Algebraicity of the Appell-Lauricella and Horn hypergeometric functions

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Abstract

We extend Schwarz' list of irreducible algebraic Gauss functions to the four classes of Appell-Lauricella functions in several variables and the 14 complete Horn functions in two variables. This gives an example of a family of functions such that for any number of variables there are infinitely many algebraic functions, namely the Lauricella F_C functions.

Contents

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1 Algebraic hypergeometric functions

1.1 Introduction

The classical Gauss hypergeometric function is

$$F(a,b,c|z)=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n n!}z^n,$$

where a, b and c are complex parameters. Here $(x)_n$ denotes the Pochhammer symbol defined by $(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}$.

There are many generalizations to hypergeometric functions in several variables. The most well-known are the Lauricella functions, introduced by Lauricella in 1893. They are defined by

$$F_A(a,\mathbf{b},\mathbf{c}|\mathbf{z})=\sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}}\frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{z}^{\mathbf{m}},$$

$$F_B(\mathbf{a},\mathbf{b},c|\mathbf{z})=\sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}}\frac{(\mathbf{a})_{\mathbf{m}}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{z}^{\mathbf{m}},$$

$$F_C(a,b,\mathbf{c}|\mathbf{z})=\sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}}\frac{(a)_{|\mathbf{m}|}(b)_{|\mathbf{m}|}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{z}^{\mathbf{m}},$$

$$F_D(a,\mathbf{b},c|\mathbf{z})=\sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}}\frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{z}^{\mathbf{m}},$$

where for $\mathbf{m}\in\mathbb{Z}^n_{\geq 0}$, $\mathbf{x}\in\mathbb{C}^n$ and $\mathbf{z}\in\mathbb{C}^n$, $(\mathbf{x})_{\mathbf{m}}$ is given by $(x_1)_{m_1}\cdot\ldots\cdot(x_n)_{m_n}$ and $\mathbf{z}^{\mathbf{m}}$ is given by $z_1^{m_1}\cdot\ldots\cdot z_n^{m_n}$. For $n=2$, these functions are the Appell F_2, F_3, F_4 and F_1 functions, respectively.

Other generalizations are the 14 so-called complete Horn series. These include the Appell series. The 10 other series are

$$G_1(a,b_1,b_2|x,y)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}}{m!n!}x^my^n,$$

$$G_2(a_1,a_2,b_1,b_2|x,y)=\sum_{m,n\geq 0}\frac{(a_1)_m(a_2)_n(b_1)_{n-m}(b_2)_{m-n}}{m!n!}x^my^n,$$

$$G_3(a_1,a_2|x,y)=\sum_{m,n\geq 0}\frac{(a_1)_{2n-m}(a_2)_{2m-n}}{m!n!}x^my^n,$$

$$H_1(a,b,c,d|x,y)=\sum_{m,n\geq 0}\frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m m!n!}x^my^n,$$

$$H_2(a,b,c,d,e|x,y)=\sum_{m,n\geq 0}\frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m m!n!}x^my^n,$$

$$H_3(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}m!n!}x^my^n,$$

$$H_4(a,b,c,d|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n m!n!}x^my^n,$$

$$H_5(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m+n}(b)_{n-m}}{(c)_n m!n!}x^my^n,$$

$$H_6(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m-n}(b)_{n-m}(c)_n}{m!n!}x^my^n,$$

$$H_7(a,b,c,d|x,y) = \sum_{m,n\geq 0} \frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m m!n!} x^m y^n.$$

In 1873, Schwarz found a list of all irreducible algebraic Gauss functions (see [Sch73]). By irreducible we mean that the monodromy group acts irreducible. This list has been extended to general one-variable hypergeometric functions ${}_{p+1}F_p$ by Beukers and Heckman (see [BH89]), to the Appell-Lauricella functions F_1 and F_D by Beazley Cohen, Wolfart and Sasaki ([BCW92]), the Appell functions F_2 and F_4 by Kato ([Kat00], [Kat97]) and the Horn G_3 function by Schipper ([Sch09]). In [BCW92], Beazley Cohen and Wolfart also gives some results on reducible algebraic F_2, F_3 and F_4 functions.

The goal of this paper is to determine the rational parameter values for which the Appell-Lauricella and Horn series are non-resonant algebraic functions over $\mathbb{C}(\mathbf{z})$ or $\mathbb{C}(x,y)$. Non-resonance is a condition that is almost equivalent to irreducibility, as will be made precise in the next section.

1.2 Some general theory

In this section, we will recall some results about GKZ-hypergeometric functions. We start with the definition of a GKZ-hypergeometric function:

Definition 1.2.1. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ be a finite subset of $\mathbb{Z}^r$ such that the $\mathbb{Z}$-span of $\mathbf{a}_1, \ldots, \mathbf{a}_N$ equals $\mathbb{Z}^r$ and there exists a linear form h on $\mathbb{R}^r$ such that $h(\mathbf{a}_i) = 1$ for all i. We assume that $\mathcal{A}$ is saturated, i.e. the $\mathbb{R}_{\geq 0}$-span of $\mathcal{A}$ intersected with $\mathbb{Z}^r$ equals the $\mathbb{Z}_{\geq 0}$-span of $\mathcal{A}$. Let $\mathbb{L} \subseteq \mathbb{Z}^N$ be the lattice of relations in $\mathcal{A}$, i.e. $\mathbb{L} = \{(l_1, \ldots, l_N) \in \mathbb{Z}^N \mid l_1\mathbf{a}_1 + \ldots + l_N\mathbf{a}_N = 0\}$. Furthermore, let $\boldsymbol{\alpha} \in \mathbb{Q}^r$ (in general, $\boldsymbol{\alpha}$ can be an element of $\mathbb{C}^r$, but we will only consider $\boldsymbol{\alpha} \in \mathbb{Q}^r$). The GKZ-system associated with $\mathcal{A}$ and $\boldsymbol{\alpha}$, denoted by $H_{\mathcal{A}}(\boldsymbol{\alpha})$, consists of the following equations:

- for every $(l_1, \ldots, l_N) \in \mathbb{L}$ the equation

$$\prod_{l_i<0} \left(\frac{\partial}{\partial z_i}\right)^{-l_i} \Phi = \prod_{l_i>0} \left(\frac{\partial}{\partial z_i}\right)^{l_i} \Phi;$$

- the system of r differential equations

$$\mathbf{a}_1 z_1 \frac{\partial \Phi}{\partial z_1} + \ldots + \mathbf{a}_N z_N \frac{\partial \Phi}{\partial z_N} = \boldsymbol{\alpha}\Phi.$$

For every $\boldsymbol{\gamma}$ such that $\gamma_1\mathbf{a}_1 + \ldots + \gamma_N\mathbf{a}_N = \boldsymbol{\alpha}$, the system of equations has a formal solution

$$\Phi(z_1, \ldots, z_N) = \sum_{(l_1,\ldots,l_N)\in\mathbb{L}} \frac{z_1^{l_1+\gamma_1} \cdot \ldots \cdot z_n^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1) \cdot \ldots \cdot \Gamma(l_N+\gamma_N+1)}. \tag{1}$$

If I is a subset of $\{1, \ldots, N\}$ such that $\{\mathbf{a}_i \mid i \in I\}$ is a maximal independent set and $\gamma_j \in \mathbb{Z}$ for all $j \notin I$, then the Laurent series has a positive radius of convergence (see [Sti07], section 3). We will always choose $\gamma_j = 0$ for all $j \notin I$.

Let $C(\mathcal{A})$ be the real positive cone generated by $\mathcal{A}$, i.e.

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^r \mid \exists \lambda_1, \ldots, \lambda_N \geq 0 : \mathbf{x} = \lambda_1\mathbf{a}_1 + \ldots + \lambda_N\mathbf{a}_N\}.$$

In this notation, $\mathcal{A}$ is saturated if $C(\mathcal{A}) \cap \mathbb{Z}^r = \mathbb{Z}_{\geq 0}\mathcal{A}$. Note that $\mathbb{Z}_{\geq 0}\mathcal{A} \subseteq C(\mathcal{A}) \cap \mathbb{Z}^r$ for all $\mathcal{A} \subseteq \mathbb{Z}^r$.

Definition 1.2.2. $H_{\mathcal{A}}(\boldsymbol{\alpha})$ is called resonant if $\boldsymbol{\alpha} + \mathbb{Z}^r$ contains a point in a face of $C(\mathcal{A})$. Otherwise $H_{\mathcal{A}}(\boldsymbol{\alpha})$ is called non-resonant.

Theorem 1.2.3. *[GKZ90, Theorem 2.11] If $H_{\mathcal{A}}(\boldsymbol{\alpha})$ is non-resonant, then it is irreducible.*

The converse is almost true:

Remark 1.2.4. Let $H_{\mathcal{A}}(\boldsymbol{\alpha})$ be resonant. Suppose that for every $i \in \{1, \ldots, N\}$ there exists $(l_1, \ldots, l_N) \in \mathbb{L}$ such that $l_i \neq 0$. Then $H_{\mathcal{A}}(\boldsymbol{\alpha})$ is reducible.

We could not find a proof of this Remark in the literature yet. However, it will be the subject of an upcoming paper by Beukers. The condition on the lattice is satisfied for all Appell-Lauricella and Horn function, as will be immediately clear from the exposition in the next sections. Therefore, one can think of irreducibility as being equivalent to non-resonance. However, we will use non-resonance in the formulation of our results. For the Gauss function and some of the Appell-Lauricella functions, irreducibility conditions can be found in the literature. For these functions, we will prove the equivalence with non-resonance, and state our results in terms of irreducibility.

Definition 1.2.5. Let $K_{\mathcal{A}}(\boldsymbol{\alpha}) = (\boldsymbol{\alpha} + \mathbb{Z}^r) \cap C(\mathcal{A})$. $\mathbf{p} \in K_{\mathcal{A}}(\boldsymbol{\alpha})$ is called an apexpoint if for every $\mathbf{q} \in K_{\mathcal{A}}(\boldsymbol{\alpha})$ such that $\mathbf{p} \neq \mathbf{q}$, it holds that $\mathbf{p} - \mathbf{q} \notin C(\mathcal{A})$. The number of apexpoints is called the signature of $\mathcal{A}$ and $\boldsymbol{\alpha}$ and is denoted by $\sigma_{\mathcal{A}}(\boldsymbol{\alpha})$.

Note that $\sigma_{\mathcal{A}}(\boldsymbol{\alpha})$ only depends on the fractional part $\{\boldsymbol{\alpha}\}$ of $\boldsymbol{\alpha}$ (where $\{\boldsymbol{\alpha}\}_i = \{\alpha_i\} = \alpha_i - \lfloor \alpha_i \rfloor$).

Lemma 1.2.6. *Suppose that $\mathbf{p} \in K_{\mathcal{A}}(\boldsymbol{\alpha})$. Then $\mathbf{p}$ is an apexpoint if and only if $\mathbf{p} - \mathbf{a}_i \notin C(\mathcal{A})$ for all $\mathbf{a}_i \in \mathcal{A}$.*

Proof. If there exists $\mathbf{a}_i \in \mathcal{A}$ such that $\mathbf{p} - \mathbf{a}_i \in C(\mathcal{A})$, then we can take $\mathbf{q} = \mathbf{p} - \mathbf{a}_i \in K_{\mathcal{A}}(\boldsymbol{\alpha})$. Then $\mathbf{p} \neq \mathbf{q}$ and $\mathbf{p} - \mathbf{q} = \mathbf{a}_i \in C(\mathcal{A})$, so $\mathbf{p}$ is not an apexpoint.

Suppose that $\mathbf{p} \in K_{\mathcal{A}}(\boldsymbol{\alpha})$ is not an apexpoint. Then there exists $\mathbf{q} \in K_{\mathcal{A}}(\boldsymbol{\alpha})$ such that $\mathbf{p} \neq \mathbf{q}$ and $\mathbf{p} - \mathbf{q} \in C(\mathcal{A})$. Since $\mathbf{q} \in C(\mathcal{A})$, there exists $\lambda_1, \ldots, \lambda_N \geq 0$ such that $\mathbf{q} = \lambda_1 \mathbf{a}_1 + \ldots + \lambda_N \mathbf{a}_N$. Define $\mathbf{v} = \mathbf{p} - \mathbf{q}$. Then $\mathbf{v} \in C(\mathcal{A}) \cap \mathbb{Z}^r$. $\mathcal{A}$ is saturated, so there exist $\mu_1, \ldots, \mu_N \in \mathbb{Z}_{\geq 0}$ such that $\mathbf{v} = \mu_1 \mathbf{a}_1 + \ldots + \mu_N \mathbf{a}_N$. It follows that $\mathbf{p} = \mathbf{q} + \mathbf{v} = \sum_{i=1}^{N} (\lambda_i + \mu_i) \mathbf{a}_i$. Since $\mathbf{v} \neq 0$, there is some i such that $\mu_i \geq 1$. Now consider $\mathbf{p} - \mathbf{a}_i$. This is clearly an element of $\boldsymbol{\alpha} + \mathbb{Z}^r$, and since $\lambda_i + \mu_i - 1 \geq 0$, it also lies in $C(\mathcal{A})$. Hence $\mathbf{p} - \mathbf{a}_i \in K_{\mathcal{A}}(\boldsymbol{\alpha})$. ☐

Proposition 1.2.7. *[Beu10, Proposition 1.9] Let $Q(\mathcal{A})$ be the convex hull of $\mathcal{A}$. Then $\sigma_{\mathcal{A}}(\boldsymbol{\alpha})$ is less than or equal to the simplex volume of $Q(\mathcal{A})$.*

The simplex volume is a normalization of the Euclidean volume, such that the simplex spanned by the standard basis has volume 1. Since the determinant function is the only linear function that gives 0 if two of the arguments are equal and maps the standard basis to 1, the simplex volume of the simplex spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ equals $|\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)|$.

In the next section, we will prove a slightly stronger version of this Proposition, under the assumption that $Q(\mathcal{A})$ has a unimodular triangulation.

Theorem 1.2.8. *[Beu10, Theorem 1.10] Suppose that the GKZ-hypergeometric system $H_{\mathcal{A}}(\boldsymbol{\alpha})$ is non-resonant. Let D be the smallest common denominator of the coordinates of $\boldsymbol{\alpha} \in \mathbb{Q}^r$. Then the solution set of $H_{\mathcal{A}}(\boldsymbol{\alpha})$ consists of algebraic solutions over $\mathbb{C}(\mathbf{z})$ if and only if $\sigma_{\mathcal{A}}(k\boldsymbol{\alpha})$ is equal to the simplex volume of $Q(\mathcal{A})$ for all integers k with $1 \leq k < D$ and $\gcd(k, D) = 1$.*

Corollary 1.2.9. *Let $\boldsymbol{\alpha} \in \mathbb{Q}^r$. If $H_{\mathcal{A}}(\boldsymbol{\alpha})$ is non-resonant, then algebraicity of the solutions of the GKZ system only depends on $\mathcal{A}$ and $\{\boldsymbol{\alpha}\}$. Furthermore, either the solution set of $H_{\mathcal{A}}(k\boldsymbol{\alpha})$ consists of algebraic functions for all k coprime to the smallest common denominator of the coordinates of $\boldsymbol{\alpha}$, or the solutions are transcendental for all k.*

By this Corollary, it suffices to consider $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}\}$. Troughout this paper, we will assume that $0 \leq \alpha_i < 1$.

The following Remark makes it possible to reduce the number of variables of a function:

Remark 1.2.10. If $f(z_1, \ldots, z_n)$ is an algebraic function over $\mathbb{C}(z_1, \ldots, z_n)$, then the function $f(z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_n)$ is algebraic over $\mathbb{C}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ for all $i \in \{1, \ldots, n\}$.

To prove that certain functions are not algebraic, we will have to find k such that $\gcd(k, D) = 1$ and $\sigma_{\mathcal{A}}(k\boldsymbol{\alpha})$ is not maximal. By the following Proposition, it suffices to find k coprime with the denominator of several, but not all, coefficients. Sometimes we can show that $\sigma_{\mathcal{A}}(k\boldsymbol{\alpha})$ is not maximal if some parameter is close to $\frac{1}{2}$. The second Proposition handles this case.

Proposition 1.2.11. *Let k, D and $\tilde{D}$ be positive integers such that $D|\tilde{D}$ and $\gcd(k, D) = 1$. Then there exists an integer l such that $l \equiv k \pmod{D}$ and $\gcd(l, \tilde{D}) = 1$.*

Proof. Let c be the minimal integer dividing $\tilde{D}$ such that $\gcd(D, \frac{\tilde{D}}{c}) = 1$. More explicitly, if $D = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ and $\tilde{D} = p_1^{l_1} \cdot \ldots \cdot p_r^{l_r} q_1^{m_1} \cdot \ldots \cdot q_s^{m_s}$, then $c = p_1^{l_1} \cdot \ldots \cdot p_r^{l_r}$. Then there exists an integer E such that $DE \equiv 1 \pmod{\frac{\tilde{D}}{c}}$. Define $l = k - (k-1)DE$. Then it is clear that $l \equiv k \pmod{D}$ and $l \equiv 1 \pmod{\frac{\tilde{D}}{c}}$, so $\gcd(l, \frac{\tilde{D}}{c}) = 1$. This implies that l is coprime with $D\frac{\tilde{D}}{c}$. Now it follows from the definition of c that l is coprime with $\tilde{D}$. □

Proposition 1.2.12. *Let $r = \frac{p}{q}$ with $\gcd(p, q) = 1$ and $q \geq 3$. Define $d = 1$ if q is odd, $d = 2$ if 4 divides q and $d = 4$ if $q \equiv 2 \pmod{4}$. Let $t \in (0, \frac{1}{2})$. If $q \geq \frac{d}{1-2t}$, then there exists $k \in \mathbb{Z}$ with $\gcd(k, q) = 1$ such that $\{kr\} \in [t, \frac{1}{2})$.*

Proof. Choose $p' \in \mathbb{Z}$ such that $pp' \equiv 1 \pmod{q}$ and take $k = \frac{q-d}{2}p'$. Then k is an integer with $\gcd(k, q) = 1$. Since $q \geq \frac{d}{1-2t}$, we have $\frac{q-d}{2q} \geq t$. Hence $\{kr\} = \{\frac{q-d}{2} \cdot \frac{pp'}{q}\} = \{\frac{q-d}{2q}\} \in [t, \frac{1}{2})$. □

Remark 1.2.13. A GKZ-function is determined by the set $\mathcal{A}$ and the parameter vector $\boldsymbol{\alpha}$. We will take the opposite approach: we will start with a Laurent series of the form (1) and read off $\mathbb{L}$ and $\boldsymbol{\gamma}$. Then we choose $\mathcal{A}$ such that $\mathbb{L}$ is the lattice of relations in $\mathcal{A}$, and compute $\boldsymbol{\alpha} = \sum_{i=1}^{N} \gamma_i \mathbf{a}_i$. Of course, we have to choose $\mathcal{A}$ such that it spans $\mathbb{Z}^r$, lies in a hyperplane $h(\mathbf{x}) = 1$ and is saturated.

The order of the vectors in $\mathcal{A}$ is important, since the relations change if we permute $\mathcal{A}$. However, permuting $\mathcal{A}$ doesn't change $C(\mathcal{A})$, $Q(\mathcal{A})$ and $K_{\mathcal{A}}(\boldsymbol{\alpha})$. Therefore, algebraicity of the functions corresponding to $\mathcal{A}$ and $\boldsymbol{\alpha}$ doesn't depend on the order of the elements of $\mathcal{A}$. Given two lattices, it is sometimes possible to choose the same $\mathcal{A}$, but in a different order. Then it suffices to determine the algebraic functions for only one of these lattices. We will use this to transfer the results from one function to another.

1.3 Triangulations

Definition 1.3.1. Let $\mathcal{A} \subseteq \mathbb{Z}^r$ such that the $\mathbb{Z}$-span of $\mathcal{A}$ equals $\mathbb{Z}^r$ and there exists a linear form h on $\mathbb{R}^r$ such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$. For a subset $V \subseteq \mathcal{A}$, let $Q(V)$ be the convex hull of the elements of V. A triangulation of $Q(\mathcal{A})$ is a finite set $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ such that each V_i is a subset of $\mathcal{A}$ containing r elements which are linearly independent (so V_i consists of the vertices of an $(r-1)$-dimensional simplex), $Q(V_i) \cap Q(V_j) = Q(V_i \cap V_j)$ for all i and j and $Q(\mathcal{A}) = \cup_{i=1}^{l} Q(V_i)$. If all $Q(V_i)$ have simplex volume 1 (i.e. the determinant is the vectors in V_i is ± 1), then the triangulation is called unimodular.

Remark 1.3.2. For all $V \subseteq \mathcal{A}$, we have the following connection between $Q(V)$ and $C(V)$: $Q(V) = C(V) \cap \{\mathbf{x} \in \mathbb{R}^r \mid h(\mathbf{x}) = 1\}$ and $C(V) = \{\lambda \mathbf{x} \mid \mathbf{x} \in Q(V), \lambda \geq 0\}$.

To prove that a set of subsets of $Q(\mathcal{A})$ is a triangulation, we will use the following Lemma:

Lemma 1.3.3. *Suppose that $V_1, \ldots, V_l$ are subsets of $\mathcal{A}$ consisting of r linearly independent vectors with determinant ± 1, such that $\mathcal{A} = \cup_{i=1}^{l} V_i$, $C(V_i) \cap C(V_j) \subseteq C(V_i \cap V_j)$ for all i and j and $\cup_{i=1}^{l} C(V_i)$ is convex. Then $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ is a unimodular triangulation of $Q(\mathcal{A})$.*

Proof. Since the determinant of the vectors in V_i is ± 1, the vectors are linearly independent and $Q(V_i)$ has volume 1. It is clear that $C(V_i \cap V_j) \subseteq C(V_i) \cap C(V_j)$, so $C(V_i) \cap C(V_j) = C(V_i \cap V_j)$. For all i and j, it follows from Remark 1.3.2 that

$$Q(V_i) \cap Q(V_j) = C(V_i) \cap C(V_j) \cap \{\mathbf{x} \in \mathbb{R}^r \mid h(\mathbf{x}) = 1\} = C(V_i \cap V_j) \cap \{\mathbf{x} \in \mathbb{R}^r \mid h(\mathbf{x}) = 1\} = Q(V_i \cap V_j).$$

It is also clear that $\cup_{i=1}^l Q(V_i) \subseteq Q(\mathcal{A})$. Note that $\cup_{i=1}^l Q(V_i) = \cup_{i=1}^l C(V_i) \cap \{\mathbf{x} \in \mathbb{R}^r \mid h(\mathbf{x}) = 1\}$ is a convex set. It contains $\mathcal{A} = \cup_{i=1}^l V_i$, so it also contains the convex hull of $\mathcal{A}$. This implies that $Q(\mathcal{A}) \subseteq \cup_{i=1}^l Q(V_i)$. □

Lemma 1.3.4. *Let $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ be a unimodular triangulation of $Q(\mathcal{A})$. Then*
(i) $vol(Q(\mathcal{A})) = l$.
(ii) $C(\mathcal{A}) = \cup_{i=1}^l C(V_i)$.
(iii) $\mathcal{A}$ is saturated.

Proof. *(i)* is clear. *(ii)* follows from Remark 1.3.2.

For *(iii)*, let $\mathbf{x} \in \mathbb{R}_{\geq 0}\mathcal{A} \cap \mathbb{Z}^r$. Then $\mathbf{x} \in C(\mathcal{A})$, so there exists $V_i = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathcal{A}$ such that $\mathbf{x} \in C(V_i)$. Then there exist $\lambda_1, \ldots, \lambda_r \geq 0$ such that $\mathbf{x} = \sum_{i=1}^r \lambda_i \mathbf{v}_i$. Hence $\mathbf{x} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)\boldsymbol{\lambda}$, so $\boldsymbol{\lambda} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)^{-1}\mathbf{x}$ (here $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ denotes the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_r$). Since the determinant of $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is ± 1, $\boldsymbol{\lambda}$ has integral coordinates. They are non-negative since $\lambda_i \geq 0$. It follows that $\mathbf{x} = (\mathbf{v}_1, \ldots, \mathbf{v}_r) \cdot \boldsymbol{\lambda}$ is an element of $\mathbb{Z}_{\geq 0}\mathcal{A}$. □

The next Remark helps in finding all apexpoints, using a triangulation of $Q(\mathcal{A})$. Proposition 1.3.6 and Corollary 1.3.7 are, in the case of a set $Q(\mathcal{A})$ with a unimodular triangulation, slightly stronger versions of Proposition 1.2.7. They will be useful in finding an interlacing condition.

Remark 1.3.5. Let $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ be a triangulation of $Q(\mathcal{A})$. If $\mathbf{p}$ is an apexpoint, then there exists i such that $\mathbf{p} \in C(V_i)$. Let $V_i = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. Since the $\mathbf{v}_j$ are linearly independent, there exist unique $\lambda_j \geq 0$ such that $\mathbf{p} = \sum_{j=1}^r \lambda_j \mathbf{v}_j$. By Lemma 1.2.6, $\mathbf{p} - \mathbf{v}_j \notin C(\mathcal{A})$. This implies that $0 \leq \lambda_j < 1$ for all j.

Proposition 1.3.6. *Let $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ be a unimodular triangulation of $Q(\mathcal{A})$. Then every $C(V_i)$ contains at most one apexpoint.*

Proof. Suppose that $\mathbf{p}$ and $\mathbf{q}$ are apexpoints in $C(V_i)$. As in Remark 1.3.5, write $\mathbf{p} = \sum_{j=1}^r \lambda_j \mathbf{v}_j$ and $\mathbf{q} = \sum_{j=1}^r \mu_j \mathbf{v}_j$ with $0 \leq \lambda_j, \mu_j < 1$. Then $(\mathbf{v}_1, \ldots, \mathbf{v}_r)(\boldsymbol{\lambda} - \boldsymbol{\mu}) = \mathbf{p} - \mathbf{q}$ has integral coordinates, so $\boldsymbol{\lambda} - \boldsymbol{\mu} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)^{-1}(\mathbf{p} - \mathbf{q})$ also has integral coordinates. This implies that $\boldsymbol{\lambda} = \boldsymbol{\mu}$, so $\mathbf{p} = \mathbf{q}$. □

Corollary 1.3.7. *Let $\mathcal{T} = \{Q(V_1), \ldots, Q(V_l)\}$ be a unimodular triangulation of $Q(\mathcal{A})$. Then the number of apexpoints is equal to the simplex volume of $Q(\mathcal{A})$ if and only if every $C(V_i)$ contains an apexpoint, and all these apexpoints are distinct.*

1.4 The Gauss hypergeometric function

The Appell-Lauricella functions and some of the Horn functions are generalizations of the Gauss hypergeometric function $F(a, b, c|z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n$. This function can be reproduced as a GKZ-function as follows: take $\mathbb{L} = \mathbb{Z}(-1, -1, 1, 1)$, $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\} \subseteq \mathbb{Z}^3$ and $\boldsymbol{\gamma} = (-a, -b, c-1, 0)$. The GKZ-system corresponding to $\mathcal{A}$ and $\boldsymbol{\alpha} = (-a, -b, c-1)$ has a formal Laurent series solution

$$G(a, b, c|z_1, z_2, z_3, z_4) = z_1^{-a} z_2^{-b} z_3^{c-1} \sum_{n \in \mathbb{Z}} \frac{(z_1^{-1} z_2^{-1} z_3 z_4)^n}{\Gamma(1-n-a)\Gamma(1-n-b)\Gamma(c+n)\Gamma(n+1)}.$$

The Laurent series converges because $\gamma_4 \in \mathbb{Z}$. Since $\Gamma(n+1)$ has a pole for $n < 0$, we only sum over $n \geq 0$. Hence G is algebraic if and only if $\tilde{G}(a, b, c|z_1, z_2, z_3, z_4) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} (z_1^{-1} z_2^{-1} z_3 z_4)^n$

is algebraic. Note that $\tilde{F}(z) = \tilde{G}(1, 1, 1, z)$, so if $\tilde{G}(z_1, z_2, z_3, z_4)$ is algebraic, then $\tilde{F}(z)$ is also algebraic by Remark 1.2.10. On the other hand, $\tilde{G}(z_1, z_2, z_3, z_4) = \tilde{F}(z_1^{-1}z_2^{-1}z_3z_4)$, so if $\tilde{F}(z)$ is algebraic, then $\tilde{G}(z_1, z_2, z_3, z_4)$ is also algebraic. It follows that the Gauss function $F(a, b, c|z)$ is algebraic if and only if the GKZ-function $G(a, b, c|z_1, z_2, z_3, z_4)$ is algebraic.

To find all algebraic Appell-Lauricella and Horn functions, we will use a reduction to algebraic Gauss functions. In 1873, Schwarz published a list of all irreducible algebraic Gauss functions ([Sch73]). To each function $F(a, b, c, |z)$, there is an associated triple $(\lambda, \mu, \nu) = (1-c, c-a-b, b-a)$. Up to permutations of $\{\lambda, \mu, \nu\}$, sign changes of each of λ, μ and ν and addition of $(l, m, n) \in \mathbb{Z}^3$ with $l + m + n$ even to (λ, μ, ν), Table 1 gives all irreducible algebraic Gauss functions.

Table 1: The tuples (λ, μ, ν) such that $F(a, b, c|z)$ is irreducible and algebraic

$(\frac{1}{2}, \frac{1}{2}, s)$	with $s \in \mathbb{Q} \setminus \mathbb{Z}$					
$(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$	$(\frac{2}{3}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$	$(\frac{2}{5}, \frac{1}{3}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5}, \frac{1}{5})$
$(\frac{1}{2}, \frac{2}{5}, \frac{1}{5})$	$(\frac{2}{5}, \frac{2}{5}, \frac{2}{5})$	$(\frac{3}{5}, \frac{1}{3}, \frac{1}{5})$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{5})$	$(\frac{4}{5}, \frac{1}{5}, \frac{1}{5})$	$(\frac{1}{2}, \frac{2}{5}, \frac{1}{3})$	$(\frac{3}{5}, \frac{2}{5}, \frac{1}{3})$

To compute all triples (a, b, c) such that $F(a, b, c|z)$ is irreducible and algebraic, notice that $(a, b, c) = (\frac{1-\lambda-\mu-\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}, 1 - \lambda)$. Since we only have to consider $(a, b, c) \pmod{\mathbb{Z}}$ and $(\lambda, \mu, \nu) + (l, m, n)$ gives the same $(a, b, c) \pmod{\mathbb{Z}}$ for all (l, m, n) such that $l + m + n$ is even, we can compute (a, b, c) for all 48 tuples obtained from (λ, μ, ν) by permutations and changes of signs, and choose (l, m, n) such that $0 \le a, b, c < 1$.

For the tuple $(\lambda, \mu, \nu) = (\frac{1}{2}, \frac{1}{2}, s)$, this turns out to give 3 possible forms for the tuple (a, b, c): either $\{b\} = \{-a\}$ and $c = \frac{1}{2}$, or $\{b\} = \{a + \frac{1}{2}\}$ and $\{c\} = \frac{1}{2}$ or $\{c\} = \{2a\}$. Writing $r = \{a\}$, this gives the triples $(r, 1 - r, \frac{1}{2}), (r, \{r + \frac{1}{2}\}, \frac{1}{2})$ and $(r, \{r + \frac{1}{2}\}, \{2r\})$. Since $s \in \mathbb{Q} \setminus \mathbb{Z}$, we have $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$. We will call all tuples of the form $(r, 1-r, \frac{1}{2}), (r, \{r+\frac{1}{2}\}, \frac{1}{2})$ or $(r, \{r+\frac{1}{2}\}, \{2r\})$ with $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ Gauss triples of type 1. This terminology is not standard, but will be used troughout this paper.

For the other 14 tuples (λ, μ, ν), we use the computer to compute (a, b, c) for all 48 tuples obtained from (λ, μ, ν). It turns out that this gives 408 triples (a, b, c). By Corollary 1.2.9, these tuples form orbits under conjugation. By conjugation we mean the action of $(\mathbb{Z}/D\mathbb{Z})^*$ by $k(\{a\}, \{b\}, \{c\}) = (\{ka\}, \{kb\}, \{kc\})$. Furthermore, the tuples come in pairs (a, b, c) and (b, a, c). In Table 2, the smallest element of each pair of orbits is given (where a fraction $\frac{p}{q}$ is considered to be smaller than another fraction $\frac{u}{v}$ if either $q < v$, or $q = v$ and $p \le u$. Tuples of fractions are ordered lexicographically.). We call these 408 tuples Gauss tuples of type 2. Notice that the denominators of a and c are at most 60 and 5, respectively, for all tuples of type 2.

We recall the usual criterion for irreduciblity and the interlacing condition for the Gauss function.

Lemma 1.4.1. *[BH89, Propositions 2.7 and 3.3] $F(a, b, c|z)$ is irreducible if and only if a, b, $c-a$ and $c - b$ are non-integral.*

As a special case of Corollary 2.1.3, we will prove that irreducibility and non-resonance are equivalent for the Gauss function. The next Theorem will be a special case of Theorem 2.1.7:

Theorem 1.4.2. *[BH89, Theorem 4.8] Suppose that $F(a, b, c|z)$ with $a, b, c \in \mathbb{Q}$ is irreducible. Then the function is algebraic if and only if for every k coprime with the denominators of a, b and c, the following interlacing condition is satisfied: either $\{ka\} \le \{kc\} < \{kb\}$ or $\{kb\} \le \{kc\} < \{ka\}$.*

Table 2: The tuples (a,b,c) of type 2 such that $F(a,b,c|z)$ is irreducible and algebraic

$(\frac{1}{2},\frac{1}{6},\frac{1}{3})$	$(\frac{1}{4},\frac{3}{4},\frac{1}{3})$	$(\frac{1}{4},\frac{7}{12},\frac{1}{2})$	$(\frac{1}{4},\frac{7}{12},\frac{1}{3})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{3})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{4})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{5})$
$(\frac{1}{6},\frac{5}{12},\frac{1}{3})$	$(\frac{1}{6},\frac{5}{12},\frac{1}{4})$	$(\frac{1}{6},\frac{11}{30},\frac{1}{3})$	$(\frac{1}{6},\frac{11}{30},\frac{1}{5})$	$(\frac{1}{10},\frac{3}{10},\frac{1}{5})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{5})$
$(\frac{1}{10},\frac{13}{30},\frac{1}{3})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{5})$	$(\frac{1}{12},\frac{5}{12},\frac{1}{4})$	$(\frac{1}{12},\frac{7}{12},\frac{1}{3})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{3})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{1}{5})$
$(\frac{1}{15},\frac{11}{15},\frac{3}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{1}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{2}{5})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{5})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{3})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{4})$
$(\frac{1}{24},\frac{17}{24},\frac{1}{2})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{4})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{2})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{3})$	$(\frac{1}{30},\frac{11}{30},\frac{1}{5})$	$(\frac{1}{30},\frac{19}{30},\frac{1}{3})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{3})$
$(\frac{1}{60},\frac{31}{60},\frac{1}{5})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{2})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{5})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{3})$		

2 The Appell-Lauricella functions

2.1 The Appell F_1 and Lauricella F_D functions

The Lauricella F_D function is given by

$$F_D(a,\mathbf{b},c|\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{z}^{\mathbf{m}}.$$

Up to a constant factor, this equals

$$\sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}} \frac{z_1^{m_1}\cdot\ldots\cdot z_n^{m_n}}{\Gamma(1-m_1-\ldots-m_n-a)\prod_{i=1}^n\Gamma(1-m_i-b_i)\Gamma(m_1+\ldots+m_n+c)\prod_{i=1}^n\Gamma(1+m_i)}.$$

Hence the lattice $\mathbb{L}$ is

$$\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\mathbf{e}_1+\mathbf{e}_{i+1}+\mathbf{e}_{n+2}+\mathbf{e}_{n+i+2}) \subseteq \mathbb{Z}^{2n+2}$$

and $\boldsymbol{\gamma} = (-a,-b_1,\ldots,-b_n,c-1,0,\ldots,0)\in\mathbb{R}^{2n+2}$. We can take

$$\mathcal{A} = \{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_{n+2},\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_{n+2},\mathbf{e}_1+\mathbf{e}_3-\mathbf{e}_{n+2},\ldots,\mathbf{e}_1+\mathbf{e}_{n+1}-\mathbf{e}_{n+2}\}\subseteq\mathbb{Z}^{n+2}$$

and $\boldsymbol{\alpha} = \sum_{i=1}^{2n+2}\gamma_i\mathbf{a}_i = (-a,-b_1,\ldots,-b_n,c-1)\in\mathbb{Q}^{n+2}$.

Lemma 2.1.1. *$\mathcal{A}$ is saturated.*

Proof. Suppose that $\mathbf{x}\in\mathbb{Z}^{n+2}$ lies in the $\mathbb{R}_{\geq 0}$-span of $\mathcal{A}$. Then there exist $\mu_1,\ldots,\mu_{2n+2}\in\mathbb{R}_{\geq 0}$ such that $\mathbf{x}=\sum_{i=1}^{2n+2}\mu_i\mathbf{a}_i$. Define $\lambda_{n+3}=\min(x_1,x_2)$, and for $n+4\leq i\leq 2n+2$, define recursively $\lambda_i=\min(x_1-\lambda_{n+3}-\ldots-\lambda_{i-1},x_{i-n-1})$. Furthermore, for $2\leq i\leq n+1$, let $\lambda_i=x_i-\lambda_{n+i+1}$. Finally, let $\lambda_1=x_1-\lambda_{n+3}-\ldots-\lambda_{2n+2}$ and $\lambda_{n+2}=x_{n+2}+\lambda_{n+3}+\ldots+\lambda_{2n+2}$. Then it is clear that $\lambda_i\in\mathbb{Z}$ for all i, and $\sum_{i=1}^{2n+2}\lambda_i\mathbf{a}_i=\mathbf{x}$. It remains to show that $\lambda_i\geq 0$ for all i. This is clear for all i except for $i=n+2$. If $\lambda_{n+i+1}=x_i$ for all $2\leq i\leq n+1$, then $\lambda_{n+2}=x_2+\ldots+x_{n+1}+x_{n+2}\geq 0$. If there exists $2\leq i\leq n+1$ such that $\lambda_{n+i+1}=x_1-\lambda_{n+3}-\ldots-\lambda_{n+i}$, then $\lambda_{n+j}=0$ for all $i+2\leq j\leq 2n+2$, so $\lambda_{n+2}=\lambda_{n+3}+\ldots+\lambda_{n+i}+x_1-\lambda_{n+3}-\ldots-\lambda_{n+i}+x_{n+2}=x_1+x_{n+2}\geq 0$. It follows that $\mathbf{x}$ lies in the $\mathbb{Z}_{\geq 0}$-span of $\mathcal{A}$. □

Lemma 2.1.2. *The positive real cone spanned by $\mathcal{A}$ is*

$$C(\mathcal{A}) = \{\mathbf{x}\in\mathbb{R}^{n+2}\mid x_1,\ldots,x_{n+1}\geq 0, x_1+x_{n+2}\geq 0, x_2+\ldots+x_{n+1}+x_{n+2}\geq 0\}.$$

Proof. It is clear that $C(\mathcal{A})$ is included in this set. Suppose that $x_1, \ldots, x_{n+1} \geq 0$, $x_1 + x_{n+2} \geq 0$ and $x_2 + \ldots + x_{n+1} + x_{n+2} \geq 0$. Define λ_i as in the proof of Lemma 2.1.1. Then it is clear that $\mathbf{x} = \sum_{i=1}^{2n+2} \lambda_i \mathbf{a}_i$, and by an argument very similar to the argument given in the proof of Lemma 2.1.1, it follows that all λ_i are non-negative. Hence $\mathbf{x} \in C(\mathcal{A})$. $\square$

Corollary 2.1.3. *$F_D(a, \mathbf{b}, c|\mathbf{z})$ is non-resonant if and only if a, $b_1, \ldots, b_n$, $c-a$, and $c-b_1-\ldots-b_n$ are non-integral.*

Remark 2.1.4. In Proposition 1 of [Sas97], Sasaki shows that $F_D(a, \mathbf{b}, c|\mathbf{z})$ function is irreducible if and only if a, $b_1, \ldots, b_n$, $c-a$, and $c - b_1 - \ldots - b_n$ are non-integral. Hence in this case, non-resonance is the same as irreducibility.

To calculate the simplex volume of the convex hull of $\mathcal{A}$, we map $\mathcal{A}$ to the hyperplane $v_{n+2} = 1$ by the invertible transformation $\mathbf{v} \mapsto (v_1, \ldots, v_{n+1}, v_1 + \ldots + v_{n+2})$. Now we can omit the last coordinate. This gives the set $\tilde{\mathcal{A}} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}, 0, \mathbf{e}_1 + \mathbf{e}_2, \ldots, \mathbf{e}_1 + \mathbf{e}_{n+1}\} \subseteq \mathbb{Z}^{n+1}$.

Lemma 2.1.5. *The convex hull of $\tilde{\mathcal{A}}$ is $\{\mathbf{x} \in \mathbb{R}^{n+1} \mid 0 \leq x_1, \ldots, x_{n+1} \leq 1, 0 \leq x_2+\ldots+x_{n+1} \leq 1\}$.*

Proof. Denote the above set by V. It is clear that V is a convex set containing $\tilde{\mathcal{A}}$, so $Q(\tilde{\mathcal{A}})$ is contained in V.

Let $\mathbf{x} \in V$. For $i \neq n+2$, define λ_i as in the proof of Lemma 2.1.1. Similar to Lemma 2.1.1, we have $\lambda_i \geq 0$ for all $i \neq n+2$. Note that $\mathbf{x} = \sum_{i=1}^{2n+2} \lambda_i \mathbf{a}_i$ for every value of λ_{n+2}. By an argument similar to the proof of Lemma 2.1.1, one can show that $\lambda_1 + \ldots + \lambda_{n+1} + \lambda_{n+3} + \ldots + \lambda_{2n+2}$ is either equal to x_1 or equal to $x_2 + \ldots + x_{n+1}$. In both cases it is smaller than 1, so we can define $\lambda_{n+2} = 1 - (\lambda_1 + \ldots + \lambda_{n+1} + \lambda_{n+3} + \ldots + \lambda_{2n+2}) \geq 0$. $\square$

The volume of $Q(\mathcal{A})$ can now be obtained by an $(n+1)$-fold integration.

Corollary 2.1.6. *The simplex-volume of $Q(\mathcal{A})$ is $n+1$, so there are at most $n+1$ apexpoints.*

Theorem 2.1.7. *Suppose that $F_D(a, \mathbf{b}, c|\mathbf{z})$ is non-resonant. Then there are $n+1$ apexpoints if and only if either $\{c\} < \{a\}$ and $\{b_1\}+\ldots+\{b_n\} \leq \{c\}$, or $\{a\} \leq \{c\}$ and $\{c\}+n-1 < \{b_1\}+\ldots+\{b_n\}$.*

Proof. Let $\mathbf{p} \in \mathbb{R}^{n+2}$. Since $\mathbf{p}$ is an apexpoint if and only if $\mathbf{p} \in K_{\mathcal{A}}(\boldsymbol{\alpha})$ and $\mathbf{p} - \mathbf{a}_i \notin C(\mathcal{A})$ for all $\mathbf{a}_i \in \mathcal{A}$, the apexpoints are precisely the points $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha}$, with $\mathbf{x} \in \mathbb{Z}^{n+2}$, satisfying the following conditions: $x_1, \ldots, x_{n+1} \geq 0$, $x_1 + x_{n+2} + \alpha_1 + \alpha_{n+2} \geq 0$ and $x_2 + \ldots + x_{n+2} + \alpha_2 + \ldots + \alpha_{n+2} \geq 0$; $x_1 = 0$ or $x_1 + x_{n+2} + \alpha_1 + \alpha_{n+2} < 1$; $x_2 + \ldots + x_{n+2} + \alpha_2 + \ldots + \alpha_{n+2} < 1$ or $x_i = 0$ for all $i \in \{2, \ldots, n+1\}$; $x_1 + x_{n+2} + \alpha_1 + \alpha_{n+2} < 1$ or $x_2 + \ldots + x_{n+2} + \alpha_2 + \ldots + \alpha_{n+2} < 1$; and $x_1 = 0$ or $x_i = 0$ for all $i \in \{2, \ldots, n+1\}$.

If $x_1 = \ldots = x_{n+1} = 0$, then $x_{n+2} + \alpha_1 + \alpha_{n+2} \geq 0$ so $x_{n+2} \geq -1$. Since we either have $x_{n+2} + \alpha_1 + \alpha_{n+2} < 1$ or $x_{n+2} + \alpha_2 + \ldots + \alpha_{n+2} < 1$, we also have $x_{n+2} \leq 0$. $x_{n+2} = -1$ gives a apexpoint if and only if $\alpha_1 + \alpha_{n+2} \geq 1$ and $\alpha_2 + \ldots + \alpha_{n+2} \geq 1$, and $x_{n+2} = 0$ gives an apexpoint in all other cases. Hence there is always exactly one apexpoint with $x_1 = \ldots = x_{n+1} = 0$.

If $x_1 = 0$ and there exists $2 \leq i \leq n+1$ with $x_i > 0$, then $x_2 + \ldots + x_{n+2} + \alpha_2 + \ldots + \alpha_{n+2} < 1$. From $x_{n+2}+\alpha_1+\alpha_{n+2} \geq 0$ it follows that $x_{n+2} \geq -1$. Now $x_2, \ldots, x_{n+1} \geq 0$ and $x_2+\ldots+x_{n+2} \leq 0$ imply that $x_i = 1$, $x_{n+2} = -1$ and $x_j = 0$ for all $j \neq i, n+2$. Hence there is at most one apexpoint, and this is indeed an apexpoint if and only if $\alpha_1 + \alpha_{n+2} \geq 1$ and $\alpha_2 + \ldots + \alpha_{n+2} < 1$. Since this condition is independent of i, there are either 0 or n apexpoints of this form.

Finally, if $x_1 > 0$, then we have $0 \leq x_1 + x_{n+2} + \alpha_1 + \alpha_{n+2} < 1$, $x_2 = \ldots = x_{n+1} = 0$ and $x_{n+2}+\alpha_2+\ldots+\alpha_{n+2} \geq 0$. It follows that $-n-1 < -(\alpha_2+\ldots+\alpha_{n+2}) \leq x_{n+2} < -(\alpha_1+\alpha_{n+2}) \leq 0$, so $-n \leq x_{n+2} \leq -1$. For every such x_{n+2}, there is exactly one x_1 which satisfies the conditions, namely $x_1 = -x_{n+2} - \lfloor \alpha_1 + \alpha_{n+2} \rfloor$. This gives at most n apexpoints, and there are n apexpoints if and only if $\alpha_1 + \alpha_{n+2} < 1$ and $\alpha_2 + \ldots + \alpha_{n+2} \geq n$.

Hence there are $n+1$ apexpoints if and only if either $\alpha_1 + \alpha_{n+2} \geq 1$ and $\alpha_2 + \ldots + \alpha_{n+2} < 1$, or $\alpha_1 + \alpha_{n+2} < 1$ and $\alpha_2 + \ldots + \alpha_{n+2} \geq n$. Since $\boldsymbol{\alpha} = (1 - \{a\}, 1 - \{b_1\}, \ldots, 1 - \{b_n\}, \{c-1\})$, this is equivalent to the condition that $\{c\} < \{a\}$ and $\{b_1\} + \ldots + \{b_n\} \leq \{c\}$, or $\{a\} \leq \{c\}$ and $\{c\} + n - 1 < \{b_1\} + \ldots + \{b_n\}$. $\square$

Now we have found an interlacing condition, we can very easily check whether a parameter vector $(a, b_1, \ldots, b_n, c)$ gives rise to an irreducible algebraic function. To find all irreducible algebraic functions, we use a reduction to Lauricella functions with less variables:

Proposition 2.1.8. *If $F_D(a, \mathbf{b}, c|\mathbf{z})$ is irreducible and algebraic, then for every $i \in \{1, \ldots, n\}$, $F_D(a, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n, c|z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ is also irreducible and algebraic.*

Proof. To simplify notation, assume that $i = n$. Write $F_{D,n-1} = F_D(a, b_1, \ldots, b_{n-1}, c|z_1, \ldots, z_{n-1})$. From the irreducibility conditions for $F_D(a, \mathbf{b}, c|\mathbf{z})$, it follows that $F_{D,n-1}$ is irreducible unless $\{c\} - \{b_1\} - \ldots - \{b_{n-1}\}$ is an integer. However, if this is an integer, then $F_D(a, \mathbf{b}, c|\mathbf{z})$ doesn't satisfy the interlacing condition. Hence $F_{D,n-1}$ is also irreducible. Algebraicity of $F_{D,n-1}$ follows from Remark 1.2.10. □

Lemma 2.1.9. *If $F_1(a, b_1, b_2, c|x, y)$ is irreducible and algebraic, then $F(a, b_1 + b_2, c|z)$ is also irreducible and algebraic.*

Proof. Since $F_1(a, b_1, b_2, c|x, y)$ is irreducible, $a, c - a$ and $c - b_1 - b_2$ are non-integral. It follows from the interlacing condition for $F_1(a, b_1, b_2, c|x, y)$ that either $0 < \{b_1\} + \{b_2\} \le \{c\} < 1$ or $1 < \{c\} + 1 < \{b_1\} + \{b_2\} < 2$, so $b_1 + b_2$ cannot be an integer. Hence $F(a, b_1 + b_2, c|z)$ is irreducible.

To prove that $F(a, b_1 + b_2, c|z)$ is algebraic, let D be the smallest common denominator of $a, b_1 + b_2$ and c and let $\tilde{D}$ be the smallest common denominator of a, b_1, b_2 and c. Then clearly $D|\tilde{D}$. Let $k \in \mathbb{Z}$ such that $1 \le k < D$ and $\gcd(k, D) = 1$. By Proposition 1.2.11, there exists $l \in \mathbb{Z}$ such that $l \equiv k \pmod D$ and $\gcd(l, \tilde{D}) = 1$. Since $F_1(a, b_1, b_2, c|x, y)$ is algebraic, either $\{lc\} < \{la\}$ and $\{lb_1\} + \{lb_2\} \le \{lc\}$, or $\{la\} \le \{lc\}$ and $\{lc\} + 1 < \{lb_1\} + \{lb_2\}$. If the first condition holds, then $\{kc\} = \{lc\} < \{la\} = \{ka\}$ and $\{k(b_1 + b_2)\} = \{l(b_1 + b_2)\} \le \{lb_1\} + \{lb_2\} \le \{lc\} = \{kc\}$. If the second condition holds, then $\{ka\} \le \{kc\}$ and $\{kc\} < \{lb_1\} + \{lb_2\} - 1 \le \{l(b_1 + b_2)\} = \{k(b_1 + b_2)\}$. In both cases, the interlacing condition for the Gauss functon is satisfied, so $F(a, b_1 + b_2, c|z)$ is algebraic. □

Remark 2.1.10. One can show that $F(a, b_1 + b_2, c|z) = F_1(a, b_1, b_2, c|z, z)$. Hence algebraicity of $F(a, b_1 + b_2, c|z)$ also follows from an argument similar to Remark 1.2.10.

Theorem 2.1.11. *$F_1(a, b_1, b_2, c|x, y)$ is irreducible and algebraic if and only if $(a, b_1, b_2, c) \pmod{\mathbb{Z}}$ is one of the following: $\pm(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$, $\pm(\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3})$, $\pm(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}, \frac{1}{3})$, $\pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$ and $\pm(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$.*

Proof. If $F_1(a, b_1, b_2, c|x, y)$ is irreducible and algebraic, then by Proposition 2.1.8 and Lemma 2.1.9, $(a, b_1, c), (a, b_2, c)$ and $(a, b_1 + b_2, c)$ are Gauss triples.

First suppose that (a, b_1, c) and (a, b_2, c) are both Gauss triples of type 1. Then there exist $r, s \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that we have $(a, b_1, c) \in \{(r, -r, \frac{1}{2}), (r, r + \frac{1}{2}, \frac{1}{2}), (r, r + \frac{1}{2}, 2r)\}$ and $(a, b_2, c) \in \{(s, -s, \frac{1}{2}), (s, s + \frac{1}{2}, \frac{1}{2}), (s, s + \frac{1}{2}, 2s)\}$ (up to congruence modulo $\mathbb{Z}$). This implies that $r = s$. This gives five possibilities for (a, b_1, b_2, c) and we obtain the four combinations

$$(a, b_1 + b_2, c) \equiv (r, -2r, \frac{1}{2}), (r, \frac{1}{2}, \frac{1}{2}), (r, 2r, \frac{1}{2}), (r, 2r, 2r) \pmod{\mathbb{Z}}$$

$(r, -2r, \frac{1}{2})$ is of type 1 if and only if $r = \pm\frac{1}{6}$. This gives the tuples $(a, b_1, b_2, c) = (\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$. The triple $(r, 2r, \frac{1}{2})$ is of type 1 if and only if $r = \pm\frac{1}{3}$, which gives the tuples $(a, b_1, b_2, c) = (\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$. In all other cases, $(a, b_1 + b_2, c)$ is of type 2, and hence the denominator of r is at most 60. This gives finitely many possibilities and for each of these possibilities, we let the computer check whether all conjugates $k(a, b_1, b_2, c)$ (with $\gcd(k, D) = 1$, where D is the smallest common denominator of a, b_1, b_2 and c) satisfy the interlacing condition. It turns out that this gives the same four solutions as we found above.

If (a, b_1, c) is a Gauss triple of type 1 and (a, b_2, c) is of type 2, then the denominator of a is at most 60. This gives finitely many possibilities for the parameter r in (a, b_1, c). Again we check these by computer, and we get the solutions $(\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{3}, \frac{1}{6}, \frac{2}{3})$.

By symmetry, if (a, b_1, c) is of type 2 and (a, b_2, c) is of type 1, the solutions are $(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3})$.

Finally, if both (a, b_1, c) and (a, b_2, c) are of type 2, we have a finite list to check. This gives two more solutions: $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})$. □

Theorem 2.1.12. *For $n = 3$, the only irreducible algebraic Lauricella F_D functions have parameters $(a, b_1, b_2, b_3, c) = (\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$ or $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ $\pmod{\mathbb{Z}}$. For $n \geq 4$, there are no irreducible algebraic Lauricella F_D functions.*

Proof. Let $n = 3$, and suppose that $F_D(a, b_1, b_2, b_3, c|z_1, z_2, z_3)$ is irreducible and algebraic. Then $F_1(a, b_1, b_2, c|x, y), F_1(a, b_1, b_3, c|x, y)$ and $F_1(a, b_2, b_3, c|x, y)$ are also irreducible and algebraic. Using Theorem 2.1.11, one easily computes that the only irreducible possibilities for (a, b_1, b_2, b_3, c) are $(a, b_1, b_2, b_3, c) = (\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})$. They form an orbit and satisfy the interlacing condition.

For $n = 4$, the only possibilities for $(a, b_1, b_2, b_3, b_4, c)$ are $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})$. However, both functions are reducible. Hence there are no irreducible algebraic functions in 4 variables. Proposition 2.1.8 implies that there are also no irreducible algebraic F_D functions in 5 or more variables. □

2.2 The Appell F_2 and Lauricella F_A functions

The Lauricella F_A function is given by

$$F_A(a, \mathbf{b}, \mathbf{c}|\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^n_{\geq 0}} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{z}^{\mathbf{m}}.$$

The lattice is $\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\mathbf{e}_1 - \mathbf{e}_{i+1} + \mathbf{e}_{n+i+1} + \mathbf{e}_{2n+i+1}) \subseteq \mathbb{Z}^{3n+1}$ and we can take

$$\begin{aligned}
\boldsymbol{\gamma} &= (-a, -b_1, \dots, -b_n, c_1 - 1, \dots, c_n - 1, 0, \dots, 0) \in \mathbb{R}^{3n+1}, \\
\mathcal{A} &= \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2n+1}, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{n+2}, \mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_{n+3}, \dots, \mathbf{e}_1 + \mathbf{e}_{n+1} - \mathbf{e}_{2n+1}\} \subseteq \mathbb{Z}^{2n+1}, \\
\boldsymbol{\alpha} &= \sum_{i=1}^{3n+1} \gamma_i \mathbf{a}_i = (-a, -b_1, \dots, -b_n, c_1 - 1, \dots, c_n - 1) \in \mathbb{Q}^{2n+1}.
\end{aligned}$$

For each $I \subseteq \{n+2, \dots, 2n+1\}$, define $\tilde{I} = \{n+2, \dots, 2n+1\} \setminus I$ and $V_I = \{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\} \cup \{\mathbf{e}_1 + \mathbf{e}_{i-n} - \mathbf{e}_i | i \in I\} \cup \{\mathbf{e}_i | i \in \tilde{I}\}$. Then the determinant of the vectors in V_I is ± 1, so the vectors are the vertices of a $2n$-dimensional simplex with volume 1.

Lemma 2.2.1. *Let $I \subseteq \{n+2, \dots, 2n+1\}$. Then*

$$\begin{aligned}
C(V_I) = \{\mathbf{x} \in \mathbb{R}^{2n+1} \mid x_1, \dots, x_{n+1} \geq 0; x_1 + \sum_{i \in I} x_i \geq 0; \\
\forall i \in \tilde{I} : x_i \geq 0; \forall i \in I : x_i \leq 0 \text{ and } x_{i-n} + x_i \geq 0\}.
\end{aligned}$$

Proof. This follows from

$$\begin{aligned}
C(V_I) = \{\mathbf{x} \in \mathbb{R}^{2n+1} \mid \exists \lambda_1, \dots, \lambda_{2n+1} \geq 0 : x_1 = \lambda_1 + \sum_{i \in I} \lambda_i; \forall i \in \tilde{I} : x_{i-n} = \lambda_{i-n} \text{ and } x_i = \lambda_i; \\
\forall i \in I : x_{i-n} = \lambda_{i-n} + \lambda_i \text{ and } x_i = -\lambda_i\}.
\end{aligned}$$

□

Corollary 2.2.2.

$$\begin{aligned}
\bigcup_I C(V_I) = \{\mathbf{x} \in \mathbb{R}^{2n+1} \mid x_1, \dots, x_{n+1} \geq 0; \text{for all } I \subseteq \{n+2, \dots, 2n+1\} : x_1 + \sum_{i \in I} x_i \geq 0; \\
\text{for all } i \in \{n+2, \dots, 2n+1\} : x_{i-n} + x_i \geq 0\}.
\end{aligned}$$

Lemma 2.2.3. *$\mathcal{T} = \{Q(V_I) \mid I \subseteq \{n+2,\ldots,2n+1\}\}$ is a triangulation of $Q(\mathcal{A})$.*

Proof. By Lemma 1.3.3, it suffices to prove that $\cup_I C(V_I)$ is convex and $C(V_I)\cap C(V_J) \subseteq C(V_I \cap V_J)$ for all $I, J \subseteq \{n+2,\ldots,2n+1\}$. The first statement follows from Corollary 2.2.2. For the second statement, let $I, J \subseteq \{n+2,\ldots,2n+1\}$. Then one can easily show that both $C(V_I)\cap C(V_J)$ and $C(V_I \cap V_J)$ equal

$$\{\mathbf{x} \in \mathbb{R}^{2n+1} \mid x_1,\ldots,x_{n+1} \geq 0; x_1 + \sum_{i\in I\cap J} x_i \geq 0; \forall i \in \tilde{I}\cap\tilde{J} : x_i \geq 0;$$
$$\forall i \in I \cap J : x_i \leq 0 \text{ and } x_{i-n} + x_i \geq 0; \forall i \in (I\cap\tilde{J}) \cup (\tilde{I}\cap J) : x_i = 0\}. \qquad \square$$

Corollary 2.2.4. *$\mathcal{A}$ is saturated, the volume of $Q(\mathcal{A})$ is 2^n and*

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^{2n+1} \mid x_1,\ldots,x_{n+1} \geq 0; \textit{for all } I \subseteq \{n+2,\ldots,2n+1\} : x_1 + \sum_{i\in I} x_i \geq 0;$$
$$\textit{for all } i \in \{n+2,\ldots,2n+1\} : x_{i-n} + x_i \geq 0\}$$

$F_A(a,\mathbf{b},\mathbf{c}|\mathbf{z})$ is non-resonant if and only if $b_1,\ldots,b_n$, $c_1 - b_1,\ldots,c_n - b_n$ and $-a + \sum_{j\in J} c_j$ are non-integral for all $J \subseteq \{1,\ldots,n\}$.

Corollary 2.2.5. *If $F_A(a,\mathbf{b},\mathbf{c}|\mathbf{z})$ is non-resonant and algebraic, then for every $i \in \{1,\ldots,n\}$, $F_A(a, b_1,\ldots,b_{i-1}, b_{i+1},\ldots,b_n, c_1,\ldots,c_{i-1}, c_{i+1},\ldots,c_n \mid z_1,\ldots,z_{i-1}, z_{i+1},\ldots,z_n)$ is non-resonant and algebraic.*

Remark 2.2.6. In section 2 of [Kat00], Kato shows that $F_2(a, b_1, b_2, c_1, c_2|x, y)$ is irreducible if and only if a, b_1, b_2, $c_1 - a$, $c_2 - a$, $c_1 + c_2 - a$, $c_1 - b_1$ and $c_2 - b_2$ are non-integral. However, we could not find similar results for F_A in the literature.

Lemma 2.2.7. *Let $I \subseteq \{n+2,\ldots,2n+1\}$. If $\mathbf{x}+\boldsymbol{\alpha} \in (\mathbb{Z}^{2n+1}+\boldsymbol{\alpha}) \cap C(V_I)$ is an apexpoint, then for all $i \in I$ either $\alpha_i = 0$ and $x_{i-n} = x_i = 0$, or $\alpha_i > 0$, $x_{i-n} \in \{0,1\}$ and $x_i = -1$. For all $i \in \tilde{I}$, it must hold that $x_{i-n} = x_i = 0$. Finally, $0 \leq x_1 \leq -\sum_{i\in I} x_i$.*

Proof. Write $\mathbf{p} = \mathbf{x}+\boldsymbol{\alpha}$. Since $\mathbf{p} \in C(V_I)$, we have $p_1,\ldots,p_{n+1} \geq 0$, $p_1 + \sum_{i\in I} p_i \geq 0$, $p_i \leq 0$ and $p_{i-n} + p_i \geq 0$ for all $i \in I$ and $p_i \geq 0$ for all $i \in \tilde{I}$. Hence

$$\mathbf{p} = (p_1 + \sum_{i\in I} p_i)\mathbf{e}_1 + \sum_{i\in I}((p_{i-n}+p_i)\mathbf{e}_{i-n} - p_i(\mathbf{e}_1 + \mathbf{e}_{i-n} - \mathbf{e}_i)) + \sum_{i\in\tilde{I}}(p_{i-n}\mathbf{e}_{i-n} + p_i\mathbf{e}_i)$$

where all coefficients are non-negative. By Remark 1.3.5, the coefficients are smaller than 1. This implies that $x_{i-n} = x_i = 0$ for all $i \in \tilde{I}$. For $i \in I$, we have $-1 < p_i \leq 0$ and $0 \leq p_{i-n} + p_i < 1$. If $\alpha_i = 0$, then $x_i = x_{i-n} = 0$. If $\alpha_i > 0$, then $x_i = -1$ and $0 \leq x_{i-n} < 1 - x_i - \alpha_{i-n} - \alpha_i < 2$. Finally, $0 \leq x_1 < 1 - \alpha_1 - \sum_{i\in I} p_i \leq 1 - \sum_{i\in I} x_i$. $\square$

Finding an interlacing condition isn't as easy as for F_D. Therefore, we will first prove an interlacing condition for the Appell function and compute all irreducible algebraic functions for $n = 2$. Using this, we can prove that there are no non-resonant algebraic functions for $n \geq 3$.

Theorem 2.2.8. *Suppose that $F_2(a, b_1, b_2, c_1, c_2|x, y)$ is irreducible. Then there are 4 apexpoints if and only if one of the following conditions holds:*

$$\{b_1\} \leq \{c_1\} \textit{ and } \{b_2\} \leq \{c_2\} \textit{ and } \{c_1\} + \{c_2\} < \{a\}$$

or

$$\{b_1\} \leq \{c_1\} \textit{ and } \{c_2\} < \{b_2\} \textit{ and } \{c_1\} < \{a\} \leq \{c_2\}$$

or

$$\{c_1\} < \{b_1\} \textit{ and } \{b_2\} \leq \{c_2\} \textit{ and } \{c_2\} < \{a\} \leq \{c_1\}$$

or

$$\{c_1\} < \{b_1\} \textit{ and } \{c_2\} < \{b_2\} \textit{ and } 1 + \{a\} \leq \{c_1\} + \{c_2\}$$

Proof. We write all apexpoints in the form $\mathbf{x}+\boldsymbol{\alpha}$ with $\mathbf{x}\in\mathbb{Z}^5$. By Corollary 1.3.7, there are four apexpoints if and only if there is an apexpoint in every $C(V_I)$ and these points are distinct. It follows from Lemma 2.2.7 that the only possible apexpoint for $I=\emptyset$ is given by $\mathbf{x}=\mathbf{0}$. In order to have another apexpoint for $I=\{4\}$, we must have $\alpha_4>0$. Similarly, $\alpha_5>0$. This gives 21 possible apexpoints, each with some conditions on $\boldsymbol{\alpha}$. For 16 of these points there exists $\boldsymbol{\alpha}$ that satisfies all conditions. They are given in Table 3.

Table 3: The possible apexpoints for $F_2(a,b_1,b_2,c_1,c_2|x,y)$

I	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
$\emptyset$	(0,0,0,0,0)	$\alpha_1+\alpha_4+\alpha_5<1$ or $((\alpha_1+\alpha_4<1$ or $\alpha_2+\alpha_4<1)$ and $(\alpha_1+\alpha_5<1$ or $\alpha_3+\alpha_5<1))$
$\{4\}$	(0,0,0,-1,0)	$\alpha_1+\alpha_4\geq 1$ and $\alpha_2+\alpha_4\geq 1$ and $(\alpha_1+\alpha_4+\alpha_5<2$ or $\alpha_3+\alpha_5<1)$
	(0,1,0,-1,0)	$\alpha_1+\alpha_4\geq 1$ and $\alpha_2+\alpha_4<1$ and $(\alpha_1+\alpha_4+\alpha_5<2$ or $\alpha_3+\alpha_5<1)$
	(1,0,0,-1,0)	$\alpha_1+\alpha_4<1$ and $\alpha_2+\alpha_4\geq 1$ and $(\alpha_1+\alpha_4+\alpha_5<1$ or $\alpha_3+\alpha_5<1)$
$\{5\}$	(0,0,0,0,-1)	$\alpha_1+\alpha_5\geq 1$ and $\alpha_3+\alpha_5\geq 1$ and $(\alpha_1+\alpha_4+\alpha_5<2$ or $\alpha_2+\alpha_4<1)$
	(0,0,1,0,-1)	$\alpha_1+\alpha_5\geq 1$ and $\alpha_3+\alpha_5<1$ and $(\alpha_1+\alpha_4+\alpha_5<2$ or $\alpha_2+\alpha_4<1)$
	(1,0,0,0,-1)	$\alpha_1+\alpha_5<1$ and $\alpha_3+\alpha_5\geq 1$ and $(\alpha_1+\alpha_4+\alpha_5<1$ or $\alpha_2+\alpha_4<1)$
$\{4,5\}$	(0,0,0,-1,-1)	$\alpha_1+\alpha_4+\alpha_5\geq 2$ and $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5\geq 1$
	(0,0,1,-1,-1)	$\alpha_1+\alpha_4+\alpha_5\geq 2$ and $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5<1$
	(0,1,0,-1,-1)	$\alpha_1+\alpha_4+\alpha_5\geq 2$ and $\alpha_2+\alpha_4<1$ and $\alpha_3+\alpha_5\geq 1$
	(0,1,1,-1,-1)	$\alpha_1+\alpha_4+\alpha_5\geq 2$ and $\alpha_2+\alpha_4<1$ and $\alpha_3+\alpha_5<1$
	(1,0,0,-1,-1)	$1\leq\alpha_1+\alpha_4+\alpha_5<2$ and $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5\geq 1$
	(1,0,1,-1,-1)	$\alpha_1+\alpha_4<1$ and $\alpha_1+\alpha_4+\alpha_5\geq 1$ and $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5<1$
	(1,1,0,-1,-1)	$\alpha_1+\alpha_5<1$ and $\alpha_1+\alpha_4+\alpha_5\geq 1$ and $\alpha_2+\alpha_4<1$ and $\alpha_3+\alpha_5\geq 1$
	(1,1,1,-1,-1)	$\alpha_1+\alpha_4<1$ and $\alpha_1+\alpha_5<1$ and $\alpha_1+\alpha_4+\alpha_5\geq 1$ and $\alpha_2+\alpha_4<1$ and $\alpha_3+\alpha_5<1$
	(2,0,0,-1,-1)	$\alpha_1+\alpha_4+\alpha_5<1$ and $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5\geq 1$

Now we have to determine the conditions on $\boldsymbol{\alpha}$ to have an apexpoint in each $C(V_I)$. For each combination of four points (with one point for each I), the simplex method can be used to determine whether there exists $\boldsymbol{\alpha}$ satisfying all conditions simultaneously. In this way one can use a computer to find the conditions on $\boldsymbol{\alpha}$ to have 4 apexpoints. We find that there are 4 apexpoints if and only if either $\alpha_1+\alpha_4+\alpha_5\geq 2$, $\alpha_2+\alpha_4<1$ and $\alpha_3+\alpha_5<1$; or $\alpha_1+\alpha_4<1$, $\alpha_1+\alpha_5\geq 1$, $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5<1$; or $\alpha_1+\alpha_4\geq 1$, $\alpha_1+\alpha_5<1$, $\alpha_2+\alpha_4<1$ and $\alpha_3+\alpha_5\geq 1$; or $\alpha_1+\alpha_4+\alpha_5<1$, $\alpha_2+\alpha_4\geq 1$ and $\alpha_3+\alpha_5\geq 1$. Since $\boldsymbol{\alpha}=(1-\{a\},1-\{b_1\},1-\{b_2\},\{c_1\},\{c_2\})$, this is equivalent to the conditions given above. □

Lemma 2.2.9. *If $F_2(a,b_1,b_2,c_1,c_2|x,y)$ is irreducible and algebraic, then $F(a-c_2,b_1,c_1|x)$ is also irreducible and algebraic.*

Proof. The proof is similar to the proof of Lemma 2.1.9 and uses the interlacing condition from Theorem 2.2.8. □

Since $F_2(a,b_1,b_2,c_1,c_2|x,y)=F_2(a,b_2,b_1,c_2,c_1|y,x)$, the algebraic functions come in pairs. Therefore, in the next Theorem we only give the smallest element of each pair of orbits of irreducible algebraic functions.

Theorem 2.2.10. *$F_2(a,b_1,b_2,c_1,c_2|x,y)$ is irreducible and algebraic if and only if (a,b_1,b_2,c_1,c_2) $\pmod{\mathbb{Z}}$ or (a,b_2,b_1,c_2,c_1) $\pmod{\mathbb{Z}}$ is conjugate to one of the tuples $(\frac{1}{2},\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{2}{3})$, $(\frac{1}{6},\frac{5}{6},\frac{5}{6},\frac{2}{3},\frac{2}{3})$, $(\frac{1}{10},\frac{7}{10},\frac{9}{10},\frac{2}{5},\frac{4}{5})$, $(\frac{1}{12},\frac{3}{4},\frac{5}{6},\frac{1}{2},\frac{2}{3})$, $(\frac{1}{12},\frac{5}{6},\frac{11}{12},\frac{2}{3},\frac{1}{2})$, $(\frac{1}{12},\frac{5}{6},\frac{7}{12},\frac{2}{3},\frac{1}{2})$ and $(\frac{1}{30},\frac{5}{6},\frac{7}{10},\frac{2}{3},\frac{2}{5})$.*

Proof. If $F_2(a,b_1,b_2,c_1,c_2|x,y)$ is irreducible and algebraic, then Corollary 2.2.5 and Lemma 2.2.9 imply that (a,b_1,c_1), (a,b_2,c_2) and $(a-c_2,b_1,c_1)$ are Gauss triples.

First suppose that (a, b_1, c_1) and (a, b_2, c_2) are both Gauss triples of type 1. Then there exist $r \in (0,1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that $(a, b_1, c_1), (a, b_2, c_2) \in \{(r, -r, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, 2r)\}$ (up to congruence modulo $\mathbb{Z}$). Hence $a - c_2, b_1 \in \{r + \frac{1}{2}, -r\} \pmod{\mathbb{Z}}$. If $(a - c_2, b_1, c_1)$ is a Gauss triple of type 1, then $a - c_2 \equiv -b_1$ or $a - c_2 \equiv b_1 + \frac{1}{2}$. However, this doesn't hold for $r \neq \frac{1}{2}$. Hence $(a - c_2, b_1, c_1)$ must be of type 2, so the denominator of $a - c_2$ is at most 60. This implies that the denominator of r is at most 60, or 2 (mod 4) and at most 120. This gives finitely many possibilities for r and using a computer it turns out that there are no solutions.

If (a, b_1, c_1) is a Gauss triple of type 1, and (a, b_2, c_2) is of type 2, then the denominator of a is at most 60 and there are again finitely many possibilities. The solutions are the 8 points in the orbits of $(\frac{1}{12}, \frac{11}{12}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3})$ and $(\frac{1}{12}, \frac{7}{12}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3})$. By symmetry, if (a, b_1, c_1) is of type 2 and (a, b_2, c_2) is of type 1, the solutions are the conjugates of $(\frac{1}{12}, \frac{5}{6}, \frac{11}{12}, \frac{2}{3}, \frac{1}{2})$ and $(\frac{1}{12}, \frac{5}{6}, \frac{7}{12}, \frac{2}{3}, \frac{1}{2})$.

Finally, when both (a, b_1, c_1) and (a, b_2, c_2) are of type 2, there are only finitely many possibilities. This gives 36 solutions, namely all conjugates of $(\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3})$, $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{2}{3}, \frac{2}{3})$, $(\frac{1}{10}, \frac{7}{10}, \frac{9}{10}, \frac{2}{5}, \frac{4}{5})$, $(\frac{1}{10}, \frac{9}{10}, \frac{7}{10}, \frac{4}{5}, \frac{2}{5})$, $(\frac{1}{12}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2}, \frac{2}{3})$, $(\frac{1}{12}, \frac{5}{6}, \frac{3}{4}, \frac{2}{3}, \frac{1}{2})$, $(\frac{1}{30}, \frac{5}{6}, \frac{7}{10}, \frac{2}{3}, \frac{2}{5})$ and $(\frac{1}{30}, \frac{7}{10}, \frac{5}{6}, \frac{2}{5}, \frac{2}{3})$. □

Theorem 2.2.11. *For $n \geq 3$, there are no non-resonant algebraic Lauricella F_A functions.*

Proof. First let $n = 3$. If $F_A(a, b_1, b_2, b_3, c_1, c_2, c_3|z_1, z_2, z_3)$ is non-resonant and algebraic, then each of the three tuples (a, b_1, b_2, c_1, c_2), (a, b_1, b_3, c_1, c_3) and (a, b_2, b_3, c_2, c_3) must give an irreducible algebraic F_2 function. From the list from Theorem 2.2.10, it can easily be seen that this gives only 2 possibilities for $(a, b_1, b_2, b_3, c_1, c_2, c_3)$: $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ and $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Hence $\boldsymbol{\alpha}$ is equal to $(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or $(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The corresponding functions are non-resonant. Using Lemma 2.2.7, we compute the number of apexpoints. For $\boldsymbol{\alpha} = (\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, there are 5 apexpoints, and $\boldsymbol{\alpha} = (\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ gives 7 apexpoints. Since the volume of $Q(\mathcal{A})$ is 8, the functions are not algebraic. By Corollary 2.2.5, this implies that there are no non-resonant algebraic functions for $n \geq 4$. □

2.3 The Appell F_3 and Lauricella F_B functions

The Lauricella F_B function is given by

$$F_B(\mathbf{a}, \mathbf{b}, c|\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^n_{\geq 0}} \frac{(\mathbf{a})_{\mathbf{m}} (\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|} \mathbf{m}!} \mathbf{z}^{\mathbf{m}}.$$

The lattice is $\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\mathbf{e}_i - \mathbf{e}_{n+i} + \mathbf{e}_{2n+1} + \mathbf{e}_{2n+i+1}) \subseteq \mathbb{Z}^{3n+1}$. We can take

$$\mathcal{A} = \{\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{n+2}, \mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_{n+3}, \ldots, \mathbf{e}_1 + \mathbf{e}_{n+1} - \mathbf{e}_{2n+1}, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{2n+1}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}\} \subseteq \mathbb{Z}^{2n+1}$$

and $\boldsymbol{\gamma} = (-a_1, \ldots, -a_n, -b_1, \ldots, -b_n, c-1, 0, \ldots, 0) \in \mathbb{R}^{3n+1}$. Then

$$\boldsymbol{\alpha} = \sum_{i=1}^{3n+1} \gamma_i \mathbf{a}_i = (c - a_1 - \ldots - a_n - 1, -a_1, \ldots, -a_n, a_1 - b_1, \ldots, a_n - b_n) \in \mathbb{Q}^{2n+1}.$$

Up to the order of the vectors, the set $\mathcal{A}$ is the same as for F_A. Therefore, we can use all results from section 2.2. The vector $\boldsymbol{\alpha} = (c - a_1 - \ldots - a_n - 1, -a_1, \ldots, -a_n, a_1 - b_1, \ldots, a_n - b_n)$, corresponding to $F_B(\mathbf{a}, \mathbf{b}, c|\mathbf{z})$, also corresponds to $F_A(\tilde{a}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}|\mathbf{z})$, where $\tilde{a}$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$ are chosen such that $\boldsymbol{\alpha} = (-\tilde{a}, -\tilde{b}_1, \ldots, -\tilde{b}_n, \tilde{c}_1 - 1, \ldots, \tilde{c}_n - 1)$. Therefore we can find all non-resonant algebraic F_B functions by computing $(a_1, \ldots, a_n, b_1, \ldots, b_n, c) = (\tilde{b_1}, \ldots, \tilde{b_n}, \tilde{b_1} - \tilde{c_1}, \ldots, \tilde{b_n} - \tilde{c_n}, \tilde{b_1} + \ldots + \tilde{b_n} - \tilde{a})$ for the non-resonant algebraic $F_A(\tilde{a}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}|\mathbf{z})$ functions from Theorems 2.2.10 and 2.2.11. However, in this case we don't have a proof that non-resonance is equivalent to irreducibility.

Lemma 2.3.1. *$F_B(\mathbf{a}, \mathbf{b}, c|\mathbf{z})$ is non-resonant if and only if $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ and $c - d_1 - \ldots - d_n$ with $d_i \in \{a_i, b_i\}$ are non-integral.*

Theorem 2.3.2. *Up to permutations of $\{a_1, b_1\}$, of $\{a_2, b_2\}$ and of $\{(a_1, b_1), (a_2, b_2)\}$, the non-resonant algebraic functions $F_3(a_1, a_2, b_1, b_2, c|x, y)$ have parameters $(\frac{1}{4}, \frac{1}{6}, \frac{3}{4}, \frac{5}{6}, \frac{1}{2})$, $(\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2})$, $(\frac{1}{6}, \frac{1}{10}, \frac{5}{6}, \frac{9}{10}, \frac{1}{2})$, $(\frac{1}{6}, \frac{3}{10}, \frac{5}{6}, \frac{7}{10}, \frac{1}{2})$, $(\frac{1}{6}, \frac{1}{12}, \frac{5}{6}, \frac{7}{12}, \frac{1}{3})$, $(\frac{1}{6}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}, \frac{2}{3})$ and $(\frac{1}{10}, \frac{3}{10}, \frac{9}{10}, \frac{7}{10}, \frac{1}{2})$. There are no non-resonant algebraic Lauricella F_B functions for $n \geq 3$.*

2.4 The Appell F_4 and Lauricella F_C functions

The Lauricella F_C function is given by

$$F_C(a,b,\mathbf{c}|\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}} \frac{(a)_{|\mathbf{m}|}(b)_{|\mathbf{m}|}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{z}^{\mathbf{m}}.$$

The lattice is $\mathbb{L} = \bigoplus_{i=1}^n \mathbb{Z}(-\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_{i+2} + \mathbf{e}_{n+i+2}) \subseteq \mathbb{Z}^{2n+2}$ and we can choose

$$\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+2}, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4, \dots, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{n+2}\} \subseteq \mathbb{Z}^{n+2}.$$

We have $\boldsymbol{\gamma} = (-a, -b, c_1 - 1, \dots, c_n - 1, 0, \dots, 0)$, so $\boldsymbol{\alpha} = (-a, -b, c_1 - 1, \dots, c_n - 1) \in \mathbb{Q}^{n+2}$.

For $I \subseteq \{3, \dots, n+2\}$, let $\tilde{I} = \{3, \dots, n+2\}\backslash I$ and $V_I = \{\mathbf{e}_1, \mathbf{e}_2\} \cup \{\mathbf{e}_i | i \in \tilde{I}\} \cup \{\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i | i \in I\}$. The determinant of the vectors in V_I equals ± 1, so the vectors are the vertices of an $(n+1)$-dimensional simplex.

Lemma 2.4.1. *For $I \subseteq \{3, \dots, n+2\}$, we have*

$$C(V_I) = \{\mathbf{x} \in \mathbb{R}^{n+2} \mid x_1, x_2 \geq 0; \forall i \in I : x_i \leq 0; \forall i \in \tilde{I} : x_i \geq 0; x_1 + \sum_{i\in I} x_i \geq 0; x_2 + \sum_{i \in I} x_i \geq 0\}.$$

Corollary 2.4.2.

$$\bigcup_I C(S_I) = \{\mathbf{x} \in \mathbb{R}^{n+2} \mid \forall I \subseteq \{3, \dots, n+2\} : x_1 + \sum_{i\in I} x_i \geq 0; x_2 + \sum_{i\in I} x_i \geq 0\}.$$

Lemma 2.4.3. *$\mathcal{T} = \{Q(V_I) \mid I \subseteq \{3, \dots, n+2\}\}$ is a triangulation of $Q(\mathcal{A})$.*

Proof. By Lemma 1.3.3, it suffices to prove that $\cup_I C(V_I)$ is convex and $C(V_I) \cap C(V_J) \subseteq C(V_I \cap V_J)$ for all $I, J \subseteq \{3, \dots, n+2\}$. The first statement follows from Corollary 2.4.2. For the second statement, let $I, J \subseteq \{3, \dots, n+2\}$. Then one can show that both $C(V_I) \cap C(V_J)$ and $C(V_I \cap V_J)$ equal

$$\{\mathbf{x} \in \mathbb{R}^{n+2} \mid x_1, x_2 \geq 0; \forall i \in I \cap J : x_i \leq 0; \forall i \in \tilde{I} \cap \tilde{J} : x_i \geq 0; \\ \forall i \in (I \cap \tilde{J}) \cup (\tilde{I} \cap J) : x_i = 0; x_1 + \sum_{i \in I\cap J} x_i \geq 0; x_2 + \sum_{i\in I \cap J} x_i \geq 0\}. \qquad \square$$

Corollary 2.4.4. *$\mathcal{A}$ is saturated, the volume of $Q(\mathcal{A})$ is 2^n and*

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^{n+2} \mid \forall I \subseteq \{3, \dots, n+2\} : x_1 + \sum_{i\in I} x_i \geq 0; x_2 + \sum_{i\in I} x_i \geq 0\}.$$

$F_C(a, b, \mathbf{c}|\mathbf{z})$ is non-resonant if and only if $-a + \sum_{i\in I} c_i$ and $-b + \sum_{i\in I} c_i$ are non-integral for all $I \subseteq \{1, \dots, n\}$.

Corollary 2.4.5. *If $F_C(a, b, \mathbf{c}|\mathbf{z})$ is non-resonant and algebraic, then for every $i \in \{1, \dots, n\}$, $F_C(a, b, c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n|z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$ is also non-resonant and algebraic.*

Remark 2.4.6. Kato shows that $F_4(a, b, c_1, c_2|x, y)$ is irreducible if and only if a, $c_1 - a$, $c_2 - a$, $c_1 + c_2 - a$, b, $c_1 - b$, $c_2 - b$ and $c_1 + c_2 - b$ are non-integral (see [Kat95a, Theorem 7.2] and [Kat95b, Theorem 1]). However, we could not find similar results for F_C in the literature.

Lemma 2.4.7. *Let $I \subseteq \{3, \dots, n+1\}$. If $\mathbf{x} + \boldsymbol{\alpha} \in (\mathbb{Z}^{n+2} + \boldsymbol{\alpha}) \cap C(V_I)$ is an apexpoint, then for all $i \in I$ either $\alpha_i = 0$ and $x_i = 0$, or $\alpha_i > 0$ and $x_i = -1$. For $i \in \tilde{I}$, it must hold that $x_i = 0$. Finally, $0 \leq x_1, x_2 \leq -\sum_{i\in I} x_i$.*

Proof. Write $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha}$. We have $p_1, p_2 \geq 0$; $\forall i \in I : p_i \leq 0$; $\forall i \in \tilde{I} : p_i \geq 0$; $p_1 + \sum_{i \in I} p_i \geq 0$ and $p_2 + \sum_{i \in I} p_i \geq 0$. We can write

$$\mathbf{p} = (p_1 + \sum_{i \in I} p_i)\mathbf{e}_1 + (p_2 + \sum_{i \in I} p_i)\mathbf{e}_2 + \sum_{i \in \tilde{I}} p_i \mathbf{e}_i - \sum_{i \in I} p_i(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i)$$

where all coefficients are non-negative. Hence all coefficients are smaller than 1. For $i \in \tilde{I}$, this gives $0 \leq p_i < 1$, so $x_i = 0$. For $i \in I$, we have $-1 < p_i \leq 0$, so either $\alpha_i = 0$ and $x_i = 0$, or $\alpha_i > 0$ and $x_i = -1$. For $k \in \{1, 2\}$, we have $0 \leq x_k \leq p_k < 1 - \sum_{i \in I} p_i \leq 1 - \sum_{i \in I} x_i$. □

Theorem 2.4.8. *Suppose that $F_4(a, b, c_1, c_2|x, y)$ is irreducible. Then there are 4 apexpoints if and only if either $\{a\} \leq \min(\{c_1\}, \{c_2\})$ and $\max(\{c_1\}, \{c_2\}) < \{b\} \leq \{c_1\} + \{c_2\} < 1 + \{a\}$, or $\{b\} \leq \min(\{c_1\}, \{c_2\})$ and $\max(\{c_1\}, \{c_2\}) < \{a\} \leq \{c_1\} + \{c_2\} < 1 + \{b\}$.*

Proof. We write $\mathbf{x} + \boldsymbol{\alpha} \in \mathbb{Z}^4 + \boldsymbol{\alpha}$ for the apexpoints. Similar to the proof of Theorem 2.2.8, it is only possible to have four distinct apexpoints, one in each $C(V_I)$, if $\mathbf{x} = \mathbf{0}$ only gives an apexpoint for $I = \emptyset$. Hence $\alpha_3, \alpha_4 > 0$. Lemma 2.4.7 gives 18 possible apexpoints. 13 of these points are indeed apexpoints if $\boldsymbol{\alpha}$ satisfies certain conditions. The points are listed in Table 4.

Table 4: The possible apexpoints for $F_4(a, b, c_1, c_2|x, y)$

I	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
$\{\emptyset\}$	(0,0,0,0)	$(\alpha_1 + \alpha_3 < 1$ or $\alpha_2 + \alpha_3 < 1)$ and $(\alpha_1 + \alpha_4 < 1$ or $\alpha_2 + \alpha_4 < 1)$
$\{3\}$	(0,0,-1,0)	$\alpha_1 + \alpha_3 \geq 1$ and $\alpha_2 + \alpha_3 \geq 1$ and $(\alpha_1 + \alpha_3 + \alpha_4 < 2$ or $\alpha_2 + \alpha_3 + \alpha_4 < 2)$
	(0,1,-1,0)	$\alpha_1 + \alpha_3 \geq 1$ and $\alpha_2 + \alpha_3 < 1$ and $\alpha_1 + \alpha_3 + \alpha_4 < 2$
	(1,0,-1,0)	$\alpha_1 + \alpha_3 < 1$ and $\alpha_2 + \alpha_3 \geq 1$ and $\alpha_2 + \alpha_3 + \alpha_4 < 2$
$\{4\}$	(0,0,0,-1)	$\alpha_1 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_4 \geq 1$ and $(\alpha_1 + \alpha_3 + \alpha_4 < 2$ or $\alpha_2 + \alpha_3 + \alpha_4 < 2)$
	(0,1,0,-1)	$\alpha_1 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + \alpha_3 + \alpha_4 < 2$
	(1,0,0,-1)	$\alpha_1 + \alpha_4 < 1$ and $\alpha_2 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_3 + \alpha_4 < 2$
$\{3, 4\}$	(0,0,-1,-1)	$\alpha_1 + \alpha_3 + \alpha_4 \geq 2$ and $\alpha_2 + \alpha_3 + \alpha_4 \geq 2$
	(0,1,-1,-1)	$\alpha_1 + \alpha_3 + \alpha_4 \geq 2$ and $1 \leq \alpha_2 + \alpha_3 + \alpha_4 < 2$
	(1,0,-1,-1)	$1 \leq \alpha_1 + \alpha_3 + \alpha_4 < 2$ and $\alpha_2 + \alpha_3 + \alpha_4 \geq 2$
	(1,1,-1,-1)	$1 \leq \alpha_1 + \alpha_3 + \alpha_4 < 2$ and $1 \leq \alpha_2 + \alpha_3 + \alpha_4 < 2$ and $(\alpha_1 + \alpha_3 < 1$ or $\alpha_2 + \alpha_3 < 1)$ and $(\alpha_1 + \alpha_4 < 1$ or $\alpha_2 + \alpha_4 < 1)$
	(1,2,-1,-1)	$\alpha_1 + \alpha_3 < 1$ and $\alpha_1 + \alpha_4 < 1$ and $\alpha_1 + \alpha_3 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_3 + \alpha_4 < 1$
	(2,1,-1,-1)	$\alpha_2 + \alpha_3 < 1$ and $\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + \alpha_3 + \alpha_4 < 1$ and $\alpha_2 + \alpha_3 + \alpha_4 \geq 1$

Using the simplex method on each combinations of points from different $C(V_I)$'s, it turns out that there are 4 apexpoints if and only if either $\alpha_1 + \alpha_3 \geq 1$, $\alpha_1 + \alpha_4 \geq 1$, $\alpha_2 + \alpha_3 < 1$, $\alpha_2 + \alpha_4 < 1$, $\alpha_1 + \alpha_3 + \alpha_4 < 2$ and $\alpha_2 + \alpha_3 + \alpha_4 \geq 1$, or $\alpha_1 + \alpha_3 < 1$, $\alpha_1 + \alpha_4 < 1$, $\alpha_2 + \alpha_3 \geq 1$, $\alpha_2 + \alpha_4 \geq 1$, $\alpha_1 + \alpha_3 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_3 + \alpha_4 < 2$. □

Lemma 2.4.9. *Suppose that $F(a, b, c_1|x)$ and $F(a, b, c_2|x)$ are irreducible and algebraic, and either $a + b \equiv c_1 + c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b - a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Then $F_4(a, b, c_1, c_2|x, y)$ is also irreducible and algebraic.*

Proof. To prove that $F_4(a, b, c_1, c_2|x, y)$ is irreducible, it suffices to show that $c_1 + c_2 - a$ and $c_1 + c_2 - b$ are non-integral.

Suppose that $a + b \equiv c_1 + c_2 \pmod{\mathbb{Z}}$. Then $c_1 + c_2 - a \equiv b$ is non-integral, and the same holds for $c_1 + c_2 - b \equiv a$. Hence $F_4(a, b, c_1, c_2|x, y)$ is irreducible. Let k be coprime with the denominators of a, b, c_1 and c_2. Then we can assume that $\{ka\} \leq \{kc_1\}, \{kc_2\} < \{kb\}$. Then $\{kc_1\} + \{kc_2\} = \{ka\} + \{kb\}$, so the interlacing condition is satisfied.

Now suppose that at least two of c_1, c_2 and $b - a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Then we can assume that $c_1 \equiv \frac{1}{2}$ and $\{ka\} \leq \frac{1}{2}, \{kc_2\} < \{kb\}$. If $c_2 \equiv \frac{1}{2}$, then $c_1 + c_2 - a \equiv -a$ and $c_1 + c_2 - b \equiv -b$ are non-integral. Furthermore, it is clear that the interlacing condition is satisfied. If $b - a \equiv \frac{1}{2}$, then $c_1 + c_2 - a \equiv c_2 - b$ and $c_1 + c_2 - b \equiv c_2 - a$ are non-integral. Since $\{ka\} \leq \frac{1}{2}, \{kc_2\} < \{ka\} + \frac{1}{2} \leq 1 + \left(\{kc_2\} - \frac{1}{2}\right) < 1 + \{ka\}$, the interlacing condition is satisfied. ☐

Theorem 2.4.10. *$F_4(a, b, c_1, c_2|x, y)$ is irreducible and algebraic if and only if (a, b, c_1) and (a, b, c_2) are Gauss triples, and either $a + b \equiv c_1 + c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b - a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Up to conjugation and permutations of $\{a, b\}$ and of $\{c_1, c_2\}$, the parameters of the irreducible algebraic functions are the tuples in Table 5.*

Proof. It suffices to find all tuples satisfying the interlacing condition and prove that to satisfy $a + b \equiv c_1 + c_2 \pmod{\mathbb{Z}}$ or at least two of c_1, c_2 and $b - a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$.

First suppose that (a, b, c_1) and (a, b, c_2) are both Gauss triples of type 1. Then we have $(a, b, c_1, c_2) \in \{(r, -r, \frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}, 2r), (r, r+\frac{1}{2}, 2r, \frac{1}{2}), (r, r+\frac{1}{2}, 2r, 2r)\}$ for some $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ (up to equivalence modulo $\mathbb{Z}$). By Lemma 2.4.9, all these tuples give algebraic functions, possibly except for $(r, r + \frac{1}{2}, 2r, 2r)$. So suppose that $(a, b, c_1, c_2) = (r, r + \frac{1}{2}, 2r, 2r)$. Write $r = \frac{p}{q}$ with $\gcd(p, q) = 1$. Then for every k coprime with $2q$ such that $\{kr\} > \frac{1}{2}$, the interlacing condition implies that $\{kr\} - \frac{1}{2} \leq 2\{kr\} - 1 < \{kr\} \leq 4\{kr\} - 2 < \{kr\} + \frac{1}{2}$, so $\{kr\} \in [\frac{2}{3}, \frac{5}{6})$. Hence for every k with $\gcd(k, 2q) = 1$, it must hold that $\{kr\} < \frac{5}{6}$. There exists k such that $kp \equiv -1 \pmod{q}$. Choose l such that $\gcd(l, 2q) = 1$ and $lp \equiv -1 \pmod{q}$. If $q \geq 6$, then $\{lr\} = \frac{q-1}{q} \geq \frac{5}{6}$. Contradiction, so for all algebraic solutions, it holds that $q < 6$. By symmetry, $r + \frac{1}{2}$ also has a denominator smaller than 6, so $r = \frac{1}{4}$ or $\frac{3}{4}$. However, this only gives solutions of the form $(r, -r, \frac{1}{2}, \frac{1}{2})$.

If (a, b, c_1) is a Gauss triple of type 1, and (a, b, c_2) is a Gauss triple of type 2, then the denominator of a is at most 60. This gives 72 solutions, which all turn out to satisfy $c_1 = \frac{1}{2}$ and $b - a \equiv \frac{1}{2} \pmod{\mathbb{Z}}$. By symmetry, if (a, b, c_1) is of type 2 and (a, b, c_2) is of type 1, we get 72 solutions which all satisfy $c_2 = \frac{1}{2}$ and $b - a \equiv \frac{1}{2} \pmod{\mathbb{Z}}$.

Finally, if (a, b, c_1) and (a, b, c_2) are both of type 1, then there are 480 irreducible algebraic functions, and all the tuples (a, b, c_1, c_2) either satisfy $a + b \equiv c_1 + c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b - a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. ☐

Table 5: The tuples (a, b, c_1, c_2) such that $F_4(a, b, c_1, c_2|x, y)$ is irreducible and algebraic

$(r, -r, \frac{1}{2}, \frac{1}{2})$	$(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(r, r+\frac{1}{2}, \frac{1}{2}, 2r)$		with $r \in (0,1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$	
$(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{6}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3})$
$(\frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4})$	$(\frac{1}{6}, \frac{5}{6}, \frac{1}{5}, \frac{4}{5})$	$(\frac{1}{6}, \frac{5}{12}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{6}, \frac{11}{30}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{10}, \frac{3}{10}, \frac{1}{5}, \frac{1}{5})$	$(\frac{1}{10}, \frac{7}{10}, \frac{2}{5}, \frac{2}{5})$
$(\frac{1}{10}, \frac{9}{10}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{10}, \frac{9}{10}, \frac{1}{5}, \frac{4}{5})$	$(\frac{1}{10}, \frac{13}{30}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{12}, \frac{5}{12}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{12}, \frac{7}{12}, \frac{1}{3}, \frac{1}{3})$
$(\frac{1}{15}, \frac{7}{15}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{15}, \frac{11}{15}, \frac{1}{5}, \frac{3}{5})$	$(\frac{1}{15}, \frac{13}{15}, \frac{1}{3}, \frac{3}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{5}, \frac{2}{5})$
$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{20}, \frac{17}{20}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{20}, \frac{17}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{4})$
$(\frac{1}{24}, \frac{13}{24}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{30}, \frac{11}{30}, \frac{1}{5}, \frac{1}{5})$
$(\frac{1}{30}, \frac{19}{30}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{5})$
$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{3})$				

Remark 2.4.11. We proved Theorem 2.4.10 by computing all tuples for which $F_4(a,b,c_1,c_2|x,y)$ is irreducible and algebraic, and checking whether for each of the tuples either $a+b \equiv c_1+c_2 \pmod{\mathbb{Z}}$, or at least two of c_1, c_2 and $b-a$ are equivalent to $\frac{1}{2}$ modulo $\mathbb{Z}$. Unfortunately, this doesn't give much insight. In [Kat97], Kato proves the same Theorem using monodromy groups, without computing all solutions explicitly. However, to find all non-resonant algebraic F_C functions in more than 2 variables, we do need to know the solutions for $n=2$ explicitly.

For the Lauricella F_D, F_A and F_B functions, from a certain number of parameters on, there are no non-resonant algebraic functions. However, for the Lauricella F_C function, the situation is totally different: for every number of parameters there are three infinite families of non-resonant algebraic functions.

Lemma 2.4.12. *Suppose that $c_1 = \ldots = c_{n-2} = \frac{1}{2}$, $0 \le c_{n-1}, c_n < 1$ and $0 \le a \le b < 1$. If for all $J \subseteq \{n-1,n\}$ the condition*

$$\sum_{j\in J} c_j < a + \frac{|J|}{2} \le \frac{1}{2} + \sum_{j\in J} c_j < b + \frac{|J|}{2} \le 1 + \sum_{j\in J} c_j$$

is satisfied, then there are 2^n apexpoints.

Proof. Note that $\boldsymbol{\alpha} = (1-a, 1-b, \frac{1}{2}, \ldots, \frac{1}{2}, c_{n-1}, c_n)$ with $\alpha_2 \le \alpha_1$. We claim that all points of the form $\mathbf{x}+\boldsymbol{\alpha}$ with $x_3,\ldots,x_{n+2} \in \{-1,0\}$, $x_1 = \lfloor -\sum_{i=3}^{n+2} \frac{x_i}{2} \rfloor$ and $x_2 = \lceil -\sum_{i=3}^{n+2} \frac{x_i}{2} \rceil$ are apexpoints. It is clear that there are 2^n of these points.

For such a point $\mathbf{x}+\boldsymbol{\alpha}$, let $I = \{i \in \{3,\ldots,n+2\} \mid x_i = -1\}$. Then $x_1 = \lfloor \frac{|I|}{2} \rfloor$ and $x_2 = \lceil \frac{|I|}{2} \rceil$. Write $I = I_1 \cup I_2$ with $I_1 \subseteq \{3,\ldots,n\}$ and $I_2 \subseteq \{n+1,n+2\}$, and define $J = \{i-2 \mid i \in I_2\}$. If $|I|$ is even, then $x_1 + \sum_{i\in I_1}(x_i+\alpha_i) + \sum_{i\in I_2} x_i = x_2 + \sum_{i\in I_1}(x_i+\alpha_i) + \sum_{i\in I_2} x_i = -\frac{|J|}{2}$. If $|I|$ is odd, then $x_1 + \sum_{i\in I_1}(x_i+\alpha_i) + \sum_{i\in I_2} x_i = \frac{-|J|-1}{2}$ and $x_2 + \sum_{i\in I_1}(x_i+\alpha_i) + \sum_{i\in I_2} x_i = \frac{-|J|+1}{2}$.

To prove that $\mathbf{x}+\boldsymbol{\alpha} \in C(\mathcal{A})$, let $I' \subseteq \{3,\ldots,n+2\}$. For $k \in \{1,2\}$, we have to prove that $x_k + \alpha_k + \sum_{i\in I'}(x_i+\alpha_i) \ge 0$. The sum is minimal if $I' = I$. If $|I|$ is even, then the sum is minimal for $k=2$, and we get $x_2 + \alpha_2 + \sum_{i\in I}(x_i+\alpha_i) = 1 - b + \sum_{j\in J} c_j - \frac{|J|}{2} \ge 0$. If $|I|$ is odd, then the sum is minimal for $k=1$, and $x_1 + \alpha_1 + \sum_{i\in I}(x_i+\alpha_i) = 1 - a + \sum_{j\in J} c_j - \frac{|J|+1}{2} \ge 0$.

It remains to show that $\mathbf{x}+\boldsymbol{\alpha}-\mathbf{a} \notin C(\mathcal{A})$ for all $\mathbf{a} \in \mathcal{A}$, so for each $\mathbf{a}$, we have to find a subset $I' \subseteq \{3,\ldots,n+2\}$ and $k \in \{1,2\}$ such that $y_k + \alpha_k + \sum_{i\in I'}(y_i+\alpha_i) < 0$, where $\mathbf{y} = \mathbf{x}-\mathbf{a}$.

For $\mathbf{a} = \mathbf{e}_1$, take $I' = I$ and $k=1$. Then $y_1 + \alpha_1 + \sum_{i\in I}(y_i+\alpha_i) = x_1 - a + \sum_{i\in I}(x_i+\alpha_i)$. This is maximal if $|I|$ is even, in which case it equals $-a + \sum_{j\in J} c_j - \frac{|J|}{2} < 0$.

For $\mathbf{a} = \mathbf{e}_2$, take $I' = I$ and $k=2$. Then $y_2 + \alpha_2 + \sum_{i\in I}(y_i+\alpha_i) = x_2 - b + \sum_{i\in I}(x_i+\alpha_i)$. This is maximal if $|I|$ is odd, in which case it equals $-b + \sum_{j\in J} c_j + \frac{-|J|+1}{2} < 0$.

Now let $\mathbf{a} = \mathbf{e}_l$ for some $l \in \{3,\ldots,n+2\}$. We take $I' = I \cup \{l\}$. If $l \in I$, then we have to find k such that $x_k + \alpha_k + \sum_{i\in I}(x_i+\alpha_i) - 1 < 0$. We just checked that this holds for both $k=1$ and $k=2$. Hence we can assume that $l \notin I$. Then we have to prove that there exists k such that $x_k + \alpha_k + \sum_{i\in I} x_i + \alpha_i - 1 + \alpha_l < 0$. If $|I|$ is even, then we take $k=2$ and get $x_2 + \alpha_2 + \sum_{i\in I} x_i + \alpha_i - 1 + \alpha_l = -b + \sum_{j\in J} c_j - \frac{|J|}{2} + \alpha_l$. If $3 \le l \le n$, then $\alpha_l = \frac{1}{2}$ and $-b + \sum_{j\in J} c_j - \frac{|J|}{2} + \frac{1}{2} < 0$. If $l \in \{n+1,n+2\}$, then $-b + \sum_{j\in J} c_j - \frac{|J|}{2} + \alpha_l = -b + \sum_{j\in J'} c_j - \frac{|J'|}{2} + \frac{1}{2} < 0$ where $J' = J \cup \{l-2\}$. Similarly, if $|I|$ is odd, we take $k=1$ and get $x_1 + \alpha_1 + \sum_{i\in I'}(x_i+\alpha_i) - 1 + \alpha_l < 0$.

Finally, let $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_l$ for some $l \in \{3,\ldots,n+2\}$. Now we take $I' = I \setminus \{l\}$. We can assume that $l \in I$. If $|I|$ is even, we take $k=2$ and get $y_2 + \alpha_2 + \sum_{i\in I'}(y_i+\alpha_i) = -b + \sum_{j\in J} c_j + 1 - \frac{|J|}{2} - \alpha_l$. If $3 \le l \le n$, then $-b + \sum_{j\in J} c_j + 1 - \frac{|J|}{2} - \alpha_l = -b + \sum_{j\in J} c_j + \frac{1}{2} - \frac{|J|}{2} < 0$ and if $l \in \{n+1,n+2\}$, then $-b + \sum_{j\in J} c_j + 1 - \frac{|J|}{2} - \alpha_l = -b + \sum_{j\in J'} c_j + \frac{1}{2} - \frac{|J'|}{2} < 0$ with $J' = J \setminus \{l\}$. Similarly, if $|I|$ is odd, we take $k=1$ and get $y_1 + \alpha_1 + \sum_{i\in I'}(y_i+\alpha_i) < 0$. $\square$

Theorem 2.4.13. *For $n \geq 3$, $F_C(a,b,\mathbf{c}|\mathbf{z})$ is a non-resonant algebraic function if and only if up to permutations of $\{a,b\}$ and permutations of $\{c_1,\dots,c_n\}$, we have $c_1 = \dots = c_{n-2} = \frac{1}{2}$ and the tuple (a,b,c_{n-1},c_n) is conjugate to one of the tuples from Table 6.*

Proof. We will call the first 3 tuples in Table 6, with a parameter r, tuples of type 1, and call the other tuples tuples of type 2.

First we show that all tuples $(a,b,\mathbf{c})$ with $c_1 = \dots = c_{n-2} = \frac{1}{2}$ and (a,b,c_{n-1},c_n) from Table 6 indeed give non-resonant algebraic functions. To prove that the functions are non-resonant, let $I \subseteq \{1,\dots,n\}$ be an arbitrary (possible empty) subset. We have to prove that $-a+\sum_{i\in I} c_i$ and $-b+\sum_{i\in I} c_i$ are non-integral. It suffices to prove that $-a+\sum_{i\in I'} c_i$ and $-b+\sum_{i\in I'} c_i$ are not equal to 0 or $\frac{1}{2}$ (mod $\mathbb{Z}$) for $I' \subseteq \{n-1,n\}$. This can easily be checked for all tuples.

To prove that the functions are algebraic, we show that all conjugates of these tuples have 2^n apexpoints by using Lemma 2.4.12. All conjugates of tuples of type 1 are again of type 1. By symmetry, we can assume that $r \leq \frac{1}{2}$. If $c_{n-1} = c_n = \frac{1}{2}$, then the condition from Lemma 2.4.12 clearly holds. It is easy to check that it also holds for $c_{n-1} = \frac{1}{2}$ and $c_n = \{2r\}$.

The 27 tuples of type 2 have 248 conjugates. However, by symmetry we can assume that $\{ka\} \leq \{kb\}$. For each of the 124 conjugates with $\{ka\} \leq \{kb\}$, we check that the condition of Lemma 2.4.12 holds for the 4 possible J's.

Now we show that, up to permutations and conjugation, all non-resonant algebraic functions are given in Table 6. For $n = 2$, we found all irreducible algebraic functions in Theorem 2.4.10. We found the tuples $(r,-r,\frac{1}{2},\frac{1}{2}), (r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2}), (r,r+\frac{1}{2},\frac{1}{2},2r), (r,r+\frac{1}{2},2r,\frac{1}{2})$ with a parameter $r \in (0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$ and 624 other tuples. We will call these tuples F_4 tuples of type 1 and type 2, respectively.

Let $n = 3$. Suppose that $F_C(a,b,c_1,c_2,c_3|z_1,z_2,z_3)$ is non-resonant and algebraic. If both (a,b,c_1,c_2) and (a,b,c_1,c_3) are of type 1, then we have, up to permutations of $\{c_1,c_2,c_3\}$, $(a,b,c_1,c_2,c_3) \in \{(r,-r,\frac{1}{2},\frac{1}{2},\frac{1}{2}), (r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), (r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2},2r), (r,r+\frac{1}{2},\frac{1}{2},2r,2r)\}$ (mod $\mathbb{Z}$). The first three give non-resonant algebraic functions. Let $(a,b,c_1,c_2,c_3) = (r,r+\frac{1}{2},\frac{1}{2},2r,2r)$. (a,b,c_2,c_3) must be an F_4 tuple, but it is not of type 1 (unless $r = \pm\frac{1}{4}$, in which case it equals $(r,-r,\frac{1}{2},\frac{1}{2})$). Hence it is of type 2, so the denominator of $2r$ is at most 5. If the denominator of r is 4, then the tuple is $(r,-r,\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and if the denominator of r equals 3 or 6, then the function will be resonant. Therefore, we can assume that r has denominator 5,8 or 10. For each of these r, we can compute whether there are 8 apexpoints using the bounds for the coordinates of the apexpoints, as found in Lemma 2.4.7. This gives 4 tuples with 8 apexpoints. After permuting the c_i, we get 12 tuples.

If (a,b,c_1,c_2) is of type 1 and (a,b,c_1,c_3) is of type 2, then the denominator of r is at most 60. Using the fact that (a,b,c_2,c_3) must also be an F_4 tuple, we get 72 tuples with 8 apexpoints. There are another 72 tuples with 8 apexpoints, such that (a,b,c_1,c_2) is of type 2 and (a,b,c_1,c_3) is of type 1.

Finally, if both (a,b,c_1,c_2) and (a,b,c_1,c_3) are of type 2, then there are finitely many possibilities. This gives another 580 tuples with 8 apexpoints.

Hence we have 736 tuples with 8 apexpoints. The corresponding functions are non-resonant, and no tuple is of the form $(r,-r,\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ or $(r,r+\frac{1}{2},\frac{1}{2},\frac{1}{2},2r)$ (up to permutations). Notice that this implies that the conjugates aren't of this form either. By checking whether all conjugates of a tuple are also one of these 736 tuples with 8 apexpoints, we find that there are 720 non-resonant algebraic F_C functions in 3 variables. The parameters of these functions are the tuples obtained by conjugation and permutations from the tuples in Table 6.

Let $n \geq 4$, and suppose that all non-resonant algebraic functions in $n-1$ variables are given in Table 6. Let $(a,b,c_1,\dots,c_n)$ correspond to a non-resonant algebraic function. Of each $n-1$ c_i's, at least $n-3$ have to be equal to $\frac{1}{2}$. Hence at least $n-2$ of $c_1,\dots,c_n$ are equal to $\frac{1}{2}$. We can assume that $c_1 = \dots = c_{n-2} = \frac{1}{2}$. Since $(a,b,c_2,\dots,c_n)$ must also give a non-resonant algebraic function, (a,b,c_{n-1},c_n) must be one of the tuples in Table 6. □

Table 6: The tuples (a, b, c_{n-1}, c_n) such that $F_C(a, b, \frac{1}{2}, \ldots, \frac{1}{2}, c_{n-1}, c_n|\mathbf{z})$ is non-resonant and algebraic

$(r, -r, \frac{1}{2}, \frac{1}{2})$	$(r, r+\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(r, r+\frac{1}{2}, \frac{1}{2}, 2r)$		with $r \in (0,1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$	
$(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3})$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{12}, \frac{7}{12}, \frac{1}{3}, \frac{1}{3})$
$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{20}, \frac{11}{20}, \frac{1}{5}, \frac{2}{5})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{20}, \frac{13}{20}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{20}, \frac{17}{20}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{20}, \frac{17}{20}, \frac{1}{2}, \frac{2}{5})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{24}, \frac{13}{24}, \frac{1}{3}, \frac{1}{4})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{24}, \frac{17}{24}, \frac{1}{2}, \frac{1}{4})$
$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{24}, \frac{19}{24}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{3})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{60}, \frac{31}{60}, \frac{1}{3}, \frac{1}{5})$	$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{2})$
$(\frac{1}{60}, \frac{41}{60}, \frac{1}{2}, \frac{1}{5})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{60}, \frac{49}{60}, \frac{1}{2}, \frac{1}{3})$			

3 The Horn G functions

3.1 The Horn G_1 function

The G_1 function is given by

$$G_1(a, b_1, b_2|x, y) = \sum_{m,n\geq 0} \frac{(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}}{m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-1, 1, -1, 1, 0) \oplus \mathbb{Z}(-1, -1, 1, 0, 1)$. We choose

$$\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$$

and $\boldsymbol{\gamma} = (-a, -b_1, -b_2, 0, 0)$. Then $\boldsymbol{\alpha} = (-a, -b_1, -b_2)$.

$\mathcal{A}$ lies in the hyperplane $x_1 + x_2 + x_3 = 1$. By projecting $\mathcal{A}$ onto the (x_1, x_2)-plane, we get the set shown in Figure 1. The thick dots represent $\mathcal{A}$, the dark gray area is the set $Q(\mathcal{A})$ and light gray area is the set $C(\mathcal{A})$. This Figure also suggest a simplex decomposition.

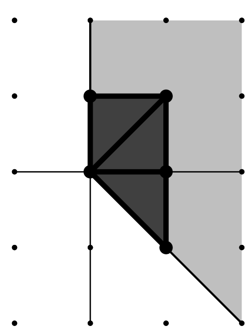

Figure 1: The sets $\mathcal{A}$, $Q(\mathcal{A})$ and $C(\mathcal{A})$ for G_1

Define $V_1 = \{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$, $V_2 = \{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$ and $V_3 = \{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3\}$. Then each V_i is a set of vectors with determinant ± 1, so it consists of the vertices of a 2-dimensional simplex with volume 1.

Lemma 3.1.1. *(i)* $C(V_1) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_2 \geq x_1 \geq 0$ *and* $x_1 + x_3 \geq 0\}$.
(ii) $C(V_2) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq x_2 \geq 0$ *and* $x_2 + x_3 \geq 0\}$.
(iii) $C(V_3) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1, x_3 \geq 0 \geq x_2$ *and* $x_1 + x_2 \geq 0$ *and* $x_2 + x_3 \geq 0\}$.

Corollary 3.1.2.

$$\bigcup_{i=1}^{3} C(V_i) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 0 \text{ and } x_1 + x_2 \geq 0 \text{ and } x_1 + x_3 \geq 0 \text{ and } x_2 + x_3 \geq 0\}.$$

Lemma 3.1.3. *$\mathcal{T} = \{Q(V_1), Q(V_2), Q(V_3)\}$ is a simplex decomposition of $Q(\mathcal{A})$.*

Proof. By Lemma 1.3.3 and Corollary 3.1.2, it suffices to show that $C(V_i) \cap C(V_j) \subseteq C(V_i \cap V_j)$ for all i and j. This can easily be checked for the three possible $\{i, j\}$. □

Corollary 3.1.4. *$\mathcal{A}$ is saturated, the volume of $Q(\mathcal{A})$ is 3 and*

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 0 \text{ and } x_1 + x_2 \geq 0 \text{ and } x_1 + x_3 \geq 0 \text{ and } x_2 + x_3 \geq 0\}.$$

$G_1(a, b_1, b_2|x, y)$ is non-resonant if and only if a, $a + b_1$, $a + b_2$ and $b_1 + b_2$ are non-integral.

Theorem 3.1.5. *Suppose that $G_1(a, b_1, b_2|x, y)$ is non-resonant. Then there are 3 apexpoints if and only if either $\{a\} + \{b_1\} \leq 1$, $\{a\} + \{b_2\} \leq 1$ and $\{b_1\} + \{b_2\} > 1$, or $\{a\} + \{b_1\} > 1$, $\{a\} + \{b_2\} > 1$ and $\{b_1\} + \{b_2\} \leq 1$.*

Proof. If $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha}$ is an apexpoint in $C(V_1)$, then $\mathbf{p} = (p_2 - p_1)\mathbf{e}_2 + (p_1 + p_3)\mathbf{e}_3 + p_1(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$. All coefficients lie in $[0, 1)$, so $x_1 = 0$ and $0 \leq x_2 + \alpha_2 - \alpha_1, x_3 + \alpha_1 + \alpha_3 < 1$. Hence $x_2 \in \{0, 1\}$ and $x_3 \in \{-1, 0\}$.

If $\mathbf{p} \in C(V_2)$ is an apexpoint, then it follows from symmetry with the previous case that $x_2 = 0$, $x_1 \in \{0, 1\}$ and $x_3 \in \{-1, 0\}$.

Finally, suppose that $\mathbf{p} \in C(V_3)$ is an apexpoint. Then $p_1, p_3 \geq 0 \geq p_2$, $p_1 + p_2 \geq 0$ and $p_2 + p_3 \geq 0$. Since $\mathbf{p} = (p_1 + p_2)\mathbf{e}_1 + (p_2 + p_3)\mathbf{e}_2 - p_2(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$, we have $-1 < p_2 \leq 0$ and $0 \leq p_1 + p_2, p_2 + p_3 < 1$. If $\alpha_2 = 0$, then $\mathbf{x} = \mathbf{0}$. If $\alpha_2 > 0$, then $x_2 = -1$ and $x_1, x_3 \in \{0, 1\}$.

This gives 10 possible apexpoints, 7 of which are indeed apexpoints under the right conditions for $\boldsymbol{\alpha}$. They are listed in Table 7, together with the $C(V_i)$ in which they can lie and the conditions on $\boldsymbol{\alpha}$.

Table 7: The possible apexpoints for $G_1(a, b_1, b_2|x, y)$

i	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
1	(0,1,-1)	$\alpha_1 + \alpha_3 \geq 1$ and $\alpha_2 + \alpha_3 < 1$
2	(1,0,-1)	$\alpha_1 + \alpha_2 < 1$ and $\alpha_1 + \alpha_3 < 1$ and $\alpha_2 + \alpha_3 \geq 1$
3	(0,-1,0)	$\alpha_1 + \alpha_2 \geq 1$ and $\alpha_2 + \alpha_3 \geq 1$
	(0,-1,1)	$\alpha_1 + \alpha_2 \geq 1$ and $\alpha_2 + \alpha_3 < 1$
	(1,-1,0)	$\alpha_1 + \alpha_2 < 1$ and $\alpha_1 + \alpha_3 < 1$ and $\alpha_2 + \alpha_3 \geq 1$
1,2	(0,0,-1)	$\alpha_1 + \alpha_3 \geq 1$ and $\alpha_2 + \alpha_3 \geq 1$
1,2,3	(0,0,0)	$\alpha_2 + \alpha_3 < 1$ or ($\alpha_1 + \alpha_2 < 1$ and $\alpha_1 + \alpha_3 < 1$)

By trying combinations of triples of these points, either by hand or using the simplex method, we find that there are 3 apexpoints if and only if either $\alpha_1 + \alpha_2 \geq 1$, $\alpha_1 + \alpha_3 \geq 1$ and $\alpha_2 + \alpha_3 < 1$, or $\alpha_1 + \alpha_2 < 1$, $\alpha_1 + \alpha_3 < 1$ and $\alpha_2 + \alpha_3 \geq 1$. By non-resonance, we have $\alpha_1 > 0$. It follows from the interlacing condition that $\alpha_2, \alpha_3 > 0$. Hence $\boldsymbol{\alpha} = (1 - \{a\}, 1 - \{b_1\}, 1 - \{b_2\})$ and the result follows. □

Lemma 3.1.6. *If $G_1(a, b_1, b_2|x, y)$ is non-resonant and algebraic, then $F(a, b_1, a + b_1 + b_2|z)$ is irreducible and algebraic.*

Proof. To show that $F(a, b_1, a + b_1 + b_2|z)$ is irreducible, it suffices to show that b_1 is non-integral. This follows immediately from the interlacing condition for $G_1(a, b_1, b_2|x, y)$.

Now we have to show that the interlacing condition for $G_1(a, b_1, b_2|x, y)$ implies the interlacing condition for $F(a, b_1, a + b_1 + b_2|z)$. If $\{a\} + \{b_1\} \leq 1$, $\{a\} + \{b_2\} \leq 1$ and $\{b_1\} + \{b_2\} > 1$, then $\{a + b_1 + b_2\} = \{a\} + \{b_1\} + \{b_2\} - 1$ and $\{a\} \leq \{a\} + \{b_1\} + \{b_2\} - 1 < \{b_2\}$.

The other case is similar. □

Theorem 3.1.7. *$G_1(a, b_1, b_2|x, y)$ is non-resonant and algebraic if and only if (a, b_1, b_2) is one of the following: $\pm(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$, $\pm(\frac{1}{6}, \frac{2}{3}, \frac{1}{2})$ and $\pm(\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$.*

Proof. By Lemma 3.1.6, we only have to consider (a, b_1, b_2) such that $(a, b_1, a+b_1+b_2)$ is a Gauss triple.

Suppose that $(a, b_1, a+b_1+b_2)$ is a Gauss triple of type 1. Then there exists $r \in (0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$ such that, up to equivalence modulo $\mathbb{Z}$, $(a, b_1, a+b_1+b_2) \in \{(r, -r, \frac{1}{2}), (r, r+\frac{1}{2}, \frac{1}{2}), (r, r+\frac{1}{2}, 2r)\}$. Hence $(a, b_1, b_2) \in \{(r, -r, \frac{1}{2}), (r, r+\frac{1}{2}, -2r), (r, r+\frac{1}{2}, \frac{1}{2})$.

If $(a, b_1, b_2) = (r, -r, \frac{1}{2})$, then $a + b_1 \in \mathbb{Z}$, so $G_1(a, b_1, b_2|x, y)$ is resonant.

Suppose that $(a, b_1, b_2) = (r, r+\frac{1}{2}, -2r)$. Then the function is non-resonant if $r \neq \frac{1}{4}$. If $r \leq \frac{1}{2}$, then the interlacing condition is satisfied if and only if $r \leq \frac{1}{4}$. If $r \geq \frac{1}{2}$, then it is satisfied if and only if $r > \frac{3}{4}$. Hence we must have $\{kr\} \leq \frac{1}{4}$ or $\{kr\} > \frac{3}{4}$ for all k coprime with the denominator of r. By Proposition 1.2.12, this is only possible for if the denominator of r is 6, so $r = \frac{1}{6}$ or $r = \frac{5}{6}$. This gives the solutions $(a, b_1, b_2) = \pm(\frac{1}{6}, \frac{2}{3}, \frac{2}{3})$.

If $(a, b_1, b_2) = (r, r+\frac{1}{2}, \frac{1}{2})$, then the interlacing condition again reduces to $\{kr\} \leq \frac{1}{4}$ or $\{kr\} > \frac{3}{4}$, so the solutions are $\pm(\frac{1}{6}, \frac{2}{3}, \frac{1}{2})$.

If (a, b_1, b_2) is a Gauss triple of type 2, then there are only finitely many possibilities. This gives two more non-resonant algebraic functions, with $(a, b_1, b_2) = \pm(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$. □

3.2 The Horn G_2 function

The G_2 function is given by

$$G_2(a_1, a_2, b_1, b_2|x, y) = \sum_{m,n\geq 0} \frac{(a_1)_m (a_2)_n (b_1)_{n-m} (b_2)_{m-n}}{m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-1, 0, 1, -1, 1, 0) \oplus \mathbb{Z}(0, -1, -1, 1, 0, 1)$. We choose

$$\mathcal{A} = \{\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4, \mathbf{e}_3\}.$$

Take $\boldsymbol{\gamma} = (-a_1, -a_2, -b_1, -b_2, 0, 0)$. Then $\boldsymbol{\alpha} = (-a_2 - b_2, -a_1, -a_2, a_2 - b_1)$.

Note that the set $\mathcal{A}$ is the same as for the F_1 function. Hence we can use all results from section 2.1. The vector $\boldsymbol{\alpha} = (-a_2-b_2, -a_1, -a_2, a_2-b_1)$ corresponding to $G_2(a_1, a_2, b_1, b_2|x, y)$ also corresponds to $F_1(\tilde{a}, \tilde{b_1}, \tilde{b_2}, \tilde{c}|x, y)$, where $\tilde{a}$, $\tilde{b_1}$, $\tilde{b_2}$ and $\tilde{c}$ are chosen so that $\boldsymbol{\alpha} = (-\tilde{a}, -\tilde{b_1}, -\tilde{b_2}, \tilde{c}-1)$. Therefore we can find all non-resonant algebraic G_2 functions by computing $(a_1, a_2, b_1, b_2) = (\tilde{b_1}, \tilde{b_2}, \tilde{b_2}-\tilde{c}, \tilde{a}-\tilde{b_2})$ for the irreducible algebraic $F_1(\tilde{a}, \tilde{b_1}, \tilde{b_2}, \tilde{c}|x, y)$ functions from Theorem 2.1.11. However, for G_2 we don't have a proof that non-resonance is equivalent to irreducibility.

Lemma 3.2.1. *$G_2(a_1, a_2, b_1, b_2|x, y)$ is non-resonant if and only if a_1, a_2, $a_1 + b_1$, $a_2 + b_2$ and $a_2 + b_2$ are non-integral.*

Theorem 3.2.2. *$G_2(a_1, a_2, b_1, b_2|x, y)$ is non-resonant and algebraic if and only if up to equivalence modulo $\mathbb{Z}$, (a_1, a_2, b_1, b_2) is one of the following: $\pm(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3})$, $\pm(\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2})$, $\pm(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3})$, $\pm(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{1}{2})$ and $\pm(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3})$.*

3.3 The Horn G_3 function

The G_3 function is given by

$$G_3(a_1, a_2|x, y) = \sum_{m,n\geq 0} \frac{(a_1)_{2n-m} (a_2)_{2m-n}}{m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(1, -2, 1, 0) \oplus \mathbb{Z}(-2, 1, 0, 1)$. To get a nice interlacing condition, we choose

$$\mathcal{A} = \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, -\mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2\}.$$

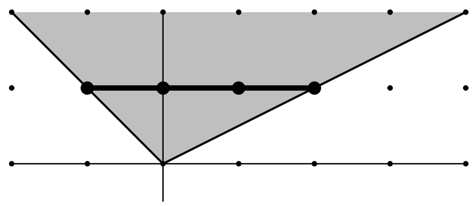

Figure 2: The sets $\mathcal{A}$, $Q(\mathcal{A})$ and $C(\mathcal{A})$ for G_3

$\boldsymbol{\gamma} = (-a_1, -a_2, 0, 0)$ gives $\boldsymbol{\alpha} = (-a_1, -a_1 - a_2)$.

In Figure 2, the thick dots represent the set $\mathcal{A}$; the thick line is $Q(\mathcal{A})$ and the gray area is the cone $C(\mathcal{A})$.

It is immediately clear that the three intervals $[-1, 0] \times \{1\}$, $[0, 1] \times \{1\}$ and $[1, 2] \times \{1\}$ form a triangulation of $Q(\mathcal{A})$. Hence $\mathcal{A}$ is saturated and the simplex volume of $Q(\mathcal{A})$ equals 3. We have

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0, 2x_2 - x_1 \geq 0\}.$$

It follows that $G_3(a_1, a_2|x, y)$ is non-resonant if and only if $2a_1 + a_2$ and $a_1 + 2a_2$ are non-integral.

Theorem 3.3.1. *Suppose that $G_3(a_1, a_2|x, y)$ is non-resonant. Then there are 3 apexpoints if and only if either $\alpha_1 + \alpha_2 < 1$ and $2\alpha_2 < \alpha_1$, or $\alpha_1 + \alpha_2 \geq 1$ and $\alpha_1 + 1 \leq 2\alpha_2$.*

Proof. Write $V_1 = \{-\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2\}$, $V_2 = \{\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$ and $V_3 = \{\mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2\}$. By Corollary 1.3.7, there are three apexpoints if and only if there is an apexpoint in every $C(V_i)$ and the apexpoints are distinct.

If $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha} \in C(V_1)$ is an apexpoint, then we have $\mathbf{p} = -p_1(-\mathbf{e}_1 + \mathbf{e}_2) + (p_1 + p_2)\mathbf{e}_2$ with $0 \leq -p_1, p_1 + p_2 < 1$. Hence either $\alpha_1 = 0$ and $\mathbf{x} = 0$, or $x_1 = -1$ and $0 \leq x_2 < 1 - x_1 - \alpha_1 - \alpha_2 < 2$, so $x_2 \in \{0, 1\}$.

If $\mathbf{p} \in C(V_2)$ is an apexpoint, then $\mathbf{p} = (p_2 - p_1)\mathbf{e}_2 + p_1(\mathbf{e}_1 + \mathbf{e}_2)$ with $0 \leq p_2 - p_1, p_1 < 1$. It follows that $x_1 = 0$ and $0 \leq x_2 < 1 - \alpha_2 + x_1 + \alpha_1 < 2$, so $x_2 \in \{0, 1\}$.

Finally, if $\mathbf{p} \in C(V_3)$ is an apexpoint, then $\mathbf{p} = (2p_2 - p_1)(\mathbf{e}_1 + \mathbf{e}_2) + (p_1 - p_2)(2\mathbf{e}_1 + \mathbf{e}_2)$ with $0 \leq 2p_2 - p_1, p_1 - p_2 < 1$. Adding this gives $x_2 \in \{0, 1\}$, and $0 \leq x_1 < 1 - \alpha_1 + x_2 + \alpha_2 < 2 + x_2$, so $0 \leq x_1 \leq x_2 + 1$.

Hence there are 7 possible apexpoints $\mathbf{x}$. 6 of these points can indeed be apexpoints. They are given in Table 8.

Table 8: The possible apexpoints for $G_3(a_1, a_2|x, y)$

i	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
1	(-1,0)	$\alpha_1 + \alpha_2 \geq 1$
	(-1,1)	$2\alpha_2 < \alpha_1$ and $\alpha_1 + \alpha_2 < 1$
3	(1,0)	$\alpha_1 + 1 \leq 2\alpha_2$
	(1,1)	$2\alpha_2 < \alpha_1$ and $\alpha_1 + \alpha_2 < 1$
2,3	(0,1)	$2\alpha_2 < \alpha_1$ and $\alpha_1 + \alpha_2 < 1$
1,2,3	(0,0)	$\alpha_1 \leq 2\alpha_2$

For each triple of possible apexpoints, we check whether the conditions on $\boldsymbol{\alpha}$ can be simultaneously satisfied. It turns out that there are 3 apexpoints if and only if the above interlacing condition is satisfied. □

Corollary 3.3.2. *Suppose that $G_3(a_1, a_2|x, y)$ is non-resonant. Then there are 3 apexpoints if and only if either $\{a_1\} + \{a_2\} < 1$, $\{a_1\} + 2\{a_2\} > 1$ and $2\{a_1\} + \{a_2\} > 1$, or $\{a_1\} + \{a_2\} \geq 1$, $\{a_1\} + 2\{a_2\} \leq 2$ and $2\{a_1\} + \{a_2\} \leq 2$.*

Figure 3 gives a graphical interpretation of the interlacing condition for $\boldsymbol{\alpha}$. There are 3 apex-points if and only if $\boldsymbol{\alpha}$ lies in the gray area.

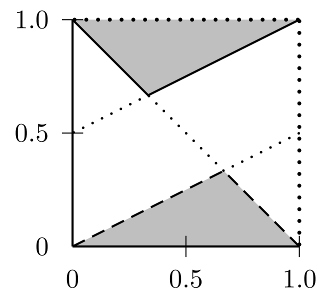

Figure 3: The interlacing condition for $\boldsymbol{\alpha}$ for G_3

This figure also gives a hint how to find all algebraic functions: if some multiple of α_2 is close enough to $\frac{1}{2}$, then the function will not be algebraic. Furthermore, if the numerator of a multiple of α_1 equals 1, then α_2 must be sufficiently small. Hence the denominator of α_1 can not be too large.

Theorem 3.3.3. *$G_3(a_1, a_2|x, y)$ is non-resonant and algebraic if and only if $a_1 + a_2 \in \mathbb{Z}$ or, up to equivalence modulo $\mathbb{Z}$, $(a_1, a_2) \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2})\}$.*

Proof. Write $\alpha_1 = \frac{p}{q}$ and $\alpha_2 = \frac{u}{v}$ with $\gcd(p, q) = \gcd(u, v) = 1$, $0 \le p < q$ and $0 \le u < v$. It follows immediately from the interlacing condition that $\alpha_1 \ne 0$. Furthermore, if $\alpha_2 = 0$ (i.e. $a_1 + a_2 \in \mathbb{Z}$), then the interlacing condition is satisfied for all α_1. If $\alpha_2 = \frac{1}{2}$, then the interlacing condition is never satisfied. Therefore, we will assume that $q \ge 2$, $v \ge 3$ and $p, u \ne 0$.

The interlacing condition is not satisfied if there exists k such that $\{k\alpha_2\} \in [\frac{1}{3}, \frac{1}{2})$ and $\gcd(k, qv) = 1$. By Proposition 1.2.12, such k exists, unless $v \in \{4, 6, 10\}$. Hence we can assume that $v \in \{4, 6, 10\}$.

Choose l such that $lp \equiv 1 \pmod{q}$ and $\gcd(l, qv) = 1$. Then $\{l\alpha_1\} = \frac{1}{q}$. Write $\{l\alpha_2\} = \frac{t}{v}$ with $0 \le t < v$. If $\frac{t}{v} < \frac{1}{3}$, then it must hold that $2\{l\alpha_2\} < \{l\alpha_1\}$, i.e. $2tq < v$. This gives that $(\{l\alpha_1\}, \{l\alpha_2\}) \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{1}{10}), (\frac{1}{3}, \frac{1}{10}), (\frac{1}{4}, \frac{1}{10})\}$. If $\frac{t}{v} \ge \frac{2}{3}$, then we must have $\{l\alpha_1\} + \{l\alpha_2\} \ge 1$, so $q \le 4$ if $v = 2$, $q \le 6$ if $v = 6$, $q \le 3$ if $\frac{t}{v} = \frac{7}{10}$ and $q \le 10$ if $\frac{t}{v} = \frac{9}{10}$. Now one easily checks that all conjugates satisfy the interlacing condition if and only if $(\{l\alpha_1\}, \{l\alpha_2\}) \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{5}{6}), (\frac{1}{3}, \frac{5}{6})\}$.

It is easy to check that these parameters, as well as $\alpha_2 = 0$, correspond to non-resonant functions. Hence the non-resonant algebraic functions are given by the orbits of these parameters. Since $\boldsymbol{\alpha} = (\{-a_1\}, \{-a_1 - a_2\})$, the parameters of the non-resonant G_3 functions are (a_1, a_2) such that $a_1 + a_2 \in \mathbb{Z}$, or $(a_1, a_2) \equiv (\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2}) \pmod{\mathbb{Z}}$. □

4 The Horn H functions

4.1 The Horn H_1 function

The H_1 function is given by

$$H_1(a, b, c, d|x, y) = \sum_{m,n \ge 0} \frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m m! n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-1, -1, 0, 1, 1, 0) \oplus \mathbb{Z}(1, -1, -1, 0, 0, 1)$ and we can take

$$\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4, -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$$

and $\boldsymbol{\alpha} = (-a, -b, -c, d - 1)$.

Define $V_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, $V_2 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_4\}$, $V_3 = \{-\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and $V_4 = \{-\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_4\}$. It is easily checked that each V_i is a set of vectors with determinant ± 1, so it consists of the vertices of a 3-dimensional simplex with volume 1.

Lemma 4.1.1. *(i)* $C(V_1) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2, x_3, x_4 \geq 0\}$.
(ii) $C(V_2) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_2, x_3 \geq 0, x_4 \leq 0, x_1 + x_4 \geq 0, x_2 + x_4 \geq 0\}$.
(iii) $C(V_3) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_2, x_3, x_4 \geq 0, x_1 \leq 0, x_1 + x_2 \geq 0, x_1 + x_3 \geq 0\}$.
(iv) $C(V_4) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_2, x_3 \geq 0, x_4 \leq 0, x_1 + x_4 \leq 0, x_1 + x_2 + 2x_4 \geq 0, x_1 + x_3 + x_4 \geq 0\}$.

Corollary 4.1.2.

$$\bigcup_{i=1}^{4} C(S_i) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_2 \geq 0, x_3 \geq 0, x_1 + x_2 \geq 0, x_1 + x_3 \geq 0,$$
$$x_2 + x_4 \geq 0, x_1 + x_2 + 2x_4 \geq 0, x_1 + x_3 + x_4 \geq 0\}.$$

Lemma 4.1.3. *$\mathcal{T} = \{Q(V_1), Q(V_2), Q(V_3), Q(V_4)\}$ is a triangulation of $Q(\mathcal{A})$.*

Corollary 4.1.4. *$\mathcal{A}$ is saturated, the volume of $Q(\mathcal{A})$ is 4 and*

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_2 \geq 0, x_3 \geq 0, x_1 + x_2 \geq 0, x_1 + x_3 \geq 0,$$
$$x_2 + x_4 \geq 0, x_1 + x_2 + 2x_4 \geq 0, x_1 + x_3 + x_4 \geq 0\}.$$

$H_1(a, b, c, d|x, y)$ is non-resonant if and only if b, c, $a+b$, $a+c$, $d-b$, $d-a-c$ and $2d-a-b$ are non-integral.

Theorem 4.1.5. *Suppose that $H_1(a, b, c, d|x, y)$ is non-resonant. Then there are 4 apexpoints if and only if one of the following conditions holds:*

$$\{a\} + \{c\} \leq \{d\} \text{ and } \{a\} + \{b\} > 1, 2\{d\}$$

or

$$\{d\} + 1 < \{a\} + \{c\} \text{ and } \{a\} + \{b\} \leq 1, 2\{d\}$$

or

$$\{a\} + \{c\} - 1, \{b\} \leq \{d\} \text{ and } 2\{d\} < \{a\} + \{b\} \text{ and } \min(\{b\}, \{c\}) \leq 1 - \{a\} < \max(\{b\}, \{c\})$$

or

$$\{d\} < \{a\} + \{c\}, \{b\} \text{ and } \{a\} + \{b\} \leq 2\{d\} \text{ and } \min(\{b\}, \{c\}) \leq 1 - \{a\} < \max(\{b\}, \{c\})$$

Proof. Suppose that $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha} \in C(V_1)$ is an apexpoint. Then $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3 + p_4\mathbf{e}_4$ with $0 \leq p_i < 1$, so $\mathbf{x} = \mathbf{0}$.

If $\mathbf{p} \in C(V_2)$ is an apexpoint, then $\mathbf{p} = (p_1+p_4)\mathbf{e}_1 + (p_2+p_4)\mathbf{e}_2 + p_3\mathbf{e}_3 - p_4(\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_4)$ with $0 \leq p_1+p_4, p_2+p_4, p_3 < 1$ and $-1 < p_4 \leq 0$. Hence $x_3 = 0$. If $\alpha_4 = 0$, then $x_4 = 0$, which implies that $\mathbf{x} = 0$. Since the four apexpoints must be distinct, we must have $\alpha_4 > 0$, so $x_4 = -1$. For $k \in \{1, 2\}$, we have $0 \leq x_k < 1 - x_4 - \alpha_k - \alpha_4 < 1 - x_4 = 2$, so $x_k \in \{0, 1\}$.

Replacing x_1 by x_3, x_3 by x_4 and x_4 by x_1 shows that $\alpha_1 > 0$ and $x_1 = -1$, $x_4 = 0$ and $x_2, x_3 \in \{0, 1\}$ for apexpoints $\mathbf{p}$ in $C(V_3)$.

Finally, if $\mathbf{p} \in C(V_4)$ is an apexpoint, then $\mathbf{p} = -(p_1+p_4)(-\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) + (p_1+p_2+2p_4)\mathbf{e}_2 + (p_1+p_3+p_4)\mathbf{e}_3 - p_4(\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_4)$ with $-1 < p_1+p_4, p_4 \leq 0$ and $0 \leq p_1+p_2+2p_4, p_1+p_3+p_4 < 1$. From $\alpha_4 > 0$ it follows that $x_4 = -1$. Since $-1 < p_1 + p_4 \leq 0$, we have $0 < x_1 + \alpha_1 + \alpha_4 \leq 1$, so $x_1 \in \{-1, 0\}$. From $0 \leq p_1 + p_2 + 2p_4 < 1$ and $p_2 \geq 0$ we get that $0 \leq x_2 < 3 - x_1$, so $0 \leq x_2 \leq 2 - x_1$. Similarly, $0 \leq x_3 \leq 1 - x_1$.

17 of the 25 points satisfying these conditions can indeed be apexpoints. They are listed in Table 9, together with the i such that they lie in $C(V_i)$ and the conditions on $\boldsymbol{\alpha}$ under which the point is an apexpoint.

Table 9: The possible apexpoints for $H_1(a,b,c,d|x,y)$

i	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
1	(0,0,0,0)	$(\alpha_1+\alpha_2<1$ or $\alpha_1+\alpha_3<1)$ and $(\alpha_2+\alpha_4<1$ or $\alpha_1+\alpha_2+2\alpha_4<2$ or $\alpha_1+\alpha_3+\alpha_4<1)$
2	(1,0,0,-1)	$\alpha_2+\alpha_4\geq 1$ and $(\alpha_1+\alpha_2+2\alpha_4<2$ or $\alpha_1+\alpha_3+\alpha_4<1)$
3	(-1,0,0,0)	$\alpha_1+\alpha_2\geq 1$ and $\alpha_1+\alpha_3\geq 1$ and $(\alpha_1+\alpha_2+2\alpha_4<3$ or $\alpha_1+\alpha_3+\alpha_4<2)$
	(-1,0,1,0)	$\alpha_1+\alpha_2\geq 1$ and $\alpha_1+\alpha_3<1$ and $\alpha_1+\alpha_2+2\alpha_4<3$
	(-1,1,0,0)	$\alpha_1+\alpha_2<1$ and $\alpha_1+\alpha_3\geq 1$ and $(\alpha_1+\alpha_2+2\alpha_4<2$ or $\alpha_1+\alpha_3+\alpha_4<2)$
4	(-1,0,0,-1)	$\alpha_1+\alpha_2+2\alpha_4\geq 3$ and $\alpha_1+\alpha_3+\alpha_4\geq 2$
	(-1,0,1,-1)	$\alpha_1+\alpha_2+2\alpha_4\geq 3$ and $\alpha_1+\alpha_3+\alpha_4<2$
	(-1,1,0,-1)	$2\leq\alpha_1+\alpha_2+2\alpha_4<3$ and $\alpha_1+\alpha_3+\alpha_4\geq 2$
	(-1,1,1,-1)	$\alpha_2+\alpha_4<1$ and $\alpha_1+\alpha_2+2\alpha_4\geq 2$ and $1\leq\alpha_1+\alpha_3+\alpha_4<2$
	(-1,1,2,-1)	$\alpha_2+\alpha_4<1$ and $\alpha_1+\alpha_2+2\alpha_4\geq 2$ and $\alpha_1+\alpha_3+\alpha_4<1$
	(-1,2,0,-1)	$1\leq\alpha_1+\alpha_2+2\alpha_4<2$ and $\alpha_1+\alpha_3+\alpha_4\geq 2$
	(0,0,1,-1)	$\alpha_2+\alpha_4\geq 1$ and $\alpha_1+\alpha_2+2\alpha_4\geq 2$ and $\alpha_1+\alpha_3+\alpha_4<1$
	(0,1,1,-1)	$\alpha_1+\alpha_2<1$ and $\alpha_2+\alpha_4<1$ and $\alpha_1+\alpha_2+2\alpha_4\geq 1$ and $\alpha_1+\alpha_3+\alpha_4<1$
	(0,2,0,-1)	$\alpha_1+\alpha_3<1$ and $\alpha_1+\alpha_2+2\alpha_4<1$ and $\alpha_1+\alpha_3+\alpha_4\geq 1$
2,4	(0,0,0,-1)	$\alpha_2+\alpha_4\geq 1$ and $\alpha_1+\alpha_2+2\alpha_4\geq 2$ and $\alpha_1+\alpha_3+\alpha_4\geq 1$ and $(\alpha_1+\alpha_2+2\alpha_4<3$ or $\alpha_1+\alpha_3+\alpha_4<2)$
	(0,1,0,-1)	$\alpha_1+\alpha_2+2\alpha_4\geq 1$ and $\alpha_1+\alpha_3+\alpha_4\geq 1$ and $(\alpha_1+\alpha_2<1$ or $\alpha_1+\alpha_3<1)$ and $(\alpha_2+\alpha_4<1$ or $\alpha_1+\alpha_2+2\alpha_4<2)$ and $(\alpha_1+\alpha_2+2\alpha_4<2$ or $\alpha_1+\alpha_3+\alpha_4<2)$

There are four apexpoints if and only if one of the following conditions is satisfied: either $\alpha_1+\alpha_2<1$, $\alpha_1+\alpha_2+2\alpha_4<2$ and $\alpha_1+\alpha_3+\alpha_4\geq 2$; or $\alpha_1+\alpha_2\geq 1$, $\alpha_2+\alpha_2+2\alpha_4\geq 2$ and $\alpha_1+\alpha_3+\alpha_4<1$; or $\alpha_1+\min(\alpha_2,\alpha_3)<1$, $\alpha_1+\max(\alpha_2,\alpha_3)\geq 1$, $\alpha_2+\alpha_4\geq 1$, $\alpha_1+\alpha_2+2\alpha_4<2$ and $\alpha_1+\alpha_3+\alpha_4\geq 1$; or $\alpha_1+\min(\alpha_2,\alpha_3)<1$, $\alpha_1+\max(\alpha_2,\alpha_3)\geq 1$, $\alpha_2+\alpha_4<1$, $\alpha_1+\alpha_2+2\alpha_4\geq 2$ and $\alpha_1+\alpha_3+\alpha_4<2$. Since $\alpha_1>0$, we have $\boldsymbol{\alpha}=(1-\{a\},1-\{b\},1-\{c\},\{d\})$. This gives the conditions stated above. □

Proposition 4.1.6. *Suppose that $H_1(a,b,c,d|x,y)$ is non-resonant and algebraic. Then $F(a,b,d|z)$ and $F(b-d,c,d-a|z)$ are irreducible and algebraic.*

Proof. For $F(a,b,d|z)$, algebraicity follows from substituting $x=z$, $y=0$. Since $H_1(a,b,c,d|x,y)$ is non-resonant, b and $d-b$ are non-integral. It follows from the interlacing condition that a and $d-a$ are also non-integral.

From the non-resonance for $H_1(a,b,c,d|x,y)$, it follows immediately that $F(b-d,c,d-a|z)$ is irreducible. Hence it suffices to show that the interlacing condition for $H_1(a,b,c,d|x,y)$ implies the interlacing condition for $F(b-d,c,d-a|z)$.

If $\{a\}+\{c\}\leq\{d\}$ and $\{a\}+\{b\}>1,2\{d\}$, then $\{d-a\}=\{d\}-\{a\}$ and $\{b\}>2\{d\}-\{a\}\geq\{d\}$, so $\{b-d\}=\{b\}-\{d\}$. Hence $\{c\}\leq\{d\}-\{a\}=\{d-a\}<\{b\}-\{d\}=\{b-d\}$.

If $\{a\}+\{c\}-1,\{b\}\leq\{d\}$, $2\{d\}<\{a\}+\{b\}$ and $\min(\{b\},\{c\})\leq 1-\{a\}<\max(\{b\},\{c\})$. Then $\{b-d\}=\{b\}-\{d\}+1$ and $\{a\}>2\{d\}-\{b\}\geq\{d\}$, so $\{d-a\}=\{d\}-\{a\}+1$. Hence $\{c\}\leq\{d\}-\{a\}+1=\{d-a\}<\{b\}-\{d\}+1=\{b-d\}$.

The other two cases are similar. □

Theorem 4.1.7. *$H_1(a,b,c,d|x,y)$ is non-resonant and algebraic if and only if (a,b,c,d) is, up to equivalence modulo $\mathbb{Z}$, one of the following: $\pm(\frac{1}{3},\frac{5}{6},\frac{1}{2},\frac{2}{3})$, $\pm(\frac{1}{4},\frac{7}{12},\frac{5}{6},\frac{1}{2})$ and $\pm(\frac{1}{4},\frac{11}{12},\frac{1}{6},\frac{1}{2})$.*

Proof. Let $H_1(a,b,c,d|x,y)$ be non-resonant and algebraic. Then (a,b,d) and $(b-d,c,d-a)$ are Gauss triples.

Suppose that (a,b,d) is of type 1. Then, up to equivalence modulo $\mathbb{Z}$, there is $r\in(0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$ such that $(a,b,d)\in\{(r,-r,\frac{1}{2}),(r,r+\frac{1}{2},\frac{1}{2}),(r,r+\frac{1}{2},2r)\}$. This implies that $(b-d,d-a)\in$

$\{(-r+\frac{1}{2},-r+\frac{1}{2}),(r,-r+\frac{1}{2}),(-r+\frac{1}{2},r)\}$. Note that $d-a\neq\frac{1}{2}$. Hence if $(b-d,c,d-a)$ is also of type 1, then there exists $s\in(0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$ such that $(b-d,c,d-a)=(s,s+\frac{1}{2},2s)$. Modulo $\mathbb{Z}$, we have $d-a\equiv 2(b-d)$. This implies that $(b-d,d-a)=(r,-r+\frac{1}{2})$ with $r=\pm\frac{1}{6}$ or $(b-d,d-a)=(-r+\frac{1}{2},r)$ with $r=\pm\frac{1}{3}$. Hence the possibilities for (a,b,c,d) are $\pm(\frac{1}{6},\frac{2}{3},\frac{2}{3},\frac{1}{2})$ and $\pm(\frac{1}{3},\frac{5}{6},\frac{2}{3},\frac{2}{3})$. However, they don't satisfy the interlacing condition. If $(b-d,c,d-a)$ is of type 2, then the denominator of $d-a$ is at most 5. Hence the denominator of r is at most 10. We check all possibilities, and find as solutions the tuples $\pm(\frac{1}{3},\frac{5}{6},\frac{1}{2},\frac{2}{3})$.

Now suppose that (a,b,d) is of type 2. It is easy to check that this implies that the denominator of $b-d$ is at most 60. If $(b-d,c,d-a)$ is of type 1, then it is of the form $(s,-s,\frac{1}{2}),(s,s+\frac{1}{2},\frac{1}{2})$ or $(s,s+\frac{1}{2},2s)$ where the denominator of s is at most 60. This gives finitely many possibilities, and there are no algebraic functions.

Finally, if both (a,b,d) and $(b-d,c,d-a)$ are of type 2, then the solutions are $\pm(\frac{1}{4},\frac{7}{12},\frac{5}{6},\frac{1}{2})$ and $\pm(\frac{1}{4},\frac{11}{12},\frac{1}{6},\frac{1}{2})$. □

4.2 The Horn H_2 function

The H_2 function is given by

$$H_2(a,b,c,d,e|x,y)=\sum_{m,n\geq 0}\frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m m!n!}x^m y^n.$$

The lattice is $\mathbb{L}=\mathbb{Z}(-1,-1,0,0,1,1,0)\oplus\mathbb{Z}(1,0,-1,-1,0,0,1)$. We can take

$$\mathcal{A}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3-\mathbf{e}_5,\mathbf{e}_5,\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_4,\mathbf{e}_4,\mathbf{e}_3\}$$

and $\boldsymbol{\gamma}=(-a,-b,-c,-d,e-1,0,0)$. Then $\boldsymbol{\alpha}=(-a-c+e-1,-b+e-1,-c,-e+1,c-d)$.

Up to the order of the vectors, the set $\mathcal{A}$ is the same as for the Appell F_2 function. Therefore, we can use all results from section 2.2. The vector $\boldsymbol{\alpha}=(-a-c+e-1,-b+e-1,-c,-e+1,c-d)$ corresponding to $H_2(a,b,c,d,e|x,y)$ also corresponds to $F_2(\tilde{a},\tilde{b_1},\tilde{b_2},\tilde{c_1},\tilde{c_2}|x,y)$, where $\tilde{a}$, $\tilde{b_1}$, $\tilde{b_2}$, $\tilde{c_1}$ and $\tilde{c_2}$ are chosen such that $\boldsymbol{\alpha}=(-\tilde{a},-\tilde{b_1},-\tilde{b_2},\tilde{c_1}-1,\tilde{c_2}-1)$. Therefore we can find all non-resonant algebraic H_2 functions by computing $(a,b,c,d,e)=(\tilde{a}-\tilde{b_2}-\tilde{c_1},\tilde{b_1}-\tilde{c_1},\tilde{b_2},\tilde{b_2}-\tilde{c_2},-\tilde{c_1})$ for the irreducible algebraic $F_2(\tilde{a},\tilde{b_1},\tilde{b_2},\tilde{c_1},\tilde{c_2}|x,y)$ functions from Theorem 2.2.10. However, for H_2 we don't have a proof that non-resonance is equivalent to irreducibility.

Lemma 4.2.1. *$H_2(a,b,c,d,e|x,y)$ is non-resonant if and only if b,c,d, $a+c,a+d$, $e-b$, $e-a-c$ and $e-a-d$ are non-integral.*

Theorem 4.2.2. *$H_2(a,b,c,d,e|x,y)$ is non-resonant and algebraic if and only if, up to equivalence modulo $\mathbb{Z}$ and permutations of $\{c,d\}$, (a,b,c,d,e) is conjugate to one of the following tuples: $(\frac{1}{2},\frac{1}{6},\frac{5}{12},\frac{11}{12},\frac{1}{3})$, $(\frac{1}{3},\frac{5}{6},\frac{1}{4},\frac{3}{4},\frac{2}{3})$, $(\frac{1}{3},\frac{5}{6},\frac{1}{6},\frac{5}{6},\frac{2}{3})$, $(\frac{1}{3},\frac{5}{6},\frac{1}{10},\frac{9}{10},\frac{2}{3})$, $(\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6},\frac{1}{2})$, $(\frac{1}{4},\frac{7}{12},\frac{1}{6},\frac{5}{6},\frac{1}{2})$, $(\frac{1}{5},\frac{7}{10},\frac{1}{6},\frac{5}{6},\frac{2}{5})$, $(\frac{1}{5},\frac{7}{10},\frac{1}{10},\frac{9}{10},\frac{2}{5})$ and $(\frac{1}{6},\frac{5}{6},\frac{5}{12},\frac{11}{12},\frac{2}{3})$.*

4.3 The Horn H_3 function

The H_3 function is given by

$$H_3(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}m!n!}x^m y^n.$$

Hence the lattice is equals $\mathbb{L}=\mathbb{Z}(-2,0,1,1,0)\oplus\mathbb{Z}(-1,-1,1,0,1)$. We can choose

$$\mathcal{A}=\{\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1-\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_3\}$$

and $\boldsymbol{\gamma}=(-a,-b,c-1,0,0)$. Then $\boldsymbol{\alpha}=(c-a-1,c-b-1,c-a-b-1)$.

The set $\mathcal{A}$ is the same as for the G_1 function, so we can use all results from section 3.1. The vector $\boldsymbol{\alpha}=(c-a-1,c-b-1,c-a-b-1)$ corresponding to $H_3(a,b,c|x,y)$ also corresponds to

$G_1(\tilde{a}, \tilde{b_1}, \tilde{b_2}|x, y)$, where $\tilde{a}$, $\tilde{b_1}$ and $\tilde{b_2}$ are chosen such that $\boldsymbol{\alpha} = (-\tilde{a}, -\tilde{b_1}, -\tilde{a} - \tilde{b_1} - \tilde{b_2})$. Therefore we can find all non-resonant algebraic H_3 functions by computing $(a, b, c) = (\tilde{a} + \tilde{b_2}, \tilde{b_1} + \tilde{b_2}, \tilde{b_2})$ for the non-resonant algebraic $G_1(\tilde{a}, \tilde{b_1}, \tilde{b_2}|x, y)$ functions from Theorem 3.1.7.

Lemma 4.3.1. *$H_3(a, b, c|x, y)$ is non-resonant if and only if a, b, $c - a$ and $2c - a - b$ are non-integral.*

Theorem 4.3.2. *$H_3(a, b, c|x, y)$ is non-resonant and algebraic if and only if (a, b, c) is one of the following: $\pm(\frac{1}{3}, \frac{5}{6}, \frac{1}{2})$, $\pm(\frac{1}{6}, \frac{2}{3}, \frac{1}{3})$ and $\pm(\frac{1}{6}, \frac{5}{6}, \frac{1}{3})$.*

4.4 The Horn H_4 function

The H_4 function is given by

$$H_4(a, b, c, d|x, y) = \sum_{m,n \geq 0} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n m!n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-2, 0, 1, 0, 1, 0) \oplus \mathbb{Z}(-1, -1, 0, 1, 0, 1)$ and we can choose

$$\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, 2\mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4\}.$$

With $\boldsymbol{\gamma} = (-a, -b, c - 1, d - 1, 0, 0)$ we get $\boldsymbol{\alpha} = (-a, -b, c - 1, d - 1)$.

Define $V_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, $V_2 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4\}$, $V_3 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, 2\mathbf{e}_1 - \mathbf{e}_3\}$ and $V_4 = \{\mathbf{e}_1, \mathbf{e}_2, 2\mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4\}$. Then the determinant of the vectors in V_i is ± 1, so the vectors are the vertices of a 3-dimensional simplex.

Lemma 4.4.1. *(i) $C(V_1) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2, x_3, x_4 \geq 0\}$.*
(ii) $C(V_2) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2, x_3 \geq 0, x_4 \leq 0, x_1 + x_4 \geq 0, x_2 + x_4 \geq 0\}$.
(iii) $C(V_3) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2, x_4 \geq 0, x_3 \leq 0, x_1 + 2x_3 \geq 0\}$.
(iv) $C(V_4) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2 \geq 0, x_3, x_4 \leq 0, x_2 + x_4 \geq 0, x_1 + 2x_3 + x_4 \geq 0\}$.

Corollary 4.4.2.

$$\bigcup_{i=1}^{4} C(V_i) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2 \geq 0, x_1 + 2x_3 \geq 0, x_1 + x_4 \geq 0, x_2 + x_4 \geq 0, x_1 + 2x_3 + x_4 \geq 0\}.$$

Lemma 4.4.3. *$\mathcal{T} = \{Q(V_1), Q(V_2), Q(V_3), Q(V_4)\}$ is a triangulation of $Q(\mathcal{A})$.*

Corollary 4.4.4. *$\mathcal{A}$ is saturated, the volume of $Q(\mathcal{A})$ is 4 and*

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1, x_2 \geq 0, x_1 + 2x_3 \geq 0, x_1 + x_4 \geq 0, x_2 + x_4 \geq 0, x_1 + 2x_3 + x_4 \geq 0\}.$$

$H_4(a, b, c, d|x, y)$ is non-resonant if and only if a, b, $2c - a$, $d - a$, $d - b$ and $2c + d - a$ are non-integral.

Theorem 4.4.5. *Suppose that $H_4(a, b, c, d|x, y)$ is non-resonant. Then there are 4 apexpoints if and only if either $\{a\} \leq \{d\} < \{b\}$ and $2\{c\} < \{a\} + 1 \leq 2\{c\} + \{d\}$, or $\{b\} \leq \{d\} < \{a\} \leq 2\{c\}$ and $2\{c\} + \{d\} < \{a\} + 1$.*

Proof. Suppose that $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha} \in C(V_1)$ is an apexpoint. From $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3 + p_4\mathbf{e}_4$ it follows that $\mathbf{x} = \mathbf{0}$.

If $\mathbf{p} \in C(V_2)$ is an apexpoint, then since $\mathbf{p} = (p_1 + p_4)\mathbf{e}_1 + (p_2 + p_4)\mathbf{e}_2 + p_3\mathbf{e}_3 - p_4(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4)$, we have $x_3 = 0$. Furthermore, $-1 < p_4 \leq 0$ and $0 \leq p_1 + p_4, p_2 + p_4 < 1$. If $\alpha_4 = 0$, then $\mathbf{x} = \mathbf{0}$. It follows that there can only be 4 apexpoints if $\alpha_4 > 0$. Then $x_4 = -1$. For $k = 1, 2$, we have $0 \leq x_k < 1 - \alpha_k - x_4 - \alpha_4 \leq 1 - x_4$, so $0 \leq x_1, x_2 \leq -x_4$.

If $\mathbf{p}$ is an apexpoint in $C(V_3)$, then $\mathbf{p} = (p_1 + 2p_3)\mathbf{e}_1 + p_2\mathbf{e}_2 + p_4\mathbf{e}_4 - p_3(2\mathbf{e}_1 - \mathbf{e}_3)$ with $x_2 = x_4 = 0$, $0 \leq p_1 + 2p_3 < 1$ and $-1 < p_3 \leq 0$. We must have $\alpha_3 > 0$, $x_3 = -1$ and $0 \leq x_1 \leq 2$.

Finally, suppose that $\mathbf{p} \in C(V_4)$ is an apexpoint. Since $\mathbf{p} = (p_1 + 2p_3 + p_4)\mathbf{e}_1 + (p_2 + p_4)\mathbf{e}_2 - p_3(2\mathbf{e}_1 - \mathbf{e}_3) - p_4(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4)$, we have $-1 < p_3, p_4 \leq 0$ and $0 \leq p_1 + 2p_3 + p_4, p_2 + p_4 < 1$. $\alpha_3 > 0$ and $\alpha_4 > 0$, so $x_3 = x_4 = -1$, $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 1$.

For 12 of these 16 points there exists $\boldsymbol{\alpha}$ such that $\mathbf{x} + \boldsymbol{\alpha}$ is an apexpoint. These 12 points are listed in table 10.

Table 10: The possible apexpoints for $H_4(a, b, c, d|x, y)$

i	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
1	(0,0,0,0)	$\alpha_1 + 2\alpha_3 < 2$ and ($\alpha_1 + \alpha_4 < 1$ or $\alpha_2 + \alpha_4 < 1$)
2	(0,0,0,-1)	$\alpha_1 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_4 \geq 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 < 3$
	(0,1,0,-1)	$\alpha_1 + \alpha_4 \geq 1$ and $\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 < 3$
	(1,0,0,-1)	$\alpha_1 + \alpha_4 < 1$ and $\alpha_2 + \alpha_4 \geq 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 < 2$
3	(0,0,-1,0)	$\alpha_1 + 2\alpha_3 \geq 2$ and ($\alpha_2 + \alpha_4 < 1$ or $\alpha_1 + 2\alpha_3 + \alpha_4 < 3$)
	(1,0,-1,0)	$1 \leq \alpha_1 + 2\alpha_3 < 2$ and ($\alpha_2 + \alpha_4 < 1$ or $\alpha_1 + 2\alpha_3 + \alpha_4 < 2$)
4	(0,0,-1,-1)	$\alpha_2 + \alpha_4 \geq 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 \geq 3$
	(0,1,-1,-1)	$\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 \geq 3$
	(1,0,-1,-1)	$\alpha_2 + \alpha_4 \geq 1$ and $2 \leq \alpha_1 + 2\alpha_3 + \alpha_4 < 3$
	(1,1,-1,-1)	$\alpha_1 + 2\alpha_3 < 2$ and $\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 \geq 2$
	(2,0,-1,-1)	$\alpha_1 + \alpha_4 < 1$ and $\alpha_2 + \alpha_4 \geq 1$ and $1 \leq \alpha_1 + 2\alpha_3 + \alpha_4 < 2$
	(2,1,-1,-1)	$\alpha_1 + 2\alpha_3 < 1$ and $\alpha_1 + \alpha_4 < 1$ and $\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 \geq 1$

There are 4 apexpoints if and only if either $\alpha_1 + 2\alpha_3 < 2$, $\alpha_1 + \alpha_4 \geq 1$, $\alpha_2 + \alpha_4 < 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 \geq 2$, or $\alpha_1 + 2\alpha_3 \geq 1$, $\alpha_1 + \alpha_4 < 1$, $\alpha_2 + \alpha_4 \geq 1$ and $\alpha_1 + 2\alpha_3 + \alpha_4 < 2$. □

Lemma 4.4.6. *If $H_4(a, b, c, d|x, y)$ is non-resonant and algebraic, then the Gauss series $F(b, a, d|z)$ and $F(b, a - 2c, d|z)$ are irreducible and algebraic.*

Proof. It is clear that $F(b, a, d|z)$ and $F(b, a - 2c, d|z)$ are irreducible.

Since we have $F(b, a, d|z) = F(a, b, d|z) = H_4(a, b, c, d|0, z)$, $F(b, a, d|z)$ is also algebraic.

If $\{a\} \leq \{d\} < \{b\}$ and $2\{c\} < \{a\} + 1 \leq 2\{c\} + \{d\}$, then we have $\{a\} - 2\{c\} + 1 > 0$, so $\{a - 2c\} \leq \{a\} - 2\{c\} + 1 < \{d\} < \{b\}$. If $\{b\} \leq \{d\} < \{a\} \leq 2\{c\}$ and $2\{c\} + \{d\} < \{a\} + 1$, then $\{a\} - 2\{c\} < 0$ (since $a - 2c \notin \mathbb{Z}$), and hence $\{b\} \leq \{d\} < \{a\} - 2\{c\} + 1 \leq \{a - 2c\}$. Hence the interlacing condition for $H_4(a, b, c, d|x, y)$ implies the interlacing condition for $F(b, a - 2c, d|z)$. □

Theorem 4.4.7. *$H_4(a, b, c, d|x, y)$ is non-resonant and algebraic if and only if (a, b, c, d) is conjugate to one of the tuples in Table 11.*

Proof. If $H_4(a, b, c, d|x, y)$ is non-resonant and algebraic, then (b, a, d) and $(b, a - 2c, d)$ are Gauss triples.

Assume that (b, a, d) and $(b, a - 2c, d)$ are both of type 1. Suppose that $d = \frac{1}{2}$. Then we have $(b, a, a - 2c, d) \in \{(r, -r, -r, \frac{1}{2}), (r, -r, r + \frac{1}{2}, \frac{1}{2}), (r, r + \frac{1}{2}, -r, \frac{1}{2}), (r, r + \frac{1}{2}, r + \frac{1}{2}, \frac{1}{2})\}$ for some $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ (up to equivalence modulo $\mathbb{Z}$). Since the interlacing condition implies that $c \notin \mathbb{Z}$, we have $(a, b, c, d) \in \{(-r, r, \frac{1}{2}, \frac{1}{2}), (-r, r, -r + \frac{1}{4}, \frac{1}{2}), (-r, r, -r + \frac{3}{4}, \frac{1}{2}), (r + \frac{1}{2}, r, r + \frac{1}{4}, \frac{1}{2}), (r + \frac{1}{2}, r, r + \frac{3}{4}, \frac{1}{2}), (r + \frac{1}{2}, r, \frac{1}{2}, \frac{1}{2})\}$. Notice that all corresponding functions are non-resonant. By checking the interlacing condition, we find six possibilities for (a, b, c, d): either $(a, b, c, d) \in \{(-r, r, \frac{1}{2}, \frac{1}{2}), (r + \frac{1}{2}, r, \frac{1}{2}, \frac{1}{2})\}$; or $r < \frac{1}{2}$ and $(a, b, c, d) \in \{(-r, r, -r + \frac{3}{4}, \frac{1}{2}), (r + \frac{1}{2}, r, r + \frac{1}{4}, \frac{1}{2})\}$; or $r > \frac{1}{2}$ and $(a, b, c, d) \in \{(-r, r, -r + \frac{1}{4}, \frac{1}{2}), (r + \frac{1}{2}, r, r + \frac{3}{4}, \frac{1}{2})\}$.

If (b, a, d) and $(b, a - 2c, d)$ are of type 1 and $d \neq \frac{1}{2}$, then there exists $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that $(b, a, a - 2c, d) = (r, r + \frac{1}{2}, r + \frac{1}{2}, 2r)$. Then $c = \frac{1}{2}$, so $(a, b, c, d) = (r + \frac{1}{2}, r, \frac{1}{2}, 2r)$, and one easily checks that the function is non-resonant and the interlacing condition is satisfied.

If (b, a, d) is of type 1, but $(b, a - 2c, d)$ is of type 2, then there exists $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that $(b, a, d) \in \{(r, -r, \frac{1}{2}), (r, r + \frac{1}{2}, \frac{1}{2}), (r, r + \frac{1}{2}, 2r)\}$, and the denominator of r is at most 60. We can check the interlacing condition for all possibilities, and find no solutions.

If (b,a,d) is of type 2 and $(b,a-2c,d)$ is of type 1, then there exists $r \in (0,1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$ such that $(b,a-2c,d) \in \{(r,-r,\frac{1}{2}),(r,r+\frac{1}{2},\frac{1}{2}),(r,r+\frac{1}{2},2r)\}$, and the denominator of r is at most 60. Again we try all possibilities and find no solutions.

Finally, if (b,a,d) and $(b,a-2c,d)$ are both of type 2, then there are only finitely many possibilities. This gives 444 non-resonant algebraic functions. The smallest parameter vector in each orbit is given in Table 11. □

Table 11: The tuples (a,b,c,d) such that $H_4(a,b,c,d|x,y)$ is non-resonant and algebraic

$(-r,r,-r+\frac{3}{4},\frac{1}{2})$		$(r+\frac{1}{2},r,r+\frac{1}{4},\frac{1}{2})$		with $r \in (0,\frac{1}{2}) \cap \mathbb{Q}$	
$(-r,r,-r+\frac{1}{4},\frac{1}{2})$		$(r+\frac{1}{2},r,r+\frac{3}{4},\frac{1}{2})$		with $r \in (\frac{1}{2},1) \cap \mathbb{Q}$	
$(-r,r,\frac{1}{2},\frac{1}{2})$	$(r+\frac{1}{2},r,\frac{1}{2},\frac{1}{2})$	$(r+\frac{1}{2},r,\frac{1}{2},2r)$		with $r \in (0,1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$	
$(\frac{1}{2},\frac{1}{6},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{2},\frac{1}{6},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{4},\frac{3}{4},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{4},\frac{7}{12},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{4},\frac{7}{12},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{2}{3})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{2}{3})$	$(\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{6},\frac{11}{30},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{6},\frac{11}{30},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{10},\frac{7}{10},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{10},\frac{7}{10},\frac{2}{5},\frac{2}{5})$
$(\frac{1}{10},\frac{9}{10},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{2},\frac{4}{5})$	$(\frac{1}{10},\frac{9}{10},\frac{1}{5},\frac{4}{5})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{10},\frac{13}{30},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{12},\frac{3}{4},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{12},\frac{3}{4},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{12},\frac{3}{4},\frac{1}{3},\frac{1}{2})$	$(\frac{1}{12},\frac{5}{6},\frac{1}{2},\frac{2}{3})$	$(\frac{1}{12},\frac{5}{6},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{12},\frac{5}{6},\frac{1}{4},\frac{2}{3})$
$(\frac{1}{12},\frac{5}{12},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{12},\frac{7}{12},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{15},\frac{7}{15},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{15},\frac{11}{15},\frac{1}{2},\frac{3}{5})$
$(\frac{1}{15},\frac{13}{15},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{15},\frac{13}{15},\frac{1}{2},\frac{3}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{20},\frac{11}{20},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{20},\frac{13}{20},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{20},\frac{17}{20},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{20},\frac{17}{20},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{24},\frac{13}{24},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{24},\frac{17}{24},\frac{1}{2},\frac{1}{4})$
$(\frac{1}{24},\frac{19}{24},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{24},\frac{19}{24},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{30},\frac{5}{6},\frac{1}{2},\frac{2}{3})$	$(\frac{1}{30},\frac{5}{6},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{30},\frac{5}{6},\frac{1}{5},\frac{2}{3})$	$(\frac{1}{30},\frac{7}{10},\frac{1}{2},\frac{1}{3})$
$(\frac{1}{30},\frac{7}{10},\frac{1}{2},\frac{2}{5})$	$(\frac{1}{30},\frac{7}{10},\frac{1}{3},\frac{2}{5})$	$(\frac{1}{30},\frac{11}{30},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{30},\frac{19}{30},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{2},\frac{1}{3})$	$(\frac{1}{60},\frac{31}{60},\frac{1}{2},\frac{1}{5})$
$(\frac{1}{60},\frac{41}{60},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{60},\frac{41}{60},\frac{1}{2},\frac{1}{5})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{60},\frac{49}{60},\frac{1}{2},\frac{1}{3})$		

4.5 The Horn H_5 function

The H_5 function is given by

$$H_5(a,b,c|x,y) = \sum_{m,n\geq 0} \frac{(a)_{2m+n}(b)_{n-m}}{(c)_n m! n!} x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-2,1,0,1,0) \oplus \mathbb{Z}(-1,-1,1,0,1)$. We can take

$$\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, 2\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$$

and $\boldsymbol{\gamma} = (-a,-b,c-1,0,0)$, so $\boldsymbol{\alpha} = (-a,-b,c-1)$.

Projecting $\mathcal{A}$ onto the plane $x_3 = 0$ gives an idea how to construct a simplex decomposition. The thick dots in Figure 4 represent $\mathcal{A}$, the dark gray area is the set $Q(\mathcal{A})$ and light gray area is the set $C(\mathcal{A})$.

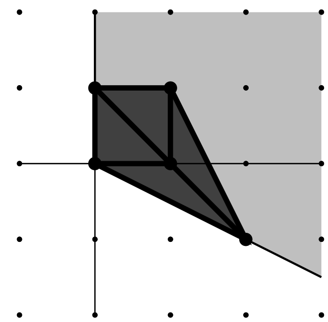

Figure 4: The sets $\mathcal{A}$, $Q(\mathcal{A})$ and $C(\mathcal{A})$ for H_5

Define the sets V_i by $V_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $V_2 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$, $V_3 = \{\mathbf{e}_1, \mathbf{e}_3, 2\mathbf{e}_1 - \mathbf{e}_2\}$ and $V_4 = \{\mathbf{e}_1, 2\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$. Then V_i is a set of vectors with determinant ± 1, so its elements are the vertices of a 2-dimensional simplex with volume 1.

Lemma 4.5.1. *(i)* $C(V_1) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\}$.
(ii) $C(V_2) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1, x_2 \geq 0, x_3 \leq 0, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0\}$.
(iii) $C(V_3) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1, x_3 \geq 0, x_2 \leq 0, x_1 + 2x_2 \geq 0\}$.
(iv) $C(V_4) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 0, x_3 \leq 0, x_2 + x_3 \leq 0, x_1 + 2x_2 + 3x_3 \geq 0\}$.

Lemma 4.5.2. $\bigcup_{i=1}^4 C(V_i) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 0, x_1 + x_3 \geq 0, x_1 + 2x_2 \geq 0, x_1 + 2x_2 + 3x_3 \geq 0\}$.

Lemma 4.5.3. $\mathcal{T} = \{Q(V_1), Q(V_2), Q(V_3), Q(V_4)\}$ *is a triangulation of* $Q(\mathcal{A})$.

Corollary 4.5.4. $\mathcal{A}$ *is saturated, the volume of* $Q(\mathcal{A})$ *is 4 and*

$$C(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 0, x_1 + x_3 \geq 0, x_1 + 2x_2 \geq 0, x_1 + 2x_2 + 3x_3 \geq 0\}.$$

$H_5(a, b, c|x, y)$ *is non-resonant if and only if* a, $a + 2b$, $c - a$ *and* $3c - a - 2b$ *are non-integral.*

Theorem 4.5.5. *Suppose that* $H_5(a, b, c|x, y)$ *is non-resonant. Then there are 4 apexpoints if and only if either* $\{a\} \leq \{c\}$, $1 < \{a\} + 2\{b\} \leq 2$ *and* $3\{c\} < \{a\} + 2\{b\} \leq 3\{c\} + 1$, *or* $\{c\} < \{a\}$, $1 < \{a\} + 2\{b\} \leq 2$ *and* $3\{c\} - 1 < \{a\} + 2\{b\} \leq 3\{c\}$.

Proof. Suppose that $\mathbf{p} = \mathbf{x} + \boldsymbol{\alpha} \in C(V_1)$ is an apexpoint. Then $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$ and by Remark 1.3.5, $0 \leq p_i < 1$ so $\mathbf{x} = \mathbf{0}$.

Suppose that $\mathbf{p} \in C(V_2)$ is an apexpoint. Since $\mathbf{p} = (p_1 + p_3)\mathbf{e}_1 + (p_2 + p_3)\mathbf{e}_2 - p_3(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$, we have $0 \leq p_1 + p_3, p_2 + p_3 < 1$ and $-1 < p_3 \leq 0$. If $\alpha_3 = 0$, then $p_3 = 0$, so $\mathbf{x} = \mathbf{0}$. Then we don't have 4 distinct apexpoints, so $\alpha_3 > 0$. Hence $x_3 = -1$ and for $k \in \{1, 2\}$, we have $0 \leq x_k < 1 - \alpha_k - x_3 - \alpha_3 < 2$, so $x_1, x_2 \in \{0, 1\}$.

If $\mathbf{p}$ is an apexpoint in $C(V_3)$, then it follows from $\mathbf{p} = (p_1 + 2p_2)\mathbf{e}_1 - p_2(2\mathbf{e}_1 - \mathbf{e} - 2) + p_3\mathbf{e}_3$ that $x_3 = 0$, $0 \leq p_1 + 2p_2 < 1$ and $-1 < p_2 \leq 0$. If $\alpha_2 = 0$, then $x_2 = 0$ and hence $\mathbf{x} = \mathbf{0}$, so we must have $\alpha_2 > 0$. This implies that $x_2 = -1$ and $0 \leq x_1 < 1 - \alpha_1 - 2x_2 - 2\alpha_2 < 3$, so $x_1 \in \{0, 1, 2\}$.

Finally, if $\mathbf{p}$ is an apexpoint in $C(V_4)$, then $\mathbf{p} = (p_1 + 2p_2 + 3p_3)\mathbf{e}_1 - (p_2 + p_3)(2\mathbf{e}_1 - \mathbf{e}_2) - p_3(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$ with $-1 < p_2 + p_3, p_3 \leq 0$ and $0 \leq p_1 + 2p_2 + 3p_3 < 1$. Since $\alpha_3 > 0$, we have $x_3 = -1$. It follows that $x_2 \in \{-1, 0\}$, and $0 \leq x_1 \leq 3 - 2x_2$.

This gives 16 possible apexpoints. It turns out that 10 of these are indeed apexpoints if $\boldsymbol{\alpha}$ satisfies certain conditions. This gives the points in Table 12.

There are two possibilities to have 4 apexpoints: either $\alpha_1 + \alpha_3 \geq 1$, $2 \leq \alpha_1 + 2\alpha_2 + 3\alpha_3 < 3$ and $1 \leq \alpha_1 + 2\alpha_2 < 2$, or $\alpha_1 + \alpha_3 > 1$, $3 \leq \alpha_1 + 2\alpha_2 + 3\alpha_3 < 4$ and $1 \leq \alpha_1 + 2\alpha_2 < 2$.

Recall that $\boldsymbol{\alpha} = (-a, -b, c - 1)$. Since the function is non-resonant, a is non-integral. We proved that $\alpha_2 > 0$, so $\boldsymbol{\alpha} = (1 - \{a\}, 1 - \{b\}, \{c\})$. Now the result follows. ∎

Table 12: The possible apexpoints for $H_5(a,b,c|x,y)$

i	$\mathbf{x}$	Conditions on $\boldsymbol{\alpha}$
1	(0,0,0)	$\alpha_1+2\alpha_2<2$ and $(\alpha_1+\alpha_3<1$ or $\alpha_1+2\alpha_2+3\alpha_3<3)$
2	(0,1,-1)	$\alpha_1+\alpha_3\geq 1$ and $\alpha_1+2\alpha_2+3\alpha_3<3$
3	(0,-1,0)	$\alpha_1+2\alpha_2\geq 2$ and $\alpha_1+2\alpha_2+3\alpha_3<5$
	(1,-1,0)	$1\leq\alpha_1+2\alpha_2<2$ and $\alpha_1+2\alpha_2+3\alpha_3<4$
4	(0,-1,-1)	$\alpha_1+2\alpha_2+3\alpha_3\geq 5$
	(1,-1,-1)	$4\leq\alpha_1+2\alpha_2+3\alpha_3<5$
	(2,-1,-1)	$\alpha_1+\alpha_3<1$ and $3\leq\alpha_1+2\alpha_2+3\alpha_3<4$
	(2,0,-1)	$\alpha_1+2\alpha_2<1$ and $\alpha_1+\alpha_3<1$ and $1\leq\alpha_1+2\alpha_2+3\alpha_3<2$
2,4	(0,0,-1)	$\alpha_1+\alpha_3\geq 1$ and $3\leq\alpha_1+2\alpha_2+3\alpha_3<5$
	(1,0,-1)	$\alpha_1+2\alpha_2<2$ and $2\leq\alpha_1+2\alpha_2+3\alpha_3<4$ and $(\alpha_1+\alpha_3<1$ or $\alpha_1+2\alpha_2+3\alpha_3<3)$

Theorem 4.5.6. *$H_5(a,b,c|x,y)$ is non-resonant and algebraic if and only if (a,b,c) is, up to equivalence modulo $\mathbb{Z}$, conjugate to one of the following: $(r,-r,\frac{1}{2})$ for some $r\in(0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$, $(\frac{1}{6},\frac{1}{2},\frac{1}{3})$, $(\frac{1}{6},\frac{2}{3},\frac{1}{3})$, $(\frac{1}{6},\frac{5}{6},\frac{1}{3})$, $(\frac{1}{10},\frac{3}{5},\frac{1}{5})$ and $(\frac{1}{12},\frac{3}{4},\frac{1}{2})$.*

Proof. Suppose that $H_5(a,b,c|x,y)$ is non-resonant and algebraic. Then $F(a,b,c|z)$ is also algebraic because $H_5(a,b,c|0,z)=F(a,b,c|z)$. Furthermore, a and $c-a$ are non-integral. It follows from the interlacing condition that b and $c-a$ are also non-integral. Hence $F(a,b,c|z)$ is irreducible. Therefore, to find all non-resonant algebraic H_5 functions, it suffices to check which Gauss triples satisfy the non-resonance and interlacing conditions for the H_5 function.

First assume that (a,b,c) is a Gauss triple of type 1. Then there exists $r\in(0,1)\cap\mathbb{Q}\setminus\{\frac{1}{2}\}$ such that up to equivalence modulo $\mathbb{Z}$, $(a,b,c)\in\{(r,-r,\frac{1}{2}),(r,r+\frac{1}{2},\frac{1}{2}),(r,r+\frac{1}{2},2r)\}$.

If $(a,b,c)=(r,-r,\frac{1}{2})$, then the function is non-resonant. If $r<\frac{1}{2}$, then $1<\{a\}+2\{b\}\leq 2$ and $3\{c\}<\{a\}+2\{b\}\leq 3\{c\}+1$, and if $r>\frac{1}{2}$, then $\{c\}<\{a\}$, $1<\{a\}+2\{b\}\leq 2$ and $3\{c\}-1<\{a\}+2\{b\}\leq 3\{c\}$. Hence the interlacing condition is satisfied. Since all conjugates are of the same form, they also satisfy the interlacing condition, so the function is algebraic.

Suppose that $(a,b,c)=(r,r+\frac{1}{2},\frac{1}{2})$. The function is non-resonant if r is not equal to $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$ or $\frac{5}{6}$. Then the interlacing condition is satisfied if and only if $r\in(\frac{1}{6},\frac{1}{3}]\cup(\frac{2}{3},\frac{5}{6}]$. All conjugates of r also have to be in this set. By choosing a conjugate with numerator 1, we get that the denominator of r can at most be 5. $\frac{1}{4}$ and $\frac{3}{4}$ give the same tuple as for $(r,-r,\frac{1}{2})$. If r has denominator 5, then $\frac{2}{5}$ is a conjugate that doesn't satisfy the condition. Hence this gives no extra algebraic functions.

Finally, suppose that $(a,b,c)=(r,r+\frac{1}{2},2r)$. Then the function is non-resonant if r is not equal to $\frac{1}{3}$ or $\frac{2}{3}$. The interlacing condition is satisfied if and only if $r\in(0,\frac{1}{3})\cup(\frac{2}{3},1)$. By Proposition 1.2.12, this implies that the denominator of r is 4, 6 or 10. If the denominator is 4, then we again get the tuple $(r,-r,\frac{1}{2})$. There are two solutions with denominator 6: $(\frac{1}{6},\frac{2}{3},\frac{1}{3})$ and $(\frac{5}{6},\frac{1}{3},\frac{2}{3})$. With denominator 10, then we find the solutions $(\frac{1}{10},\frac{3}{5},\frac{1}{5})$, $(\frac{3}{10},\frac{4}{5},\frac{3}{5})$, $(\frac{7}{10},\frac{1}{5},\frac{2}{5})$ and $(\frac{9}{10},\frac{2}{5},\frac{4}{5})$. For all these tuples, the interlacing condition is indeed satisfied.

If (a,b,c) is a Gauss triple of type 2, then there are only finitely many possibilities. We use the computer to check the conditions, and find 8 solutions: $\pm(\frac{1}{6},\frac{1}{2},\frac{1}{3})$, $\pm(\frac{1}{6},\frac{5}{6},\frac{1}{3})$, $\pm(\frac{1}{12},\frac{3}{4},\frac{1}{2})$ and $\pm(\frac{5}{12},\frac{3}{4},\frac{1}{2})$. □

4.6 The Horn H_6 function

The H_6 function is given by

$$H_6(a,b,c|x,y)=\sum_{m,n\geq 0}\frac{(a)_{2m-n}(b)_{n-m}(c)_n}{m!n!}x^m y^n.$$

Hence the lattice is $\mathbb{L} = \mathbb{Z}(-2,1,0,1,0) \oplus \mathbb{Z}(1,-1,-1,0,1)$. We choose

$$\mathcal{A} = \{\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3\}$$

and $\boldsymbol{\gamma} = (-a,-b,-c,0,0)$. Then $\boldsymbol{\alpha} = (-a-b,-b,a-b-c)$.

Up to the order of the vectors, we have the same set $\mathcal{A}$ as for the G_1 function. Hence we can use all results from section 3.1. The vector $\boldsymbol{\alpha} = (-a-b,-b,a-b-c)$ corresponding to the function $H_6(a,b,c|x,y)$ also corresponds to $G_1(\tilde{a},\tilde{b_1},\tilde{b_2}|x,y)$, where $\tilde{a}$, $\tilde{b_1}$ and $\tilde{b_2}$ are chosen such that $\boldsymbol{\alpha} = (-\tilde{a},-\tilde{b_1},-\tilde{a}-\tilde{b_1}-\tilde{b_2})$. Therefore we can find all non-resonant algebraic H_3 functions by computing $(a,b,c) = (\tilde{a}-\tilde{b_1},\tilde{b_1}+\tilde{b_2})$ for the non-resonant algebraic $G_1(\tilde{a},\tilde{b_1},\tilde{b_2}|x,y)$ functions from Theorem 3.1.7.

Lemma 4.6.1. *$H_6(a,b,c|x,y)$ is non-resonant if and only if $a+b$, $a+2b$, c and $a+c$ are non-integral.*

Theorem 4.6.2. *$H_6(a,b,c|x,y)$ is non-resonant and algebraic if and only if up to equivalence modulo $\mathbb{Z}$, (a,b,c) is one of the following: $\pm(\frac{1}{2},\frac{1}{3},\frac{2}{3})$, $\pm(\frac{1}{2},\frac{1}{3},\frac{5}{6})$ and $\pm(\frac{1}{3},\frac{1}{2},\frac{5}{6})$.*

4.7 The Horn H_7 function

The H_7 function is given by

$$H_7(a,b,c,d|x,y) = \sum_{m,n\geq 0} \frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m m!n!} x^m y^n.$$

The lattice is $\mathbb{L} = \mathbb{Z}(-2,0,0,1,1,0) \oplus \mathbb{Z}(1,-1,-1,0,0,1)$. We can take

$$\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_3, 2\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2\}$$

and $\boldsymbol{\gamma} = (-a,-b,-c,d-1,0,0)$. Then $\boldsymbol{\alpha} = (-a-b,-b,d-1,b-c)$.

Up to the order of the vectors, the set $\mathcal{A}$ is the same as for the H_4 function. Therefore, we use all results from section 4.4. The vector $\boldsymbol{\alpha} = (-a-b,-b,d-1,b-c)$ corresponding to $H_7(a,b,c,d|x,y)$ also corresponds to $H_4(\tilde{a},\tilde{b},\tilde{c},\tilde{d},|x,y)$, where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are chosen such that $\boldsymbol{\alpha} = (-\tilde{a},-\tilde{b},\tilde{c}-1,\tilde{d}-1)$. Therefore we can find all non-resonant algebraic H_7 functions by computing $(a,b,c,d,e) = (\tilde{a}-\tilde{b},\tilde{b},\tilde{b}-\tilde{d},\tilde{c})$ for the non-resonant algebraic $H_4(\tilde{a},\tilde{b},\tilde{c},\tilde{d},|x,y)$ functions from Theorem 4.4.7.

Lemma 4.7.1. *$H_7(a,b,c,d|x,y)$ is non-resonant if and only if b, c, $a+b$, $a+c$, $2d-a-b$ and $2d-a-c$ are non-integral.*

Theorem 4.7.2. *$H_7(a,b,c,d|x,y)$ is non-resonant and algebraic if and only if at least one of (a,b,c,d) or (a,c,b,d) is conjugate to one of the tuples in Table 13.*

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Table 13: The tuples (a, b, c, d) such that $H_7(a, b, c, d|x, y)$ is non-resonant and algebraic

$(\frac{1}{2}, r, r+\frac{1}{2}, r+\frac{1}{4})$		$(-2r, r, r+\frac{1}{2}, -r+\frac{3}{4})$		with $r \in (0, \frac{1}{2}) \cap \mathbb{Q}$	
$(\frac{1}{2}, r, r+\frac{1}{2}, r+\frac{3}{4})$		$(-2r, r, r+\frac{1}{2}, -r+\frac{1}{4})$		with $r \in (\frac{1}{2}, 1) \cap \mathbb{Q}$	
$(\frac{1}{2}, r, -r, \frac{1}{2})$	$(\frac{1}{2}, r, r+\frac{1}{2}, \frac{1}{2})$	$(-2r, r, r+\frac{1}{2}, \frac{1}{2})$		with $r \in (0, 1) \cap \mathbb{Q} \setminus \{\frac{1}{2}\}$	
$(\frac{1}{2}, \frac{1}{4}, \frac{7}{12}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{20}, \frac{13}{20}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{24}, \frac{17}{24}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{24}, \frac{19}{24}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{60}, \frac{41}{60}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{60}, \frac{49}{60}, \frac{1}{2})$
$(\frac{1}{3}, \frac{1}{2}, \frac{5}{6}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{4}, \frac{11}{12}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{3})$
$(\frac{1}{3}, \frac{5}{6}, \frac{1}{12}, \frac{1}{2})$	$(\frac{1}{3}, \frac{5}{6}, \frac{1}{30}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{10}, \frac{9}{10}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{10}, \frac{9}{10}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{10}, \frac{23}{30}, \frac{1}{2})$	$(\frac{1}{3}, \frac{5}{12}, \frac{11}{12}, \frac{1}{2})$
$(\frac{1}{3}, \frac{2}{15}, \frac{11}{15}, \frac{1}{2})$	$(\frac{1}{3}, \frac{5}{24}, \frac{17}{24}, \frac{1}{2})$	$(\frac{1}{3}, \frac{5}{24}, \frac{23}{24}, \frac{1}{2})$	$(\frac{1}{3}, \frac{11}{30}, \frac{29}{30}, \frac{1}{2})$	$(\frac{1}{3}, \frac{11}{60}, \frac{41}{60}, \frac{1}{2})$	$(\frac{1}{3}, \frac{11}{60}, \frac{59}{60}, \frac{1}{2})$
$(\frac{1}{4}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2})$	$(\frac{1}{4}, \frac{1}{6}, \frac{5}{6}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{6}, \frac{11}{12}, \frac{1}{2})$	$(\frac{1}{4}, \frac{7}{12}, \frac{11}{12}, \frac{1}{2})$	$(\frac{1}{4}, \frac{7}{24}, \frac{19}{24}, \frac{1}{2})$	$(\frac{1}{4}, \frac{7}{24}, \frac{23}{24}, \frac{1}{2})$
$(\frac{1}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2})$	$(\frac{1}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{5})$	$(\frac{1}{5}, \frac{1}{6}, \frac{29}{30}, \frac{1}{2})$	$(\frac{1}{5}, \frac{1}{10}, \frac{9}{10}, \frac{1}{2})$	$(\frac{1}{5}, \frac{1}{10}, \frac{9}{10}, \frac{1}{5})$	$(\frac{1}{5}, \frac{7}{10}, \frac{9}{10}, \frac{1}{2})$
$(\frac{1}{5}, \frac{9}{10}, \frac{7}{30}, \frac{1}{2})$	$(\frac{1}{5}, \frac{4}{15}, \frac{13}{15}, \frac{1}{2})$	$(\frac{1}{5}, \frac{4}{15}, \frac{14}{15}, \frac{1}{2})$	$(\frac{1}{5}, \frac{8}{15}, \frac{13}{15}, \frac{1}{2})$	$(\frac{1}{5}, \frac{7}{20}, \frac{17}{20}, \frac{1}{2})$	$(\frac{1}{5}, \frac{7}{20}, \frac{19}{20}, \frac{1}{2})$
$(\frac{1}{5}, \frac{9}{20}, \frac{19}{20}, \frac{1}{2})$	$(\frac{1}{5}, \frac{19}{30}, \frac{29}{30}, \frac{1}{2})$	$(\frac{1}{5}, \frac{19}{60}, \frac{49}{60}, \frac{1}{2})$	$(\frac{1}{5}, \frac{19}{60}, \frac{59}{60}, \frac{1}{2})$		

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