN}$ *given by*

$$\left[\sum_{\gamma\in\Gamma,s(\gamma)=v}\begin{matrix}X_i i_{\mathcal{Q}_{r(\gamma)}}\left(I_{\mathcal{Y}_{r(\gamma)}}\otimes(Z^{(i)})^\gamma e_{i'}^{(v)}e_{j'}^{(v)*}(Z^{(j)})^{\gamma*}\right)i^*_{\mathcal{Q}_{r(\gamma)}}X_j^*+\\ -Y_i i_{\mathcal{Q}_{r(\gamma)}}\left(I_{\mathcal{U}_{r(\gamma)}}\otimes(Z^{(i)})^\gamma e_{i'}^{(v)}e_{j'}^{(v)*}(Z^{(j)})^{\gamma*}\right)i^*_{\mathcal{Q}_{r(\gamma)}}Y_j^*\end{matrix}\right]_{(i,i'),(j,j')}$$

where (i,i') and (j,j') range over $\{1,\dots,N\}\times\{1,\dots,\kappa_v\}$, is positive semidefinite.

2. *Assume that $\mathcal{C}$ is separable, and that $\{e_1,\dots,e_\kappa\}$ is an orthonormal basis for $\mathcal{C}$ (with possibly $\kappa=\infty$). Then a solution to the* **QLTRD-NP** *interpolation problem exists if and only if the Pick matrix $\mathbb{P}_{\mathrm{QLTRD}}\in\mathcal{L}(\mathcal{Z})^{\kappa N\times\kappa N}$, for which the entry corresponding to the pairs $(i,i'),(j,j',)\in\{1,\dots,N\}\times\{1,\dots,\kappa\}$ is given by*

$$\begin{aligned}&\left[\ \mathbb{P}_{\mathrm{QLTRD}}\ \right]_{(i,i'),(j,j')}\\&\quad=\sum_{\gamma\in\Gamma} i_{\mathcal{Z}_{s(\gamma)}}(Z^{(i)})^{\gamma*}i^*_{\mathcal{Z}_{r(\gamma)}}(X_i^*e_{i'}e_{j'}^*X_j-Y_i^*e_{i'}e_{j'}^*Y_j)i_{\mathcal{Z}_{r(\gamma)}}(Z^{(j)})^{\gamma}i^*_{\mathcal{Z}_{s(\gamma)}}\end{aligned}$$

is positive semidefinite.

3. *A solution to the* **QLTOA-NP** *interpolation problem exists if and only if the Pick matrix*

$$\mathbb{P}_{\mathrm{QLTOA}}=\left[\ \sum_{\gamma\in\Gamma} i_{\mathcal{X}_{s(\gamma)}}(T^{(i)})^{\gamma}(X^{(i)}_{r(\gamma)}X^{(j)*}_{r(\gamma)}-Y^{(i)}_{r(\gamma)}Y^{(j)*}_{r(\gamma)})(T^{(j)})^{\gamma*}i^*_{\mathcal{X}_{s(\gamma)}}\ \right]_{i,j=1}^N$$

is positive semidefinite.

We conclude this subsection with an example for a concrete quiver; an example also considered in [60]. Let $G=\{Q_0,Q_1,s,r\}$ be the quiver with two vertices $Q_0=\{a,b\}$, two arrows $Q_1=\{\alpha,\beta\}$, and source and range map given by

$$s(\alpha)=r(\alpha)=a,\quad s(\beta)=a\quad\text{and}\quad r(\beta)=b.$$

With the vertices a and b we associate Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$, and we consider the Toeplitz algebra $\mathfrak{L}_\Gamma(\mathcal{A}\oplus\mathcal{B})$. Here Γ is the path semigroupoid of G which equals

$$\Gamma=\{\alpha^n,\beta\alpha^n,b\colon n\in\mathbb{Z}_+\},$$

where α^n and $\beta\alpha^n$ are abbreviations for $(\alpha,\dots,\alpha)$ (with length n) and $(\beta,\alpha,\dots,\alpha)$ (with length $n+1$), respectively, and $\alpha^0=a$. The elements in the Toeplitz algebra are then given by infinite tuples

$$R=(V_n,\,W_n,\,B_0\colon n\in\mathbb{Z}_+,V_n\in\mathcal{L}(\mathcal{A}),W_n\in\mathcal{L}(\mathcal{A},\mathcal{B}),B_0\in\mathcal{L}(\mathcal{B}))\tag{3.11}$$

with the property that the infinite operator matrix

$$\begin{bmatrix} V_0&0&0&0&0&0&\cdots\\ 0&B_0&0&0&0&0&\cdots\\ V_1&0&V_0&0&0&0&\cdots\\ W_0&0&0&B_0&0&0&\cdots\\ V_2&0&V_1&0&V_0&0&\cdots\\ W_1&0&W_0&0&0&B_0&\\ \vdots&\vdots&\vdots&\vdots&\vdots&&\ddots\end{bmatrix}\tag{3.12}$$

defines a bounded operator on $\ell^2_{\mathcal{A}\oplus\mathcal{B}}(\mathbb{Z}_+)$; the norm of R in $\mathfrak{L}_\Gamma(\mathcal{A}\oplus\mathcal{B},\mathcal{A}\oplus\mathcal{B})$ is equal to the operator norm of the operator matrix in $\mathcal{L}(\ell^2_{\mathcal{A}\oplus\mathcal{B}}(\mathbb{Z}_+))$. After rearranging rows and columns it follows that the operator norm of (3.12) is the same as that of

$$\begin{bmatrix} B_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & V_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & W_0 & B_0 & 0 & 0 & 0 & \cdots \\ 0 & V_1 & 0 & V_0 & 0 & 0 & \cdots \\ 0 & W_1 & 0 & W_0 & B_0 & 0 & \cdots \\ 0 & V_2 & 0 & V_1 & 0 & V_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

as an element in $\mathcal{L}(\mathcal{B} \oplus \ell^2_{\mathcal{A}\oplus\mathcal{B}}(\mathbb{Z}_+))$, and one can even leave out the first row and column. Thus R can be identified with the functions $V \in H^\infty_{\mathcal{L}(\mathcal{A})}(\mathbb{D})$ and $W \in H^\infty_{\mathcal{L}(\mathcal{A},\mathcal{B})}(\mathbb{D})$ given by

$$V(\lambda) = \sum_{n=0}^{\infty} \lambda^n V_n, \quad W(\lambda) = \sum_{n=0}^{\infty} \lambda^n W_n, \tag{3.13}$$

and the constant function with value B_0, which we also denote by B_0. The norm of R is then equal to the norm of the multiplication operator

$$\begin{bmatrix} M_V & 0 \\ M_W & M_{B_0} \end{bmatrix} \text{ on } \begin{bmatrix} H^2_{\mathcal{A}}(\mathbb{D}) \\ H^2_{\mathcal{B}}(\mathbb{D}) \end{bmatrix}, \tag{3.14}$$

where M_V, M_W and M_{B_0} denote the multiplication operators for the functions V, W and B_0, respectively. The Toeplitz algebra $\mathfrak{L}_\Gamma(\mathcal{A} \oplus \mathcal{B})$ can thus be identified with the algebra

$$\begin{bmatrix} H^\infty_{\mathcal{A}}(\mathbb{D}) & 0 \\ H^\infty_{\mathcal{L}(\mathcal{A},\mathcal{B})}(\mathbb{D}) & \mathcal{L}(\mathcal{B}) \end{bmatrix},$$

(with $\mathcal{L}(\mathcal{B})$ identified with the space of constant functions), which is easily seen to be isometrically isomorphic to the algebra

$$\begin{bmatrix} H^\infty_{\mathcal{A}}(\mathbb{D}) & 0 \\ H^\infty_{\mathcal{L}(\mathcal{A},\mathcal{B}),0}(\mathbb{D}) & \mathcal{L}(\mathcal{B}) \end{bmatrix}.$$

Here $H^\infty_{\mathcal{L}(\mathcal{A},\mathcal{B}),0}(\mathbb{D})$ is the Banach space of functions $W \in H^\infty_{\mathcal{L}(\mathcal{A},\mathcal{B})}(\mathbb{D})$ with $W(0) = 0$. The identification with the latter algebra was already obtained in [60].

We first consider Nevanlinna-Pick interpolation for the Riesz-Dunford functional calculus, i.e., when $\mathcal{A} = \mathcal{B} = \mathbb{C}$. Let $\mathcal{Z} = \mathcal{Z}_a \oplus \mathcal{Z}_b$ be an auxiliary Hilbert space. The generalized unit disk $\mathbb{D}_{G,\mathcal{Z}}$ then consists of all pairs of operators (Z_α, Z_β) with $Z_\alpha \in \mathcal{L}(\mathcal{Z}_a)$ and $Z_\beta \in \mathcal{L}(\mathcal{Z}_a, \mathcal{Z}_b)$ such that $\left[\begin{smallmatrix} Z_a \\ Z_b \end{smallmatrix}\right]$ is a strict contraction. Given a point (Z_α, Z_β) in $\mathbb{D}_{G,\mathcal{Z}}$ and an

$$R = (v_n, w_n, b_0 \in \mathbb{C} \colon n \in \mathbb{Z}_+) \in \mathfrak{L}_\Gamma,$$

the point evaluation of R in (Z_α, Z_β) is given by

$$R(Z_\alpha, Z_\beta) = \begin{bmatrix} \sum_{n=0}^\infty v_n Z_\alpha^n & 0 \\ Z_\beta \sum_{n=0}^\infty w_n Z_\alpha^n & b_0 I_{\mathcal{Z}_b} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Z}_a \\ \mathcal{Z}_b \end{bmatrix}.$$

Assume we are given data for the **QLTRD-NP** problem:

$$(Z_\alpha^{(i)}, Z_\beta^{(i)}) \in \mathbb{D}_{G,\mathcal{Z}},\ X^{(i)} = \begin{bmatrix} X_a^{(i)} & X_b^{(i)} \end{bmatrix},\ Y^{(i)} = \begin{bmatrix} Y_{a^{(i)}} & Y_b^{(i)} \end{bmatrix} \text{ for } i = 1, \dots, N,$$

with $X^{(i)}, Y^{(i)} \in \mathcal{L}(\mathcal{Z}, \mathcal{C})$, where $\mathcal{C}$ is some separable Hilbert space with orthonormal basis $\{e_1, \dots, e_\kappa\}$. It then follows from Theorem 3.3 that there exists an $S \in \mathfrak{L}_\Gamma$ with $\|S\| \leq 1$ such that

$$X^{(i)} S(Z_\alpha^{(i)}, Z_\beta^{(i)}) = Y^{(i)} \text{ for } i = 1, \dots, N$$

if and only if the Pick matrices $\mathbb{P}^{(1)}_{\text{QLTRD}}$ and $\mathbb{P}^{(2)}_{\text{QLTRD}}$, with $\mathbb{P}^{(1)}_{\text{QLTRD}}$ given by

$$\left[\sum_{n=0}^{\infty} (Z_\alpha^{(i)})^{n*} \begin{pmatrix} X_a^{(i)*} e_{i'} e_{j'}^* X_a^{(j)} + Z_\beta^{(i)*} X_b^{(i)*} e_{i'} e_{j'}^* X_b^{(j)} Z_\beta^{(j)} + \\ -Y_a^{(i)*} e_{i'} e_{j'}^* Y_a^{(j)} - Z_\beta^{(i)*} Y_b^{(i)*} e_{i'} e_{j'}^* Y_b^{(j)} Z_\beta^{(j)} \end{pmatrix} (Z_\alpha^{(j)})^n \right]_{(i,i'),(j,j')}$$

and

$$\mathbb{P}^{(2)}_{\text{QLTRD}} = \begin{bmatrix} X_b^{(i)*} e_{i'} e_{j'}^* X_b^{(j)} - Y_b^{(i)*} e_{i'} e_{j'}^* Y_b^{(j)} \end{bmatrix}_{(i,i'),(j,j')},$$

are both positive semidefinite. The range of the pairs (i, i') and (j, j') in the definition of the Pick matrices is $\{1, \dots, N\} \times \{1, \dots, \kappa\}$.

Now assume that $S = (v_n, w_n, b_0 \in \mathbb{C} \colon n \in \mathbb{Z}_+) \in \mathfrak{L}_\Gamma$ is a solution. Then b_0 necessarily satisfies $|b_0| \leq 1$ and $b_0 X_b^{(i)} = Y_b^{(i)}$ for $i = 1, \dots, N$. The existence of a number b_0 with these properties turns out to be equivalent to the positive semidefiniteness of the pick matrix $\mathbb{P}^{(2)}_{\text{QLTRD}}$, which is the content of the following lemma.

Lemma 3.4. *For $i = 1, \dots, N$ let $X_i, Y_i \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\mathcal{H}$ is separable with orthonormal basis $\{e_1, \dots, e_\kappa\}$. Then*

$$\begin{bmatrix} X_i e_{i'} e_{j'} X_j^* - Y_i e_{i'} e_{j'} Y_j^* \end{bmatrix}_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}} \tag{3.15}$$

is a positive semidefinite if and only if there exist a $\delta \in \mathbb{C}$ with $|\delta| \leq 1$ such that $\delta X_i = Y_i$ for all $i = 1, \dots, N$.

Proof. Set

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}.$$

Then (3.15) is unitarily equivalent, via a permutation matrix, to

$$\text{col}_{i' \in \{1,\dots,\kappa\}}(X e_{i'}) (\text{col}_{i' \in \{1,\dots,\kappa\}}(X e_{i'}))^* - \text{col}_{i' \in \{1,\dots,\kappa\}}(Y e_{i'}) (\text{col}_{i' \in \{1,\dots,\kappa\}}(Y e_{i'}))^*. \tag{3.16}$$

Thus positive semidefiniteness of (3.15) corresponds to positive semidefiniteness of (3.16). It follows right away from the Douglas factorization lemma [43] that (3.16) being positive semidefinite is equivalent to the existence of a $\delta \in \mathbb{C}$ with $|\delta| \leq 1$ such that $\delta \text{col}_{i' \in \{1,\dots,\kappa\}}(X e_{i'}) = \text{col}_{i' \in \{1,\dots,\kappa\}}(Y e_{i'})$, which is the same as $\delta X_i = Y_i$ for all $i = 1, \dots, N$. □

Next assume that both Pick matrices $\mathbb{P}^{(1)}_{\rm QLTRD}$ and $\mathbb{P}^{(2)}_{\rm QLTRD}$ are positive semidefinite. According to Lemma 3.4 there exists a b_0 with $|b_0| \le 1$ such that $b_0 X_b^{(i)} = Y_b^{(i)}$ for $i = 1, \dots, N$. Using this number b_0 we can rewrite $\mathbb{P}^{(2)}_{\rm QLTRD}$ as

$$\left[\sum_{n=0}^{\infty} (Z_\alpha^{(i)})^{n*} \left(\begin{array}{l} X_a^{(i)*} e_{i'} e_{j'}^* X_a^{(j)} - Y_a^{(i)*} e_{i'} e_{j'}^* Y_a^{(j)} + \\ \qquad +(1-|b_0|^2) Z_\beta^{(i)*} X_b^{(i)*} e_{i'} e_{j'}^* X_b^{(j)} Z_\beta^{(j)} \end{array} \right) (Z_\alpha^{(j)})^n \right]_{(i,i'),(j,j')},$$

where again the range of the pairs (i, i') and (j, j') in the definition of the Pick matrices is $\{1, \dots, N\} \times \{1, \dots, \kappa\}$. We then distinguish between two case, namely (1) $|b_0| = 1$ and (2) $|b_0| < 1$. In case $|b_0| = 1$, for $S = (v_n, w_n, b_0 \in \mathbb{C} \colon n \in \mathbb{Z}_+) \in \mathfrak{L}_\Gamma$ to be a solution it is necessary that $w_n = 0$ for all $n \in \mathbb{Z}_+$ due to the norm constraint $\|S\| \le 1$. On the other hand, the Pick matrix $\mathbb{P}^{(1)}_{\rm QLTRD}$ reduces to

$$\left[\sum_{n=0}^{\infty} (Z_\alpha^{(i)})^{n*} \left(X_a^{(i)*} e_{i'} e_{j'}^* X_a^{(j)} - Y_a^{(i)*} e_{i'} e_{j'}^* Y_a^{(j)} \right) (Z_\alpha^{(j)})^n \right]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}}$$

which is a Pick matrix of the form appearing in Part 2 of Theorem 2.3. In fact, it is the pick matrix for the **LTRD-NP** problem for the scalar-valued Schur class $\mathcal{S}$ with data

$$Z_i = Z_\alpha^{(i)},\ X_i = X_a^{(i)},\ Y_i = Y_a^{(i)}, \quad i = 1, \dots, N.$$

Applying Theorem 2.3 to this data set we obtain a $v \in \mathcal{S}$ with $X_a^{(i)} v(Z_\alpha^{(i)}) = Y_a^{(i)}$ for $i = 1, \dots, N$. Let $v_0, v_1, \dots$ be the Taylor coefficients of v. It is then not difficult to see that $S = (v_n, w_n, b_0 \in \mathbb{C} \colon n \in \mathbb{Z}_+,\ w_n = 0)$ is a solution. The case that $|b_0| < 1$ does not reduce to a **LTRD-NP** problem, but rather to a left-tangential tensor functional-calculus Nevanlinna-Pick (**LTT-NP**) problem which is a type of problem we discuss in Subsection 4.8 below. The problem can then be solved directly using some of the techniques developed there, but we will not work out the details here.

Next we specify the left-tangential operator-argument Nevanlinna-Pick result for our example. Let $\mathcal{X} = \mathcal{X}_\alpha \oplus \mathcal{X}_\beta$ be a given Hilbert space. in this case, points are elements of the generalized disk $\mathbb{D}_{\widetilde{G},\mathcal{X}}$ (with $\widetilde{G}$ the transposed quiver of G), which corresponds to the set of pairs (T_α, T_β) with $T_\alpha \in \mathcal{L}(\mathcal{X}_a, \mathcal{X}_a)$ and $T_\beta \in \mathcal{L}(\mathcal{X}_b, \mathcal{X}_a)$ such that the row operator $\left[\begin{smallmatrix} T_\alpha & T_\beta \end{smallmatrix}\right]$ is a strict contraction. Now let $R \in \mathfrak{L}_\Gamma(\mathcal{A} \oplus \mathcal{B})$ be given by (3.11) and $(T_\alpha, T_\beta) \in \mathbb{D}_{\widetilde{G},\mathcal{X}}$. If in addition we are also given a block diagonal operator $X = \operatorname{diag}(X_a, X_b)$ with $X_a \in \mathcal{L}(\mathcal{A}, \mathcal{X}_a)$ and $X_b \in \mathcal{L}(\mathcal{B}, \mathcal{X}_b)$, then the left-tangential operator-argument point evaluation $(XR)^{\wedge L}(T_\alpha, T_\beta)$ is given by

$$(XR)^{\wedge L}(T_\alpha, T_\beta) = \left[\begin{array}{cc} \sum_{n=0}^{\infty} T_\alpha^n (X_a V_n + T_\beta X_b W_n) & 0 \\ 0 & X_b B_0 \end{array} \right].$$

The data for the **QLTOA-NP** problem in then given by

$$(T_\alpha^{(i)}, T_\beta^{(i)}) \in \mathbb{D}_{\widetilde{G},\mathcal{X}},\ X^{(i)} = \operatorname{diag}(X_a^{(i)}, X_b^{(i)}),\ Y^{(i)} = \operatorname{diag}(Y_a^{(i)}, Y_b^{(i)}) \text{ for } i = 1, \dots, N$$

with $X_a^{(i)}, Y_a^{(i)} \in \mathcal{L}(\mathcal{A}, \mathcal{X}_a)$ and $X_b^{(i)}, Y_b^{(i)} \in \mathcal{L}(\mathcal{B}, \mathcal{X}_b)$, and it follows from Part 3 of Theorem 3.3 that there exists an $S \in \mathfrak{L}_\Gamma(\mathcal{A} \oplus \mathcal{B})$ with $\|S\| \leq 1$ such that

$$(X^{(i)}S)^{\wedge L}(T_\alpha^{(i)}, T_\beta^{(i)}) = Y^{(i)} \quad \text{for} \quad i = 1, \ldots, N \tag{3.17}$$

if and only if the Pick matrix $\mathbb{P}_{\text{QLTOA}}$ in Part 3 of Theorem 3.3 is positive semidefinite. In this case, after rearranging columns and rows, $\mathbb{P}_{\text{QLTOA}}$ can be identified with

$$\mathbb{P}_{\text{QLTOA}} = \begin{bmatrix} \mathbb{P}^{(1)}_{\text{QLTOA}} & 0 \\ 0 & \mathbb{P}^{(2)}_{\text{QLTOA}} \end{bmatrix}$$

where $\mathbb{P}^{(1)}_{\text{QLTOA}}$ is the Pick matrix given by

$$\left[\sum_{n=0}^{\infty} (T_\alpha^{(i)})^n (X_a^{(i)} X_a^{(j)*} + T_\beta^{(i)} X_b^{(i)} X_b^{(j)*} T_\beta^{(j)*} - Y_a^{(i)} Y_a^{(j)*} - T_\beta^{(i)} Y_b^{(i)} Y_b^{(j)*} T_\beta^{(j)*})(T_\alpha^{(j)})^{n*} \right]_{i,j=1}^{N}$$

and

$$\mathbb{P}^{(2)}_{\text{QLTOA}} = \left[\ X_b^{(i)} X_b^{(j)*} - Y_b^{(i)} Y_b^{(j)*} \ \right]_{i,j=1}^{N}.$$

To see the sufficiency of this Pick matrix criterion, assume that $\mathbb{P}_{\text{QLTOA}}$ is positive semidefinite, and thus, equivalently, that $\mathbb{P}^{(1)}_{\text{QLTOA}}$ and $\mathbb{P}^{(2)}_{\text{QLTOA}}$ are positive semidefinite. Notice that $\mathbb{P}^{(2)}_{\text{QLTOA}}$ can also be written as

$$\begin{bmatrix} X_b^{(1)} \\ \vdots \\ X_b^{(N)} \end{bmatrix} \begin{bmatrix} X_b^{(1)*} & \cdots & X_b^{(N)*} \end{bmatrix} - \begin{bmatrix} Y_b^{(1)} \\ \vdots \\ Y_b^{(N)} \end{bmatrix} \begin{bmatrix} Y_b^{(1)*} & \cdots & Y_b^{(N)*} \end{bmatrix}.$$

Hence the positive semidefiniteness of $\mathbb{P}^{(2)}_{\text{QLTOA}}$, again using Douglas factorization lemma, corresponds to the existence of a contraction $B_0 \in \mathcal{L}(\mathcal{B})$ with

$$X_b^{(i)} B_0 = Y_b^{(i)} \quad \text{for} \quad i = 1, \ldots, N.$$

Let $D_{B_0^*}$ denote the defect operator of B_0^*, that is, $D_{B_0^*}$ is the positive square root of $I_\mathcal{B} - B_0 B_0^*$. We can then rewrite the first Pick matrix $\mathbb{P}^{(1)}_{\text{QLTOA}}$ as

$$\left[\sum_{n=0}^{\infty} (T_\alpha^{(i)})^n (X_a^{(i)} X_a^{(j)*} + T_\beta^{(i)} X_b^{(i)} D^2_{B_0^*} X_b^{(j)*} T_\beta^{(j)*} - Y_a^{(i)} Y_a^{(j)*})(T_\alpha^{(j)})^{n*} \right]_{i,j=1}^{N}.$$

In this form $\mathbb{P}^{(1)}_{\text{QLTOA}}$ is a Pick matrix of the type appearing in Part 1 of Theorem 2.2. In fact, it is the Pick matrix for the **LTOA-NP** problem for functions from the Schur class $\mathcal{S}(\mathcal{A}, \mathcal{A} \oplus \mathcal{B})$ with data

$$T_i = T_\alpha^{(i)},\ X_i = \begin{bmatrix} X_a^{(i)} & T_\beta^{(i)} X_b^{(i)} D_{B_0^*} \end{bmatrix},\ Y_i = Y_a^{(i)}, \ \text{for } i = 1, \ldots, N. \tag{3.18}$$

Applying Theorem 2.2 to this data set, we obtain a function

$$H = \begin{bmatrix} V \\ \widetilde{W} \end{bmatrix} \in \mathcal{S}(\mathcal{A}, \mathcal{A} \oplus \mathcal{B}) \tag{3.19}$$

with

$$(X_iH)^{\wedge L}(T_i) = Y_i \quad \text{for} \quad i = 1, \dots, N.$$

In other words, the Taylor coefficients $V_0, V_1, \dots$ of V and $\widetilde{W}_0, \widetilde{W}_1, \dots$ of $\widetilde{W}$ satisfy

$$\sum_{n=0}^{\infty} (T_\alpha^{(i)})^n (X_a^{(i)} V_n + T_\beta^{(i)} X_b^{(i)} D_{B_0^*} \widetilde{W}_n) = Y_a^{(i)} \text{ for } i = 1, \dots, N.$$

Now set $W_n = D_{B_0^*}\widetilde{W}_n$ for each $n \in \mathbb{Z}_+$. It follows that $S = (V_n, W_n, B_0 \colon n \in \mathbb{Z}_+)$ is in $\mathfrak{L}_\Gamma(\mathcal{A} \oplus \mathcal{B})$ and satisfies the interpolation conditions (3.17), and it is not difficult to see that $\|S\| \le 1$. Thus S is a solution to the **QLTOA-NP** problem for the quiver G considered in this example. It is also possible to provide a direct proof of the necessity of the Pick matrix condition; we leave the details as an exercise for the interested reader.

3.4. The polydisk setting: commutative and noncommutative

For the setting of the polydisk $\mathbb{D}^d = \{\lambda = (\lambda_1, \dots, \lambda_d) \colon |\lambda_k| < 1 \text{ for } k = 1, \dots, d\}$, the results concerning Nevanlinna-Pick-like interpolation are of a different flavor. We define what is now called the d-variable *Schur-Agler class* $\mathcal{SA}_d$ to consist of those holomorphic complex-valued functions $s(\lambda) = \sum_{n \in \mathbb{Z}_+^d} s_n \lambda^n$ on $\mathbb{D}^d$ with the property that, for every commutative d-tuple $Z = (Z_1, \dots, Z_d)$ of strict contraction operators ($\|Z_k\| < 1$ for $k = 1, \dots, d$) on a Hilbert space $\mathcal{K}$, it happens that the resulting operator

$$s(Z) = \sum_{n \in \mathbb{Z}_+^d} s_n Z^n$$

(d-variable Riesz-Dunford functional calculus) has $\|s(Z)\| \le 1$. The special choice $Z = (\lambda_1 I_\mathcal{K}, \dots, \lambda_d I_\mathcal{K})$ with $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{D}^d$ shows that the Schur-Agler class is a subset of the Schur class (defined to be the set of holomorphic functions mapping $\mathbb{D}^d$ into the closed unit disk $\overline{\mathbb{D}}$). The converse holds for the cases $d = 1$ and $d = 2$ as a consequence of the von Neumann inequality holding for these two cases, as can be seen from the Sz.-Nagy dilation theorem for the case $d = 1$ and from the Andô dilation theorem for the case $d = 2$ – see, e.g., [6].

In fact, as a consequence of the Drury-von Neumann inequality/dilation theorem for commutative row contractions $Z = (Z_1, \dots, Z_d)$ [47] and the Popescu-von Neumann inequality/dilation theorem [82], a Schur-Agler type characterization also holds for the Drury-Arveson Schur-multiplier class $\mathcal{S}_d$ and the free-semigroup algebra $\mathcal{L}_d$: *a holomorphic function s on the ball $\mathbb{B}^d$ is in the Drury-Arveson Schur-multiplier class $\mathcal{S}_d$ if and only if $\|s(Z)\| \le 1$ for all commutative d-tuples $Z = (Z_1, \dots, Z_d)$ for which the block row matrix $\mathbf{Z} = \begin{bmatrix} Z_1 & \cdots & Z_d \end{bmatrix}$ is a strict contraction*, and, similarly, *a formal power series $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ is in the noncommutative Schur-multiplier class $\mathcal{S}_{nc,d}$ if and only if $s(Z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma Z^\gamma$ has $\|s(Z)\| \le 1$ for any (not necessarily commutative) d-tuple $Z = (Z_1, \dots, Z_d)$ for which the row matrix $\mathbf{Z} := \begin{bmatrix} Z_1 & \cdots & Z_d \end{bmatrix}$ is a strict contraction.*

The main result on interpolation for the Schur-Agler class is the following result of Agler.

Theorem 3.5. (*See* [2, 3, 4, 6].) *Suppose that we are given a subset X of $\mathbb{D}^d$ and a function $f\colon X \to \mathbb{C}$. Then there exists a function $s\colon \mathbb{D}^d \to \mathbb{C}$ in the Schur-Agler class $\mathcal{SA}_d$ such that*

$$s|_X = f$$

if and only if there exist d positive kernels $K_1, \dots, K_d$ on $X \times X$ so that

$$1 - f(\lambda)\overline{f(\zeta)} = \sum_{k=1}^{d} (1 - \lambda_k \overline{\zeta_k}) K_k(\lambda, \zeta) \text{ for all } \lambda, \zeta \in X. \tag{3.20}$$

We mention that an independent proof valid for a more general function-algebra setting has been given by Paulsen [76].

Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, the operator-valued version $\mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$ of the Schur-Agler class can be defined via tensor functional calculus as follows. Given a holomorphic function $S(\lambda) = \sum_{n \in \mathbb{Z}_+^d} S_n \lambda^n$ with coefficients (and hence also values) in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ and given a commutative d-tuple $Z := (Z_1, \dots, Z_d)$ of operators on another auxiliary Hilbert space $\mathcal{K}$ such that each Z_k is a strict contraction for $k = 1, \dots, d$, we define $S(Z)$ via

$$S(Z) = \sum_{n \in \mathbb{Z}_+^d} S_n \otimes Z^n \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K}),$$

using the (commutative) multivariable notation for Z^n defined in Part 3 of Theorem 3.1. We then say that $S \in \mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$ if $\|S(Z)\| \le 1$ whenever $Z = (Z_1, \dots, Z_d)$ is a commutative d-tuple of strict contraction operators on $\mathcal{K}$. We will not state the results precisely here, but rather merely mention that the extensions to left- and right-tangential interpolation have been given in [28]. The theory can be generalized to an arbitrary domain $\mathcal{D}$ with polynomial-matrix defining function ($\mathcal{D} = \mathcal{D}_Q = \{\lambda \in \mathbb{C}^d \colon \|Q(\lambda)\| < 1\}$ for a fixed matrix polynomial $Q(\lambda)$) – see [10, 18]; note that the polydisk corresponds to the case

$$Q(\lambda) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}.$$

In this general setting the analysis of the left- and right-tangential Nevanlinna-Pick problem with operator-argument (using an adaptation of the Taylor-Vasilescu functional calculus) has been worked out in [19].

A noncommutative version of the Schur-Agler class can be defined as follows. Given coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ and a formal power series S of the form $S(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma$, where for each γ in the free semigroup $\mathcal{F}_d$ defined in Subsection 3.2 the coefficient S_γ is in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$, we say that S is in the noncommutative operator-valued d-variable Schur-Agler class $\mathcal{SA}_{nc,d}(\mathcal{U},\mathcal{Y})$ if, for any (not necessarily commutative) d-tuple $Z = (Z_1, \dots, Z_d)$ of strict contraction operators, it is the case that

$$S(Z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma \otimes Z^\gamma$$

has $\|S(Z)\| \leq 1$. Here we use the noncommutative multivariable notation of Subsection 3.1: $Z^\gamma = Z_{i_N} \cdots Z_{i_1}$ if $\gamma = i_N \dots i_1$. It is one of the results of [9] that in fact one need only check $\|S(Z)\| \leq 1$ for $Z = (Z_1, \dots, Z_d)$ a d-tuple of matrices of finite size $\kappa \times \kappa$ for arbitrary $\kappa = 1, 2, \dots$.

We then have the following result for left-tangential Nevanlinna-Pick interpolation with operator-argument on the noncommutative polydisk.

Theorem 3.6. (1) Left-Tangential interpolation with Operator-Argument for the noncommutative polydisk: *Suppose that we are given N d-tuples of (not necessarily commutative) strict contraction operators*

$$T^{(1)} = (T_1^{(1)}, \dots, T_d^{(1)}), \dots, T^{(N)} = (T_1^{(N)}, \dots, T_d^{(N)})$$

on a Hilbert space $\mathcal{C}$ together with operators $X_1, \dots, X_N$ in $\mathcal{L}(\mathcal{Y},\mathcal{C})$ and $Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{U},\mathcal{C})$. Then there exists a formal power series $S(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma$ in the noncommutative d-variable operator-valued Schur class $\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ such that

$$(X_i S)^{\wedge L}(T^{(i)}) := \sum_{\gamma \in \mathcal{F}_d} (T^{(i)})^{\gamma^\top} X_i S_\gamma = Y_i \text{ for } i = 1, \dots, N$$

if and only if there exist d positive semidefinite block matrices $K_1, \dots, K_d$ with entries in $\mathcal{L}(\mathcal{C})$ of the form

$$K_k = [K_k(i,j)]_{i,j=1,\dots,N}$$

so that the noncommutative Agler decomposition

$$X_i X_j^* - Y_i Y_j^* = \sum_{k=1}^d \Big(K_k(i,j) - T_k^{(i)} K_k(i,j) (T_k^{(j)})^* \Big)$$

holds.

(2) Riesz-Dunford interpolation for the noncommutative polydisk. *Suppose that $\mathcal{Z}$ is a separable Hilbert space with orthonormal basis $\{e_1, \dots, e_\kappa\}$ (with possibly $\kappa = \infty$) and that we are given N (not necessarily commutative) d-tuples of strictly contractive operators $Z^{(1)} = (Z_1^{(1)}, \dots, Z_d^{(1)})$, $\dots$, $Z^{(N)} = (Z_1^{(N)}, \dots, Z_d^{(N)})$ in $\mathcal{L}(\mathcal{Z})$. Assume also that we are given operators $W_1, \dots, W_N$ in $\mathcal{L}(\mathcal{Z})$. Then there exists a formal power series $s(z) = \sum_{\gamma \in \mathcal{F}_d} s_\gamma z^\gamma$ in the scalar noncommutative Schur-Agler class $\mathcal{SA}_{nc,d}$ such that*

$$s(Z^{(i)}) := \sum_{\gamma \in \mathcal{F}_d} s_\gamma (Z^{(i)})^{\gamma^\top} = W_i \text{ for } i = 1, \dots, N$$

if and only if there exist d positive semidefinite operator matrices $K_1, \dots, K_d$ of size $(N \cdot \kappa) \times (N \cdot \kappa)$ with entries in $\mathcal{L}(\mathcal{Z})$ written in the form

$$K_k = [K_k((i,i'),\,(j,j'))]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}}$$

so that the following noncommutative Agler decomposition holds:

$$e_{i'}e_{j'}^* - W_i e_{i'} e_{j'}^* W_j^* = \sum_{k=1}^{d} \Big(K_k\left((i,i'),(j,j')\right) - Z_k^{(i)} K_k\left((i,i'),(j,j')\right)(Z_k^{(j)})^* \Big).$$

Proof. Statement (1) is a particular case of [20, Theorem 7.9]. Statement (2) then follows from statement (1) in the same way as was done for the single-variable case in Section 2. □

As an example we next discuss how the Schur interpolation problem for the noncommutative polydisk can be handled as an application of Theorem 3.6. We first say that a subset Γ of $\mathcal{F}_d$ is *lower inclusive* if, whenever $\gamma \in \Gamma$ and γ factors as $\gamma = \gamma'\gamma''$, then also $\gamma' \in \Gamma$. The Carathéodory-Fejér interpolation problem for the noncommutative polydisk can be formulated as follows: *Given a lower inclusive subset Γ of $\mathcal{F}_d$ and given a collection of operators $\{F_\gamma \colon \gamma \in \Gamma\} \subset \mathcal{L}(\mathcal{U},\mathcal{Y})$, find a formal power series $S(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma$ in the noncommutative operator-valued Schur class $\mathcal{SA}_{nc,d}(\mathcal{U},\mathcal{Y})$ such that*

$$S_\gamma = F_\gamma \text{ for } \gamma \in \Gamma.$$

In [20, Section 7.4] it is shown how to choose the data set

$$\mathfrak{D}: \quad T^{(1)}, \dots, T^{(N)}, \quad X_1, \dots, X_d, \quad Y_1, \dots, Y_d$$

so that the associated left-tangential interpolation problem with operator-argument handled by statement (1) in Theorem 3.6 is equivalent to the Carathéodory-Fejér interpolation problem for the noncommutative polydisk. We note that this Carathéodory-Fejér problem was handled directly earlier in [57].

Just as was the case for the commutative case, the theory for the noncommutative polydisk can be extended to more general noncommutative domains. This is done in [26] for noncommutative operator domains with a certain type of linear defining function Q: $Z = (Z_1, \dots, Z_d) \in \mathcal{D}_Q$ if $\|Q(Z)\| < 1$ where $Q(z) = Q_1 z_1 + \cdots + Q_d z_d$ and $Q(Z) = Q_1 \otimes Z_1 + \cdots + Q_d \otimes Z_d$. The results on Left (and/or Right) Tangential interpolation with Operator-Argument in [20] are actually given for this level of generality.

4. W^*-correspondence Nevanlinna-Pick theorems

4.1. Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and E a linear space. We say that E is an $(\mathcal{A},\mathcal{B})$-*correspondence* when E is a bi-module with respect to a given right $\mathcal{B}$-action and a left $\mathcal{A}$-action, and E is endowed with a $\mathcal{B}$-valued inner product $\langle\ ,\ \rangle_E$ satisfying the following axioms: For any $\lambda, \mu \in \mathbb{C}$, $\xi, \eta, \zeta \in E$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$

1. $\langle \lambda\xi + \mu\zeta, \eta\rangle_E = \lambda\langle \xi, \eta\rangle_E + \mu\langle \zeta, \eta\rangle_E$;
2. $\langle \xi \cdot b, \eta\rangle_E = \langle \xi, \eta\rangle_E b$;
3. $\langle a \cdot \xi, \eta\rangle_E = \langle \xi, a^* \cdot \eta\rangle_E$;
4. $\langle \xi, \eta\rangle_E^* = \langle \eta, \xi\rangle_E$;
5. $\langle \xi, \xi\rangle_E \geq 0$ (in $\mathcal{B}$);
6. $\langle \xi, \xi\rangle_E = 0$ implies that $\xi = 0$;

and such that E is a Banach space with respect to the norm $\| \, \|_E$ defined by

$$\|\xi\|_E = \|\langle \xi, \xi\rangle_E\|_{\mathcal{B}}^{\frac{1}{2}} \quad (\xi \in E),$$

where $\| \, \|_{\mathcal{B}}$ denotes the norm of $\mathcal{B}$. We also impose that

$$(\lambda\xi) \cdot b = \xi \cdot (\lambda b) \quad \text{and} \quad (\lambda a) \cdot \xi = a \cdot (\lambda\xi) \quad (\lambda \in \mathbb{C}, a \in \mathcal{A}, b \in \mathcal{B}, \xi \in E).$$

In practice we usually write $\langle \; , \; \rangle$ and $\| \; \|$ for the inner product and norm on E, and in case $\mathcal{A} = \mathcal{B}$ we say that E is an $\mathcal{A}$-correspondence.

Given two $(\mathcal{A}, \mathcal{B})$-correspondences E and F, the set of bounded linear operators from E to F is denoted by $\mathcal{L}(E, F)$. It may happen that a $T \in \mathcal{L}(E, F)$ is not adjointable, i.e., it is not necessarily the case that

$$\langle T\xi, \gamma\rangle = \langle \xi, T^*\gamma\rangle \quad (\xi \in E, \gamma \in F)$$

for some $T^* \in \mathcal{L}(F, E)$. We will write $\mathcal{L}^a(E, F)$ for the set of adjointable operators in $\mathcal{L}(E, F)$. As usual we have the abbreviations $\mathcal{L}(E)$ and $\mathcal{L}^a(E)$ in case $F = E$.

The third inner-product axiom implies that the left $\mathcal{A}$-action can be identified with a $*$-homomorphism φ of $\mathcal{A}$ into the C^*-algebra $\mathcal{L}^a(E)$. In case this $*$-homomorphism φ is specified we will occasionally write $\varphi(a)\xi$ instead of $a \cdot \xi$.

Furthermore, an operator $T \in \mathcal{L}(E, F)$ is said to be a *right $\mathcal{B}$-module map* if

$$T(\xi \cdot b) = T(\xi) \cdot b \quad (\xi \in E, b \in \mathcal{B}),$$

and a *left $\mathcal{A}$-module map* whenever

$$T(a \cdot \xi) = a \cdot T(\xi) \quad (\xi \in E, a \in \mathcal{A}).$$

It is easily checked that an adjointable map $T \in \mathcal{L}^a(E, F)$ is automatically a right module map. Occasionally we leave out the $\mathcal{B}$ and $\mathcal{A}$, and just say left or right module map. In case T is both a left and right module map we also say that T is a *bi-module map*. Notice that the product of two left (right) module maps is again a left (right) module map, and the adjoint of an adjointable left (right) module map, is also a left (right) module map.

We will have a need for various constructions which create new correspondences out of given correspondences.

Given two $(\mathcal{A}, \mathcal{B})$-correspondences E and F, we define the *direct-sum correspondence* $E \oplus F$ to be the direct-sum vector space $E \oplus F$ together with the diagonal left $\mathcal{A}$-action and right $\mathcal{B}$-action and the direct-sum $\mathcal{B}$-valued inner-product defined by setting for each $\xi, \xi' \in E$, $\gamma, \gamma' \in F$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$:

$$a \cdot (\xi \oplus \gamma) = (a \cdot \xi) \oplus (a \cdot \gamma), \qquad (\xi \oplus \gamma) \cdot b = (\xi \cdot b) \oplus (\gamma \cdot b),$$
$$\langle \xi \oplus \gamma, \xi' \oplus \gamma'\rangle_{E \oplus F} = \langle \xi, \xi'\rangle_E + \langle \gamma, \gamma'\rangle_F.$$

Bounded linear operators between direct-sum correspondences admit operator matrix decompositions in precisely the same way as in the Hilbert space case ($\mathcal{B} = \mathbb{C}$), while adjointability and the left and right module map property of such an operator corresponds to the operators in the decomposition being adjointable, or left or right module maps, respectively.

Now suppose that we are given three C^*-algebras $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ together with an $(\mathcal{A}, \mathcal{B})$-correspondence E and a $(\mathcal{B}, \mathcal{C})$-correspondence F. Then we define the *tensor-product correspondence* $E \otimes F$ to be the completion of the linear span of all tensors $\xi \otimes \gamma$ (with $\xi \in E$ and $\gamma \in F$) subject to the identification

$$(\xi \cdot b) \otimes \gamma = \xi \otimes (b \cdot \gamma) \quad (\xi \in E, \gamma \in F, b \in \mathcal{B}), \tag{4.1}$$

with left $\mathcal{A}$-action, right $\mathcal{C}$-action and the $\mathcal{C}$-valued inner-product defined by setting for each $\xi, \xi' \in E$, $\gamma, \gamma' \in F$, $a \in \mathcal{A}$ and $c \in \mathcal{C}$:

$$a \cdot (\xi \otimes \gamma) = (a \cdot \xi) \otimes \gamma, \qquad (\xi \otimes \gamma) \cdot c = \xi \otimes (\gamma \cdot c),$$
$$\langle \xi \otimes \gamma, \xi' \otimes \gamma' \rangle_{E \otimes F} = \langle \langle \xi, \xi' \rangle_E \cdot \gamma, \gamma' \rangle_F.$$

In case the left action on F is given by the $*$-homomorphism φ we occasionally emphasize this by writing $E \otimes_\varphi F$ for $E \otimes F$.

It is more complicated to characterize the bounded linear operators between tensor-product correspondences. One way to construct such operators is as follows. Let E and E' be $(\mathcal{A}, \mathcal{B})$-correspondences and F and F' $(\mathcal{B}, \mathcal{C})$-correspondences, for C^*-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Furthermore, let $X \in \mathcal{L}(E, E')$ be a right module map and $Y \in \mathcal{L}(F, F')$ a left module map. Then we write $X \otimes Y$ for the operator in $\mathcal{L}(E \otimes F, E' \otimes F')$ which is determined by

$$X \otimes Y(\xi \otimes \gamma) = (X\xi) \otimes (Y\gamma) \quad (\xi \otimes \gamma \in E \otimes F). \tag{4.2}$$

The module map properties are needed to guarantee that the balancing in the tensor-product (see (4.1)) is respected by the operator $X \otimes Y$.

If, in addition, X is also a left module map, then $X \otimes Y$ is a left module map, while Y also being a right module map guarantees that $X \otimes Y$ is a right module map. Moreover, $X \otimes Y$ is adjointable in case X and Y are both adjointable operators, with $(X \otimes Y)^* = X^* \otimes Y^*$.

Notice that the left action on $E \otimes F$ can now be written as $a \mapsto \varphi(a) \otimes I_F \in \mathcal{L}^a(E \otimes F)$, where $I_F \in \mathcal{L}^a(F)$ is the identity operator on F.

4.2. The Fock space $\mathcal{F}^2(E)$ and the Toeplitz algebra $\mathcal{F}^\infty(E)$

In this section we shall consider the situation where $\mathcal{A} = \mathcal{B}$, i.e, E is an $\mathcal{A}$-correspondence. We also restrict our attention to the case where $\mathcal{A}$ is a von Neumann algebra and let E be an $\mathcal{A}$-W^*-correspondence. This means that E is an $\mathcal{A}$-correspondence which is also *self-dual* in the sense that any right $\mathcal{A}$-module map $\rho\colon E \to \mathcal{A}$ is given by taking the inner-product against some element e_ρ of E:

$$\rho(e) = \langle e, e_\rho \rangle_E \in \mathcal{A}. \tag{4.3}$$

It is easily seen that such maps are adjointable with adjoint $\rho^*\colon \mathcal{A} \to E$ given by

$$\rho^*(a) = e_\rho \cdot a \tag{4.4}$$

and hence also *any* right module map $\nu\colon \mathcal{A} \to E$ has the form $\nu = \rho^*$ as in (4.4). Moreover, the space $\mathcal{L}^a(E)$ of adjointable operators on the W^*-correspondence E is in fact a W^*-algebra, i.e., it is the abstract version of a von Neumann algebra with an ultra-weak topology (see [69]).

Since E is an $\mathcal{A}$-correspondence, we may define the self-tensor-product $E^{\otimes 2} = E \otimes E$ to get another $\mathcal{A}$-correspondence, and, inductively, an $\mathcal{A}$-correspondence $E^{\otimes n} = E \otimes (E^{\otimes (n-1)})$ for each $n = 1, 2, \ldots$. If we use $a \mapsto \varphi(a)$ to denote the left $\mathcal{A}$-action $\varphi(a)e = a \cdot e$ on E, we denote the left $\mathcal{A}$-action on $E^{\otimes n}$ by $\varphi^{(n)}$:

$$\varphi^{(n)}(a)\colon \xi_n \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1 \mapsto (\varphi(a)\xi_n) \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1.$$

Note that, using the notation in (4.2), we may write $\varphi^{(n)}(a) = \varphi(a) \otimes I_{E^{\otimes n-1}}$. We formally set $E^{\otimes 0} = \mathcal{A}$. Then the *Fock space* $\mathcal{F}^2(E)$ is defined to be

$$\mathcal{F}^2(E) = \oplus_{n=0}^{\infty} E^{\otimes n} \tag{4.5}$$

and is also an $\mathcal{A}$-W^*-correspondence. The left $\mathcal{A}$-action on $\mathcal{F}^2(E)$ is given by φ_∞:

$$\varphi_\infty(a)\colon\ \oplus_{n=0}^{\infty} \xi^{(n)} \mapsto \oplus_{n=0}^{\infty}(\varphi^{(n)}(a)\xi^{(n)}) \text{ for } \ \oplus_{n=0}^{\infty} \xi^{(n)} \in \oplus_{n=0}^{\infty} E^{\otimes n}, \tag{4.6}$$

or, more succinctly,

$$\varphi_\infty(a) = \operatorname{diag}(a, \varphi^{(1)}(a), \varphi^{(2)}(a), \ldots).$$

In addition to the operators $\varphi_\infty(a) \in \mathcal{L}^a(\mathcal{F}^2(E))$, we introduce the so-called *creation operators* on $\mathcal{F}^2(E)$ given, for each $\xi \in E$, by the subdiagonal (or shift) block matrix

$$T_\xi = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ T_\xi^{(0)} & 0 & 0 & \cdots \\ 0 & T_\xi^{(1)} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

where the block entry $T_\xi^{(n)}\colon E^{\otimes n} \to E^{\otimes n+1}$ is given by

$$T_\xi^{(n)}\colon \xi_n \otimes \cdots \otimes \xi_1 \mapsto \xi \otimes \xi_n \otimes \cdots \otimes \xi_1.$$

The operator T_ξ is also in $\mathcal{L}^a(\mathcal{F}^2(E))$. In summary, both T_ξ and $\varphi_\infty(a)$ are right $\mathcal{A}$-module maps with respect to the right $\mathcal{A}$-action on $\mathcal{F}^2(E)$ for each $\xi \in E$ and $a \in \mathcal{A}$. Moreover, one easily checks that

$$\varphi_\infty(a)T_\xi = T_{a\xi} = T_{\varphi(a)\xi} \quad \text{and} \quad T_\xi\varphi_\infty(a) = T_{\xi a} \quad \text{for each } a \in \mathcal{A} \text{ and } \xi \in E.$$

We let $\mathcal{F}^\infty(E)$ denote the *Toeplitz algebra* equal to the weak-$*$ closed algebra generated by the linear span of the collection of operators

$$\{\varphi_\infty(a),\, T_\xi\colon a \in \mathcal{A} \text{ and } \xi \in E\}$$

in the W^*-algebra $\mathcal{L}^a(\mathcal{F}(E))$. The justification for the term "Toeplitz algebra" comes from the following proposition, which is a variation on Proposition 4.2 in [16].

Proposition 4.1. *If $R \in \mathcal{L}^a(\mathcal{F}^2(E))$ is in the Toeplitz algebra $\mathcal{F}^\infty(E)$ with matrix representation*

$$R = [R_{i,j}]_{i,j=0,1,2,\dots} \tag{4.7}$$

where $R_{i,j} \in \mathcal{L}^a(E^{\otimes j}, E^{\otimes i})$, then there exists a sequence $\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \dots$ with $\xi^{(n)} \in E^{\otimes n}$ such that

$$R_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ T^{(0)}_{\xi^{(i-j)}} \otimes I_{E^{\otimes j}} & \text{if } i \geq j, \end{cases} \tag{4.8}$$

where for $n = 0, 1, 2, \dots$ the operator $T^{(0)}_{\xi^{(n)}} \in \mathcal{L}^a(\mathcal{A}, E^{\otimes n})$ is given by

$$T^{(0)}_{\xi^{(n)}} a = \xi^{(n)} a \quad (a \in \mathcal{A}). \tag{4.9}$$

In particular, R is completely determined by the entries of its first column. Conversely, if $\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \dots$ is a sequence with $\xi^{(n)} \in E^{\otimes n}$ such that the infinite operator matrix R given by (4.7), (4.8) *and* (4.9) *induces a bounded operator on $\mathcal{F}^2(E)$, then R is in $\mathcal{F}^\infty(E)$.*

Proof. Let $\mathcal{F}^{\infty\prime}(E)$ be the set of all adjointable operators on $\mathcal{F}^2(E)$ with matrix representation $R = [R_{i,j}]_{i,j=0,1,\dots}$ given by (4.8) and (4.9). Note first that $\mathcal{F}^{\infty\prime}(E)$ is an algebra; indeed if S and R in $\mathcal{F}^{\infty\prime}(E)$ are given by the sequences $\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \dots$ and $\zeta^{(0)}, \zeta^{(1)}, \zeta^{(2)}, \dots$ with $\xi^{(n)}, \zeta^{(n)} \in E^{\otimes n}$, respectively, then it is straightforward to check that SR is the element of $\mathcal{F}^{\infty\prime}(E)$ given by the sequence $\rho^{(0)}, \rho^{(1)}, \rho^{(2)}, \dots$ with $\rho^{(n)} \in E^{\otimes n}$ equal to

$$\rho^{(n)} = \sum_{k=0}^{n} \xi^{(k)} \otimes \zeta^{(n-k)}.$$

To show that $\mathcal{F}^\infty(E) \subset \mathcal{F}^{\infty\prime}(E)$, it therefore suffices to check that (1) each of the generators $\varphi_\infty(a)$ and T_ξ (for $a \in \mathcal{A}$ and $\xi \in E$) is in $\mathcal{F}^{\infty\prime}(E)$, (2) that $\mathcal{F}^{\infty\prime}(E)$ is closed under addition, and that (3) $\mathcal{F}^{\infty\prime}(E)$ is weak-$*$ closed. These verifications are straightforward and are left to the reader.

Conversely, to show that $\mathcal{F}^{\infty\prime}(E) \subset \mathcal{F}^\infty(E)$, since $\mathcal{F}^\infty(E)$ is weak-$*$ closed by definition, it suffices to consider that case of an operator $R \in \mathcal{F}^{\infty\prime}(E)$ with support only on a subdiagonal: $R_{i,j} = 0$ unless $i - j = k$ for some $k \geq 0$. Moreover, we can restrict to the case that $R_{k,0} \in \mathcal{L}^a(\mathcal{A}, E^{\otimes k})$ is defined by some pure tensor $\xi_k \otimes \cdots \otimes \xi_1$ in $E^{\otimes k}$ via $R_{k,0}a = \xi_k \otimes \cdots \otimes \xi_2 \otimes (\xi_1 a)$ for $a \in \mathcal{A}$. To see that this is the case, first note that, since $E^{\otimes k}$ is an $\mathcal{A}$-W^*-correspondence, $R_{k,0}$ is of the form $R_{k,0}a = \xi^{(k)} a$ for some $\xi^{(k)} \in E^{\otimes k}$. The claim then follows since any element $\xi^{(k)} \in E^{\otimes k}$ can be approximated by linear combinations of pure tensors in $E^{\otimes k}$, and because $\mathcal{F}^\infty(E)$ is weak-$*$ closed. Finally, assume that R is only supported on the k^{th} diagonal and that $R_{k,0}$ is defined by the pure tensor $\xi_k \otimes \cdots \otimes \xi_1 \in E^{\otimes k}$ as described above. It is then straightforward to check that $R = T_{\xi_k} \cdots T_{\xi_1}$. Thus R is in $\mathcal{F}^\infty(E)$. ∎

4.3. Correspondence-representation pairs and their dual

In addition to the von Neumann algebra $\mathcal{A}$ and the $\mathcal{A}$-correspondence E, suppose that we are also given an auxiliary Hilbert space $\mathcal{E}$ and a representation (meaning a nondegenerate $*$-homomorphism) $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$; as this will be the setting for much of the analysis to follow, we refer to such a pair (E, σ) as a *correspondence-representation pair*. We further assume that σ is faithful (injective) and normal (σ-weakly continuous). Then the Hilbert space $\mathcal{E}$ equipped with σ becomes an $(\mathcal{A}, \mathbb{C})$-correspondence with left $\mathcal{A}$-action given by σ:

$$a \cdot y = \sigma(a)y \text{ for all } a \in \mathcal{A} \text{ and } y \in \mathcal{E}.$$

Thus we can form the tensor-product $(\mathcal{A}, \mathbb{C})$-correspondence $E \otimes_\sigma \mathcal{E}$. With E^σ we denote the set of all bounded linear operators $\mu\colon \mathcal{E} \to E \otimes_\sigma \mathcal{E}$ which are also left $\mathcal{A}$-module maps:

$$E^\sigma = \{\mu\colon \mathcal{E} \to E \otimes_\sigma \mathcal{E}\colon \mu\sigma(a) = (\varphi(a) \otimes I_\mathcal{E})\mu\}. \tag{4.10}$$

It turns out that E^σ is itself a W^*-correspondence (the correspondence *dual* to E^σ; see [69, Section 3]), not over $\mathcal{A}$ but over the W^*-algebra

$$\sigma(\mathcal{A})' = \{b \in \mathcal{L}(\mathcal{E})\colon b\sigma(a) = \sigma(a)b \text{ for all } a \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{E})$$

(the *commutant* of the image $\sigma(\mathcal{A})$ of the representation σ in $\mathcal{L}(\mathcal{E})$) with left and right $\sigma(\mathcal{A})'$-action and $\sigma(\mathcal{A})'$-valued inner-product $\langle\cdot,\cdot\rangle_{E^\sigma}$ given by

$$\mu \cdot b = \mu b, \quad b \cdot \mu = (I_\mathcal{E} \otimes b)\mu,$$
$$\langle \mu, \nu\rangle_{E^\sigma} = \nu^*\mu.$$

The intertwining relations of elements μ and ν in E^σ with $\sigma(a)$ and $a \otimes I_\mathcal{E}$ for $a \in \mathcal{A}$ imply that $\nu^*\mu$ is indeed in $\sigma(\mathcal{A})'$.

Notice that elements $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$ define operators on the $(\mathcal{A}, \mathbb{C})$-correspondence

$$\mathcal{F}^2(E, \sigma) := \mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = \oplus_{n=0}^\infty E^{\otimes n} \otimes_\sigma \mathcal{E}$$

via the operator matrices

$$I_{\mathcal{F}^2(E)} \otimes b = \operatorname{diag}(b, I_E \otimes b, I_{E^{\otimes 2}} \otimes b, \dots)$$

and

$$I_{\mathcal{F}^2(E)} \otimes \mu = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \mu & 0 & 0 & 0 & \cdots \\ 0 & I_E \otimes \mu & 0 & 0 & \cdots \\ 0 & 0 & I_{E^{\otimes 2}} \otimes \mu & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here, in interpreting $I_{E^{\otimes n}} \otimes \mu$, we use the identification

$$E^{\otimes n} \otimes_\sigma \mathcal{E} = E^{\otimes n-1} \otimes (E \otimes_\sigma \mathcal{E}). \tag{4.11}$$

The operators $I_{E^{\otimes n}} \otimes b$ and $I_{E^{\otimes n}} \otimes \mu$ are well defined because b and μ are left $\mathcal{A}$-module maps.

The following theorem provides another way of characterizing the elements of $\mathcal{F}^\infty(E)$ using the operators $I_{\mathcal{F}^2(E)} \otimes b$ and $I_{\mathcal{F}^2(E)} \otimes \mu$. The result follows directly from the combination of Theorems 3.9 and 3.10 in [69].

Proposition 4.2. *Given an operator $X \in \mathcal{L}(\mathcal{F}^2(E,\sigma))$, there exists an $R \in \mathcal{F}^\infty(E)$ so that $X = R \otimes I_\mathcal{E}$ if and only if X commutes with $I_{\mathcal{F}^2(E)} \otimes b$ and $I_{\mathcal{F}^2(E)} \otimes \mu$ for each $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$. Moreover, if $X = R \otimes I_\mathcal{E}$ for some $R \in \mathcal{F}^\infty(E)$, then $\|X\| = \|R\|$.*

We can then prove the following concrete version of the C^*-correspondence commutant lifting theorem [67, Theorem 4.4].

Theorem 4.3. *Given subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{F}^2(E,\sigma)$ that are both invariant under $I_{\mathcal{F}^2(E)} \otimes b$ and $I_{\mathcal{F}^2(E)} \otimes \mu^*$ for all $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$, and a contractive operator X from $\mathcal{M}$ into $\mathcal{N}$ such that*

$$X^*(I_{\mathcal{F}^2(E)} \otimes b)|_\mathcal{N} = (I_{\mathcal{F}^2(E)} \otimes b)X^* \quad \textit{and} \quad X^*(I_{\mathcal{F}^2(E)} \otimes \mu^*)|_\mathcal{N} = (I_{\mathcal{F}^2(E)} \otimes \mu^*)X^*,$$

for all $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$, there exists an $S \in \mathcal{F}^\infty(E)$ with $\|S\| \leq 1$ so that $(S^* \otimes I_\mathcal{E})\mathcal{N} \subset \mathcal{M}$ and $X = P_\mathcal{N}(S \otimes I_\mathcal{E})|_\mathcal{M}$.

Proof. Using the C^*-correspondence commutant lifting theorem [67, Theorem 4.4] and the intertwining relations of X it follows that X can be lifted to a contractive operator Y on $\mathcal{F}^2(E,\sigma)$ that commutes with $I_{\mathcal{F}^2(E)} \otimes b$ and $I_{\mathcal{F}^2(E)} \otimes \mu$ for each $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$, with the property that $Y^*\mathcal{N} \subset \mathcal{M}$ and $X = P_\mathcal{N} Y|_\mathcal{M}$. The claim then follows immediately with Proposition 4.2. □

4.4. The generalized disk $\mathbb{D}((E^\sigma)^*)$ and the first Muhly-Solel point-evaluation

For the definition of point-evaluations to follow, however, the important object is $(E^\sigma)^*$, the set of adjoints of elements of E^σ (which are also left $\mathcal{A}$-module maps):

$$(E^\sigma)^* = \{\eta : E \otimes_\sigma \mathcal{E} \to \mathcal{E} \colon \eta^* \in E^\sigma\}. \tag{4.12}$$

For a given $\eta \in (E^\sigma)^*$ and a positive integer n, we may define the generalized power $\eta^n \colon E^{\otimes n} \otimes_\sigma \mathcal{E} \to \mathcal{E}$ by

$$\eta^n = \eta(I_E \otimes \eta) \cdots (I_{E^{\otimes n-1}} \otimes \eta)$$

where we again use the identification (4.11) in this definition. We also set $\eta^0 = I_\mathcal{E} \in \mathcal{L}(\mathcal{E})$. Again the fact that η is a left $\mathcal{A}$-module map ensures that $I_{E^{\otimes k}} \otimes \eta$ is a well-defined operator in $\mathcal{L}(E^{\otimes k+1} \otimes_\sigma \mathcal{E}, E^{\otimes k} \otimes_\sigma \mathcal{E})$. The defining $\mathcal{A}$-module property of η in (4.12) then extends to the generalized powers η^n in the form

$$\eta^n(\varphi^{(n)}(a) \otimes I_\mathcal{E}) = \sigma(a)\eta^n, \tag{4.13}$$

i.e., η^n is also an $\mathcal{A}$-module map.

Denote by $\mathbb{D}((E^\sigma)^*)$ the set of strictly contractive elements of $(E^\sigma)^*$:

$$\mathbb{D}((E^\sigma)^*) = \{\eta \in (E^\sigma)^* \colon \|\eta\| < 1\}.$$

We then consider elements $R \in \mathcal{F}^\infty(E)$ as functions $\widehat{R}$ on $\mathbb{D}((E^\sigma)^*)$ with values in $\mathcal{L}(\mathcal{E})$ according to the formula for the *first Muhly-Solel point evaluation*

$$\widehat{R}(\eta) = \sum_{n=0}^{\infty} \eta^n (R_{n,0} \otimes I_{\mathcal{E}}) \text{ where } R = [R_{i,j}] \in \mathcal{F}^\infty(E). \tag{4.14}$$

From the facts that $\|\eta\| < 1$ and $\|R_{n,0}\| \leq M < \infty$ (since R is bounded on $\mathcal{F}^2(E)$), one can see that the series in (4.14) converges in operator norm.

Remark 4.4. The definition of the first Muhly-Solel point-evaluation in [69, 70, 71] is actually more elaborate. There it is observed that an element $\eta \in \overline{\mathbb{D}}((E^\sigma)^*)$ induces an ultraweakly continuous completely contractive bimodule map $\widehat{\eta}\colon E \to \mathcal{L}(\mathcal{E})$ via the formula

$$\widehat{\eta}(\xi)e = \eta(\xi \otimes e) \text{ for } \xi \in E,\ e \in \mathcal{E}. \tag{4.15}$$

The bimodule property means that

$$\widehat{\eta}(\varphi(a)\xi b) = \sigma(a)\widehat{\eta}(\xi)\sigma(b) \text{ for all } a, b \in \mathcal{A},\ \xi \in E.$$

Conversely, if $S\colon E \to \mathcal{L}(\mathcal{E})$ is an ultraweakly continuous completely contractive bimodule map, then the same formula (4.15) turned around

$$\widetilde{S}(\xi \otimes e) = S(\xi)e \tag{4.16}$$

can be used to define an element $\widetilde{S}$ of the closed generalized disk $\overline{\mathbb{D}}((E^\sigma)^*)$. The pair $(\widehat{\eta}, \sigma)$ (or (S, σ)) is said to be a *covariant representation* of the correspondence E. Given a covariant representation (S, σ), in case $\|S\| < 1$ (or, equivalently, $\|\widetilde{S}\| < 1$), there is an associated completely contractive representation $S \times \sigma$ of the Toeplitz algebra $\mathcal{F}^\infty(E)$ defined on generators via

$$\begin{aligned} (S \times \sigma)(\varphi_\infty(a)) &= \sigma(a), \\ (S \times \sigma)(T_\xi) &= S(\xi). \end{aligned} \tag{4.17}$$

Conversely, if ρ is any ultraweakly continuous completely contractive $\mathcal{L}(\mathcal{E})$-valued representation of $\mathcal{F}^\infty(E)$, the same formulas (4.17) can be turned around

$$\begin{aligned} \sigma(a) &= \rho(\varphi_\infty(a)), \\ S(\xi) &= \rho(T_\xi) \end{aligned} \tag{4.18}$$

to define a covariant representation (S, σ) of E. Given any covariant representation (S, σ) of E, the formulas (4.17) can always be extended to define a representation of the *uniform closure* $\mathcal{T}^\infty(E)$ of the algebra generated by $\varphi_\infty(a)$ and T_ξ ($a \in \mathcal{A}$ and $\xi \in E$). It is an unsolved problem in the theory to identify which covariant representations (S, σ) have the property that (4.17) can be extended to define an ultraweakly continuous completely contractive representation of the weak-$*$ closed Toeplitz algebra $\mathcal{F}^\infty(E)$.

If we fix a representation $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$, and let $\eta \in \mathbb{D}((E^\sigma)^*)$ and $R \in \mathcal{F}^\infty(E)$, then the value $\widehat{R}(\eta)$ of R at η is defined to be the element of $\mathcal{L}(\mathcal{E})$ assigned to R by the representation $\widehat{\eta} \times \sigma$:

$$\widehat{R}(\eta) = (\widehat{\eta} \times \sigma)(R).$$

In the end this definition agrees with the definition (4.14). The formula (4.14) for the point-evaluation is called the *Cauchy transform* in [69].

Following [70] (see also [16]), given a correspondence-representation pair (E, σ), we define the *Schur class* $\mathcal{S}_{E,\sigma}$ to be the class of all functions $S\colon \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{E})$ which can be expressed in the form $S(\eta) = \widehat{R}(\eta)$ for some $R \in \mathcal{F}^\infty(E)$ with $\|R\| \le 1$. The associated left-tangential Nevanlinna-Pick interpolation problem is: *Given a subset* Ω *of* $\mathbb{D}((E^\sigma)^*)$ *and two functions* $F\colon \Omega \to \mathcal{L}(\mathcal{E})$ *and* $G\colon \Omega \to \mathcal{L}(\mathcal{E})$, *determine when there exists a function* S *in the Schur class* $\mathcal{S}_{E,\sigma}$ *so that* $GS|_\Omega = F$. We shall explain the solution to this problem due to Muhly-Solel [69] in Subsection 4.6. For the moment we note the following two extreme cases:

- **(I)** $\Omega = \mathbb{D}((E^\sigma)^*)$ **and** $G(\eta) = I_{\mathcal{E}}$. In this case the function F is defined on all of $\mathbb{D}((E^\sigma)^*)$ and we seek a test to decide if $F \in \mathcal{S}_{E,\sigma}$.
- **(II)** Ω **finite.** In this case Ω consists of finitely many points, say $\eta_1, \dots, \eta_N \in \mathbb{D}((E^\sigma)^*)$, and we are given operators $Y_1 = F(\eta_1), \dots, Y_N = F(\eta_N)$ and $Y_1 = F(\eta_1), \dots, Y_N = F(\eta_N)$ in $\mathcal{L}(\mathcal{E})$. We seek a test to decide if there is a function S (or, more ambitiously, a description of all such functions S) in the Schur class $\mathcal{S}_{E,\sigma}$ satisfying the interpolation conditions

$$X_j S(\eta_j) = Y_j \text{ for } j = 1, \dots, N.$$

4.5. Positive and completely positive kernels/maps

The solution of the Nevanlinna-Pick interpolation problem in [69] involves the notion of a completely positive map while the characterization of the Schur class $\mathcal{S}_{E,\sigma}$ in [70] involves the notion of *completely positive kernel* introduced in [31].

In the following discussion $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are C^*-algebras (in particular, possibly W^*-algebras) and Ω is a set. We begin with the notion of *positive kernel* which goes back at least to Aronszajn [12]; a function $K\colon \Omega \times \Omega \to \mathcal{C}$ is said to be a *positive kernel* if

$$\sum_{i,j=1}^{n} c_i^* K(\omega_i, \omega_j) c_j \ge 0$$

for all choices of $\omega_1, \dots, \omega_n \in \Omega$ and $c_1, \dots, c_n \in \mathcal{C}$ for $n = 1, 2, \dots$. Equivalently, for every choice of n points $\omega_1, \dots, \omega_n \in \Omega$, the matrix

$$[K(\omega_i, \omega_j)]_{i,j=1,\dots,n}$$

is a positive element of $\mathcal{C}^{n\times n}$ where $n = 1, 2, \dots$. Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, a map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is said to be a *positive map* if

$$a \ge 0 \text{ in } \mathcal{A} \Longrightarrow \varphi(a) \ge 0 \text{ in } \mathcal{B}.$$

Such a map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is said to be *completely positive* if

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \succeq 0 \text{ in } \mathcal{A}^{n\times n} \Longrightarrow \begin{bmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1n}) \\ \vdots & & \vdots \\ \varphi(a_{n1}) & \dots & \varphi(a_{nn}) \end{bmatrix} \succeq 0 \text{ in } \mathcal{B}^{n\times n} \quad (4.19)$$

for every $n = 1, 2, \dots$. If the implication (4.19) holds for a fixed n in $\mathbb{Z}_+$, then we say that φ is *n-positive*. In case $\mathcal{A} = \mathbb{C}^{N\times N}$, a result of Choi [38] (see also [77, Theorem 3.14]) says that $\varphi\colon \mathcal{A} \to \mathcal{B}$ *is completely positive if and only if the single block matrix*

$$\begin{bmatrix} \varphi(e_{11}) & \dots & \varphi(e_{1N}) \\ \vdots & & \vdots \\ \varphi(e_{N1}) & \dots & \varphi(e_{NN}) \end{bmatrix}$$

is positive in $\mathcal{B}^{N\times N}$, where e_{ij} are the standard matrix units

$$[e_{ij}]_{\alpha,\beta} = \delta_{i,\alpha}\delta_{j,\beta}$$

in $\mathbb{C}^{N\times N}$ (here we make use of the Kronecker delta – $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$). There is an alternative characterization in terms of positive kernels (see [103]): *A map* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *is completely positive if and only if the kernel* $k_\varphi\colon \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ *defined by*

$$k_\varphi(a, a') = \varphi(a^* a')$$

is a positive kernel. Barreto-Bhat-Liebscher-Skeide in [31] (with motivation from quantum physics which need not concern us here) combined these notions as follows: we say that a kernel $\mathbb{K}\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ is *completely positive* if the associated kernel $k\colon (\Omega \times \mathcal{A}) \times (\Omega \times \mathcal{A}) \to \mathcal{B}$ given by

$$k((\omega, a), (\omega', a')) = \mathbb{K}(\omega, \omega')[a^* a']$$

is a positive kernel, i.e., if, for all choices (ω_1, a_1), $\dots$, (ω_n, a_n) in $\Omega \times \mathcal{A}$ and for all choices b_1, $\dots$, b_n in $\mathcal{B}$ it is the case that

$$\sum_{i,j=1}^{n} b_i^* \mathbb{K}(\omega_i, \omega_j)[a_i^* a_j] b_j \geq 0 \text{ in } \mathcal{B} \quad (4.20)$$

for $n = 1, 2, \dots$. Note that the notion of completely positive kernel contains the notions of positive kernel and of completely positive map as special cases: in case the set Ω consists of a single point, then $\mathbb{K}$ can be considered simply as a linear map from $\mathcal{A}$ to $\mathcal{B}$, and the condition that $\mathbb{K}$ be a completely positive kernel collapses to the condition that $\mathbb{K}$ be a completely positive map from $\mathcal{A}$ to $\mathcal{B}$ by the positive-kernel formulation of completely positive map mentioned above. On the other extreme, if $\mathcal{A}$ is just the complex numbers $\mathbb{C}$, then the condition that $\mathbb{K}$ be completely positive collapses to the condition that the kernel $k(\cdot, \cdot) := \mathbb{K}(\cdot, \cdot)[1]$ be a positive kernel.

A number of equivalent characterizations of complete positivity for a kernel $\mathbb{K}\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ is given in [31]. Let us mention some of these which will be convenient for our analysis of various generalized Nevanlinna-Pick interpolation problems.

Proposition 4.5. *Suppose that* $\mathbb{K}$ *is a function from* $\Omega \times \Omega$ *into* $\mathcal{L}(\mathcal{A}, \mathcal{B})$*. Then the following are equivalent:*

1. $\mathbb{K}$ *is a completely positive kernel, i.e., the kernel* $k\colon (\Omega \times \mathcal{A}) \times (\Omega \times \mathcal{A}) \to \mathcal{B}$ *given by*
$$k((\omega, a), (\omega', a')) = \mathbb{K}(\omega, \omega')[a^* a']$$
is a positive kernel.
2. *For every choice of* n *points* $\omega_1, \dots, \omega_n$ *in* Ω*, the map* $\varphi_{\omega_1,\dots,\omega_n}\colon \mathcal{A}^{n\times n} \to \mathcal{B}^{n\times n}$ *given by*
$$\varphi_{\omega_1,\dots,\omega_n}([a_{ij}]_{i,j=1,\dots,n}) = [\mathbb{K}(\omega_i, \omega_j)[a_{ij}]]_{i,j=1,\dots,n} \tag{4.21}$$
is a completely positive map for any $n = 1, 2, \dots$*.*
3. *For every choice of* n *points* $\omega_1, \dots, \omega_n$ *in* Ω*, the map* $\varphi_{\omega_1,\dots,\omega_n}\colon \mathcal{A}^{n\times n} \to \mathcal{B}^{n\times n}$ *given by* (4.21) *is a positive map for any* $n = 1, 2, \dots$*.*

Proof. The equivalence of (1), (2) and (3) correspond to the equivalence of (1), (4) and (5) in Lemma 3.2.1 from [31]. □

An important example of a completely positive kernel for our purposes here appearing implicitly in the work of Muhly-Solel [69, 70] and developed explicitly in [16] is the Szegő kernel associated with a correspondence-representation pair (E, σ) and defined as follows. We suppose that we are given an $\mathcal{A}$-W^*-correspondence E together with a representation $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$. We then define the associated *Szegő kernel*
$$\mathbb{K}_{E,\sigma}\colon \mathbb{D}((E^\sigma)^*) \times \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\sigma(\mathcal{A})', \sigma(\mathcal{A})')$$
by
$$\mathbb{K}_{E,\sigma}(\eta, \zeta)[b] = \sum_{n=0}^{\infty} \eta^n (I_{E^{\otimes n}} \otimes b) \zeta^{n*}. \tag{4.22}$$
Then it can be seen that $\mathbb{K}_{E,\sigma}$ is a completely positive kernel and in fact is the reproducing kernel for a reproducing kernel correspondence $H^2(E, \sigma)$ whose elements are $\mathcal{E}$-valued functions on the generalized disk $\mathbb{D}((E^\sigma)^*)$ which also carries a representation ι_∞ of $\sigma(\mathcal{A})'$. This construction is based on yet another and perhaps the most fundamental characterization of completely positive kernel which is not mentioned in Proposition 4.5, namely: $\mathbb{K}$ *is completely positive if and only if* $\mathbb{K}$ *has a* Kolmogorov decomposition *in the sense of* [31]; the reproducing-kernel correspondence associated with the completely positive kernel can then be taken to be the middle space in its Kolmogorov decomposition and the Schur class $\mathcal{S}_{E,\sigma}$ can alternatively be characterized as the space of contractive-multiplier $\sigma(\mathcal{A})'$-module maps acting on $H^2(E, \sigma)$. For further details on these ideas, we refer to [31].

In short there are at least two points of view to the Schur class $\mathcal{S}_{E,\sigma}$. The first is the connection with the representation theory for $\mathcal{F}^\infty(E)$ as developed in the work of Muhly-Solel [69, 70, 71], while the second is the view of the Schur class as contractive-multiplier module maps from [16]. Our purpose here is to ignore all this finer structure and use the most direct definition (4.14) of the point-evaluation to make explicit how the Muhly-Solel Nevanlinna-Pick interpolation theorem hooks up with the existing literature on assorted generalizations (in particular, to multivariable contexts) of Nevanlinna-Pick interpolation.

4.6. A W^*-correspondence Nevanlinna-Pick interpolation theorem

The solution to the W^*-correspondence version of the Nevanlinna-Pick interpolation problem is as follows. In the statement of the result we make use of the Szegő kernel $\mathbb{K}_{E,\sigma}$ defined as in (4.22).

Theorem 4.6. *Suppose that we are given a correspondence-representation pair (E,σ) together with a subset Ω of $\mathbb{D}((E^\sigma)^*)$ and two functions $F\colon \Omega \to \mathcal{L}(\mathcal{E})$ and $G\colon \Omega \to \mathcal{L}(\mathcal{E})$. Then the following are equivalent.*

(1) *There exists a function $S\colon \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{E})$ in the Schur class $\mathcal{S}_{E,\sigma}$ such that*
$$GS|_\Omega = F.$$
(2) *The kernel $\mathbb{K}_F\colon \Omega\times\Omega \to \mathcal{L}(\sigma(\mathcal{A})', \mathcal{L}(\mathcal{E}))$ given by*
$$\mathbb{K}_F(\eta,\eta')[b] = G(\eta)\mathbb{K}_{E,\sigma}(\eta,\eta')[b]G(\zeta)^* - F(\eta)\mathbb{K}_{E,\sigma}(\eta,\zeta)[b]F(\zeta)^*$$
is completely positive.

(3) *For each choice of $\eta_1,\dots,\eta_n \in \omega$, the map $\varphi_{\eta_1,\dots,\eta_n}$ from $(\sigma(\mathcal{A})')^{n\times n}$ to $\mathcal{L}(\mathcal{E})^{n\times n}$ given by*
$$\begin{aligned}&\varphi_{\eta_1,\dots,\eta_n}\left([b_{ij}]_{i,j=1}^n\right)\\ &\quad = [G(\eta_i)\mathbb{K}_{E,\sigma}(\eta_i,\eta_j)[b_{ij}]G(\eta_j)^* - F(\eta_i)\mathbb{K}_{E,\sigma}(\eta_i,\eta_j)[b_{ij}]F(\eta_j)^*]_{i,j=1}^n\end{aligned}$$
is a completely positive (or even just a positive) map for any $n = 1,2,3,\dots$.

Remark 4.7. In case Ω is finite, say $\Omega = \{\eta_1,\dots,\eta_N\}$, and $X_1 = G(\eta_1),\dots,X_N = G(\eta_N)$ and $Y_1 = F(\eta_1),\dots,Y_N = F(\eta_N)$, we can always assume n in condition (3) to be of the form $n = kN$ for some positive integer k and the sequence of points from X to be the sequence $\eta_1,\dots,\eta_N$ repeated k times. It then follows that condition (3) can be written as: *The map $\varphi : \sigma(\mathcal{A})'^{N\times N} \to \mathcal{L}(\mathcal{E})^{N\times N}$ given by*
$$\varphi\left(\left[\ b_{i,j}\ \right]_{i,j=1}^N\right) = \left[X_i\mathbb{K}_{E,\sigma}(\eta_i,\eta_j)[b_{ij}]X_j^* - Y_i\mathbb{K}_{E,\sigma}(\eta_i,\eta_j)[b_{ij}]Y_j^*\right]_{i,j=1}^N \tag{4.23}$$
is a completely positive map.

Proof of Theorem 4.6. Note that the equivalence of (2) and (3) is just a particular case of Proposition 4.5. The equivalence of (1) and (3) for the case that Ω is finite is given in [69] (see also Remark 4.7). For the case where $\Omega = \mathbb{D}((E^\sigma)^*)$ and $G(\eta) = I_\mathcal{E}$ for each $\eta\in\Omega$ the equivalence of (1) and (2) is given in [70] (see also [16] for the reproducing-kernel point of view). The case of a general Ω can be seen to follow from the case of a finite Ω via a standard weak-$*$ compactness argument; one way to organize this argument is as an application of Kurosh's Theorem (see [6, page 30]). ☐

4.7. The second Muhly-Solel point-evaluation and associated Nevanlinna-Pick theorem

In the recent paper [71] Muhly and Solel introduced a Poisson kernel, and applied this object to define a second point-evaluation for elements of the Toeplitz algebra $H^\infty(E)$. As pointed out in [71], this point-evaluation has certain characteristics that resemble those of the point-evaluation used in discrete-time time-varying interpolation and system theory, as developed in the 1990s; cf. [23, 8, 53]. In [52] it was observed that many time-varying interpolation problems can be recast as classical interpolation problems with an operator argument. In this section we prove a Nevanlinna-Pick interpolation theorem for the alternative point-evaluation of [71]; in the examples to follow (see Subsections 4.8 and 4.9 below) we show that this Nevanlinna-Pick theorem indeed corresponds to the operator-argument versions in the settings considered there.

For the second point-evaluation, the points are formed by pairs (ζ, a), where a is from the W^*-algebra $\mathcal{A}$ and ζ is from the set

$$\mathbb{D}(E^*) := \{\zeta \colon \zeta^* \in E,\ \|\zeta^*\|_E < 1\}.$$

Given such a pair (ζ, a) we set

$$\zeta^*_{(n)} := \zeta^* \otimes \cdots \otimes \zeta^* \in E^{\otimes n} \text{ for } n = 1, 2, \ldots,$$

and define

$$W_{\zeta^*,a^*} = \varphi_\infty(a^*) \begin{bmatrix} 1_{\mathcal{A}} \\ T^{(0)}_{\zeta^*_{(1)}} \\ T^{(0)}_{\zeta^*_{(2)}} \\ \vdots \end{bmatrix} \in \mathcal{L}^a(\mathcal{A}, \mathcal{F}^2(E)),$$

where $\varphi_\infty(a^*)$ and $T^{(0)}_{\zeta^*_{(n)}}$ are given by (4.6) and (4.9), respectively. The Poisson kernel defined in [71] is the special case of W_{ζ^*,a^*} with $a^* = T^{(0)*}_{\zeta^*} T^{(0)}_{\zeta^*}$. In that case W_{ζ^*,a^*} is well defined even if $\|\zeta^*\| = 1$, and W_{ζ^*,a^*} is a contractive operator. It is easy to see that for each $\zeta \in \mathbb{D}(E^*)$ and $a \in \mathcal{A}$ the operator W_{ζ^*,a^*} still defines a bounded operator, but not necessarily contractive.

For an $R \in \mathcal{F}^\infty(E)$ we define the evaluation of R in a point (ζ, a) from $\mathbb{D}(E^*) \times \mathcal{A}$ by

$$\widehat{R}(\zeta, a) = W^*_{\zeta^*,a^*} R|_{\mathcal{A}} \in \mathcal{L}^a(\mathcal{A}) = \mathcal{A}.$$

Here we used the fact that for any W^*-algebra $\mathcal{A}$, considered in the standard way as a W^*-correspondence over itself, we can identify $\mathcal{L}^a(\mathcal{A})$ with the C^*-algebra $\mathcal{A}$ itself. If $R \in \mathcal{F}^\infty(E)$ is given by a sequence of elements of $E^{\otimes n}$, $n = 0, 1, \ldots$, as in Proposition 4.1, then $\widehat{R}(\zeta, a)$ can be written more concretely.

Proposition 4.8. *Let $R \in \mathcal{F}^\infty(E)$ be given by (4.7)–(4.9) with $\xi^{(n)} \in E^{\otimes n}$ for $n = 0, 1, 2, \ldots$. Then for any $(\zeta, a) \in \mathbb{D}(E^*) \times \mathcal{A}$ we have*

$$\widehat{R}(\zeta, a) = \sum_{n=0}^{\infty} \langle a\xi^{(n)}, \zeta^*_{(n)} \rangle.$$

Proof. The statement follows directly from the fact that for any $a' \in \mathcal{A}$ we have

$$\begin{aligned}\widehat{R}(\zeta, a)a' &= \sum_{n=0}^{\infty} T^{(0)*}_{\zeta^*_{(n)}} \varphi_n(a) T^{(0)}_{\xi^{(n)}} a' = \sum_{n=0}^{\infty} T^{(0)*}_{\zeta^*_{(n)}} a \cdot \xi^{(n)} \cdot a' \\ &= \sum_{n=0}^{\infty} \langle a \cdot \xi^{(n)} \cdot a', \zeta^*_{(n)} \rangle = \sum_{n=0}^{\infty} \langle a \cdot \xi^{(n)}, \zeta^*_{(n)} \rangle a'. \end{aligned}$$

□

The following lemma will be useful in the sequel.

Lemma 4.9. *For any $(\zeta, a) \in \mathbb{D}(E^*) \times \mathcal{A}$ and $R \in \mathcal{F}^\infty(E)$ we have*

$$W^*_{\zeta^*, a^*} R = W^*_{\zeta^*, \widehat{R}(\zeta, a)^*}. \tag{4.24}$$

Proof. Let $(\zeta, a) \in \mathbb{D}(E^*) \times \mathcal{A}$ and $R \in \mathcal{F}^\infty(E)$. Let $\xi^{(n)} \in E^{\otimes n}$, for $n = 0, 1, 2, \ldots$, such that R is given by (4.7)–(4.9). First observe that for $n, k = 0, 1, 2, \ldots$ and for any $\rho \in E^{\otimes k}$ we have

$$\begin{aligned} T^{(0)*}_{\zeta^*_{(n+k)}} \varphi_{n+k}(a)(T^{(0)}_{\xi^{(n)}} \otimes I_{E^{\otimes k}})\rho &= T^{(0)*}_{\zeta^*_{(n+k)}} (a \cdot \xi^{(n)}) \otimes \rho \\ &= \langle (a \cdot \xi^{(n)}) \otimes \rho, \zeta^*_{(n+k)} \rangle \\ &= \langle (a \cdot \xi^{(n)}) \otimes \rho, \zeta^*_{(n)} \otimes \zeta^*_{(k)} \rangle \\ &= \langle \langle a \cdot \xi^{(n)}, \zeta^*_{(n)} \rangle \cdot \rho, \zeta^*_{(k)} \rangle \\ &= T^{(0)*}_{\zeta^*_{(k)}} \varphi_k(\langle a \cdot \xi^{(n)}, \zeta^*_{(n)} \rangle)\rho. \end{aligned}$$

It then follows that the k^{th} entry in the infinite bock row matrix $W^*_{\zeta^*, a^*} R$ is equal to

$$\begin{aligned} \sum_{n=0}^{\infty} T^{(0)*}_{\zeta^*_{(n+k)}} \varphi_{n+k}(a)(T^{(0)}_{\xi^{(n)}} \otimes I_{E^{\otimes k}}) &= \sum_{n=0}^{\infty} T^{(0)*}_{\zeta^*_{(k)}} \varphi_k(\langle a \cdot \xi^{(n)}, \zeta^*_{(n)} \rangle) \\ &= T^{(0)*}_{\zeta^*_{(k)}} \varphi_k \left(\sum_{n=0}^{\infty} \langle a \cdot \xi^{(n)}, \zeta^*_{(n)} \rangle \right) \\ &= T^{(0)*}_{\zeta^*_{(k)}} \varphi_k(\widehat{R}(\zeta, a)). \end{aligned}$$

This proves our claim. □

One application of Lemma 4.9 is the following multiplicative law for elements in the Toeplitz algebra $\mathcal{F}^\infty(E)$ with respect to the point-evaluation considered in the present subsection.

Proposition 4.10. *Let $R, S \in \mathcal{F}^\infty(E)$ and $(\zeta, a) \in \mathbb{D}(E^*) \times \mathcal{A}$. Then*

$$(\widehat{RS})(\zeta, a) = \widehat{S}(\zeta, \widehat{R}(\zeta, a)).$$

Proof. It follows from Lemma 4.9 that

$$(\widehat{RS})(\zeta, a) = W^*_{\zeta, a^*} RS|_{\mathcal{A}} = W^*_{\zeta, \widehat{R}(\zeta,a)^*} S|_{\mathcal{A}} = \widehat{S}(\zeta, \widehat{R}(\zeta, a)). \qquad \square$$

It follows, in particular, from this proposition that the alternative point-evaluation is not multiplicative; unlike the first Muhly-Solel point-evaluation discussed in Subsection 4.4; cf., [16, Proposition 4.4]. The multiplicative law obtained in this case shows more resemblance to that appearing in the operator-argument functional calculus; cf., formula I.2.7 in [53].

We now prove the following Nevanlinna-Pick interpolation theorem.

Theorem 4.11. *Suppose that we are given $(\zeta_1, a_1), \ldots, (\zeta_N, a_N) \in \mathbb{D}(E^*) \times \mathcal{A}$ and $w_1, \ldots, w_N \in \mathcal{A}$. Then there exists an $S \in \mathcal{F}^\infty(E)$ with $\|S\| \le 1$ and $\widehat{S}(\zeta_k, a_k) = w_k$ for $k = 1, \ldots, N$ if and only if the operator matrix*

$$\left[\sum_{n=0}^{\infty} \langle (a_i a_j^* - w_i w_j^*) \zeta^*_{j\,(n)}, \zeta^*_{i\,(n)} \rangle_{E^{\otimes n}} \right]_{i,j=0}^{N} \tag{4.25}$$

is a positive element of $\mathcal{A}^{N\times N}$.

Proof. Assume we have an $S \in \mathcal{F}^\infty(E)$ with $\|S\| \le 1$ and $\widehat{S}(\zeta_k, a_k) = w_k$ for $k = 1, \ldots, N$. Then, by Lemma 4.9, $W^*_{\zeta_k^*, a_k^*} S = W^*_{\zeta_k^*, \widehat{S}(\zeta_k, a_k)^*} = W^*_{\zeta_k^*, w_k^*}$ for $k = 1, \ldots, N$, and thus

$$S^* \begin{bmatrix} W_{\zeta_1^*, a_1^*} & \cdots & W_{\zeta_N^*, a_N^*} \end{bmatrix} = \begin{bmatrix} W_{\zeta_1^*, w_1^*} & \cdots & W_{\zeta_N^*, w_N^*} \end{bmatrix}.$$

Since $\|S\| \le 1$, this implies that

$$\begin{aligned} &\begin{bmatrix} W^*_{\zeta_1^*, a_1^*} \\ \vdots \\ W^*_{\zeta_N^*, a_N^*} \end{bmatrix} \begin{bmatrix} W_{\zeta_1^*, a_1^*} \cdots W_{\zeta_N^*, a_N^*} \end{bmatrix} - \begin{bmatrix} W^*_{\zeta_1^*, w_1^*} \\ \vdots \\ W^*_{\zeta_N^*, w_N^*} \end{bmatrix} \begin{bmatrix} W_{\zeta_1^*, w_1^*} \cdots W_{\zeta_N^*, w_N^*} \end{bmatrix} \\ &\quad = \begin{bmatrix} W^*_{\zeta_1^*, a_1^*} \\ \vdots \\ W^*_{\zeta_N^*, a_N^*} \end{bmatrix} (I - SS^*) \begin{bmatrix} W_{\zeta_1^*, a_1^*} \cdots W_{\zeta_N^*, a_N^*} \end{bmatrix} \succeq 0 \quad (\text{in } \mathcal{A}^{N\times N}). \end{aligned} \tag{4.26}$$

Next observe that for any $(\zeta, a), (\zeta', a') \in \mathbb{D}(E^*) \times \mathcal{A}$ we have

$$W^*_{\zeta^*, a^*} W_{\zeta'^*, a'^*} = \sum_{n=0}^{\infty} \langle a a'^* \zeta'^*_{(n)}, \zeta^*_{(n)} \rangle \in \mathcal{A} = \mathcal{L}^a(\mathcal{A}).$$

From this computation it follows that the left-hand side of the identity in (4.26) is equal to the Pick matrix (4.25). Thus (4.25) is a positive element of $\mathcal{A}^{N\times N}$.

Conversely, assume that (4.25) is positive in $\mathcal{A}^{N\times N}$. Let $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ be a faithful (i.e., injective) representation of $\mathcal{A}$ into $\mathcal{L}(\mathcal{E})$ for some Hilbert space $\mathcal{E}$. Fix $(\zeta, a) \in \mathbb{D}(E^*) \times \mathcal{A}$. For notational convenience we set $\widetilde{W}_{\zeta^*,a^*} = W_{\zeta^*,a^*} \otimes I_{\mathcal{E}} \in \mathcal{L}^a(\mathcal{E}, \mathcal{F}^2(E,\sigma))$, where $\mathcal{F}^2(E,\sigma) := \mathcal{F}^2(E) \otimes_\sigma \mathcal{E}$. We claim that for each $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$ we have

$$\begin{aligned} \widetilde{W}^*_{\zeta^*,a^*}(I_{\mathcal{F}^2(E)} \otimes b) &= b\widetilde{W}^*_{\zeta^*,a^*}, \\ \widetilde{W}^*_{\zeta^*,a^*}(I_{\mathcal{F}^2(E)} \otimes \mu) &= ((T^{(0)*}_{\zeta^*} \otimes I_{\mathcal{E}})\mu)\widetilde{W}^*_{\zeta^*,a^*}. \end{aligned} \tag{4.27}$$

The first identity follows directly from the computation

$$\widetilde{W}^*_{\zeta^*,a^*}(I_{\mathcal{F}^2(E)} \otimes b) = (W^*_{\zeta^*,a^*} \otimes I_{\mathcal{E}})(I_{\mathcal{F}^2(E)} \otimes b) = (I_{\mathcal{A}} \otimes b)(W^*_{\zeta^*,a^*} \otimes I_{\mathcal{E}}) = b\widetilde{W}^*_{\zeta^*,a^*}.$$

Here we used the fact that $\mathcal{A} \otimes_\sigma \mathcal{E}$ can be identified with $\mathcal{E}$. To see that the second identity in (4.27) holds we show that the n^{th} entry

$$(T^{(0)*}_{\zeta^*_{(n+1)}}\varphi_{n+1}(a) \otimes I_{\mathcal{E}})(I_{E^{\otimes n}} \otimes \mu) : E^{\otimes n} \otimes_\sigma \mathcal{E} \to \mathcal{A}$$

in the infinite block row matrix $\widetilde{W}^*_{\zeta^*,a^*}(I_{\mathcal{F}^2(E)} \otimes \mu)$ is equal to

$$((T^{(0)*}_{\zeta^*} \otimes I_{\mathcal{E}})\mu)(T^{(0)*}_{\zeta^*_{(n)}}\varphi_n(a) \otimes I_{\mathcal{E}}).$$

It suffices to check the equality for elements from $E^{\otimes n} \otimes_\sigma \mathcal{E}$ of the form $\xi^{(n)} \otimes e$ with $\xi^{(n)} \in E^{\otimes n}$ and $e \in \mathcal{E}$. For such elements $\xi^{(n)} \otimes e \in E^{\otimes n} \otimes_\sigma \mathcal{E}$ and for each $e' \in \mathcal{E}$ we obtain that the

$$\begin{aligned} &\langle (T^{(0)*}_{\zeta^*_{(n+1)}}\varphi_{n+1}(a) \otimes I_{\mathcal{E}})(I_{E^{\otimes n}} \otimes \mu)(\xi^{(n)} \otimes e), e'\rangle_{\mathcal{E}} \\ &\quad = \langle (a \cdot \xi^{(n)}) \otimes \mu e, \zeta^*_{(n+1)} \otimes e'\rangle = \langle \langle a \cdot \xi^{(n)}, \zeta^*_{(n)}\rangle \cdot \mu e, \zeta^* \otimes e'\rangle \\ &\quad = \langle \mu\sigma(\langle a \cdot \xi^{(n)}, \zeta^*_{(n)}\rangle)e, \zeta^* \otimes e'\rangle = \langle ((T^{(0)*}_{\zeta^*_{(1)}} \otimes I_{\mathcal{E}})\mu)\sigma(T^{(0)*}_{\zeta^*_{(n)}}\varphi_n(a)\xi^{(n)})e, e'\rangle \\ &\quad = \langle ((T^{(0)*}_{\zeta^*_{(1)}} \otimes I_{\mathcal{E}})\mu)(T^{(0)*}_{\zeta^*_{(n)}}\varphi_n(a) \otimes I_{\mathcal{E}})(\xi^{(n)} \otimes e), e'\rangle. \end{aligned}$$

Thus our claim follows. Now set

$$\mathcal{H}_{\zeta,a} := \overline{\operatorname{Im} \widetilde{W}_{\zeta,a}} \subset \mathcal{F}^2(E,\sigma) := \mathcal{F}^2(E) \otimes_\sigma \mathcal{E}.$$

Using the intertwining relations (4.27) we see that $\mathcal{H}_{\zeta,a}$ is an invariant subspace for the operators $I_{\mathcal{F}^2(E)} \otimes \mu^*$ and $I_{\mathcal{F}^2(E)} \otimes b$ for each $\mu \in E^\sigma$ and $b \in \sigma(\mathcal{A})'$.

Identifying $\mathcal{A}$ with its image in $\mathcal{L}(\mathcal{E})$ under the representation σ, we see that the Pick matrix (4.25) defines a positive semidefinite element in $\mathcal{L}(\mathcal{E})^{N\times N}$. Moreover, from the first part of the proof we see that, equivalently, the left-hand side of the identity in (4.26), with $W^*_{\zeta^*_k,a^*_k}$ replaced by $\widetilde{W}^*_{\zeta^*_k,a^*_k}$, defines positive semidefinite element in $\mathcal{L}(\mathcal{E})^{N\times N}$. Thus, by the Douglas factorization lemma [43], we can define a contraction

$$V : \mathcal{M} \to \mathcal{N} \quad \text{by} \quad \widetilde{W}_{\zeta^*_k,a^*_k}V = \widetilde{W}^*_{\zeta^*_k,w^*_k} \text{ for } k = 1,\dots,N,$$

where we set

$$\mathcal{M} := \bigvee_{k=1}^{N} \operatorname{Im} \widetilde{W}_{\zeta_k^*, w_k^*} \quad \text{and} \quad \mathcal{N} := \bigvee_{k=1}^{N} \operatorname{Im} \widetilde{W}_{\zeta_k^*, a_k^*}.$$

Now, for $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$, let T_b, $\widetilde{T}_b$ and T_μ, $\widetilde{T}_\mu$ be the compressions of $I_{\mathcal{F}^2(E)} \otimes b$ and $I_{\mathcal{F}^2(E)} \otimes \mu$ to $\mathcal{M}$ and $\mathcal{N}$, respectively. Then, from the intertwining relations (4.27) it follows for $k = 1, \ldots, N$ that

$$\widetilde{W}^*_{\zeta_k^*, a_k^*} T_b V = b \widetilde{W}^*_{\zeta_k^*, a_k^*} V = b \widetilde{W}^*_{\zeta_k^*, w_k^*} = \widetilde{W}^*_{\zeta_k^*, w_k^*} \widetilde{T}_b = \widetilde{W}^*_{\zeta_k^*, a_k^*} V \widetilde{T}_b,$$

and with a similar computation $\widetilde{W}^*_{\zeta_k^*, a_k^*} T_\mu V = \widetilde{W}^*_{\zeta_k^*, a_k^*} V \widetilde{T}_\mu$. Since these identities hold for all $k = 1, \ldots, N$, we obtain that V intertwines T_b with $\widetilde{T}_b$ and T_μ with $\widetilde{T}_\mu$ for each $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$. It then follows from Theorem 4.3 that there exists an $S \in \mathcal{F}^\infty(E)$ with $\|S\| \leq 1$, such that $(S^* \otimes I_\mathcal{E})\mathcal{N} \subset \mathcal{M}$ and $V = P_\mathcal{N}(S \otimes I_\mathcal{E})|_\mathcal{M}$. Hence for $k = 1, \ldots, N$ we have

$$\begin{aligned} \sigma(\widehat{S}(\zeta_k, a_k)) &= \widehat{S}(\zeta_k^*, a_k) \otimes I_\mathcal{E} = (W^*_{\zeta_k^*, a_k^*} S|_\mathcal{A}) \otimes I_\mathcal{E} = \widetilde{W}^*_{\zeta_k^*, a_k^*} (S \otimes I_\mathcal{E})|_\mathcal{E} \\ &= \widetilde{W}^*_{\zeta_k^*, w_k^*}|_\mathcal{E} = \sigma(w_k). \end{aligned}$$

Thus $\widehat{S}(\zeta_k, a_k) = w_k$ for $k = 1, \ldots, N$, because σ is injective. □

4.8. Example: Operator-valued Nevanlinna-Pick interpolation on the unit disk $\mathbb{D}$

As a first example we consider the case where $E = \mathcal{A} = \mathcal{L}(\mathcal{V})$ for $\mathcal{V}$ some Hilbert space. Then E is a W^*-$\mathcal{A}$-correspondence with left and right actions given by the usual left and right multiplication in $\mathcal{L}(\mathcal{V})$ and with $\mathcal{L}(\mathcal{V})$-valued inner product given by

$$\langle R, Q \rangle = Q^* R \text{ for all } R, Q \in \mathcal{L}(\mathcal{V}).$$

The preliminaries of this case have been spelled out in Subsection 6.1 of [16]. We recall here the results needed in the sequel. The balancing (4.1) in the tensor products causes the tensor spaces $E^{\otimes n}$ to reduce to $E = \mathcal{L}(\mathcal{V})$ via the identification

$$R_n \otimes \cdots \otimes R_1 = I_\mathcal{V} \otimes \cdots \otimes I_\mathcal{V} \otimes (R_n \cdots R_1) \equiv R_n \cdots R_1. \tag{4.28}$$

Then the Fock space $\mathcal{F}^2(E)$ is equal to $\mathcal{L}(\mathcal{V}, \ell^2_\mathcal{V}(\mathbb{Z}_+))$, and the Toeplitz algebra $\mathcal{F}^\infty(E)$ is the algebra of block Toeplitz matrices

$$R = \begin{bmatrix} R_0 & 0 & 0 & \cdots \\ R_1 & R_0 & 0 & \cdots \\ R_2 & R_1 & R_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad R_j \in \mathcal{L}(\mathcal{V}) \tag{4.29}$$

that induce bounded operators on $\mathcal{L}(\mathcal{V}, \ell^2_{\mathcal{V}}(\mathbb{Z}_+))$ via matrix multiplication: if we write $X \in \mathcal{L}(\mathcal{V}, \ell^2_{\mathcal{V}}(\mathbb{Z}_+))$ in block column-matrix form

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix} \text{ with } X_k \in \mathcal{L}(\mathcal{V}) \text{ for } k = 1, 2, \dots,$$

then $R\colon X \mapsto R \cdot X$ where the matrix for R is given by (4.29). By the closed-graph theorem, for any such fixed X, $R \cdot X \in \mathcal{L}(\mathcal{V}, \ell^2_{\mathcal{V}}(\mathbb{Z}_+))$ if and only if $(R \cdot X) \cdot v = R \cdot (X \cdot v) \in \ell^2_{\mathcal{V}}(\mathbb{Z}_+)$ for each $v \in \mathcal{V}$. As X and v are arbitrary, an equivalent condition is that $R \cdot w \in \ell^2_{\mathcal{V}}(\mathbb{Z}_+)$ for each $w \in \ell^2_{\mathcal{V}}(\mathbb{Z}_+)$, i.e., (again by the closed-graph theorem) R is bounded when viewed as an ordinary Toeplitz matrix acting on $\ell^2_{\mathcal{V}}(\mathbb{Z}_+)$. Furthermore one can check that the norm of R viewed as an element of $\mathcal{L}(\mathcal{L}(\mathcal{V}, \ell^2_{\mathcal{V}}(\mathbb{Z}_+)))$ is equal to its norm when viewed as an element of $\mathcal{L}(\ell^2_{\mathcal{V}}(\mathbb{Z}_+))$. Thus a sequence of coefficients $\{R_n\}_{n\in\mathbb{Z}}$ with $R_n \in \mathcal{L}(\mathcal{V})$ and $R_n = 0$ for $n < 0$ gives rise to an element $[R_{i-j}]_{i,j\in\mathbb{Z}_+}$ in $\mathcal{F}^\infty(E)$ exactly when $R(\lambda) := \sum_{n=0}^\infty R_n\lambda^n$ is in the operator-valued Schur class $\mathcal{S}(\mathcal{V})$.

Let $\mathcal{G}$ be another Hilbert space. We take $\mathcal{E}$ to be the tensor product Hilbert space $\mathcal{V} \otimes \mathcal{G}$, and define a representation $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ by

$$\sigma(R) = R \otimes I_{\mathcal{G}} \text{ for } R \in \mathcal{L}(\mathcal{V}) = \mathcal{A}.$$

Then $(E \otimes_\sigma \mathcal{E})^*$ can be identified with $\mathcal{E}$ via the identification

$$R \otimes_\sigma (v \otimes g) = I_{\mathcal{U}} \otimes_\sigma ((Rv) \otimes g) \equiv (Rv) \otimes g \text{ for } R \in \mathcal{L}(\mathcal{V}),\ v \in \mathcal{V},\ g \in \mathcal{G}.$$

We identify E^σ as

$$E^\sigma = \{\eta \in \mathcal{L}(\mathcal{E})\colon \eta(R \otimes I_{\mathcal{G}}) = (R \otimes I_{\mathcal{G}})\eta \text{ for all } R \in \mathcal{L}(\mathcal{V})\} = I_{\mathcal{V}} \otimes \mathcal{L}(\mathcal{G}),$$

and hence the generalized disk $\mathbb{D}((E^\sigma)^*)$ can be identified with the set of strict contraction operators on $\mathcal{G}$. For $\eta \in \mathbb{D}((E^\sigma)^*)$ identified with a strict contraction operator $Z \in \mathcal{L}(\mathcal{G})$ and for $R = [R_{i-j}]_{i,j\in\mathbb{Z}_+} \in \mathcal{F}^\infty(E)$, one can work out that the associated first point-evaluation $\widehat{R}(Z)$ is given by a tensor functional-calculus

$$\widehat{R}(Z) = \sum_{n=0}^\infty R_n \otimes Z^n.$$

Thus, the correspondence Nevanlinna-Pick interpolation problem for this setting becomes: *Given strict contraction operators $Z_1, \dots, Z_N$ in $\mathcal{L}(\mathcal{G})$ and operators $X_1, \dots, X_N$ and $Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{V} \otimes \mathcal{G})$, find a Schur class function $S(\lambda) = \sum_{n=0}^\infty S_n\lambda^n \in \mathcal{S}(\mathcal{V})$ so that*

$$X_i S(Z_i) := X_i \sum_{n=0}^\infty S_n \otimes Z_i^n = Y_i \text{ for } i = 1, \dots, n. \tag{4.30}$$

We call this the *left-tangential tensor functional-calculus Nevanlinna-Pick interpolation problem* (**LTT-NP**). Note that this contains the **LT-NP** (take $\mathcal{G} = \mathbb{C}$) and

LTRD-NP (take $\mathcal{V} = \mathbb{C}$) problems discussed in Subsections 2.1 and 2.3, respectively, as special cases. The solution to the **LTT-NP** problem is readily obtained as a direct application of Theorem 4.6.

Theorem 4.12. *Suppose that we are given the data set*

$$\mathfrak{D}: \quad Z_1, \dots, Z_N \in \mathcal{L}(\mathcal{G}), \quad X_1, \dots, X_N, Y_1, \dots, Y_N \in \mathcal{L}(\mathcal{V} \otimes \mathcal{G})$$

for an **LTT-NP** *problem. Then a solution $S \in \mathcal{S}(\mathcal{V})$ exists if and only if any of the following equivalent conditions holds:*

(1) *The kernel $\mathbb{K}_{\mathfrak{D}}$ mapping $\{1, \dots, N\} \times \{1, \dots, N\}$ into $\mathcal{L}^a(\mathcal{L}(\mathcal{G}), \mathcal{L}(\mathcal{V} \otimes \mathcal{G}))$ given by*

$$\mathbb{K}_{\mathfrak{D}}(i,j)[B] = \sum_{n=0}^{\infty} X_i \left(I_{\mathcal{V}} \otimes \left(Z_i^n B Z_j^{*n}\right)\right) X_j^* - Y_i \left(I_{\mathcal{V}} \otimes \left(Z_i^n B Z_j^{*n}\right)\right) Y_j^*$$

is completely positive.

(2) *The map $\varphi\colon \mathcal{L}(\mathcal{G})^{N \times N} \to \mathcal{L}(\mathcal{V} \otimes \mathcal{G})^{N \times N}$ given by*

$$\varphi\left([B_{ij}]_{i,j=1,\dots,N}\right) = \tag{4.31}$$
$$= \left[\sum_{n=0}^{\infty} X_i \left(I_{\mathcal{V}} \otimes \left(Z_i^n B_{ij} Z_j^{*n}\right)\right) X_j^* - Y_i \left(I_{\mathcal{V}} \otimes \left(Z_i^n B_{ij} Z_j^{*n}\right)\right) Y_j^*\right]_{i,j=1}^{N}$$

is a completely positive map.

Note that criteria of Theorem 4.12 give seemingly different criteria than the ones obtained in Section 2. We now show how, after some reductions, one can see directly the equivalence of the criteria in Theorem 4.12 with the Pick-matrix criteria of Section 2.

First assume that $\mathcal{V}$ and $\mathcal{G}$ admit direct sum decompositions

$$\mathcal{V} = \mathcal{U} \oplus \mathcal{Y} \oplus \mathcal{C} \quad \text{and} \quad \mathcal{G} = \mathcal{Z} \oplus \mathcal{D},$$

and that the operators in the data set $\mathfrak{D}$ are actually operators

$$Z_i \in \mathcal{L}(\mathcal{Z}), \quad X_i \in \mathcal{L}(\mathcal{Y} \otimes \mathcal{Z}, \mathcal{C} \otimes \mathcal{D}), \quad Y_i \in \mathcal{L}(\mathcal{U} \otimes \mathcal{Z}, \mathcal{C} \otimes \mathcal{D}) \text{ for } i = 1, \dots, N,$$

identified with operators in $\mathcal{L}(\mathcal{G})$ and $\mathcal{L}(\mathcal{E})$, respectively, by adding zero-operators in the operator matrix decompositions. A solution to the "non-square" **LTT-NP** problem: *Given strict contraction operators $Z_1, \dots, Z_N$ in $\mathcal{L}(\mathcal{Z})$, and operators $X_1, \dots, X_N \in \mathcal{L}(\mathcal{Y} \otimes \mathcal{Z}, \mathcal{C} \otimes \mathcal{D})$ and $Y_1, \dots, Y_N \in \mathcal{L}(\mathcal{U} \otimes \mathcal{Z}, \mathcal{C} \otimes \mathcal{D})$, find a Schur class function $S(\lambda) = \sum_{n=0}^{\infty} S_n \lambda^n \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ satisfying* (4.30), can then be obtained from a solution $\tilde{S} \in \mathcal{S}(\mathcal{E})$ just by assigning $S(\lambda)$ to be the compression of $\tilde{S}(\lambda)$ to $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Conversely, a solution $S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ to the "non-square" problem defines a solution $\tilde{S} \in \mathcal{L}(\mathcal{E})$ to the "square" problem by embedding the values $S(\lambda) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ into $\mathcal{L}(\mathcal{E})$. It thus follows that a solution to the "non-square" problem exists if and only if the map φ in criterion 2 of Theorem 4.12 (which now can also be seen as a map from $\mathcal{L}(\mathcal{Z})^{N \times N}$ into $\mathcal{L}(\mathcal{Z} \otimes \mathcal{D})^{N \times N}$) is completely positive.

Transforming a result from a "square" Nevanlinna-Pick problem to one for Schur class functions that take "non-square" values can even be performed on the level of W^*-correspondence Nevanlinna-Pick interpolation using techniques developed in [70].

Standard functional calculus Nevanlinna-Pick interpolation. Assume that $\mathcal{G} = \mathbb{C}$. Hence the data for our Nevanlinna-Pick interpolation is of the **LT-NP** form:

$$\mathfrak{D}: \ \lambda_1, \dots, \lambda_N \in \mathbb{D}, \ X_1, \dots, X_N \in \mathcal{L}(\mathcal{Y}, \mathcal{C}), \ Y_1, \dots, Y_N \in \mathcal{L}(\mathcal{U}, \mathcal{C}).$$

In that case, the kernel $\mathbb{K}_{\mathfrak{D}} : \{1, \dots, N\} \times \{1, \dots, N\} \to \mathcal{L}(\mathbb{C}, \mathcal{L}(\mathcal{L}(\mathcal{V})))$ reduces to

$$\mathbb{K}_{\mathfrak{D}}(i,j)[c] = c \cdot \frac{X_i X_j^* - Y_i Y_i^*}{i - \lambda_i \overline{\lambda}_j}.$$

It is then straightforward to see that complete positivity of the kernel $\mathbb{K}_{\mathfrak{D}}$ collapses to positive semidefiniteness of the standard Pick matrix

$$\left[\ \frac{X_i X_j^* - Y_i Y_i^*}{i - \lambda_i \overline{\lambda}_j} \ \right]_{j,i=1}^N .$$

Thus we recover the criterion of Theorem 2.1 (Part 2).

Riesz-Dunford functional calculus Nevanlinna-Pick interpolation. Next we consider the case that $\mathcal{V} = \mathbb{C}$. In that case the data for our Nevanlinna-Pick interpolation problem takes the **LTRD-NP** form:

$$\mathfrak{D}: \ Z_1, \dots, Z_N \in \mathcal{L}(\mathcal{Z}), \ X_1, \dots, X_N, Y_1, \dots, Y_N \in \mathcal{L}(\mathcal{Z}, \mathcal{C}).$$

To see that Theorem 4.12 also contains the result of Theorem 2.3 (Part 2), it is convenient to introduce a third criterion.

Theorem 4.13. *In case $\mathcal{V} = \mathbb{C}$, in addition to the two conditions* (1) *and* (2) *in Theorem* 4.12*, a third condition equivalent to the existence of an $S \in \mathcal{S}(\mathbb{C}, \mathbb{C})$ that satisfies* (4.30) *is that the map φ_* from $\mathcal{L}(\mathcal{C})^{N \times N}$ to $\mathcal{L}(\mathcal{C})^{N \times N}$ given by*

$$\varphi_* \left([C_{ij}]_{i,j=1}^N \right) = \left[\sum_{n=0}^{\infty} Z_i^{*n} (X_i^* C_{ij} X_j - Y_i^* C_{ij} Y_j) Z_j^n \right]_{i,j=1}^N \tag{4.32}$$

be a completely positive map.

Proof. We prove that positivity of φ is equivalent to positivity of φ_*. The proof of the equivalence for k-positivity with $k \in \mathbb{Z}_+$ arbitrary (and thus for complete positivity) goes analogously. The map φ in (4.31) being positive implies that

$$\sum_{i,j=1}^N \operatorname{trace} \left(C_{ji} \left(\sum_{n=0}^{\infty} X_i Z_i^n B_{ij} Z_j^{*n} X_j^* - Y_i Z_i^n B_{ij} Z_j^{*n} Y_j^* \right) \right) \geq 0 \tag{4.33}$$

for all $\mathbf{B} = [B_{ij}]_{i,j=1}^N \succeq 0$ and $\mathbf{C} = [C_{ij}]_{i,j=1}^N \succeq 0$ in $\mathcal{L}(\mathcal{G})^{N \times N}$ with $\mathbf{B}$ also in the trace class. Using the invariance of the trace under cyclic permutations, we may rewrite (4.33) as

$$\sum_{i,j=1}^{N} \mathrm{trace}\left(\left(\sum_{n=0}^{\infty} Z_j^{*n}(X_j^* C_{ji} X_i - Y_j^* C_{ji} Y_i) Z_i^n\right) B_{ij}\right) \geq 0. \tag{4.34}$$

From this it follows that φ_* is a positive map. To see the converse implication one just reverses the above computation but now with $\mathbf{C}$ in the trace class. □

In order to transform the completely positive map criterion of Theorem 4.13 into one of checking positivity of a (possibly infinite) operator matrix we need the following extension of Choi's theorem [38].

Theorem 4.14. *A weak-$*$ continuous map $\psi : \mathcal{L}(\mathcal{H}) \to \mathcal{A}$, where $\mathcal{H}$ a separable Hilbert space with orthonormal basis $\{e_1, \ldots, e_\kappa\}$ (with possibly $\kappa = \infty$) and $\mathcal{A}$ is a C^*-algebra, is completely positive if and only if the (possibly infinite) block matrix*

$$\left[\ \psi(e_i e_j^*)\ \right]_{i,j=1}^{\kappa} \tag{4.35}$$

is a positive element of $\mathcal{A}^{\kappa\times\kappa}$ (i.e., if $\kappa = \infty$, then all M-finite sections are positive in $\mathcal{A}^{M\times M}$ for each $M \in \mathbb{Z}_+$).

Proof. If $\kappa < \infty$, then the statement is just Choi's theorem [38]. So assume that $\kappa = \infty$. Complete positivity of ψ means that for each $N \in \mathbb{Z}_+$ the map

$$\left[\ B_{i,j}\ \right]_{i,j=1}^{N} \mapsto \left[\ \psi(B_{i,j})\ \right]_{i,j=1}^{N}$$

from $\mathcal{L}(\mathcal{H})^{N\times N}$ into $\mathcal{A}^{N\times N}$ is positive. In particular, using $B_{i,j} = e_i e_j^*$ it follows directly that ψ being completely positive implies that the block matrix (4.35) is a positive element of $\mathcal{A}^{\kappa\times\kappa}$.

Now assume that the block matrix (4.35) is positive. For each $M \in \mathbb{Z}_+$, let P_M denote the projection onto $\mathcal{H}_M := \mathrm{span}\{e_1, \ldots, e_M\}$. Using Choi's theorem and the fact that the M-finite section $\left[\ \varphi(e_i e_j^*)\ \right]_{i,j=1}^{M}$ of (4.35) is a positive element of $\mathcal{A}^{M\times M}$ we obtain that the map $\tilde{B} \mapsto \psi(P_M \tilde{B} P_M)$ from $\mathcal{H}_M$ into $\mathcal{A}$ is completely positive. Fix a $N \in \mathbb{Z}_+$ and a positive $\left[\ B_{i,j}\ \right]_{i,j=1}^{N}$ in $\mathcal{L}(\mathcal{H})^{N\times N}$. Then for each $M \in \mathbb{Z}_+$ we have $\left[\ \psi(P_M B_{i,j} P_M)\ \right]_{i,j=1}^{N} \succeq 0$. Moreover, $P_M B_{i,j} P_M$ converges to $B_{i,j}$ in the weak-$*$ topology as $M \to \infty$, and thus, by hypothesis $\psi(P_M B_{i,j} P_M) \to \psi(B_{i,j})$ in the weak-$*$ topology as $M \to \infty$ for all $i,j = 1, \ldots, N$. This implies that

$$\left[\ \psi(P_M B_{i,j} P_M)\ \right]_{i,j=1}^{N} \to \left[\ \psi(B_{i,j})\ \right]_{i,j=1}^{N} \text{ weak-}* \text{ as } M \to \infty.$$

Since positivity is preserved under weak-$*$ convergence it follows that $\left[\ \psi(B_{i,j})\ \right]_{i,j=1}^{N}$ is a positive element of $\mathcal{A}^{N\times N}$, and thus, because N was chosen arbitrarily, that ψ is a completely positive map. □

Now assume that $\mathcal{C}$ is a separable Hilbert space with orthonormal basis $\{e_1, \ldots, e_\kappa\}$. It then follows from Theorem 4.14 that complete positivity of the maps φ_* given by (4.32) reduces to positivity of the Pick matrix

$$\left[\sum_{n=0}^{\infty} Z_i^{*n}(X_i^* e_{i'} e_{j'}^* X_j - Y_i^* e_{i'} e_{j'}^* Y_j) Z_j^n \right]_{(i,i'),(j,j')\in\{1,\dots,N\}\times\{1,\dots,\kappa\}}.$$

Hence we recover Part 2 of Theorem 2.3. Indeed, the one thing left to verify is that the map φ_* is weak-$*$ continuous, which we leave as an exercise to the reader.

Operator-argument functional calculus Nevanlinna-Pick interpolation. Finally, we show that the second point-evaluation for $\mathcal{F}^\infty(E)$ discussed in Subsection 4.7 gives us the operator-argument Nevanlinna-Pick theorem for the operator-valued Schur class. To see that this is the case, note that the points are elements of

$$\mathbb{D}(E^*) \times \mathcal{A} = \{T \in \mathcal{L}(\mathcal{V})\colon \|T\| < 1\} \times \mathcal{L}(\mathcal{V}).$$

For each $\zeta \in \mathbb{D}(E^*)$ identified with a strict contraction operator $T \in \mathcal{L}(\mathcal{V})$ the element $\zeta_{(n)}^* = \zeta^* \otimes \cdots \otimes \zeta^*$ of $E^{\otimes n}$ corresponds to T^{*n} (via the identification (4.28)), and the operator $T^{(0)}_{\zeta_{(n)}^*}$ in (4.9) is just multiplication with T^{*n}. Thus, we obtain that for $(T,X) \in \mathbb{D}(E^*) \times \mathcal{A}$ and $R = \left[R_{i-j} \right]_{i,j\in\mathbb{Z}_+} \in \mathcal{F}^\infty(E)$ the value of R in (T,X) is equal to the left-tangential point-evaluation:

$$\widehat{R}(T,X) = \sum_{n=0}^{\infty} T^n X R_n = (XR)^{\wedge L}(T),$$

where we also use R to indicate the Schur-class function in $\mathcal{S}(\mathcal{V})$ corresponding to $R \in \mathcal{F}^\infty(E)$. So, in this case, the Nevanlinna-Pick problem with data $\zeta_k = T_k \in \mathbb{D}(E^*)$, $a_k = X_k, w_k = Y_k \in \mathcal{A}$, for $k = 1,\dots,N$, considered in Subsection 4.7 is the **LTOA** problem of Subsection 2.2. One easily computes that the operator matrix (4.25) reduces to the Pick matrix $\mathbb{P}_{\rm LTOA}$ in (2.8). Thus, Theorem 4.11 here gives us precisely Part 1 of Theorem 2.2 (for the case $\mathcal{U} = \mathcal{Y} = \mathcal{C}$; the non-square case can be obtained as explained above).

4.9. Example: Nevanlinna-Pick interpolation for Toeplitz algebras associated with directed graphs

Next, following Section 5 in [70], we show how the Toeplitz algebra associated with a directed graph/quiver can be seen as an example of the W^*-correspondence formalism, and we prove Theorem 3.3. We follow the notation and terminology used in Subsection 3.3. Let $G = \{Q_0, Q_1, s, r\}$ be a quiver and $\mathcal{V}$ a given Hilbert space that admits an orthogonal decomposition $\mathcal{V} = \oplus_{v\in Q_0}\mathcal{V}_v$. For each nonnegative integer n we associate with Q_n, the set of paths of length n, the space $C_{G,\mathcal{V}}(Q_n)$ of continuous $\mathcal{L}(\mathcal{V})$-valued functions f on Q_n, where $f(\gamma)$ maps $\mathcal{V}_{s(\gamma)}$ into $\mathcal{V}_{r(\gamma)}$ and $f(\gamma)|_{\mathcal{V}_{s(\gamma)}^\perp} = 0$ for each $\gamma \in Q_n$; of course the continuity is automatic as Q_n is a finite set with the discrete topology – this notation is used for consistency with more general settings where Q_n is a more general topological space as in [60, 61]. Usually we will just write $C_G(Q_n)$, rather than $C_{G,\mathcal{V}}(Q_n)$, for notational convenience.

The space $C_G(Q_n)$ can be seen as a W^*-$C_G(Q_0)$-correspondence with left and right multiplication and $C_G(Q_0)$-valued inner-product given by:

$$\begin{aligned}(f\cdot\xi)(\gamma)&=f(r_n(\gamma))\xi(\gamma)\\(\xi\cdot f)(\gamma)&=\xi(\gamma)f(s_n(\gamma))\\\langle\xi,\eta\rangle(v)&=\sum_{s_n(\gamma')=v}\overline{\eta(\gamma')}\xi(\gamma')\end{aligned}\qquad(\xi,\eta\in C_G(Q_n),f\in C_G(Q_0),\gamma\in Q_n,v\in Q_0).$$

Tensoring $C_G(Q_n)$ with $C_G(Q_m)$ gives $C_G(Q_{n+m})$, hence, in particular, we obtain for each $n\in\mathbb{Z}_+$ that $C_G(Q_n)=C_G(Q_1)^{\otimes n}$. More precisely, for $\xi_n,\dots,\xi_1\in C_G(Q_1)$ we can identify $\xi_1\otimes\cdots\otimes\xi_n\in C_G(Q_1)^{\otimes n}$ with the element $\xi^{(n)}\in C_G(Q_n)$ given by

$$\xi^{(n)}(\gamma):=\xi_n(\alpha_n)\cdots\xi_1(\alpha_1)\quad\text{for}\quad\gamma=(\alpha_n,\dots,\alpha_1)\in Q_n.\tag{4.36}$$

To see that this is the case, note that, rather than viewing elements of $C_G(Q_n)$ as functions, they are also given by tuples $F=(F_\gamma\in\mathcal{L}(\mathcal{V}_{s(\gamma)},\mathcal{V}_{r(\gamma)})\colon\gamma\in Q_n)$ that we usually identify with operator matrices

$$\mathbf{F}=\left[\ F_{\gamma,v}\ \right]_{\gamma\in Q_n,v\in Q_0}:\bigoplus_{v\in Q_0}\mathcal{V}_v\to\bigoplus_{\gamma\in Q_n}\mathcal{V}_{r(\gamma)},\ \text{where } F_{\gamma,v}=\begin{cases}F_\gamma&\text{if } s_n(\gamma)=v,\\0&\text{otherwise.}\end{cases}\tag{4.37}$$

The advantage of the operator matrix representation is that for $F,F'\in C_G(Q_n)$ the norm $\|F\|_{C_G(Q_n)}$ is equal to the operator norm of $\mathbf{F}$, and $\langle F,F'\rangle$ can be identified with $\mathbf{F}'^*\mathbf{F}$; the operator matrix $\mathbf{F}'^*\mathbf{F}\in\mathcal{L}(\mathcal{V})$ is block diagonal and, for each $v\in Q_0$, the diagonal entry from $\mathcal{L}(\mathcal{V}_v)$ corresponds to the value of $\langle F,F'\rangle(v)$. Note that $C_G(Q_0)$ corresponds to the W^*-algebra of block diagonal operators $\operatorname{diag}_{v\in Q_0}(A_v)$ on $\mathcal{V}=\oplus_{v\in Q_0}\mathcal{V}_v$. Moreover, the operator $T_F^{(0)}$ defined by (4.9) mapping $C_G(Q_0)$ (block diagonal operators) into $C_G(Q_n)$ can be identified with multiplication with $\mathbf{F}$. Subject to this identification, the operator $\mathbf{F}\otimes I_{C_G(Q_1)}$ from $C_G(Q_1)$ to $C_G(Q_{n+1})$ corresponds to multiplication with the block operator matrix

$$\mathbf{F}\otimes I_{C_G(Q_1)}=\left[\ F_{\gamma,\alpha}\ \right]_{\gamma\in Q_{n+1},\alpha\in Q_1},\ \text{where } F_{\gamma,\alpha}=\begin{cases}F_{\gamma'}&\text{if }\gamma=(\gamma',\alpha),\\0&\text{otherwise.}\end{cases}$$

Analogously, one obtains formulas for $\mathbf{F}\otimes I_{C_G(Q_1)^{\otimes n}}=\mathbf{F}\otimes I_{C_G(Q_n)}$ for each $n\in\mathbb{Z}_+$. Taking a product of such operators $\mathbf{F_1},\mathbf{F_2}\otimes I_{C_G(Q_1)},\dots,\mathbf{F_n}\otimes I_{C_G(Q_1)^{\otimes n}}$, with $F_1,F_2,\dots,F_n\in C_G(Q_1)$, we finally arrive at (4.36).

We now take E to be the W^*-$C_G(Q_0)$-correspondence $C_G(Q_1)$. Elements of the Fock space $\mathcal{F}^2(C_G(Q_1))=\oplus_{n\in\mathbb{Z}_+}C_G(Q_n)$ are then given by tuples $F=(F_\gamma\in\mathcal{L}(\mathcal{V}_{s(\gamma)},\mathcal{V}_{r(\gamma)})\colon\gamma\in\Gamma)$, where Γ is the collection of all finite paths of whatever length, such that the operator matrix

$$\mathbf{F}=\left[\ F_{\gamma,v}\ \right]_{v\in Q_0,\gamma\in\Gamma}$$

is in $\mathcal{L}(\mathcal{V},\oplus_{\gamma\in\Gamma}\mathcal{V}_{r(\gamma)})$ (i.e., bounded). Here $F_{\gamma,v}$ is as defined in (4.37). As in Subsection 3.3 we denote the Hilbert space $\oplus_{\gamma\in\Gamma}\mathcal{V}_{r(\gamma)}$ by $\ell^2_{\mathcal{V}}(\Gamma)$. The Toeplitz algebra $\mathcal{F}^\infty(E)$, seen as a subalgebra of $\mathcal{L}(\ell^2_{\mathcal{V}}(\Gamma))$, is then precisely the Toeplitz algebra $\mathfrak{L}_\Gamma(\mathcal{V},\mathcal{V})$ defined in Subsection 3.3. That is, an element R in $\mathcal{F}^\infty(E)$ is given by a tuple $R = (R_\gamma \in \mathcal{L}(\mathcal{V}_{s(\gamma)}, \mathcal{V}_{r(\gamma)})\colon \gamma \in \Gamma)$ with the property that the operator matrix

$$\mathbf{R} = \left[\ R_{\gamma,\gamma'}\ \right]_{\gamma,\gamma'\in\Gamma}, \text{ where } R_{\gamma,\gamma'} = R_{\gamma\gamma'^{-1}} \text{ (with } R_{\text{undefined}} = 0),$$

induces a bounded operator on $\ell^2_{\mathcal{V}}(\Gamma)$. Here $\gamma\gamma'^{-1}$ is defined by (3.7).

In addition to $\mathcal{G}$ we assume that we are given a Hilbert space $\mathcal{G}$ that also has an orthogonal-sum decomposition of the form $\mathcal{G} = \oplus_{v\in Q_0}\mathcal{G}_v$. We then set $\mathcal{E}_v = \mathcal{V}_v \otimes \mathcal{G}_v$ for each $v \in Q_0$ and $\mathcal{E} = \oplus_{v\in Q_0}\mathcal{E}_v$. Now let $\sigma : C_G(Q_0) \to \mathcal{L}(\mathcal{E})$ be the representation of $C_G(Q_0)$ given by

$$\sigma(\operatorname{diag}_{v\in Q_0}(A_v)) = \operatorname{diag}_{v\in Q_0}(A_v \otimes I_{\mathcal{G}_v}).$$

Then, for each $n \in \mathbb{Z}_+$, we have $C_G(Q_n) \otimes_\sigma \mathcal{E} = \oplus_{\gamma\in Q_n}(\mathcal{V}_{r_n(\gamma)} \otimes \mathcal{G}_{s_n(\gamma)})$ and thus

$$\mathcal{F}^2(E,\sigma) := \mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = \oplus_{\gamma\in\Gamma}(\mathcal{V}_{r(\gamma)} \otimes \mathcal{G}_{s(\gamma)}).$$

It is straightforward that

$$\sigma(C_G(Q_0))' = \left\{\operatorname{diag}_{v\in Q_0}(I_{\mathcal{V}_v} \otimes B_v)\colon B_v \in \mathcal{L}(\mathcal{G}_v)\right\}.$$

Moreover, the space $C_G(Q_1)^\sigma$ consists of those operator matrices mapping $\mathcal{E} = \oplus_{v\in Q_0}(\mathcal{V}_v \otimes \mathcal{G}_v)$ into $C_G(Q_1) \otimes \mathcal{E} = \oplus_{\alpha\in Q_1}(\mathcal{V}_{r(\alpha)} \otimes \mathcal{G}_{s(\alpha)})$ that are of the form

$$\left[\ I_{\mathcal{V}_{r(\alpha)}} \otimes K_{\alpha,v}\ \right]_{\alpha\in Q_1, v\in Q_0}$$

with $K_{\alpha,v} \in \mathcal{L}(\mathcal{G}_{r(\alpha)}, \mathcal{G}_{s(\alpha)})$ and $K_{\alpha,v} = 0$ in case $r(\alpha) \neq v$. Leaving out the identity operators $I_{\mathcal{V}_v}$, we can identify $\sigma(C_G(Q_0))'$ with $C_{\widetilde{G},\mathcal{G}}(Q_0)$ and $C_G(Q_1)^\sigma$ with $C_{\widetilde{G},\mathcal{G}}(Q_1)$, where $\widetilde{G} = \{Q_0, Q_1, r, s\}$ is the transposed quiver of G (i.e., with the source and range maps interchanged). Note that the generalized disk $\mathbb{D}((C_G(Q_1)^\sigma)^*)$ of strictly contractive operators with adjoint in $C_G(Q_1)^\sigma$ can be identified with the set $\mathbb{D}_{G,\mathcal{G}}$ defined in Subsection 3.3. Next observe that for $Z = (Z_\alpha \in \mathcal{L}(\mathcal{G}_{s(\alpha)}, \mathcal{G}_{r(\alpha)})\colon \alpha \in Q_1) \in \mathbb{D}_{G,\mathcal{G}}$, the n^{th} generalized power Z^n of Z then corresponds to the tuple $(Z_\gamma \in \mathcal{L}(\mathcal{G}_{s_n(\gamma)}, \mathcal{G}_{r_n(\gamma)})\colon \gamma \in Q_n)$ with $Z_\gamma = Z^\gamma$, where Z^γ is defined by (3.8).

Now let $R = (R_\gamma\colon \gamma \in \Gamma) \in \mathcal{F}^\infty(E) = \mathfrak{L}_\Gamma(\mathcal{V},\mathcal{V})$ and $Z = (Z_\alpha\colon \alpha \in Q_1) \in \mathbb{D}_{G,\mathcal{G}}$. It then follows that the first Muhly-Solel point-evaluation $\widehat{R}(Z)$ of R in Z is given by the tensor-product functional-calculus for $\mathfrak{L}_\Gamma(\mathcal{V},\mathcal{V})$:

$$\widehat{R}(Z) = \sum_{\gamma\in\Gamma} i_{\mathcal{V}_{r(\gamma)}\otimes\mathcal{G}_{r(\gamma)}}(R_\gamma \otimes Z^\gamma) i^*_{\mathcal{V}_{s(\gamma)}\otimes\mathcal{G}_{s(\gamma)}},$$

where, as in Subsection 3.3, we use the general notation: for a subspace $\mathcal{H}$ of a Hilbert space $\mathcal{K}$ we write $i_{\mathcal{H}}$ for the canonical embedding of $\mathcal{H}$ into $\mathcal{K}$. Thus the W^*-correspondence Schur class $\mathcal{S}(E,\sigma)$ corresponds to the free semigroupoid algebra Schur class $\mathcal{S}_G(\mathcal{V},\mathcal{V})$ in combination with the point-evaluation in the generalized disk $\mathbb{D}_{G,\mathcal{G}}$.

The W^*-correspondence Nevanlinna-Pick problem in this case thus turns out to be the left-tangential tensor-product functional-calculus free semigroupoid algebra Nevanlinna-Pick problem (**QLTT-NP**): *Given a data set*

$$\mathfrak{D} : Z^{(1)}, \dots, Z^{(N)} \in \mathbb{D}_{G,\mathcal{G}},\ X_1, \dots, X_N, Y_1, \dots, Y_N \in \mathcal{L}(\mathcal{E}), \tag{4.38}$$

determine when there exists a Schur class function $S \in \mathcal{S}_G(\mathcal{V}, \mathcal{V})$ *such that*

$$X_i S(Z^{(i)}) = Y_i \quad \text{for} \quad i = 1, \dots, N. \tag{4.39}$$

As in the "unit disk" example of Subsection 4.8, one can deduce from the solution to this problem the analogous result for the "non-square" case considered in Subsection 3.3, but we will not work out those details here.

In order to state the solution we remark that the Szegő kernel (4.22) specified for this setting has the form

$$\mathbb{K}_{C_G(Q_1),\sigma} \colon \mathbb{D}_{G,\mathcal{G}} \times \mathbb{D}_{G,\mathcal{G}} \to \mathcal{L}^a(C_{\widetilde{G}}(Q_0), \mathcal{L}(\mathcal{E}))$$

and can be written as

$$\mathbb{K}_{C_G(Q_1),\sigma}(Z, Z')[B] = \sum_{\gamma \in \Gamma} i_{\mathcal{E}_{r(\gamma)}} \left(I_{\mathcal{V}_{r(\gamma)}} \otimes Z^{\gamma} i^*_{\mathcal{G}_{s(\gamma)}} B i_{\mathcal{G}_{s(\gamma)}} (Z'^{\gamma})^* \right) i^*_{\mathcal{E}_{r(\gamma)}}. \tag{4.40}$$

An application of the general Theorem 4.6 then leads to the following solution of the problem.

Theorem 4.15. *Suppose we are given data as in* (4.38)*. Then there exists a solution* $S \in \mathcal{S}_G(\mathcal{V}, \mathcal{V})$ *to the* **QLTT-NP** *interpolation problem if and only if one of the following equivalent conditions holds:*

(1) *the kernel* $\mathbb{K}_{\mathfrak{D}} \colon \{1, \dots, N\} \times \{1, \dots, N\} \to \mathcal{L}^a(C_{\widetilde{G}}(Q_0), \mathcal{L}(\mathcal{E}))$ *given by*

$$\mathbb{K}_{\mathfrak{D}}(i,j)[B] = X_i \mathbb{K}_{C_G(Q_1),\sigma}(Z^{(i)}, Z^{(j)})[B] X_j^* - Y_i K_{C_G(Q_1),\sigma}(Z^{(i)}, Z^{(j)})[B] Y_j^*$$

is a completely positive kernel.

(2) *the map* φ *from* $C_{\widetilde{G}}(Q_0)^{N \times N}$ *to* $\mathcal{L}(\mathcal{E})^{N \times N}$ *given by*

$$\varphi\left([B_{ij}]_{i,j=1}^N\right) = \left[\ \mathbb{K}_{\mathfrak{D}}(i,j)[B_{i,j}]\ \right]_{i,j=1}^N$$

is completely positive.

As a consequence of the extension of Choi's theorem (see Theorem 4.14) we obtain the following third condition.

Theorem 4.16. *Assume that* $\mathcal{V}$ *is separable, and that for each* $v \in Q_0$ *we have an orthonormal basis* $\{e_1^{(v)}, \dots, e_{\kappa_v}^{(v)}\}$ *for* $\mathcal{V}_v$*. Then, in addition to the two conditions* (1) *and* (2) *in Theorem* 4.15*, a third condition equivalent to the existence of an* $S \in \mathcal{S}_G(\mathcal{V}, \mathcal{V})$ *that satisfies* (4.39) *is that for each* $v \in Q_0$ *the operator matrix*

$$\left[\ \mathbb{K}_{\mathfrak{D}}(i,j)[e_{i'}^{(v)} e_{j'}^{(v)*}]\ \right]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa_v\}} \in \mathcal{L}(\mathcal{E})^{\kappa_v N \times \kappa_v N} \tag{4.41}$$

is positive. Here $\mathbb{K}_{\mathfrak{D}}$ *is the kernel defined in Part* 1 *of Theorem* 4.15.

If one writes out the definition of the kernel $\mathbb{K}_{\mathfrak{D}}$ (and that of $\mathbb{K}_{C_G(Q_1),\sigma}$) the operator matrix (4.41) turns out to be exactly the Pick matrix $\mathbb{P}_{QLTT}$ of Part 1 of Theorem 3.3. Thus we obtain the first solution criterion of Theorem 3.3.

In order to prove Theorem 4.16 it is convenient to first prove two lemmas.

Lemma 4.17. *Let $\mathcal{H}$ be a Hilbert space with orthogonal sum decomposition $\mathcal{H} = \oplus_{i=1}^N \mathcal{H}_i$. Then the map $\psi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ defined by*

$$\psi(\left[\begin{array}{c} B_{i,j} \end{array}\right]_{i,j=1}^N) = \operatorname{diag}_{i=1}^N (B_{i,i}),$$

where $\left[\begin{smallmatrix} B_{i,j} \end{smallmatrix}\right]_{i,j=1}^N \in \mathcal{L}(\mathcal{H})$ is an operator matrix with $B_{i,j} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, is a completely positive map.

More generally, the statement remains true for any conditional expectation operator $\psi : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is an arbitrary C^*-algebra [104] (see also [105]), of which the map ψ in Lemma 4.17 is just a particular example. We give the following independent proof.

Proof of Lemma 4.17. It is immediately clear that ψ is positive. Let $M \in \mathbb{Z}_+$. Assume that we have an operator matrix

$$\mathbb{B} = \left[[B_{\alpha,i;\beta,j}]_{i,j=1,\dots,N} \right]_{\alpha,\beta=1,\dots,M}$$

that is a positive element of $\mathcal{L}(\oplus_{\alpha=1}^M \mathcal{H})$, where each $B_{\alpha,i;\beta,j} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. The assumption that $\mathbb{B}$ is positive implies that the matrix

$$\widetilde{\mathbb{B}}_i = [B_{\alpha,i;\beta,i}]_{\alpha,\beta=1,\dots,M}$$

is positive for each $i = 1, \dots, N$, since $\widetilde{\mathbb{B}}_i$ is a principal submatrix of $\mathbb{B}$. On the other hand by definition we have

$$\psi(\mathbb{B}) = \left[\operatorname{diag}_{i=1}^N [B_{\alpha,i;\beta,i}] \right]_{\alpha,\beta=1,\dots,M}.$$

which is unitarily equivalent via a permutation matrix to

$$\operatorname{diag}_{i=1}^N \left([B_{\alpha,i;\beta,i}]_{\alpha,\beta=1,\dots,M} \right) = \operatorname{diag}_{i=1}^N (\widetilde{\mathbb{B}}_i).$$

Thus positivity of $\mathbb{B}$ implies positivity of $\psi(\mathbb{B})$ as required. □

Now observe that we can extend the Szegő kernel $\mathbb{K}_{C_G(Q_1),\sigma}$ to a kernel $\overline{\mathbb{K}}_{C_G(Q_1),\sigma}$ of the form

$$\overline{\mathbb{K}}_{C_G(Q_1),\sigma} : \mathbb{D}_{G,\mathcal{G}} \times \mathbb{D}_{G,\mathcal{G}} \to \mathcal{L}(\mathcal{L}(\mathcal{G}), \mathcal{L}(\mathcal{E}))$$

using the same formula, i.e., in the right-hand side of (4.40) we allow B to be in $\mathcal{L}(\mathcal{G})$ rather than just a block diagonal operator. We can then also extend the kernel $\mathbb{K}_{\mathfrak{D}}$ in condition (1) and the map φ in condition (2) of Theorem 4.15 to a kernel $\overline{\mathbb{K}}_{\mathfrak{D}}$ and a map $\overline{\varphi}$ of the form

$$\overline{\mathbb{K}}_{\mathfrak{D}} : \{1, \dots, N\} \times \{1, \dots, N\} \to \mathcal{L}^a(\mathcal{L}(\mathcal{G}), \mathcal{L}(\mathcal{E})), \quad \overline{\varphi} : \mathcal{L}(\mathcal{G})^{N \times N} \to \mathcal{L}(\mathcal{E})^{N \times N},$$

simply by replacing $\mathbb{K}_{C_G(Q_1),\sigma}$ by $\overline{\mathbb{K}}_{C_G(Q_1),\sigma}$ in the definitions of $\overline{\mathbb{K}}_{\mathfrak{D}}$ and φ.

Lemma 4.18. *The map φ defined in Part* 2 *of Theorem* 4.15 *is completely positive if and only if the map $\overline{\varphi}$ is completely positive.*

Proof. Let ψ be the completely positive map of Lemma 4.17 with $\mathcal{H}$ replaced by $\mathcal{G}$, relative to the orthogonal decomposition $\mathcal{G} = \oplus_{v \in Q_0} \mathcal{G}_v$. Notice that complete positivity of $\overline{\varphi}$ automatically implies complete positivity of φ. To see that the converse is also true, observe that for any $B \in \mathcal{L}(\mathcal{G})$ and any $Z, Z' \in \mathbb{D}_{G,\mathcal{G}}$ we have $\overline{\mathbb{K}}_{C_G(Q_1),\sigma}(Z,Z')[B] = \mathbb{K}_{C_G(Q_1),\sigma}(Z,Z')[\psi(B)]$, and thus also $\overline{\varphi}[[\, B_{i,j}\,]_{i,j=1}^N] = \varphi[[\, \psi(B_{i,j})\,]_{i,j=1}^N]$. Hence the converse statement follows from Lemma 4.17 and the fact that compositions of completely positive maps are again completely positive maps. □

Proof of Theorem 4.16. Let $\{e_1, \dots, e_\kappa\}$ be a reordering of the orthonormal basis $\{e_i^{(v)} \colon v \in Q_0,\ i = 1, \dots, \kappa_v\}$ of $\mathcal{V}$. Since $\overline{\varphi}$ is defined on $\mathcal{L}(\mathcal{V})^{N \times N} = \mathcal{L}(\mathcal{V}^N)$, we can apply Theorem 4.14 to $\overline{\varphi}$, obtaining that $\overline{\varphi}$ is completely positive if and only the operator matrix

$$\left[\ \overline{\mathbb{K}}_{\mathfrak{D}}(i,j)[e_{i'}e_{j'}^*]\ \right]_{(i,i'),(j,j') \in \{1,\dots,N\} \times \{1,\dots,\kappa\}} \tag{4.42}$$

is positive. Next observe that $\overline{\mathbb{K}}_{\mathfrak{D}}(i,j)[e_{i'}e_{j'}^*] = \mathbb{K}_{\mathfrak{D}}(i,j)[e_{i'}e_{j'}^*]$ in case $e_{i'}, e_{j'} \in \{e_1^{(v)}, \dots, e_{\kappa_v}^{(v)}\}$ for some $v \in Q_0$, and that $\overline{\mathbb{K}}_{\mathfrak{D}}(i,j)[e_{i'}e_{j'}^*] = 0$ otherwise. Thus, after a reordering in the basis $\{e_1, \dots, e_\kappa\}$, we can identify the operator matrix (4.42) with the block diagonal operator matrix with the operator matrices (4.41) on the diagonal. This proves our claim. □

We now show how the solution criterion obtained above can be used to derive the results for the Riesz-Dunford and operator-argument functional calculus listed in Subsection 3.3.

Riesz-Dunford functional calculus Nevanlinna-Pick interpolation. The Riesz-Dunford functional calculus is the special case of the tensor functional calculus with $\mathcal{V}_v = \mathbb{C}$ for each $v \in Q_0$, i.e., $\mathcal{E} = \mathcal{G}$. In this case, the same argument as used in the proof of Theorem 4.13 (i.e., invariance under cyclic permutations of the trace) now applied to the map $\overline{\varphi}$, in combination with Theorem 4.14, gives us the following result.

Theorem 4.19. *In case $\mathcal{V}_v = \mathbb{C}$ for each $v \in Q_0$, then there exists an $S \in \mathcal{S}_G(\mathbb{C}, \mathbb{C})$ that satisfies* (4.39) *if and only if the map φ_* from $\mathcal{L}(\mathcal{G})^{N \times N}$ to $\mathcal{L}(\mathcal{G})^{N \times N}$ given by*

$$\varphi_*\left([C_{ij}]_{i,j=1}^N\right) \tag{4.43}$$
$$= \left[\ \sum_{\gamma \in \Gamma} i_{\mathcal{G}_{s(\gamma)}}((Z^{(i)})^\gamma)^* i_{\mathcal{G}_{r(\gamma)}}^* (X_i^* C_{i,j} X_j - Y_i^* C_{i,j} Y_j) i_{\mathcal{G}_{r(\gamma)}} (Z^{(j)})^\gamma i_{\mathcal{G}_{s(\gamma)}}^*\ \right]_{i,j=1}^N$$

is a completely positive map. If $\mathcal{G}$ is separable and $\{e_1, \dots, e_\kappa\}$ is an orthonormal basis for $\mathcal{G}$, then complete positivity of φ_* is equivalent to positivity of the operator matrix $\mathbb{P}_{\mathrm{QLTRD}} \in \mathcal{L}(\mathcal{G})_{\kappa N \times \kappa N}$ for which the entry corresponding to the pairs

$(i,i'),(j,j') \in \{1,\dots,N\}\times\{1,\dots,\kappa\}$ *is given by*

$$\begin{aligned}\left[\ \mathbb{P}_{\text{QLTRD}}\ \right]_{(i,i'),(j,j')} \\ = \sum_{\gamma\in\Gamma} i_{\mathcal{G}_{s(\gamma)}}((Z^{(i)})^{\gamma})^* i^*_{\mathcal{G}_{r(\gamma)}}(X_i^* e_i e_{j'}^* X_j - Y_i^* e_i e_{j'}^* Y_j) i_{\mathcal{G}_{r(\gamma)}}(Z^{(j)})^{\gamma} i^*_{\mathcal{G}_{s(\gamma)}}.\end{aligned} \tag{4.44}$$

The definition of $\mathbb{P}_{\text{QLTRD}}$ in Theorem 4.16 is the same as that in Theorem 3.3. Thus we obtain the second statement of Theorem 3.3.

Operator-argument functional calculus Nevanlinna-Pick interpolation. Note that in the case of the second Muhly-Solel point-evaluation of Subsection 4.7, specified for the setting considered here, the points are pairs $(T,A)\in\mathbb{D}(C_G(Q_1)^*)\times C_G(Q_0)$, i.e., T corresponds to a tuple $(T_\alpha\in\mathcal{L}(\mathcal{V}_{r(\alpha)},\mathcal{V}_{s(\alpha)})\colon \alpha\in Q_1)$ so that $F=(T_\alpha^*\colon \alpha\in Q_1)$ is in $C_G(Q_1)$ and the operator matrix $\mathbf{F}$ corresponding to F is a strict contraction. Thus T is an element of the set $\mathbb{D}_{\widetilde{G},\mathcal{V}}$ defined in Subsection 3.3. The element $T^*_{(n)} = T^*\otimes\cdots\otimes T^* = F\otimes\cdots\otimes F \in C_G(Q_n)$ is then given by the tuple $(T_\gamma^*\colon \gamma\in Q_n)$ with $T_\gamma = T^\gamma$, following the notation (3.10), and the operator $T^{(0)}_{T^*_{(n)}}$ (as in (4.9)) corresponds to multiplication with the operator matrix $\mathbf{T}^*_{(\mathbf{n})}$ associated with $T^*_{(n)}$.

It is then not difficult to see that the evaluation of an element $R=(R_\gamma\colon\gamma\in\Gamma)$ of the Toeplitz algebra $\mathcal{F}^\infty(E)=\mathfrak{L}_\Gamma(\mathcal{V},\mathcal{V})$ in a pair $(T,A)\in C_G(Q_0)\times\mathbb{D}_{G,\mathbf{OA}}$ is given by the left-tangential operator-argument functional calculus:

$$\hat{R}(T,A) = \sum_{\gamma\in\Gamma} i_{s(\gamma)} T^{\gamma^\top} A_{r(\gamma)} R_\gamma i^*_{s(\gamma)},$$

where $A=\operatorname{diag}_{v\in Q_0}(A_v)$ and $T=(T_\alpha\colon\alpha\in Q_1)$. The corresponding Nevanlinna-Pick problem of Subsection 4.7 thus turns out to be the **QLTOA-NP** problem considered in Subsection 3.3, and one easily sees that the Pick matrix criterion of Theorem 4.11 is exactly the Pick matrix criterion of Part 3 of Theorem 3.3.

4.10. Still more examples

There are still more seemingly different examples of generalized Nevanlinna-Pick interpolation covered by the Muhly-Solel correspondence-representation formalism. We mention in particular the semicrossed product algebras of Peters [78] (see Example 2.6 in [67]). A particular instance of this setup yields as the Toeplitz algebra $\mathcal{F}^\infty(E)$ the algebra of operators on $\ell^2(\mathbb{Z})$ having lower-triangular matrix representation with points equal to strictly contractive bilateral weighted shift operators on $\ell^2(\mathbb{Z})$; this algebra can also be seen as the Toeplitz algebra associated with the infinite quiver:

$$\begin{aligned} &Q_0=\{v_k\colon k\in\mathbb{Z}\},\quad Q_1=\{\alpha_k\colon k\in\mathbb{Z}\},\\ &s(\alpha_k)=v_k,\quad r(\alpha_k)=v_{k+1}.\end{aligned}$$

The point-evaluation $\widehat{R}(\eta)$ in this context has a neat interpretation in terms of the realization of the lower triangular operator R as the input-output operator for a conservative time-varying linear system

$$\Sigma\colon \begin{cases} x(n+1) & = & A(n)x(n) + B(n)u(n) \\ y(n) & = & C(n)x(n) + D(n)u(n) \end{cases}$$

(see [16, Section 6.3]). In order to recover the time-varying interpolation theory related to time-varying H^∞-control and model reduction carried out in the 1990s [8, 23, 41, 98, 42], one needs to work with the second Muhly-Solel point-evaluation specified for this setting. We leave details to another occasion.

5. More general Schur classes

For all the classes of Schur functions discussed to this point with the exception of the (commutative and noncommutative) polydisk examples in Subsection 3.4, the Schur class can be isometrically identified with the space of contractive multipliers between two reproducing kernel Hilbert spaces. Although there are some issues with different kinds of point-evaluations, one can say that the most general of these is the Muhly-Solel correspondence-representation setup in Section 4, and that all these are tied down to the setting of a *complete Pick kernel* (see [6]). An alternative noncommutative version of a complete Pick kernel with a rather complete theory of Nevanlinna-Pick interpolation has been worked out by Popescu in [90]. There are a number of generalized Hardy algebras which go beyond these limitations. We mention a few:

Powers of complete Pick kernels: As mentioned above, there is a good theory of transfer-function realization and Nevanlinna-Pick interpolation for the generalized Schur class defined to be the set of contractive multipliers on a reproducing Hilbert space $\mathcal{H}(k)$ with complete Pick kernel (see [5, 29, 6]). More generally, one can consider the generalized Schur-Agler class associated with the set of powers $k(z,w)^q$, $m = 1, \dots, m$ for some integer m, of some complete Pick kernel k (see [91] for the commutative setting and [89] for the noncommutative setting). While there is a good operator-model and dilation theory as well as a von Neumann inequality, there does not appear to be any approach to transfer-function realization or Nevanlinna-Pick interpolation for this generalized Schur-Agler class.

Higher-rank graph algebras: These include direct products of the graph algebras considered in Subsection 3.3 above and much more – see [62]. Whether these more general algebras are of interest for robust control theory, as those coming from SNMLs (see [27]), remains to be seen. For this general setting it remains to work out the nature of possible point-evaluations, the Schur class and a Schur-Agler type interpolation theory.

Hardy algebras associated with product systems over more general semigroups: We mention that the Fock space $\mathcal{F}^2(E)$ of Section 4 is a product decomposition over the semigroup $\mathbb{Z}_+$. Similar constructions but over more general semigroups, such as $\mathbb{Z}_+^n$, pick up the higher-rank graphs of [62] as examples. Here also it remains to work out the point-evaluations, Schur class and interpolation theory. These results should include the commutative and noncommutative interpolation theory for the polydisk discussed in Subsection 3.4.

Schur classes based on a family of test functions: In this approach we assume that we are given a set X and a family Ψ of functions on X. We then say that a positive kernel $k\colon X \times X \to \mathbb{C}$ is *admissible* whenever $k_\psi(x,x') := (1 - \psi(x)\overline{\psi(x')})k(x,x')$ is also a positive kernel; we denote the class of all admissible kernels by $\mathcal{K}_\Psi$. A function φ on X is then said to be in the Schur-Agler class $\mathcal{SA}_\Psi$ associated with Ψ if $k_\varphi(x,x') := (1 - \varphi(x)\overline{\varphi(x')})k(x,x')$ is a positive kernel whenever $k \in \mathcal{K}_\Psi$. If one takes $X = \mathbb{D}^d$ and $\Psi = \{\psi_k(\lambda) := \lambda_k \colon k = 1, \dots, d\}$, then the associated Schur-Agler class $\mathcal{SA}_\Psi$ is just the Schur-Agler class $\mathcal{SA}_d$ defined in Subsection 3.4. In addition to the Schur-Agler class on the polydisk as discussed in Subsection 3.4, it has been known since the paper of Abrahamse [1] that the Schur class over a finitely-connected planar domain fits into this framework, but with an infinite family of test functions. It turns out that many of the original ideas of Agler giving rise to transfer-function realization and Nevanlinna-Pick interpolation theorems via Agler decompositions go through for this general setting (see [39, 46, 64]); in the case of finitely-connected planar domains, one even gets a continuous analogue of the Agler decomposition (3.20) (see [45]). The paper [44] handles a more general scenario where the underlying set is a semigroupoid (satisfying some additional hypotheses) and pointwise multiplication of functions is replaced by semigroupoid convolution. The setup gives a unified formalism (an alternative to the **LTOA/RTOA-NP** or **RTRD-NP** setup discussed in Section 2) for the simultaneous encoding of interpolation problems of Nevanlinna-Pick and of Carathéodory-Fejér type inspired by the ideas of Jury [56]. For this class to include examples of interest (e.g., the Hardy algebras associated with the higher-rank graph algebras mentioned above, the (unit balls of) Toeplitz algebras $\mathcal{F}^\infty(E)$ appearing in Section 4, Schur classes associated with a complete Pick kernel [5, 29, 90], as well as the more general commutative Schur-Agler classes in [19, 20] and noncommutative Schur-Agler classes in [26]), one must identify the appropriate collection Ψ of test functions to get started. To handle these examples, the theory from [44, 46, 64] must be extended to handle matrix- or operator-valued test functions. Even after this is done, it appears that something more must be incorporated in the test-function approach in order to handle the Muhly-Solel tensor-type point-evaluation. Work has begun on finding a single formalism containing all these examples as special cases (see [15]).

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