The authors propose some new log-bilinear models for the analysis of asymmetry in square contingency tables. In these models the logarithm of the expected frequency is split up into two parts: (a) a symmetric or a quasi-symmetric part, and (b) a skew-symmetric part. The skew-symmetric part is decomposed using the so-called Gower decomposition of skew-symmetric matrices. It is shown how graphical representations of this decomposition can simplify the interpretation.

Some New Log-Bilinear Models for the Analysis of Asymmetry in a Square Contingency Table

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1. INTRODUCTION

Square contingency tables with corresponding row and column categories occur in many substantive research areas. For example, in sociology, social mobility tables show how the profession of a son is related to the profession of his father; in psychology, confusion data show how often people mix up stimuli; in marketing, product-switching data show how often people change from one product to another; in bibliometrics, citation data show how often journals cite each other, and so on. The modeling of such square tables has long since been an active research domain.

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In the modeling of square contingency tables, several aspects can be of interest. We focus here on the asymmetry in the contingency table. Symmetry in square contingency tables has been studied with one log-linear model, namely the symmetry model. The quasi-symmetry model was proposed to deal with asymmetry due to heterogeneous margins, where the association was assumed to be symmetric. This model is a popular tool in many research areas. For example, in sociology, substantive interpretations to the parameters of the quasi-symmetry model are proposed by Sobel, Hout, and Duncan (1985). In psychometric models for confusion data, the unconstrained similarity choice model is a special case of quasi-symmetry (see Takane and Shibayama 1986, for an overview). However, in spite of its popularity, the quasi-symmetry model regularly does not fit in these areas (see, e.g., de Soete 1984 for confusion data and Yamaguchi 1990 for mobility data). This calls for different models that can take account of the asymmetry in the frequencies. For this purpose, we propose some different models that can be useful when the quasi-symmetry model does not fit adequately.

Earlier proposals for such modelings are summarized by Yamaguchi (1990). He begins with the quasi-symmetry model, and proposes models to study the departure from quasi-symmetry by introducing additional parameters. We will begin with both the symmetry model and the quasi-symmetry model, and propose a specific parametrization for the asymmetry in the table using properties of skew-symmetric matrices. When we begin with the symmetry model, it will turn out that this proposal has quasi-symmetry as a special case.

We discuss one example. It is an intergenerational occupational mobility table for the United States. The models proposed in this article show some new results for this data set.

2. PRELIMINARIES

Let n_{ij} ($i = 1, \ldots, I$; $j = 1, \ldots, I$) be the frequency in cell (i,j), and let m_{ij} be the expected frequency for cell (i,j). One way to write the saturated log-linear model is

$$\log m_{ij} = u + u_{1(i)} + u_{2(j)} + u_{ij} \qquad (1)$$

with identifying restrictions $\Sigma_i u_{1(i)} = \Sigma_j u_{2(j)} = \Sigma_i u_{ij} = \Sigma_j u_{ij} = 0$. The symmetry model can be written as

$$\log m_{ij} = u + u_{1(i)} + u_{2(j)} + u_{ij} \qquad (2)$$

with

$$u_{1(i)} = u_{2(i)} \text{ and } u_{ij} = u_{ji}.$$

The model has homogeneous marginal terms and symmetric association parameters. It has $I(I - 1)/2$ degrees of freedom. A less restrictive model is the quasi-symmetry model (Caussinus 1965). This model also assumes that the association is symmetric, but allows for different row and column margins. It can be written as

$$\log m_{ij} = u + u_{1(i)} + u_{2(j)} + u_{ij} \qquad (3)$$

with

$$u_{ij} = u_{ji}.$$

It has $(I - 1)(I - 2)/2$ degrees of freedom. This model allows for asymmetric frequencies $m_{ij} \neq m_{ji}$ resulting from marginal heterogeneity. In fact, Sobel et al. (1985) and Sobel (1988) show that if and only if model (3) is true the differences $u_{1(i)} - u_{2(i)}$ represent the asymmetry due to marginal heterogeneity.

In this article, we discuss models that can be useful when quasi-symmetry does not adequately represent the asymmetry in the table. We deal with this asymmetry in two ways. First, we will start off from the symmetry model and add extra parameters to this model. These parameters constitute a skew-symmetric matrix. Therefore, we will name this model symmetry plus skew-symmetry (SSK). It will turn out that this model has the quasi-symmetry model as a special case. A second way to approach this problem is by adding extra parameters to the quasi-symmetry model (cf. Yamaguchi 1990). These extra parameters also constitute a skew-symmetric matrix. Therefore this model is named quasi-symmetry plus skew-symmetry (QSSK). Another notation seems to be more useful for the introduction of these models.

We will reparametrize the saturated log-linear model in two ways. For the extension of the symmetry model to SSK we reparametrize the saturated model as

$$\log m_{ij} = s_{ij} + t_{ij} \quad (4a)$$

with

$$s_{ij} = s_{ji} \text{ and } t_{ij} = -t_{ji},$$

that is, the asymmetric elements m_{ij} are transformed by a logarithmic transformation and decomposed into symmetric elements s_{ij} and a skew-symmetric t_{ij} (cf. Sobel et al. 1985; Sobel 1988; Yamaguchi 1990). In terms of the saturated model (1), $s_{ij} = u + u_i + u_j + (u_{ij} + u_{ji})/2$, where $u_i \equiv (u_{1(i)} + u_{2(i)})/2$. Let $a_i \equiv u_{1(i)} - u_i \equiv -(u_{2(i)} - u_i)$, then $t_{ij} = (a_i - a_j) + (u_{ij} - u_{ji})/2$. Thus the symmetry model holds when $t_{ij} = 0$, whereas the quasi-symmetry model holds when $t_{ij} = (a_i - a_j)$, so under the quasi-symmetry model the differences $u_{1(i)} - u_{2(i)}$ are used to form a skew-symmetric matrix. The parameters a_i are directly interpretable as marginal shift parameters (Sobel 1988). In the general case the elements t_{ij} constitute both asymmetry due to marginal heterogeneity as well as asymmetry not due to marginal heterogeneity.

For the extension of the quasi-symmetry model to QSSK, we reparametrize the saturated model as

$$\log m_{ij} = q_{ij} + t_{ij} \quad (4b)$$

with

$$q_{ij} \text{ quasi-symmetric and } t_{ij} = -t_{ji}$$

(cf. Yamaguchi 1990). In terms of the saturated model (1), $q_{ij} = u + u_{1(i)} + u_{2(j)} + (u_{ij} + u_{ji})/2$, and $t_{ij} = (u_{ij} - u_{ji})/2$. Notice that in (4b), $\Sigma_i t_{ij} = \Sigma_j t_{ij} = 0$. In this model we are not concerned with the interpretation of $u_{1(i)} - u_{2(i)}$. In fact, we cannot interpret $u_{1(i)} - u_{2(i)}$ to measure marginal heterogeneity in a model that introduces parameters for skew-symmetry.

Keeping the findings of Sobel et al. (1985) and Sobel (1988) in mind, we should conclude the following about (4a) and (4b): if we are concerned with both marginal heterogeneity and asymmetric association, we should model the departure from symmetry, using (4a) as a starting point. However, if we are concerned only with asymmetric association that is not confounded with marginal heterogeneity, we should model the departure from quasi-symmetry, using (4b) as a starting point.

Thus far the elements t_{ij} have been unconstrained. We will now propose a constrained model for the skew-symmetric elements t_{ij}. We do this by using the so-called Gower decomposition of skew-symmetric matrices.

3. THE GOWER DECOMPOSITION OF SKEW-SYMMETRIC MATRICES

We propose to study the skew-symmetry in (4a) and (4b) by finding an optimal reduced rank approximation of a matrix with elements t_{ij}, factorizing it, and studying the factorization by means of graphical displays. In this section we study the properties of rank decompositions of skew-symmetric matrices that are studied extensively by Gower (1977, 1980), Constantine and Gower (1978, 1982), and Harshman (1981) (some uniqueness properties are discussed in Appendix A).

Let **H** be an I × I skew-symmetric matrix, with elements $h_{ij} = -h_{ji}$. Then the singular value decomposition of **H** has the following form:

$$\mathbf{H} = \mathbf{P}\boldsymbol{\Omega}\mathbf{Q}' = \mathbf{P}\boldsymbol{\Omega}\mathbf{Z}\mathbf{P}' \quad (5)$$

with $\mathbf{P}'\mathbf{P} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}$, where **I** is the identity matrix; $\boldsymbol{\Omega}$ is a diagonal matrix with singular values in nonincreasing order; **Z** is a block-diagonal matrix

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 & & & 0 \\ & \mathbf{Z}_2 & & \\ & & \ddots & \\ 0 & & & \mathbf{Z}_W \end{pmatrix}$$

with

$$\mathbf{Z}_w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (6)$$

and if I is uneven, then element (I, I) of **Z** is unity. The special property is, first, that $\mathbf{Q} = \mathbf{P}\mathbf{Z}'$, that is, the first (third, fifth, etc.) column of **Q** is equal to −1 times the second (fourth, sixth, etc.) column of **P**, and the second (fourth, sixth, etc.) column of **Q** is equal to the first (third, fifth, etc.) column of **P**. The second special property is that the singular values are ordered in pairs, that is, $\omega_1 = \omega_2$, $\omega_3 = \omega_4$, $\omega_5 = \omega_6$, and so on; if I is uneven, then $\omega_I = 0$. It shows that the subsequent pairs of columns in **P** are unidentified. This also shows that the rank of skew-symmetric matrices is always even. We index the values ω by w, the pairs of dimensions. The values ω_w can always be chosen to be nonnegative (see Appendix A).[1]

We now show how such a decomposition can be interpreted. This is most easily done by making a graphical display for each pair of dimensions. For this purpose first define

$$p^*_{iw(m)} = \omega_w^{1/2} p_{iw(m)}, \tag{7}$$

where $p_{iw(m)}$ the element of matrix **P** in row i and the *m*th column (m = 1, 2) of column pair w. For the paired dimension w a graphical display is made of the I categories by using the parameter pair $(p^*_{iw(1)}, p^*_{iw(2)})$ as coordinates for the point of category i in pair of dimensions w. Harshman (1981) coins such a pair of dimensions a "bimension." Gower (1977, 1980), Constantine and Gower (1978, 1982), and Harshman (1981) describe in detail how such graphical displays should be interpreted. The main results are these: In the graphical display the skew-symmetries are represented by twice the area of the triangle formed from the origin, category point i and category point j; the skew-symmetry is positive if the rotation from i to j is clockwise, and it is negative if the rotation from i to j is counterclockwise. So we do not interpret inner products, but areas of triangles.

Basically, this gives us enough information for understanding how to interpret such representations further: Categories farther away from the origin have larger skew-symmetries with other categories, and those categories that fall in the origin have no skew-symmetries with other categories. Given the distances of points to the origin, the area is maximal when points have a right angle with the origin. Points that lie on a line through the origin have no skew-symmetry with each other, and the pattern of skew-symmetries with the other categories is proportionally different where the skew-symmetries of the points farther away from the origin are larger. If all points are in a half plane, there are no circular triads in this approximation, in the sense that if A has a positive skew-symmetry with B, and B with C, then A must have a positive skew-symmetry with C.

It may take some time to get used to interpretation of the graphical displays, but after this the graphical displays can simplify the interpretation of skew-symmetric matrices considerably, especially when the number of categories is large and it is possible to represent the skew-symmetry adequately in two or four dimensions. Then the graphical displays can simplify the interpretation of the skew-symmetry considerably.

4. *APPLYING THE GOWER DECOMPOSITION TO SQUARE CONTINGENCY TABLES*

These properties suggest that we use decomposition (5) in models (4a) and (4b) to model the skew-symmetric elements t_{ij}. First, we work this out for the symmetry plus skew-symmetry model (SSK), then we work this out for the quasi-symmetry plus skew-symmetry model (QSSK).

SYMMETRY PLUS SKEW-SYMMETRY MODEL (SSK)

Assume first that one paired dimension is sufficient to model the skew-symmetry adequately. Then model (4a) becomes

$$\log m_{ij} = s_{ij} + \phi(v_{i(2)}v_{j(1)} - v_{i(1)}v_{j(2)}) \tag{8}$$

with

$$s_{ij} = s_{ji}$$

with identifying restrictions $\Sigma_i\, v_{i(m)}v_{i(n)} = \delta^{mn}$, where $\delta^{mn} = 1$ if m = n, zero else.[2]

Therefore an I × 2 matrix of parameters is estimated, and this matrix has elements $v_{i(1)}$ in the first column and elements $v_{i(2)}$ in the second column. The identifying restrictions impose variances of 1 for each dimension, and ensure that the scores for different dimensions are uncorrelated. The model has I(I − 1)/2 − (2I − 3) degrees of freedom, where (2I – 3) independent skew-symmetric parameters are fitted (see Appendix B).

The terms $t_{ij} = \phi(v_{i(2)}v_{j(1)} - v_{i(1)}v_{j(2)})$ constitute a skew-symmetric matrix. A graphical representation can be made to simplify the interpretation of the parameter estimates. For this purpose the estimates $(\hat{\phi}^{1/2}\hat{v}_{i(1)}, \hat{\phi}^{1/2}\hat{v}_{i(2)})$ are to be used as coordinates for the point of category i. The resulting graph of I points has to be interpreted in terms of areas of triangles, in the way described in section 3. Because it can be difficult to estimate the size of the areas by eye, it is often also useful to calculate $\hat{t}_{ij}$ from $\hat{\phi}$, $\hat{v}_{i(1)}$ and $\hat{v}_{i(2)}$. Because $m_{ij} = \exp(s_{ij})\exp(t_{ij})$, these estimates $\hat{t}_{ij}$ can also be used to measure the size of the skew-symmetry for the cell pair (i, j) – (j, i).

The likelihood equations for s_{ij} are $n_{ij} + n_{ji} = m_{ij} + m_{ji}$, and for $v_{i(1)}$ they are

$$\sum_{j=1}^{I} m_{ij}\phi v_{j(1)} - \sum_{i=1}^{I} m_{ij}\phi v_{i(1)} = \sum_{i=1}^{I} n_{ij}\phi v_{i(1)} - \sum_{j=1}^{I} n_{ij}\phi v_{j(1)}. \quad (9)$$

This shows that, in general, $n_{i+} \neq m_{i+}$ and $n_{+j} \neq m_{+j}$. This is unlike the quasi-symmetry model, which has likelihood equations $n_{i+} = m_{i+}$ and $n_{+i} = m_{+i}$. The reason is that in SSK (i.e., [8]) we model the departure from the symmetry model and not from the quasi-symmetry model. However, if quasi-symmetry is true, then SSK is true. In this case $t_{ij} = (a_i - a_j)$, and we may choose $v_{i(1)} = 1$ and $\phi v_{i(2)} = a_i$. So, if the quasi-symmetry model is true, the skew-symmetry is attributable to marginal heterogeneity only, and this yields a special skew-symmetric matrix of rank 2. In the graphical representation of the skew-symmetry, the I points fall on a straight line. Lines or approximate lines in graphical displays of skew-symmetric matrices have received considerable attention from Gower (1977, 1980), Constantine and Gower (1978, 1982), and Harshman (1981), but thus far it was not linked to the quasi-symmetry model in this way. Constantine and Gower (1982) noticed that if data y_{ij} follow model $E(y_{ij}) = \mu_{ij}\beta_j$ with $\mu_{ij} = \mu_{ji}$, then an analysis of the skew part of log y_{ij} would result in a line, but they did not mention that $\mu_{ij}\beta_j$ is one way to write the quasi-symmetry model (see Caussinus 1965). In general if SSK holds, the points will not lie on a straight line.

A more general model, which has SSK as a special case, is

$$\log m_{ij} = s_{ij} + \sum_{w=1}^{W} \phi_w (v_{iw(2)}v_{jw(1)} - v_{iw(1)}v_{jw(2)}) \quad (10)$$

with

$$s_{ij} = s_{ji}$$

with identifying restrictions $\Sigma_i v_{iw(m)}v_{iz(n)} = \delta^{wz}\delta^{mn}$. We name this model SSK(W). In this model W paired dimensions are used for the skew-symmetry. Now, interpretation becomes more complicated because for studying the skew-symmetry between i and j, W graphical displays have to be studied simultaneously. It follows from the properties discussed in section 3 that SSK(W) corresponds to the saturated model when W = I/2 when I is even, and for W = (I – 1)/2 when I is uneven. The model has I(I – 1)/2 – W(2I – 2W – 1) degrees of freedom, where W(2I – 2W – 1) independent skew-symmetric parameters are fitted.

QUASI-SYMMETRY PLUS SKEW-SYMMETRY MODEL (QSSK)

We introduce the QSSK model in the same way as the SSK model (we emphasize the differences). We start from model (4b). Assume first that one paired dimension is sufficient to model the skew-symmetry in (4b) adequately. Then model (4b) becomes the QSSK model, defined as

$$\log m_{ij} = q_{ij} + \phi(v_{i(2)}v_{j(1)} - v_{i(1)}v_{j(2)}) \quad (11)$$

with

q_{ij} quasi-symmetric

and with identifying restrictions $\Sigma_i v_{i(m)}v_{i(n)} = \delta^{mn}$, and additional identifying restrictions $\Sigma_i v_{i(m)} = 0$. The latter identifying restrictions ensure that the scores are centered around zero. Thus, if we collect the elements $t_{ij} = \phi(v_{i(2)}v_{j(1)} - v_{i(1)}v_{j(2)})$ in a matrix, the elements of this matrix add up to zero both rowwise as well as columnwise. The approach to interpret the parameter estimates by making graphical displays, and studying the sizes of triangles remains basically the same as in the SSK model. However, under the QSSK model the points in the diagrams will be centered around the origin because $\Sigma_i v_{i(m)} = 0$. Therefore, whereas it was possible under SSK that all points would lie in a half plane, indicating that there are no intransitivities in the data (see section 3), under QSSK the parameter estimates will always reveal intransitivities. The diagrams cannot be interpreted in terms of marginal heterogeneity, but instead reflect why the quasi-symmetry model did not fit. Compared with SSK, QSSK has additional likelihood equations $n_{i+} = m_{i+}$ and $n_{+i} = m_{i+}$. The model has $(I-1)(I-2)/2 - (2I-5)$ degrees of freedom, with $(2I-5)$ independent skew-symmetric parameters (see Appendix B).

Similar to the SSK(W) model, a general model QSSK(W) can be defined that has QSSK as a special case:

$$\log m_{ij} = q_{ij} + \sum_{w=1}^{W} \phi_w (v_{iw(2)} v_{jw(1)} - v_{iw(1)} v_{jw(2)}) \qquad (12)$$

with

q_{ij} quasi-symmetric

and with identifying restrictions $\Sigma_i v_{iw(m)} v_{iz(n)} = \delta^{wz}\delta^{mn}$ and $\Sigma_i v_{iw(m)} = 0$. Appendix B shows that QSSK(W) has $(I-1)(I-2)/2 - W(2I-2W-3)$ degrees of freedom, where $W(2I-2W-3)$ independent skew-symmetric parameters are fitted. QSSK(W) is equivalent to the saturated model if $W = (I-1)/2$ when I is uneven or $W = (I-2)/2$ when I is even.

MODEL CHOICE

If the quasi-symmetry model is true, there is no need to investigate the fit of SSK(W) or QSSK(W). If quasi-symmetry fails to fit the data adequately, both SSK(W) or QSSK(W) could be used as starting points to model the departure from quasi-symmetry. SSK(W) seems most appropriate when interest goes out simultaneously to marginal heterogeneity and asymmetry not due to marginal heterogeneity. QSSK(W) seems most appropriate when interest goes out to asymmetry that is not confounded with marginal heterogeneity.

For the purpose of determining the fit of models, the likelihood ratio test and the Pearson chi-square test can be used to test the model against the data; both tests follow asymptotically a chi-square distribution with the degrees of freedom associated with the model when this model is true. The use of conditional tests is more complicated.

First, notice that the models symmetry, quasi-symmetry, SSK, and QSSK are nested, where QSSK is least restrictive and symmetry is most restrictive. We will now show that the likelihood ratio chi-square test can be used as a conditional test between any pair chosen from these four models. If the most restrictive of the two models chosen is true, then the chi-square test follows a chi-square distribution asymptotically. The reason is that these four models only differ in terms of equality constraints or fixed value constraints. Starting from QSSK, by imposing homogeneity restrictions $u_{1(i)} = u_{2(i)}$ we get SSK. From SSK, by imposing $v_{k(1)} = 1$ and $\phi v_{k(2)} = a_i$ (see above), we obtain quasi-symmetry. From quasi-symmetry, by imposing homogeneity restrictions $u_{1(i)} = u_{2(i)}$, we get symmetry.

In a similar way, the conditional test of model SSK(W) against QSSK(W) for $W > 1$ also follows asymptotically a chi-square distribution if SSK(W) is true. However, for models SSK(W) and QSSK(W) when $W > 1$ conditional tests for W against $W - k$ ($W \geq k > 0$) do not follow asymptotically a chi-square distribution if the model with $W - k$ is true, because under the model with $W - k$ the parameter $\phi_W = 0$, which makes the parameters $v_{iw(m)}$ undetermined (see Haberman 1981).

We think that in many cases the quasi-symmetry model, SSK(W), and QSSK(W) will nicely supplement each other, and we will illustrate this in the next section by showing an example.

ESTIMATION

Assuming a multinomial distribution for the frequencies, the models are fitted by maximum likelihood. We used unidimensional Newton. Application of this algorithm is straightforward and described in Goodman (1979). After convergence the parameter estimates have to be rescaled so that they follow the identifying restrictions of the model. How this is to be done is described in Appendix C.

5. EXAMPLE

Our example deals with the analysis of an intergenerational occupational mobility table of the United States obtained from the 1973 Occupational Changes in a Generation (OCG-II) survey. The sample of 18,237 men aged 20-64 is given in Table 1. It was analyzed previously by Yamaguchi (1987) and Xie (1992). Given the large sample size, the quasi-symmetry model fits reasonably well (G^2 = 46.23, *df* = 6), but by applying the chi-square test in a strict way, the model has to be rejected. The same holds for the SSK model: It has a fit of G^2 = 20.16 (*df* = 3). The QSSK model fits adequately (G^2 = .08, *df* = 1). For illustrative purposes we discuss the relevant parameter estimates for all three models. This will show that the results of the three analyses nicely supplement each other.

In Figures 1a-1c, we find graphical representations of the parameter estimates. In Figure 1a, we find the graphical display for the quasi-symmetry model, and the relevant parameter estimates are given in Table 2a. We will start with a discussion of the graphical display made from parameter estimates under quasi-symmetry, that is, of the pairs of estimates $(\hat{v}_{i(1)}, \hat{v}_{i(2)}) = (1, \alpha_i)$ (for reasons of space the display is rotated over 90 degrees). This is the simplest display of a Gower decomposition, and it may be helpful in obtaining a better understanding of more complicated displays. Triangles are to be made of pairs of points with the origin, and as an example the triangle is made for the pair of points 5 (farm) and 4 (lower manual). The arrow indicates into which direction the skew-symmetry is positive. Under the quasi-symmetry model the skew-symmetry from farm to lower manual is positive; it is of the same size but negative from lower manual to farm. This means that there are more sons having "origin farmer to destination lower nonmanual" than sons having "origin lower manual who become farmers."

Under this model the asymmetry is attributable to marginal heterogeneity only, so Figure 1a shows the shift in occupational structure across generations. Areas of triangles between two occupations and the origin specify the magnitudes of the skew-symmetry in the quasi-symmetry model. To make the interpretation easier, the estimates of skew-symmetries $t_{ij} = a_i - a_j$ are given in Table 2a. Figure 1a and the parameter estimates in Table 2a show that the occupations are nicely ordered from farmer (5) to upper nonmanual (1). So the general shift is away from the farm category (5) and toward the nonmanual occupations 1 and 2, with the manual occupations being in between. This general shift is well known, and not typical for the United States (see, e.g., Wong 1989, chap. 5). The quasi-symmetry model is quite restric-

TABLE 1: Intergenerational Occupational Mobility Table for the United States Obtained From the 1973 Occupational Changes in a Generation (OCG-II) Survey

	Son's Status					
Father's Status	*Upper Nonmanual*	*Lower Nonmanual*	*Upper Manual*	*Lower Manual*	*Farm*	*Total*
Upper nonmanual	1275	364	274	272	17	2202
Lower nonmanual	1055	597	394	443	31	2520
Upper manual	1043	587	1045	951	47	3673
Lower manual	1159	791	1323	2046	52	5371
Farm	666	496	1031	1632	646	4471
Total	5198	2835	4067	5344	793	18237

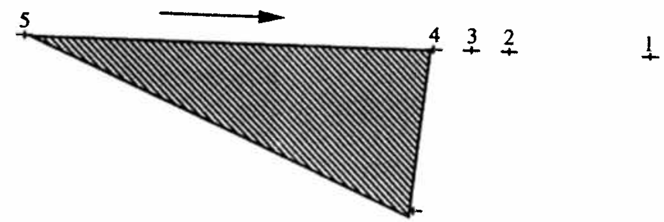

Figure 1a: Graphical Display of Skew-Symmetries Under Quasi-Symmetry

tive. This is also evident from the representation of the model chosen here. Because the occupations are placed on a straight line, areas are additive. For example, the skew-symmetry to go from farmer (5) to upper manual (3) is equal to the skew-symmetry to go from farmer (5) to lower manual (4) plus the skew-symmetry to go from lower manual (4) to upper manual (3). Such additivities can be checked in Table 2a (1.609 = 1.481 + .128). In the quasi-symmetry model the shift away from the occupation farm is spread evenly over all the other occupations. However, the quasi-symmetry model does not fit adequately, and therefore this simple interpretation is not completely correct.

Under SSK the skew-symmetry is the result of both marginal heterogeneity (shifts in occupational structure across generations) as well as asymmetric association. In this model these two sources of skew-symmetry are confounded. In the context of occupational mobility this can be a useful model,[3] in particular if we theoretically

TABLE 2a: Parameter Estimates and Skew-Symmetries t_{ij} Under the Quasi-Symmetry Model

Parameter Estimate $u_{1(i)} - u_{2(i)}$	Elements $t_{ij} = (u_{1(i)} - u_{2(j)})/2$: 1	2	3	4	5
1. 1.67	—	–0.498	–0.644	–0.722	–2.253
2. 0.67		—	–0.146	–0.274	–1.755
3. 0.38			—	–0.128	–1.609
4. 0.12				—	–1.481
5. –2.84					—

NOTE: Only the upper triangle of the table is given because the lower triangle can be derived from the relation $t_{ij} = -t_{ji}$.

assume that asymmetry in mobility is generated from asymmetric labor market supplies for each pair of origin and destinations (in other words, if the difference between the supply of those having origin A for the destination B and the supply of those who have origin B for the destination A in the labor market generated asymmetric mobility, and consequently, both asymmetric marginal distributions as well as asymmetric association of positions). On the other hand, we can also theoretically assume instead that the structural changes in occupational compositions over time are exogenously imposed due to changes in the demands for jobs over time. Then the skew-symmetry in model SSK may be considered to reflect the consequences of the two distinct forces, that is, the consequences of changes in the distributions of occupations over time and those of asymmetric association between the destination positions and persons with different origins under these structural constraints on the distribution of destination positions. In both cases it is interesting to model the asymmetric association, and this is what is done in the SSK model.

Figure 1b shows the parameter estimates for $(\phi^{1/2}v_{i(1)}, \phi^{1/2}v_{i(2)})$ for the SSK model (see also Table 2b). Here the occupations do not fall on a straight line. By making triangles of two occupations with the origin, and going clockwise from one to the other occupation, the skew-symmetry is positive. For example, the area of the triangle of farmer (5) with lower manual (4) with the origin is positive (and hence the skew-symmetry is positive) indicating, as in the analysis of the quasi-symmetry model, that there are more sons becoming lower manual (4) whose father was a farmer (5) than the other way around. In Figure

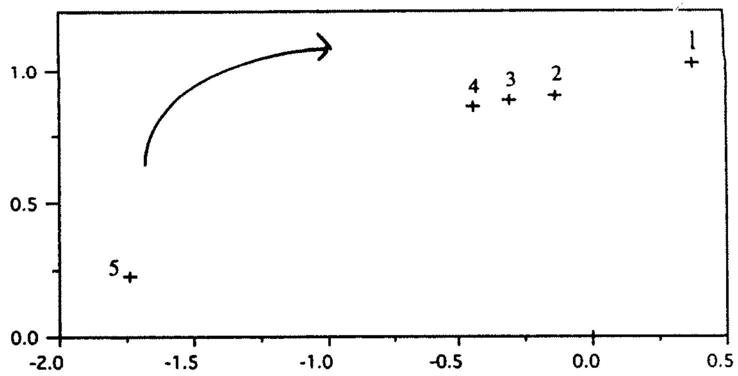

Figure 1b: Graphical Display of Skew-Symmetry Under SSK

TABLE 2b: Parameter Estimates and Skew-Symmetries t_{ij} Under the SSK Model

Parameter Estimate $\phi^{1/2}v_{i(1)}$	$\phi^{1/2}v_{i(2)}$	Elements $t_{ij} = \phi(v_{i(2)}v_{j(1)} - v_{i(1)}v_{j(2)})$: 1	2	3	4	5
1. –0.38	–1.03	—	–0.487	–0.656	–0.797	–1.704
2. 0.14	–0.90		—	–0.150	–0.285	–1.598
3. 0.31	–0.89			—	–0.139	–1.627
4. 0.45	–0.86				—	–1.616
5. 1.74	0.23					—

NOTE: Only the upper triangle of the table is given because the lower triangle can be derived from the relation $t_{ij} = -t_{ji}$.

1b, the general order from 5 to 1 is the same in Figure 1a. This shows that the structural changes in mobility are much stronger than the asymmetric association. We can then notice how the points in Figure 1b depart from a straight line. It seems that lower manual (4), upper manual (3), lower nonmanual (2), and upper nonmanual (1) are more or less on a straight line, and that their respective distances are very similar to those in Figure 1a. This is different for the occupation farmer (5), which falls below this line. Compared to Figure 1a, this position below the line through 1-4 makes the triangles of farmer with these other occupations different.

A comparison of the estimates of t_{ij} in Table 2a with those in Table 2b shows us that the position of farmer with respect to the other occupations is changed in the sense that the skew-symmetry between farmers (5) and upper nonmanuals (1) becomes much less extreme in Table 2b (relative to the solution for quasi-symmetry; see Table 2a), a bit less extreme between farmer (5) and lower nonmanual (2), and the skew-symmetry between farmers (5) and manuals (3 and 4) becomes a bit more extreme than the skew-symmetries under the quasi-symmetry model. Compared to Figure 1a, the largest new effect shown by the SSK model in Figure 1b is that, by assuming symmetric association, the quasi-symmetry model overestimates mobility from farmers to nonmanuals (2.253 and 1.755 under quasi-symmetry compared with 1.704 and 1.598 under SSK), and consequently underestimates mobility from nonmanuals to farm occupations (–2.253 and –1.755 under quasi-symmetry compared with –1.704 and –1.598 under SSK). (Notice that t_{ij} should always be interpreted as a pair of t_{ij} and $t_{ji} = -t_{ij}$.) So in the quasi-symmetry model these estimates of $t_{ij} = -t_{ji}$ are farther away from zero. Similarly, by assuming symmetric association, the quasi-symmetry model underestimates mobility from farmers to lower manuals (1.481 under quasi-symmetry compared with 1.616 under SSK) and consequently overestimates mobility from lower manuals to farmers (–1.481 under quasi-symmetry compared with –1.616 under SSK). The SSK model corrects this.

The comparison of Figure 1a with 1b leads to the question about the asymmetric associations, controlling for structural mobility and the symmetric association. The QSSK model shows this. The results are displayed in Figure 1c (see Table 2c for the parameter estimates). As in Figure 1b, in Figure 1c the sign of the estimates of t_{ij} is positive when we rotate clockwise (with respect to the origin) from i to j, and the sign is negative when we rotate counterclockwise from i to j. When we take the farm occupation as a starting point, we find that, controlling for structural mobility and symmetric association, the destination of farmers is less often nonmanual occupations (1) and (2) (estimates are –.210 and –.186, respectively) and more often manual occupations (.082 for upper manual and .315 for lower manual, respectively). Conversely, controlling for structural mobility and symmetric association, the origin of farmers is more often nonmanual occupations (1) and (2) (estimates are .210 and .186, respectivley) and less often manual occupations. For manual and nonmanual occupations we find that, by only considering the structural mobility and

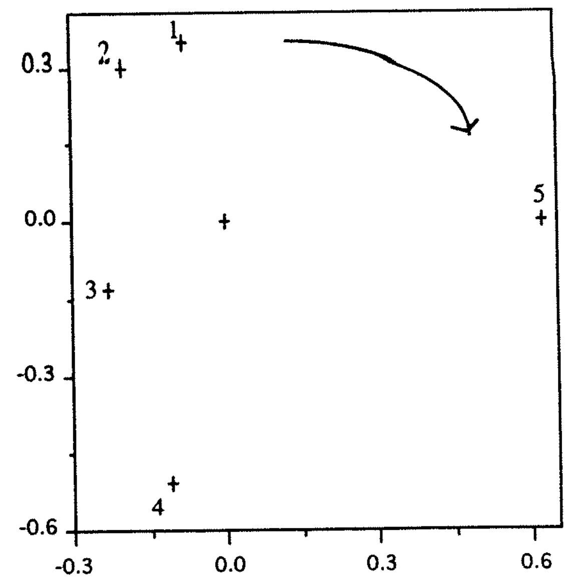

Figure 1c: Graphical Display of Skew-Symmetry Under QSSK

TABLE 2c: Parameter Estimates and Skew-Symmetries t_{ij} Under the QSSK Model

Parameter Estimate		*Elements $t_{ij} = \phi(v_{i(2)}v_{j(1)} - v_{i(1)}v_{j(2)})$*				
$\phi^{1/2}v_{i(1)}$	$\phi^{1/2}v_{i(2)}$	*1*	*2*	*3*	*4*	*5*
1. –0.08	0.34	—	–0.044	–0.090	–0.077	0.210
2. –0.20	0.30		—	–0.096	–0.134	0.186
3. –0.23	–0.13			—	–0.104	–0.082
4. –0.11	–0.51				—	–0.315
5. 0.62	0.00					—

NOTE: Only the upper triangle of the table is given because the lower triangle can be derived from the relation $t_{ij} = -t_{ji}$.

symmetric association in the quasi-symmetry model, upward mobility among nonfarm occupations is slightly underestimated and, conversely, downward mobility among them is slightly overestimated.

Of the two findings we could make from the QSSK model, namely, (a) skew-symmetry between farm and other occupations and (b) the characteristics of skew-symmetry among nonfarm occupations, the skew-symmetry between farm and other occupations is captured by the SSK model as well as the QSSK model. On the other hand, the characteristics of skew-symmetry among nonfarm occupations is revealed only by the QSSK model and not by the SSK model, and this is the reason why the QSSK model fits better than the SSK model for this data set.

6. DISCUSSION

In this article the asymmetry in square contingency tables is modeled by transforming the frequencies by a logarithmic transformation, where the transformed frequencies are split up into either a symmetric or a quasi-symmetric part and a skew-symmetric part. The skew-symmetric part is constrained by choosing lower rank approximations of a Gower decomposition. The quasi-symmetric part is left unconstrained in this article, but this part can in principle be constrained in various ways (c.f. Yamaguchi 1990).

Our approach to model the skew-symmetry was inspired by Goodman's (1985) RC(M) association model. This model provides a rank M approximation of the matrix of interaction parameters u_{ij}. Similarly, the models SSK, SSK(W), QSSK, and QSSK(W) provide a lower rank approximation of the matrix with skew-symmetric parameters t_{ij}. Yamaguchi (1990) proposed many models that were motivated from different points of view. Models (8) and (10) can be considered as elements of what he called skew-symmetric association models.[4] In the context of the symmetry and the quasi-symmetry model, the notion of skew-symmetry was earlier introduced by van der Heijden, de Falguerolles, and de Leeuw (1989). By making use of the properties of the singular value decomposition of skew-symmetric matrices described in this article, they proposed an exploratory correspondence analysis procedure to study the residuals from the symmetry and the quasi-symmetry model. Several objections are raised against such procedures (see the discussion of van der Heijden et al. 1989). For example, it is argued that it is better to use one model for the whole procedure instead of starting off with a log-linear model and then doing a (generalized) correspondence analysis to decompose the residuals. Another objection is that the adequateness of low-dimensional correspondence analysis procedures cannot be assessed by formal fit statistics. The models proposed in this article provide a solution to these objections.

APPENDIX A
Uniqueness of Skew-Symmetric Decomposition

We study the uniqueness of a skew-symmetric matrix **T** made up of one pair of dimensions, that is, $\mathbf{T} = \phi\mathbf{VZV}'$, with **T** a skew-symmetric matrix of order I × I and rank 2, ϕ nonnegative, **V** an I × 2 matrix, and **Z** a matrix as in (6). We prove that we can find the same matrix **T** by using the matrix $-\mathbf{Z}$. There are two ways to find a decomposition with a matrix $-\mathbf{Z}$ instead of **Z**. First, this can be accomplished by multiplying both ϕ as well as **Z** with −1. Second, this can be accomplished by multiplying either the first or the second column of **V** with −1 and multiplying **Z** with −1. That is, if we multiply the second column of **V** with −1, we can rewrite $\mathbf{T} = \phi\mathbf{VZV}'$ as

$$\mathbf{T} = \phi\mathbf{V}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{V}' = \phi\left\{\mathbf{V}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\right\}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{V}\right\}' = \phi\mathbf{V}^{-}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{V}^{-\prime}$$

where $\mathbf{V}^{-}$ is equal to **V** with the second column multiplied by −1. So ϕ can always be chosen to be positive, and **Z** as in (6). This choice for **Z** implies that, going clockwise around the origin, the area of the triangle spanned by the origin, point i and point j, gives a positive skew-symmetry t_{ij}, and counterclockwise it gives a negative skew-symmetry $t_{ji} = -t_{ij}$.

The matrix **V** is not uniquely determined because the same value ϕ is associated with both columns. Let $\mathbf{V}^{*} = \mathbf{VR}$ with $\mathbf{V'V} = \mathbf{I}$. **V** is unique if $\mathbf{VRZR'V'} = \mathbf{VZ{+}V'}$ only holds if $\mathbf{R} = \mathbf{I}$. There exists an $\mathbf{R} \neq \mathbf{I}$ so that $\mathbf{RZR'} = \mathbf{Z}$, namely

$$\mathbf{R} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}$$

The conclusion is that one additional restriction is needed to identify **V**. This proof holds for a rank 2 matrix (i.e., for one pair of dimensions). It is easy to see that for each pair of dimensions one extra restriction is needed to identify **V** uniquely.

APPENDIX B
The Number of Degrees of Freedom

The number of degrees of freedom is equal to the number of cells minus the number of independent parameters. The number of cells is I^2. In the symmetry model, $t_{ij} = 0$, and $I(I + 1)/2$ parameters are fit for the general mean, the homogeneous margins and the symmetric association. This gives $I(I - 1)/2$ degrees of freedom. The quasi-symmetry model has $(I-1)$ parameters more to allow for marginal heterogeneity, so it has $(I-2)(I-1)/2$ degrees of freedom. We first derive the number of degrees of freedom for the model SSK (symmetry plus skew-symmetry), and then for the model QSSK (quasi-symmetry plus skew-symmetry).

Degrees of Freedom for SSK

By fitting $t_{ij} = \Sigma_w \phi_w (v_{iw(2)} v_{jw(1)} - v_{iw(1)} v_{jw(2)})$, the number of independent parameters $n(T)$ is determined as follows: The parameters $v_{iw(1)}$ and $v_{iw(2)}$ are collected in a matrix V of order $I \times 2W$, W being the number of pairs of dimensions, and we define a matrix $\mathbf{T} = \mathbf{V\Phi ZV'}$ with Φ a diagonal $(2W \times 2W)$ matrix with elements ϕ_w, $\mathbf{V'V} = \mathbf{I}$ and $\mathbf{1'V} = \mathbf{0}$. Notice that $\mathbf{ZZ'} = \mathbf{I}$. Due to the properties of skew-symmetric matrices, $\phi_1 = \phi_2$, $\phi_3 = \phi_4$, and so on. The maximal rank of $\mathbf{T}$ is I if I is uneven and $I-1$ if I is even.

When I is even $W \le I/2$, when I is uneven $W \le (I-1)/2$. The number of parameters of $\mathbf{P}$ is $2IW$, the number of parameters ϕ is equal to W. The number of restrictions for $\mathbf{P}$ is $2W(2W+1)/2$ for $\mathbf{P'P} = \mathbf{I}$. There is one additional restriction for each pair of dimensions (see Appendix A for α), so there are W additional restrictions. It follows that $n(\mathbf{T})$ is

$$n(T) = 2WI + W - 2W(2W+1)/2 - W = W(2I - 2W - 1).$$

It follows that the SSK model has $I^2 - I(I+1)/2 - W(2I-2W-1) = I(I-1)/2 - W(2I-2W-1)$ degrees of freedom. For maximal rank of $\mathbf{T}$ (i.e., $W = I/2$ or $W = (I-1)/2$) we find $I(I-1)/2$ independent parameters, which is equal to the number of degrees of freedom for the symmetry model.

Degrees of Freedom for QSSK

By fitting $t_{ij} = \Sigma_w \phi_w (v_{iw(2)} v_{jw(1)} - v_{iw(1)} v_{jw(2)})$ in the QSSK model, the number of independent parameters $n(T)$ is determined in a slightly different way because there is the additional restriction that $\mathbf{1'P}$. This gives $2W$ extra restrictions, so that

$$n(T) = 2WI + W - 2W - 2W(2W+1)/2 - W = W(2I - 2W - 3).$$

It follows that the QSSK model has $I^2 - I(I+1)/2 - (I-1) - W(2I-2W-3) = (I-1)(I-2)/2 - W(2I-2W-3)$ degrees of freedom. Because $\Sigma_i t_{ij} = \Sigma_j t_{ij} = 0$, the maximal rank of $\mathbf{T}$ is now $I-1$ if I is even and $I-2$ if I is uneven. For maximal rank of $\mathbf{T}$ (i.e., $W = (I-1)/2$ or $W = (I-2)/2$) we find $(I-1)(I-2)/2$ independent parameters, which is equal to the number of degrees of freedom for the quasi-symmetry model.

APPENDIX C
Rescaling the Parameter Estimates

It is not necessary to rescale the parameters during the iterative process. This can be postponed until convergence is attained. Because QSSK(W) has the most identi-

fying restrictions, we take this model as an example. There are two types of rescaling: first, the parameters $v_{iw(m)}$ have to be rescaled so that they will have average zero ($\Sigma_i v_{iw(m)} = 0$). Second, scores of distinct dimensions have to be made orthogonal ($\Sigma_i v_{iw(m)} v_{iz(n)} = \delta^{wz}\delta^{mn}$).

Assume that we have the unidentified ML estimates of the association parameters $v_{iw(m)}$. Let the parameters $v^{+}_{iw(1)}$ and $v^{+}_{iw(2)}$ be parameters to be identified, and let $v^{++}_{iw(1)}$ and $v^{++}_{iw(2)}$ be the identified parameters. Define $v_{w(1)} \equiv (\Sigma_i v^{+}_{iw(1)})/I$ and $v_{w(2)} \equiv (\Sigma_i v^{+}_{iw(2)})/I$. Then $v^{++}_{iw(1)} = v^{+}_{iw(1)} - v_{w(1)}$ and $v^{++}_{iw(2)} = v^{+}_{iw(2)} - v_{w(2)}$. By working out $\Sigma_w (v^{+}_{iw(2)} v^{+}_{jw(1)} - v^{+}_{iw(1)} v^{+}_{jw(2)})$, we find

$$u^{++}_{1(i)} = u^{+}_{1(i)} + (v_{w(1)} v^{++}_{iw(2)} - v_{w(2)} v^{++}_{iw(1)})$$

$$u^{++}_{2(j)} = u^{+}_{2(j)} + (v_{w(2)} v^{++}_{jw(1)} - v_{w(1)} v^{++}_{jw(2)}).$$

As a second step, collect the parameters $v^{++}_{iw(1)}$ and $v^{++}_{iw(2)}$ in a matrix V of order $I \times (2W)$. Then derive the singular value decomposition of the matrix $\mathbf{W} = \mathbf{VZV'} = \mathbf{K\Omega ZK'} = \mathbf{V^{*\prime}ZV^{*}}$, where $\mathbf{\Omega}$ is a diagonal matrix with singular values in decreasing order, $\mathbf{K'K} = \mathbf{I}$, and $\mathbf{V^{*}} = \mathbf{K\Omega}^{1/2}$. Then the orthogonal parameters we are looking for are in $\mathbf{V^{*}}$.

NOTES

1. Another way to understand this decomposition is that $\pm \omega_w$, where i is the imaginary number and ω_w are the singular values, are eigenvalues of H and, because they are purely imaginary numbers, we get them in pairs and the first dimension of W captures skew-symmetry that corresponds to the pair of dimensions with the largest absolute value for the eigenvalue.

2. We have chosen to use uniform weights so that the skew-symmetric association will not be confounded with the marginal distributions. For a discussion of using nonuniform weights in other log-bilinear models, see Becker and Clogg (1989) and Goodman (1991).

3. We are grateful to Professor Yamaguchi for pointing this out to us.

4. We will now present model (8) in a similar way as Yamaguchi (1990) presented his skew-symmetric association models. For $i < j$, let θ_{ij} be the odds ratio $\theta_{ij} = n_{ij} n_{(i+1)(j+1)} / n_{i(j+1)} n_{(i+1)j}$. Instead of modeling association by modeling log θ_{ij}, as is done by Goodman (1979) and Clogg (1982), Yamaguchi (1990) models $\Phi_{ij} \equiv (\log \theta_{ij}/\theta_{ji})$, which becomes under the saturated model (4) $\Phi_{ij} \equiv 2(\rho_{ij} + r_{(i+1)(j+1)} - r_{(i+1)j} - r_{i(j+1)})$, where in his presentation $r_{ij} = (u_{ij} - u_{ji})/2$ (compare [4]). Quasi-symmetry is attained if and only if $\Phi_{ij} = 0$ holds true whenever i and j are different. Yamaguchi then proposes three models: (a) the uniform skew-symmetric association model, where $\Phi_{ij} \equiv 2\Phi$ for $i < j$, and this leads to $r_{ij} = [\delta^{i<j} i(j-i) - \delta^{i>j} j(i-j)]\Phi$; (b) the log-linear row and column effect skew-symmetric association model, where $\Phi_{ij} = 2(v_{i+1} - v_i) + 2(v_{j+1} - v_j)$ for $i < j$, and $\Sigma_i v_i = 0$, and this leads to $r_{ij} = \delta^{i<j}[i(v_j - v_i) + v_i(j-i)] - \delta^{i>j}[j(v_i - v_j) + v_j(i-j)]$; and (c) the log-bilinear row and column effect skew-symmetric association model, where $\Phi_{ij} = 2\phi_v(v_{i+1} - v_i)(v_{j+1} - v_j)$ for $i < j$, and $\Sigma_i v_i = 0$ and $\Sigma_i v_i^2 = 1$, and this leads to $r_{ij} = \delta^{i<j}\phi_v v_i(v_j - v_i) - \delta^{i>j}\phi_v v_j(v_i - v_j)$.

Model (8) has $t_{ij} = \phi(v_{i(2)}v_{j(1)} - v_{j(2)}v_{i(1)})$, and this leads to $\Phi_{ij} = 2\phi[(v_{i(2)} - v_{i+1(2)})(v_{j(1)} - v_{j+1(1)}) - (v_{i(1)} - v_{i+1(1)})(v_{j(2)} - v_{j+1(2)})]$. This shows that the basic structure of t_{ij} of model (8) is similar to, but more complicated, than the structure of r_{ij} of Yamaguchi's model (c). Model (9) uses more parameters than model (c), but on the other hand, model (c) is meant for modeling r_{ij} (i.e., the departure from quasi-symmetry), whereas (8) is meant for modeling t_{ij} (i.e., the departure from symmetry).

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