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INTERSECTION THEORY ON DELIGNE–MUMFORD COMPACTIFICATIONS [after Witten and Kontsevich]

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A MATHEMATICIAN'S APOLOGY

Physicists have developed two approaches to quantum gravity in dimension two. One involves an a priori ill-defined integral over all conformal structures on a surface which after a suitable renormalization procedure produces a well-defined integral over moduli spaces of curves. In another they consider a weighted average over piecewise flat metrics on that surface and take a suitable limit of such expressions. The belief that these two approaches yield the same answer led Witten to make a number of conjectures about the intersection numbers of certain natural classes that live on the moduli space of stable pointed curves. One of these conjectures has been rigourously proved by Kontsevich.

In this talk I will mainly focus on Kontsevich' proof and on some results that are immediately related to it. For lack of competence I have not discussed the physical part of the story and as a result, this account is a rather one-sided one. This is regrettable, since developments of the last decade have taught geometers that the imagery and intuition that comes with quantum field theory is a powerful heuristic tool for their field, too. (I leave it to the reader to speculate whether an algebraic geometer would have ever come up with the Witten conjecture.) An overview which does the physical background more justice is the one by Dijkgraaf [3]. That paper also contains a more complete list of references.

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1. THE WITTEN CONJECTURE

Let us agree that an *n-pointed* curve is a complex projective curve with n given ordered distinct points on its smooth part. The isomorphism classes of smooth n-pointed curves of genus g are naturally parametrized by a variety $\mathcal{M}_g^n$ * of dimension $3g-3+n$, assuming, as we always will, that $n \geq 3$ if $g = 0$, and $n \geq 1$ if $g = 1$. An n-pointed curve $(\Sigma; x_1, \ldots, x_n)$ is said to be *stable* if is has only simple crossings as singularities and has finite automorphism group. (The last condition is equivalent to: the normalization of every irreducible component of genus zero must contain at least three distinct points that map to a singular point or to an x_i.) According to Deligne–Mumford–Knudsen, a projective completion $\overline{\mathcal{M}}_g^n$ of $\mathcal{M}_g^n$ is obtained by including the isomorphism classes of stable n-pointed curves of arithmetic genus g. This completion is in a natural way an orbifold, i.e., is locally in a natural way the quotient of a smooth space by a finite group. By taking the cotangent space at the i-th point we obtain a line bundle $\mathcal{L}_i$ (in the orbifold sense) over $\overline{\mathcal{M}}_g^n$ $(i = 1, \ldots, n)$. Denote its first (rational) Chern class by τ_i. For every sequence $(d_1, \ldots, d_n)$ of nonnegative integers we have a "characteristic number"

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_g^n} \tau_1^{d_1} \tau_2^{d_2} \ldots \tau_n^{d_n}.$$

Since we are in an orbifold setting, this is a rational number, not necessarily an integer. By a theorem of Arakelov $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g$ is always ≥ 0. Of course it vanishes when $\sum_i d_i \neq 3g-3+n$. One may interpret these numbers as follows: the topological vector bundle underlying $\oplus_i \mathcal{L}_i$ has a classifying map $\overline{\mathcal{M}}_g^n \to BU$. Knowing the image of the fundamental class is equivalent to knowing the $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g$'s.

Witten [22] conjectured a formula for these characteristic numbers, which he expressed in terms of the generating function

$$\sum_{g,n} \frac{1}{n!} \sum_{(d_1,\ldots,d_n)} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g t_{d_1} t_{d_2} \ldots t_{d_n}.$$

The conjecture is easier to state if we pass to the variables $T_1, T_2, \ldots$, where $t_i = (2i+1)!!T_{2i+1}$ (recall that $(2i+1)!! = (2i+1)(2i-1)\cdots .3.1$). We denote the

* Some authors denote this variety by $\mathcal{M}_{g,n}$. Our notation is in agreement with the topologist's use of Γ_g^n for the corresponding mapping class group (whereas $\Gamma_{g,n}$ refers to a different—but closely related—group).

resulting expression by $F(T_1, T_2, \ldots)$. Thus F is independent of the T_{2i}'s. Witten's conjecture—now a theorem of Kontsevich—says:

Theorem 1 [16] *The expansion* $\exp(F) \in \mathbf{Q}[[T_1, T_2, T_3, \ldots]]$ *is the* τ*-function for the KdV hierarchy whose initial value (all times zero) is* $(d/dx)^2 + 2x$.

A more informative statement is given in theorem 10. We shall recall the definition of a τ-function in section 5. Such functions have already turned up in the theory of Riemann surfaces through the Krichever construction, but thus far no relation between the two appearances has been found.

Preceding the proof of this conjecture, Kontsevich had obtained an explicit expansion of $\exp(F)$. To state it, choose a positive integer N, let $\mathcal{H}_N$ be the space of hermitian $N \times N$-matrices, and let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathcal{H}_N$ be positive definite. Then $\mathrm{tr}\,(\Lambda X^2)$ defines a positive definite inner product on $\mathcal{H}_N$, and there is a unique Gaussian probability measure $d\mu_\Lambda$ on $\mathcal{H}_N$ of the form $d\mu \exp \mathrm{tr}(-\frac{1}{2}\Lambda X^2)$, where $d\mu$ is a Haar measure.

Theorem 2 [15] *If we substitute* $T_k = \frac{1}{k}\mathrm{tr}\,((-\Lambda)^{-k})$, *then* $\exp(F)$ *is the asymptotic expansion at* $\Lambda = \mathrm{diag}(\infty, \ldots, \infty)$ *of*

$$\int_{\mathcal{H}_N} d\mu_\Lambda(X) \exp \mathrm{tr}(\frac{\sqrt{-1}}{6} X^3).$$

Since for $k = 1, \ldots, N$, the expressions $\frac{1}{k}\mathrm{tr}\,(\Lambda^k)$ are formally independent, and since we may choose N as large as we want, this equality completely determines F. As we will see in section 5, it is quite natural for a τ-function to make a substitution as above. We therefore make it a rule to set for any automorphism X of a finite dimensional vector space,

$$T_k(X) := \frac{1}{k}\mathrm{tr}\,(X^{-k}), \quad k = 1, 2, \ldots.$$

(These are called the *Miwa coordinates.*)

There is yet another characterization, more suited for the purpose of computing the coefficients of $\exp(F)$, which says that F satisfies certain differential equations. Define L_n, $n \geq -1$ by

$$L_{-1} := \tfrac{1}{4}T_1^2 - \tfrac{1}{2}\frac{\partial}{\partial T_1} + \tfrac{1}{2}\sum_{k=1}^{\infty}(k+2)T_{k+2}\frac{\partial}{\partial T_k},$$

$$L_0 := -\tfrac{1}{2}\frac{\partial}{\partial T_3} + \tfrac{1}{2}\sum_{k=1}^{\infty} kT_k\frac{\partial}{\partial T_k} + \tfrac{1}{16},$$

$$L_n := -\tfrac{1}{2}\frac{\partial}{\partial T_{2n+3}} + \tfrac{1}{2}\sum_{k=1}^{\infty} kT_k\frac{\partial}{\partial T_{2n+k}} + \tfrac{1}{4}\sum_{i+j=2n}\frac{\partial^2}{\partial T_i \partial T_j}, \quad (n \geq 1).$$

One verifies that they satisfy the relations $[L_n, L_m] = (n-m)L_{n+m}$. So their linear span is a Lie algebra, and $L_n \mapsto -z^{n+1}d/dz$ establishes an isomorphism of this Lie algebra with the Lie algebra of algebraic vector fields on the affine line.

Theorem 3 [4],[6] *The function F is annihilated by the operators L_n, $n \geq -1$.*

The observation that L_{-1} and L_0 kill F was made by Witten; as we will see below this has a simple algebro-geometric origin. The annihilation by L_{-1} is called the *string equation*. Dijkgraaf and the Verlindes [4], and independently, Fukama–Kawai–Nakayama [6], showed that the annihilation by the other L_k's is a formal consequence of the string equation and the property that $\exp(F)$ is a τ-function. Subsequently other variants of their proof have appeared; the proof that we will sketch in section 6 is basically that of Kac–Schwartz [13].

If we write out these differential equations, we find recursive relations among the coefficients of F which determine F up to scalar factor. So this property of F is useful for doing explicit calculations.

The numbers $\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_g$ can be expressed by means of the "tautological" classes on $\overline{\mathcal{M}}_g^0$. To see this, we first observe that there is a morphism

$$\pi : \overline{\mathcal{M}}_g^{\geq 1} \to \overline{\mathcal{M}}_g^{\geq 0}.$$

which forgets the last point [14]. (Here $\mathcal{M}_g^{\geq s}$ denotes the disjoint union of the $\mathcal{M}_g^k$, $k \geq s$.) To be concrete, notice that if $(\Sigma; x_1, \ldots, x_{n+1})$ is a stable pointed curve, then so is $(\Sigma; x_1, \ldots, x_n)$ unless the irreducible component C containing x_{n+1} is a smooth rational curve which contains exactly two special points other than x_{n+1}, i.e., C meets the other components in 1 (resp. 2) points and contains precisely one (resp. no) $x_i \neq x_{n+1}$. In either case collapsing C to a point yields a stable n-pointed curve of the same arithmetic genus. This pointed curve then represents the image of $(\Sigma; x_1, \dots, x_{n+1})$ under π. The inverse procedure shows us that over $\overline{\mathcal{M}}_g^n$ the morphism π comes with n sections $x_1, \dots, x_n$; this restriction can be regarded as the universal n-pointed genus g-curve (again, in an orbifold sense).

Let K be the first Chern class of the relative dualizing sheaf of π, so that $\tau_i = x_i^* K$ on $\overline{\mathcal{M}}_g^{\geq i}$. The integration of $\tau_1^{d_1} \tau_2^{d_2} \dots \tau_{n+1}^{d_{n+1}}$ over $\overline{\mathcal{M}}_g^{n+1}$ can first be carried out over the fibres of π, and then over $\overline{\mathcal{M}}_g^n$. A somewhat delicate (but not really difficult) computation gives the following equality of rational cohomology classes on $\overline{\mathcal{M}}_g^n$:

(1)

$$\pi_*(\tau_1^{d_1} \tau_2^{d_2} \dots \tau_{n+1}^{d_{n+1}}) = \tau_1^{d_1} \tau_2^{d_2} \dots \tau_n^{d_n} \pi_*(K^{d_{n+1}}) + \sum_{\{i : d_i + d_{n+1} > 0\}} \tau_1^{d_1} \tau_2^{d_2} \dots \tau_i^{d_i + d_{n+1} - 1} \dots \tau_n^{d_n}.$$

In fact, one can show with induction on n that if we integrate $\tau_1^{d_1} \tau_2^{d_2} \dots \tau_{n+1}^{d_{n+1}}$ along the fibres of $\pi^{n+1} : \overline{\mathcal{M}}_g^{n+1} \to \overline{\mathcal{M}}_g^0$, the resulting class on $\overline{\mathcal{M}}_g^0$ is a polynomial in the tautological classes $\kappa_i := \pi_*(K^{i+1}) | \overline{\mathcal{M}}_g^0$. It should be worthwhile to exhibit this polynomial and to express theorem 2 accordingly.

If we take $d_{n+1} = 0$ resp. 1 in (1) we get

$$\pi_*(\tau_1^{d_1} \tau_2^{d_2} \dots \tau_n^{d_n}) = \sum_{\{i : d_i + d_{n+1} > 0\}} \tau_1^{d_1} \tau_2^{d_2} \dots \tau_i^{d_i - 1} \tau_n^{d_n} \quad \text{resp.}$$

$$\pi_*(\tau_1^{d_1} \tau_2^{d_2} \dots \tau_n^{d_n} \tau_{n+1}) = (2g - 2 + n) \tau_1^{d_1} \tau_2^{d_2} \dots \tau_n^{d_n}.$$

These give rise to relations that express the fact that F is annihilated by the differential operators L_{-1} and L_0. Since the Lie algebra spanned by the L_k's is generated by L_{-1} and L_2, theorem 3 would follow if one could somehow directly show that $L_2(F) = 0$.

2. A CELLULAR DECOMPOSITION

A *ribbon graph* is a finite graph without isolated vertices such that for every vertex a cyclic order on its set of outgoing edges is given. This cyclic structure is often depicted by a general projection of the graph in an oriented plane such that the cyclic structure is induced by the orientation of the plane. If we take a regular neighborhood of the image and then separate it near the self-intersections we obtain an oriented surface which contains the graph as a deformation retract and has the property that the cyclic structure at the vertices comes from the orientation of the surface. If the graph is connected, then the classification of surfaces implies that the surface will be homeomorphic to a closed oriented surface of a certain genus minus a finite number of points. Each of the removed points determines a circular oriented graph Z made up of oriented edges of Γ and a graph map $Z \to \Gamma$. We call the pair $(Z, Z \to \Gamma)$ a *boundary cycle* of the ribbon graph. Each oriented edge of Γ becomes after orientation part of a unique boundary cycle.

It is worthwhile to proceed more formally. Given a ribbon graph Γ, let $A = A(\Gamma)$ denote the set of oriented edges of Γ. This set has two distinguished automorphisms σ_0, σ_1: the first sends the oriented edge a leaving the vertex p to its successor (with respect to the cyclic ordering on the set of oriented edges leaving p) and the second is the involution that reverses orientation. (Conversely, a finite set A coming with an automorphism σ_0 and a fixed point free involution σ_1 defines a ribbon graph.) Put $\sigma_2 := (\sigma_1\sigma_0)^{-1}$, and denote the set of orbits of σ_i in A by Γ_i. Then for $i = 0, 1, 2$, Γ_i bijectively labels the set of vertices of Γ, resp. the set of edges of Γ, resp. the set of boundary cycles of Γ. If $(Z_i \to \Gamma)_{i \in \Gamma_2}$ is the set of boundary cycles, then we define the space $F(\Gamma)$ as what we get by attaching the cylinders $Z_i \times \mathbf{R}_{\geq 0}$ to Γ via the attaching map $Z_i \times \{0\} \cong Z_i \to \Gamma$. This is a piecewise-linear oriented surface which contains Γ as a deformation retract.

A *metric* on Γ is simply a map l which assigns to every edge a positive real number l_s. Such a metric induces a metric in every boundary cycle Z_i. If we give $Z_i \times \mathbf{R}_{\geq 0}$ the product metric, then $F(\Gamma)$ becomes a metric space. This metric is Riemannian except at the vertices of Γ. But it is easily seen that the underlying conformal structure extends over $F(\Gamma)$ (uniquely), and so turns $F(\Gamma)$ into a Riemann surface. By examining this structure on a halfcylinder we see immediately that as a Riemann surface, $F(\Gamma)$ is isomorphic to a compact Riemann surface minus a finite set of points.

Let $\mathcal{G}_g^n$ denote the set of isomorphism classes of connected ribbon graphs of genus g with n numbered boundary cycles and with the property that every vertex has degree ≥ 3. The formula for the euler characteristic $2 - 2g - n = |\Gamma_0| - |\Gamma_1|$

and the inequality $|\Gamma_0| \le \frac{2}{3}|\Gamma_1|$ imply that this is a finite set. Let $\mathcal{M}_g^{n,\mathrm{comb}}$ denote the set of isomorphism classes of the same objects endowed with metrics. We have a decomposition

$$\mathcal{M}_g^{n,\mathrm{comb}} = \bigcup_{\Gamma\in\mathcal{G}_g^n} e(\Gamma), \quad \text{with} \quad e(\Gamma) := \mathrm{Aut}\,(\Gamma)\backslash \mathbf{R}_{>0}^{\Gamma_1}.$$

The set $\mathcal{M}_g^{n,\mathrm{comb}}$ has a natural topology. It is locally compact Hausdorff, and if (Γ, l) is a metrized graph, $s \in \Gamma_1$ a non-loop, and $l(t)$ the metric which differs from l in that it assigns the value tl_s to s, then letting t go to zero makes $(\Gamma, l(t))$ go to the metrized graph obtained from (Γ, l) by collapsing the edge s. (In fact, these properties characterize the topology.) As a space, $\mathcal{M}_g^{n,\mathrm{comb}}$ has the structure of a topological orbifold.

The codimension of the "orbicell" $e(\Gamma)$ in $\mathcal{M}_g^{n,\mathrm{comb}}$ is $\sum_{v\in\Gamma_0}(\deg(v)-3)$. In particular, the cells of top dimension are labeled by the graphs Γ whose vertices all have degree 3. The dual of such a Γ in the closed surface $\overline{F}(\Gamma)$ is a triangulation of $\overline{F}(\Gamma)$ with n vertices. So these cells are also labeled by the isomorphism classes of triangulations of a closed oriented genus g surface with n vertices.

Let $p_i : \mathcal{M}_g^{n,\mathrm{comb}} \to \mathbf{R}_{>0}$ assign to a (Γ, l) the length of the i-th boundary cycle. This is a piecewise linear function. The earlier discussion furnishes a map $\mathcal{M}_g^{n,\mathrm{comb}} \to \mathcal{M}_g^n$. Combining these defines a map

$$h : \mathcal{M}_g^{n,\mathrm{comb}} \to \mathcal{M}_g^n \times \mathbf{R}_{>0}^n.$$

Theorem 5 *The map h is a homeomorphism of orbifolds.*

The proof of this theorem is based on results of Jenkins and Strebel [20] on the trajectory structure of a quadratic differential. Whereas these results date back to the sixties (see also [10]), the observation that they lead to a stratication of Teichmüller space (because they imply the existence of the inverse of h) was only made around 1981, and was the result of an interaction between mainly Harer, Mumford and Thurston. (It appeared in [9] as Theorem 1.3.) Penner later gave an alternate approach based on hyperbolic geometry and introduced the ribbon graph description of the cells [18].

There is a natural extension $\overline{\mathcal{M}}_g^{n,\mathrm{comb}} \supset \mathcal{M}_g^{n,\mathrm{comb}}$. It is defined in a similar manner as $\mathcal{M}_g^{n,\mathrm{comb}}$; only now we allow more edges to have zero length, but we insist that the length of each boundary cycle remains positive. To be more precise, let for $\Gamma \in \mathcal{G}_g^n$, $\overline{e}(\Gamma)$ be the space of $\mathrm{Aut}(\Gamma)$-orbits in the space of $l : \Gamma_1 \to \mathbf{R}_{\geq 0}$ with the property that each boundary cycle has positive length. Then $\overline{\mathcal{M}}_g^{n,\mathrm{comb}}$ is the space obtained by glueing these orbit spaces together via the maps induced by collapsing edges of zero length. It is a locally compact Hausdorff space that comes with a projection $p : \overline{\mathcal{M}}_g^{n,\mathrm{comb}} \to \mathbf{R}_{>0}^n$. This projection is proper and has the structure of a cellular bundle. A description of the points of $\overline{\mathcal{M}}_g^{n,\mathrm{comb}}$ is implicit in the following theorem (which is stated in [16] without proof).

Theorem 6 [16] *The inverse of the homeomorphism of theorem 5 extends to an identification mapping $\overline{\mathcal{M}}_g^n \times \mathbf{R}_{>0}^n \to \overline{\mathcal{M}}_g^{n,\mathrm{comb}}$. Two n-pointed stable curves together with a fixed $p \in \mathbf{R}_{>0}^n$ map to the same element of $\overline{\mathcal{M}}_g^{n,\mathrm{comb}}$ if and only if there exists a homeomorphism between them that takes the i-th point to the i-th point, and is complex-analytic on every irreducible component that contains one of the (n) distinguished points.*

In the above description the $U(1)$-bundle associated to the line bundle $\mathcal{L}_i$ is easy to give: the fibre over (Γ, l) is given by the i-th boundary cycle. This allows Kontsevich to give a an explicit representative of its first Chern class as a piecewise smooth form ω_i. Its restriction to the cell $e(\Gamma)$ is given as follows: number the edges of the i-th boundary cycle of Γ in a cyclic oriented order $s_0 \dots, s_{k-1}$ and let $l_j : e(\Gamma) \to \mathbf{R}_{>0}$ be the length of s_j so that $p_i = \sum_j l_j$. Then the restriction of ω_i to the cell $e(\Gamma)$ will be

$$\omega_i(\Gamma) := \sum_{j<j'} d(l_j/p_i) \wedge d(l_{j'}/p_i)$$

It is easily seen that this form only depends on Γ, and that $\omega_i(\Gamma)$ induces on a cell $e(\Gamma')$ in the closure of $e(\Gamma)$ the form $\omega_i(\Gamma')$.

Consider the piecewise smooth 2-form

$$\Omega := \sum_{i=1}^{n} p_i^2 \omega_i.$$

The restriction of Ω to any fibre of $p|e(\Gamma)$ is a nondegenerate symplectic form with *constant* coefficients. Thus we obtain an orientation of the fibres of $p|e(\Gamma)$ and hence also of $e(\Gamma)$ itself. We give $e(\Gamma)$ this orientation.

Suppose that $\mathbf{m} = (m_1, m_2, \ldots)$ is a sequence of nonnegative integers which are zero for almost all k. Let Γ be a connected ribbon graph with n boundary cycles with the property that its number of degree l vertices is zero if l even or 1 and equal to m_k if $l = 2k+1 \geq 3$. Then clearly

$$
\begin{aligned}
|\Gamma_0| &= \sum_k m_k, \\
\dim e(\Gamma) = |\Gamma_1| &= \frac{1}{2}\sum_k m_k(2k+1),
\end{aligned} \tag{2}
$$

and so

$$
\operatorname{codim} e(\Gamma) = 6g - 6 + 3n - \dim e(\Gamma) = 2|\Gamma_1| - 3|\Gamma_0| = 2\sum_k m_k(k-1)
$$

The collection of such Γ defines a chain on $\overline{\mathcal{M}}_g^{n,\mathrm{comb}}$ with closed support. The following lemma is not difficult to prove.

Lemma $\overline{\mathcal{M}}_g^{n,\mathrm{comb}}(\mathbf{m})$ *is a cycle in degree* $\frac{1}{2}\sum_k m_k(2k+1)$ *with closed support.*

So $\overline{\mathcal{M}}_g^{n,\mathrm{comb}}(\mathbf{m})$ defines via h a class with closed support on $\overline{\mathcal{M}}_g^n \times \mathbf{R}^n_{>0}$. This corresponds to a class $Z_g^n(\mathbf{m})$ on $\overline{\mathcal{M}}_g^n$ in a degree which is n units smaller. Its Poincaré dual defines a cohomology class on $\overline{\mathcal{M}}_g^n$ of degree $2\sum_k m_k(k-1)$. In this way we produce in a given degree as many elements as we get as pull-backs of monomials of the Mumford–Miller–Morita classes. Kontsevich conjectures that the former are linear combinations of the latter. In particular, the single classes in degree 2 should be proportional, and this has been verified by Penner.

Using these cycles we may form

$$
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{\mathbf{m},g} := \int_{Z_g^n(\mathbf{m})} \tau_1^{d_1} \cdots \tau_n^{d_n}
$$

and a corresponding generating function

$$
\sum_{g,n} \frac{1}{n!} \sum_{d_1,\ldots,d_n} \sum_{s_1,s_2,\ldots} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{\mathbf{m},g} t_{d_1} t_{d_2} \ldots t_{d_n} s_1^{m_1} s_2^{m_2} \cdots .
$$

Let $\tilde{F} \in \mathbf{Q}[[T_1, T_2, \ldots; s_1, s_2, \ldots]]$ be the expansion obtained by making the substitution $t_i = (2i+1)!!T_{2i+1}$. It reduces to Witten's generating function F if we put $s_1 = 1$ and $s_i = 0$ for $i > 1$, for clearly

$$\pm F(T_1, T_2, \ldots) = \tilde{F}(T_1, T_2, \ldots; 1, 0, 0, \ldots),$$

where the sign is the degree of of h, and by comparing the signs of the coefficients on both sides, we see that in fact the plus sign holds. Kontsevich proved a corresponding result for $\tilde{F}$, thus generalizing theorem 2:

Theorem 7 [16] *We have*

$$\exp \tilde{F}(T_1(-\Lambda), T_2(-\Lambda), \ldots; s_1, s_2, \ldots) = \langle \exp \operatorname{tr}\big(\sqrt{-1}\sum_{k=1}^{\infty}(\tfrac{-1}{2})^k s_k \frac{X^{2k+1}}{2k+1}\big)\rangle_\Lambda,$$

where $T_k(\Lambda) = \frac{1}{k}\operatorname{tr}(\Lambda^{-k})$.

This will be proved in section 4.

Remark. Grothendieck [7] has made the amazing observation that there is a natural action of the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of isomorphism classes of ribbon graphs. It is defined as follows. Given a ribbon graph Γ, then triangulate $\overline{F}(\Gamma)$ by regarding $\overline{F}(\Gamma)$ as obtained from Γ by putting a cone (in the simplicial category) over each boundary cycle. If we metrize Γ by assigning to each edge unit length, then we get a distinguished complex structure on $F(\Gamma)$ and its compactification $\overline{F}(\Gamma)$ (and hence also a distinguished element of the orbicell $e(\Gamma)$ if Γ is connected). Denote the corresponding Riemann surfaces by $X(\Gamma)$ resp. $\overline{X}(\Gamma)$. Let $I \subset \mathbf{P}^1(\mathbf{R})$ the closed interval containing 0 whose end points are ∞ and 1. There is a unique holomorphic map $f_\Gamma : \overline{X}(\Gamma) \to \mathbf{P}^1$ which maps the vertices of Γ to 0, the midpoints of the edges of Γ to 1, the points of $\overline{X}(\Gamma) - X(\Gamma)$ to ∞ and the interior of every triangle conformally onto the complement of I. Notice that (i) f_Γ is unramified over $\mathbf{P}^1 - \{0, 1, \infty\}$ and (ii) that every point over $1 \in \mathbf{P}^1$ has ramification index 2. The connected components of $f_\Gamma^{-1}(\mathbf{P}^1 - I)$ are naturally indexed by the elements of $A(\Gamma)$ and via this labeling the permutations σ_0 resp. σ_1 simply correspond to the monodromy of the covering along a simple loop around 0 resp. 1; in particular the group generated by these two elements can be identified with the monodromy group of f_Γ.

Conversely, any covering of $\mathbf{P}^1$ with properties (i) and (ii) is thus obtained and the isomorphism type of the corresponding ribbon graph is unique. Since $\mathbf{P}^1 - \{0, 1, \infty\}$ is a normal $\mathbf{Q}$-scheme, the Galois group $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ "acts" on the profinite completion of its fundamental group via outer automorphisms. This induces an action of $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of isomorphism classes of the coverings of $\mathbf{P}^1$ unramified over $\mathbf{P}^1 - \{0, 1, \infty\}$. It is easily seen that the set of isomorphism classes of coverings of $\mathbf{P}^1$ satisfying (ii) is stable under this action and so we obtain in fact an action of $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of isomorphism classes of ribbon graphs. It turns out to be quite difficult to make this action explicit [19].

3. KONTSEVICH' MAIN IDENTITY

The form Ω introduced in the previous section is closed relative p and its pull-back under the map h represents the smooth family of classes $\sum_i p_i^2 \tau_i$. So integration of $\exp(\Omega)$ over the fibres of $p|\mathcal{M}_g^{n,\mathrm{comb}}(\mathbf{m})$ is the function on $\mathbf{R}^n_{>0}$ defined by

$$\text{(3)} \qquad \int_{p_*} \frac{1}{d!}\Omega^d |\mathcal{M}_g^n(\mathbf{m}) = \sum_{\sum d_i = d} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{\mathbf{m},g} \prod_{i=1}^n \frac{p_i^{2d_i}}{d_i!},$$

where d must be half the degree of $Z_g^n(\mathbf{m})$. We apply to both sides the Laplace transform $\int_{\mathbf{R}^n_{>0}} dp_1 \cdots dp_n \exp(-\sum_i \lambda_i p_i)$ $(\lambda_i > 0)$. The right-hand side becomes

$$\text{(4)} \qquad \begin{aligned} &\sum_{\sum d_i = d} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{\mathbf{m},g} \prod_i \frac{(2d_i)!}{d_i!} \lambda_i^{-(2d_i+1)} \\ =&2^d \sum_{\sum d_i = d} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{\mathbf{m},g} \prod_i (2d_i - 1)!! \lambda_i^{-(2d_i+1)}. \end{aligned}$$

The Laplace transform of the left-hand side of (3) gets a contribution $I(\Gamma)$ for each open cell $e(\Gamma)$ in $\mathcal{M}_g^n(\mathbf{m})$. It is easy to see from the definition of $\Omega(\Gamma)$ that $\frac{1}{d!} dy_1 \wedge \cdots \wedge dy_n \wedge \Omega^d(\Gamma)$ has constant coefficients. So this is a constant times the natural measure on $e(\Gamma)$ defined by its coordinates l_s, $s \in \Gamma_1$. A rather intricate computation shows that this constant is equal to 2^{2d+1-g}. Given an edge s of Γ, then each orientation of s makes it part of a boundary cycle; if these have number i and i', let $\lambda_s := \frac{1}{2}(\lambda_i + \lambda_{i'})$. Thus

$$\sum_i \lambda_i p_i = 2 \sum_{s\in\Gamma_1} \lambda_s l_s$$

and hence

$$I(\Gamma) = \frac{1}{|\mathrm{Aut}\,(\Gamma)|} 2^{d+1-g-|\Gamma_1|} \prod_{s\in\Gamma_1} \lambda_s^{-1}.$$

Using the identities $2d = |\Gamma_1| - n$, $2 - 2g = |\Gamma_0| - |\Gamma_1| + n$ and the formulas (2) we get the following beautiful identity:

$$(5) \quad \sum_{\sum d_i = d} \langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_{\mathbf{m},g} \prod_i (2d_i - 1)!!\lambda_i^{-(2d_i+1)} = \sum_\Gamma \frac{2^{-(\sum_k m_k k)}}{|\mathrm{Aut}\,(\Gamma)|} \prod_{s\in\Gamma_1} \lambda_s^{-1},$$

where on the right the sum is over the open cells of $\mathcal{M}_g^n(\mathbf{m})$. This equality implies the remarkable fact that the right-hand side is a polynomial in the variables λ_i^{-1}.

We now wish to vary g, n and $\mathbf{m}$. For this, we fix a positive integer N, and express the above equality in variables $\lambda_1, \ldots, \lambda_N$ as follows. For every map $i : \{1, \ldots, n\} \to \{1, \ldots, N\}$ write down the above equality with variables $\lambda_{i(1)}, \ldots, \lambda_{i(n)}$, and sum over all such maps i. On the right-hand side this will then an amount to a sum over connected ribbon graphs of genus g with n numbered boundary cycles together with a map from the set of boundary cycles to $\{1, \ldots, N\}$. To this last datum we shall refer by saying that the ribbon graph is *N-colored*. If we do not care about the numbering of the boundary cycles, then we must divide by $n!$. Summation over g and n then gives

$$\tilde{F}(T_1(\Lambda), T_2(\Lambda), \ldots; s_1, s_2, \ldots) = \sum_\Gamma \frac{1}{|\mathrm{Aut}\,(\Gamma)|} \prod_{s\in\Gamma_1} \lambda_s^{-1} . \prod_k (2^{-k} s_k)^{m_k}.$$

Since $\sum_k m_k(2k+1) = 2|\Gamma_1|$, we find

Theorem 8 [16] *The expression* $\tilde{F}(T_1(-\Lambda), T_2(-\Lambda), \ldots; s_1, s_2, \ldots)$ *is equal to*

$$\sum_\Gamma \frac{1}{|\mathrm{Aut}\,(\Gamma)|} \prod_{s\in\Gamma_1} \lambda_s^{-1} . \prod_k ((-2)^{-k}\sqrt{-1}s_k)^{\mathrm{vrt}_{2k+1}(\Gamma)},$$

where the sum is over all isomorphism classes of connected N-colored ribbon graphs whose vertices have odd degree ≥ 3, and $\mathrm{vrt}_l(\Gamma)$ *denotes the number of vertices of* Γ *of degree* l.

All other properties of $\tilde{F}$ and F will be proved via this identity. Notice that since $T_1, \ldots, T_N$ are formally independent expressions, this theorem enables us to compute the coefficients of $\tilde{F}$ inductively.

4. A HERMITIAN ONE-MATRIX MODEL

Let E be a euclidean space of dimension d. Then there is a unique Gaussian probability measure $d\mu_E$ on E of the form $d\mu \exp(-\frac{1}{2}x \cdot x)$, where $d\mu$ is a Haar measure. If f is a function on E which is integrable with respect to this measure, then we write $\langle f \rangle$ for $\int f \, d\mu_E$. Wick's lemma gives the value of $\langle f \rangle$ in case f is a product of linear forms.

Let $\gamma : \mathrm{Hom}\,(E, \mathbf{C}) \times \mathrm{Hom}\,(E, \mathbf{C}) \to \mathbf{C}$ be the $\mathbf{C}$-bilinear extension of the inner product on the dual of E.

Wick's lemma *Given a collection* $\{\phi_a \in \mathrm{Hom}\,(E, \mathbf{C})\}_{a \in A}$, *where A is a finite set, then*

$$\langle \prod_a \phi_a \rangle = \sum_P \prod_{\{a,a'\} \in P} \gamma(\phi_a, \phi_{a'}),$$

where the sum is taken over all pairings P on A.

(A *pairing* on A is a partition of A in two-element subsets.) So if $|A|$ is odd, then $\langle \prod_a \phi_a \rangle = 0$. For the (elementary) proof, we refer to [1].

We shall apply this to the case where the underlying vector space is the space $\mathcal{H}_N$ of hermitian endomorphisms of $\mathbf{C}^N$. The matrix coefficients $X_{i,j}$ make up a $\mathbf{C}$-basis of $\mathrm{Hom}\,(\mathcal{H}_N, \mathbf{C})$ and satisfy $\overline{X}_{i,j} = X_{j,i}$. The inner product on $\mathcal{H}_N$ shall depend on a positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ in $\mathcal{H}_N$ and is given by $\mathrm{tr}\,(\Lambda X^2)$. The equality $\mathrm{tr}\,(\Lambda X^2) = \sum_{i,j} \lambda_i X_{i,j} X_{j,i} = \sum_{i,j} \lambda_i |X_{i,j}|^2$ shows that this form is positive definite. We also find that

$$\gamma(X_{i,j}, X_{k,l}) = \frac{2}{\lambda_i + \lambda_j} \delta_{i,l} \delta_{j,k}. \tag{6}$$

With the help of Wick's lemma we are going to compute

$$\langle \exp \operatorname{tr}(\sum_{k\geq 1} \frac{u_k X^k}{k})\rangle_\Lambda$$

as a formal expansion in the u_k's. First notice that

$$\operatorname{tr}(X^k) = \sum_\alpha X_{\alpha(0),\alpha(1)} X_{\alpha(1),\alpha(2)} \cdots X_{\alpha(k-1),\alpha(0)} ,$$

where the sum is over all maps α from the cyclic group $\mathbf{Z}/k$ to $\underline{N}$. (An underlined positive integer stands for the set of positive integers smaller or equal to it.)

More generally, an expression of the form $\operatorname{tr}(X^1)^{d_1} \operatorname{tr}(X^2)^{d_2} \cdots \operatorname{tr}(X^K)^{d_K}$ can be written out as a sum of monomials in the $X_{i,j}$'s naturally labeled by the set of maps $\phi : \cup_{k=1}^K \underline{d_k} \times \mathbf{Z}/k \to \underline{N}$. To be precise, put $A := \cup_{k=1}^K \underline{d_k} \times \mathbf{Z}/k$ and denote the automorphism of A that sends (i,j) to $(i,j+1)$ by σ_0. Given $\phi : A \to \underline{N}$, let $\phi_a := X_{\phi(a),\phi\sigma_0(a)}$, where $a \in A$. Then the expression at issue is the sum (over ϕ) of the monomials $\prod_{a\in A} \phi_a$.

According to Wick's lemma

$$\langle \prod_{a\in A} \phi_a \rangle_\Lambda = \sum_P \prod_{\{a,a'\}\in P} \gamma(\phi_a, \phi_{a'}), \tag{7}$$

where the sum runs over all pairings P on A. A pairing P on A is the same thing as a fixed point free involution on A, and as we explained in section 2, it therefore determines with σ_0 a ribbon graph $\Gamma(P)$ whose set of oriented edges is bijectively labeled by the elements of A. Notice that this graph has exactly d_k vertices of degree k. Formula (6) shows that the general term of the right-hand side of (7) can be nonzero only if the following condition is fulfilled: if P pairs a and a', then $\phi(a) = \phi\sigma_0(a')$ and $\phi(a') = \phi\sigma_0(a)$; the value then being the reciprocal of the average of $\lambda_{\phi(a)}$ and $\lambda_{\phi(a')}$. In terms of the ribbon graph, this means that ϕ factors through an N-coloring $\bar\phi$ of its boundary cycles and thus defines an N-colored ribbon graph $\Gamma(P, \bar\phi)$. Any N-colored ribbon graph with d_k vertices of degree k is so obtained.

How often do we get the same colored ribbon graph? To answer this, consider the group G of permutations of the index set A that commute with σ_0. It can be identified with the direct product of semi-direct products

$$\prod_{k=1}^K (\mathcal{S}_{v_k}.(\mathbf{Z}/k)^{d_k}).$$

In particular, it has order $\prod_k d_k! k^{d_k}$. This group acts on in an obvious way on the set of pairs $(P, \bar\phi)$ as above. It is easy to see that two pairs define isomorphic colored ribbon graphs iff they are in the same G-orbit of $(P, \bar\phi)$. Moreover, the G-stabilizer of $(P, \bar\phi)$ can be identified with the automorphism group of $\Gamma(P, \bar\phi)$. Applying this to the expansion

$$\langle \exp \operatorname{tr}(\sum_{k\geq 1} \frac{u_k X^k}{k})\rangle_\Lambda = \sum_{(d_1, d_2 \ldots)} (\prod_k \frac{u_k^{d_k}}{d_k! k^{d_k}}) \langle \prod_k \operatorname{tr}^{d_k}(X^k)\rangle_\Lambda,$$

we find:

Theorem 9 [16]

$$\langle \exp \operatorname{tr}(\sum_{k\geq 1} \frac{u_k X^k}{k})\rangle_\Lambda = \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{s\in\Gamma_1} \lambda_s^{-1} \prod_{k\geq 1} u_k^{\operatorname{vrt}_k(\Gamma)},$$

where the sum is over all isomorphism classes of N-colored ribbon graphs and $\operatorname{vrt}_k(\Gamma)$ *is the number of vertices of* Γ *of degree* k. *If we limit the summation in the right-hand side to connected graphs, then we get the logarithm of the left-hand side.*

The last clause follows in a straightforward manner from the first. The expansion converges if we take the u_k's purely imaginary and almost all zero. Combining this theorem with 8 proves theorem 7.

Matrix models were introduced in physics by 't Hooft. Bessis–Itzykson–Zuber [1] gave it a simple mathematical treatment. The matrix model discussed here (due to Kontsevich) generalizes their work (these authors took $\Lambda = \mathbf{1}$).

5. THE KdV AND GELFAND–DIKII HIERARCHIES

This section will very briefly review some aspects of the KdV hierarchies. A nice introduction to the subject is the paper by Segal–Wilson [21]. (But in this paper the definition of the relevant Grassmannian differs from the one used here in that these authors impose an L_2-condition.) Fix an integer $p \geq 2$ and consider a general monic differential operator of order p

$$L = \partial^p + \sum_{i=0}^{p-1} u_{p-i}(x)\partial^i$$

in a single variable x with smooth coefficients u_i (∂ stands for differentiation). Conjugating L with respect to the multiplication by a suitable function makes the coefficient u_1 disappear; we shall therefore assume that this is already the case. We ask ourselves which monic differential operators P_k of order k have the property that $[P_k, L]$ is of order $\leq p-2$. This is easily answered in terms of pseudo-differential operators: First find a pseudo-differential operator K of the form $1+\sum_{i=1}^{\infty} k_i(x)\partial^{-i}$ such that $L = K\partial^p K^{-1}$. Then $Q := K\partial K^{-1}$ is a p-th root of L which has the form $\partial + \sum_{i=1}^{\infty} q_i(x)\partial^{-i}$. Write Q^k as a sum of a differential operator $(Q^k)_+$ and a pseudo differential operator $(Q^k)_-$ of order ≤ -1. In the equality

$$[(Q^k)_+, L] = [L, (Q^k)_-]$$

the left-hand side is a differential operator and the right-hand side has order $\leq p-2$. Hence $P_k := (Q^k)_+$ solves our problem. Notice that if k is a multiple of p, then $P_k = L^{k/p}$, so that $[P_k, L] = 0$. The p-th *Gelfand–Dikii hierarchy* is the system of differental equations

$$\frac{\partial L}{\partial T_k} = [P_k, L] \quad k = 1, 2, \ldots . \tag{8}$$

(For $p = 2$ this is also called the *Korteweg–de Vries* hierarchy.) Here we ask for solutions $u_i = u_i(x; T_1, T_2, \ldots)$, $i = 1, \ldots, p-2$. Each member of the system should be thought of as a vector field on the (infinite-dimensional) affine space of monic differential operators of order p without ∂^{p-1}-term. We can only hope for a solution of the above system if these vector fields commute, but this turns out to be the case. It is clear that a solution $L(\mathbf{T})$ of (8) will not depend on the variables T_{kp}, $k = 1, 2, \ldots$. Since $P_1 = \partial$, the flow defined by the T_1-variable is given by translation $x \mapsto x + T_1$. So we may write $u_i = u_i(x + T_1, T_2, \ldots)$.

There is beautiful geometric method, initiated by M. Sato, which converts this into a system of commuting vector fields on an infinite Grassmannian that comes at least formally already in integrated form. In one direction (which we will not describe) one attaches to L a space $V(L)$ of eigenfunctions on the spectrum of L which is invariant under multiplication by the function λ^p.

In the opposite direction one proceeds as follows. Consider complex-linear subspaces V of the algebra of Laurent expansions in λ^{-1}, $\mathbf{C}((\lambda^{-1}))$, which admit a basis $(f_k)_{k=0}^{\infty}$ such that for k sufficiently large the expansion of f_k ends with λ^k

(or equivalently, such that the projection $V \to \mathbf{C}[\lambda]$ which omits the polar part be Fredholm of zero index). The collection of these subspaces is called the *Sato Grassmannian*. Let us denote it by Gr and let Gr^0 be the set of $V \in Gr$ with the property that V supplements $\lambda^{-1}\mathbf{C}[[\lambda^{-1}]]$, or equivalently, that V admits a basis $(f_k)_{k=0}^{\infty}$ (a "standard basis") such that the expansion of f_k ends with λ^k for *all* k; it is the "big cell" of Gr. We will see in a moment that the complement $Gr - Gr^0$ is very much like a divisor in Gr.

For $\mathbf{T} = (T_1, T_2, T_3, \ldots)$, let $M(\mathbf{T})$ be the operator in $\mathbf{C}((\lambda^{-1}))$ given by multiplication by $\exp(\sum_{k=1}^{\infty} T_k \lambda^k)$. This operator is defined whenever $M(\mathbf{T}) \in \mathbf{C}((\lambda^{-1}))$; this is so when $M(\mathbf{T})$ is the expansion at ∞ of a rational function. This defines in a formal sense an action of $\mathbf{C}^{\mathbf{N}}$ on $\mathbf{C}((\lambda^{-1}))$; it is this action we were alluding to. The passage from this formal action to a solution of the Gelfand–Dikii hierarachy is somewhat indirect.

Let us first define the τ-fuction of a $V \in Gr^0$. If $(f_k)_{k=1}^{\infty}$ is a standard basis of $V \in Gr^0$, then clearly, $(M(-\mathbf{T})(f_k))_k$ is a basis of $M(-\mathbf{T})(V)$. One defines $\tau_V(\mathbf{T})$ as the determinant of the $\mathbf{C}[\lambda]$-parts of $(M(-\mathbf{T})(f_k))_k$ with respect to the basis $(\lambda^k)_{k\geq 0}$ of $\mathbf{C}[\lambda]$; as the notation indicates this is independent of the choice of a basis. Of course, τ_V is here not defined as a function, but as a formal expansion in $\mathbf{T}$ (and even that is not immediately clear). Notice that the constant term of τ_V equals 1. Saying that $M(-\mathbf{T})(V) \notin Gr^0$ is equivalent to $\tau_V(\mathbf{T}) = 0$ (when this makes sense). The function τ_V is characterized by the following property:

Lemma *For* $\Lambda = (\lambda_1, \ldots, \lambda_N)$, *we have*

$$\tau_V(-T_k(\Lambda)_{k\geq 1}) = \frac{\det(f_{i-1}(\lambda_j))}{\det(\lambda_j^{i-1})}.$$

The case $N = 1$ is Lemma 5.15 of [21] and the proof of the general assertion is outlined in [16].

One associates to τ_V in a canonical way a formal family of formal pseudo differential operators $Q(\mathbf{T})$ with $Q(\mathbf{T})_+ = \partial$ and such that for all k,

$$\frac{\partial Q}{\partial T_k} = [Q_+^k, Q].$$

If V is stable under multiplication by λ^p, then this property is preserved under the flow, $L := Q^p$ is a formal differential operator, and $L(\mathbf{T})$ satisfies (8). Both $\tau_V(\mathbf{T})$ and $L(\mathbf{T})$) will be independent of $T_p, T_{2p}, \ldots$.

The coefficients of L can be expressed explicitly in terms of τ. For instance, if $p = 2$, then

$$L(\mathbf{T}) = \partial^2 + 2\frac{\partial^2}{\partial T_1^2} \log \tau_V(T_1 + x, T_2, T_3, \ldots). \tag{9}$$

In general one has the formula

$$\mathrm{Res}\,(Q^k) = \frac{\partial^2}{\partial T_1 \partial T_k} \log \tau_V(T_1 + x, T_2, T_3, \ldots), \quad k = 1, 2. \ldots \tag{10}$$

(The residue of a pseudo differential operator is simply the coefficient of ∂^{-1}.) For $k = 1, \ldots, p-1$, $\mathrm{Res}\,(Q^k)$ is of the form $\frac{p-k}{p} u_{k+1}$ plus a differential polynomial in the u_l, $l < k$, so that the coefficients of L can be computed inductively from this.

6. A CONSTRUCTION OF τ-FUNCTIONS

We shall describe a rather general procedure for constructing τ-functions. It is due to Kontsevich and leads to a proof that the expression in theorem 2 defines a τ-function. Let be given $u, v \in \mathbf{R}[x]$. Then the integral

$$a(x) := \int_{-\infty}^{\infty} dy \exp \sqrt{-1}(xu(y) + v(y)), \quad x \in \mathbf{R},$$

converges in the sense that the integral over $[-n, n]$ has a finite limit as $n \to \infty$). An alternative way to give this integral a meaning is to move the path of integration in the complex domain according to the method of steepest descend; the corresponding integral will then converge absolutely. It is in this sense that similar integrals appearing here are to be understood. An N-variable analogue of a is

$$A(\mathbf{x}) := \int_{\mathcal{H}_N} dY \exp \mathrm{tr} \sqrt{-1}(\mathbf{x}u(Y) + v(Y)), \quad \mathbf{x} = \mathrm{diag}(x_1, \ldots, x_N) \in \mathcal{H}_N$$

Remarkably, this integral can be reduced to an expression involving only a and its derivatives. To see this, we make the substitution $Y = U\mathbf{y}U^{-1}$, where U is unitary and $\mathbf{y} = \mathrm{diag}(y_1, \ldots, y_N)$. The corresponding jacobian is $dU d\mathbf{y}\, \Delta(\mathbf{y})^2$, where dU

is a Haar measure on U_N. An ancient formula due to Harish-Chandra [8] states that

$$\Delta(\mathbf{x})\Delta(\mathbf{y}) \int_{U_N} dU \exp \mathrm{tr}\sqrt{-1}(U\mathbf{y}U^{-1}\mathbf{x}) = \mathrm{const}\,.\det(\exp(\sqrt{-1}x_i y_j)),$$

where const stands (here as well as below) for an expression which only depends on N and

$$\Delta(\mathbf{x}) = \det(x_i^{j-1}) = \sum_{\sigma\in\mathcal{S}_N} \mathrm{sign}(\sigma) x_1^{\sigma(1)-1}\cdots x_N^{\sigma(N)-1}$$

is the Vandermonde determinant. Feeding this into the integral gives

$$A(\mathbf{x}) = \mathrm{const}\,.\Delta(\mathbf{x})^{-1} \int_{\mathbf{R}^N} d\mathbf{y}\,\Delta(\mathbf{y}) \det(\exp(\sqrt{-1}x_j u(y_i))\exp \mathrm{tr}\, v(\mathbf{y})$$

If we develop the determinants, the variables separate and the right-hand side becomes

(11)

$$\begin{aligned}
&\mathrm{const}\,.\Delta(\mathbf{x})^{-1} \sum_{\sigma,\tau\in\mathcal{S}_N} \mathrm{sign}(\sigma\tau) \prod_{i=1}^{N} \int dy_i\, y_i^{\tau(i)-1} \exp\sqrt{-1}(x_{\sigma(i)}u(y_i) + v(y_i))\\
=&\mathrm{const}\,.\Delta(\mathbf{x})^{-1} \sum_{\sigma\in\mathcal{S}_N} \mathrm{sign}(\sigma) a_0(x_{\sigma(1)})\cdots a_{N-1}(x_{\sigma(N)})\\
=&\mathrm{const}\,.\det(x_i^{j-1})^{-1}\det(a_{j-1}(x_i)),
\end{aligned}$$

where

(12) $$a_k(x) := \int_{-\infty}^{\infty} dy\, y^k \exp\sqrt{-1}(xu(y)+v(y)).$$

This makes $A(-\mathbf{x})$ resemble a matricial τ-function. (To be one, the expansion of a_k in x^{-1} should begin with x^k.)

Now take $u(y) = y$, $v(y) = \frac{1}{3}y^3$, so that

$$a(x) = \int_{-\infty}^{\infty} dy\, \exp\sqrt{-1}(yx + \frac{y^3}{3}).$$

(Moving the line of integration over a positive distance in the upper half plane will make the integral converge absolutely.) This the classical Airy function. It clearly satisfies the differential equation $a''(x) = xa(x)$ and we have

$$a_k(x) = (-\sqrt{-1}\frac{d}{dx})^k a(x).$$

It can be written (for x real and positive) as

$$a(x) = x^{-\frac{1}{4}} \exp(-\frac{2}{3}x^{\frac{3}{2}}) f(x^{\frac{3}{2}}),$$

where f has an asymptotic expansion at $+\infty$ of the form

$$f(x) \asymp f_0 + f_1 x^{-1} + f_2 x^{-2} \cdots$$

with $f_0 \neq 0$. (NB: This expansion has no positive radius of convergence.)

A simple estimate shows that in the integral defining $A(\mathbf{x})$ the domain of integration may be replaced by $\mathcal{H}_N + \mathbf{x}\sqrt{-1}$ when $\mathbf{x} \geq 0$. If we make the substitution $Y \mapsto 2^{-\frac{1}{3}}Y + \mathbf{x}\sqrt{-1}$ and take $\mathbf{x} := 2^{\frac{1}{3}}\Lambda$, where now $\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_N) > 0$, we find

$$A(2^{-\frac{2}{3}}\Lambda^2) = \exp \mathrm{tr}(-\frac{1}{3}\Lambda^3) \int_{\mathcal{H}_N} dY \exp \mathrm{tr}(-\frac{1}{2}\Lambda Y^2 + \frac{\sqrt{-1}}{6} Y^3). \tag{13}$$

The integral on the left can be expressed in terms of $\exp F(-\mathbf{T}(\Lambda))$, with $T_k(\Lambda) = \frac{1}{k}\mathrm{tr}\,(\Lambda^{-k})$. Following theorem 2,

$$\exp F(-T(\Lambda)) = \frac{\int_{\mathcal{H}_N} dY \exp \mathrm{tr}\left(-\frac{\Lambda Y^2}{2} + \frac{\sqrt{-1}}{6} Y^3\right)}{\int_{\mathcal{H}_N} dY \exp \mathrm{tr}\left(-\frac{\Lambda Y^2}{2}\right)}. \tag{14}$$

The denominator appearing on the right hand is the Gaussian integral associated to the quadratic form $\sum_{i<j}(\lambda_i + \lambda_j)|X_{ij}|^2 + \sum_i \lambda_i X_{ii}^2$, and so proportional to

$$\prod_{i<j} \frac{1}{\lambda_i + \lambda_j} \prod_i \lambda_i^{-\frac{1}{2}} = \frac{\Delta(\Lambda)}{\Delta(\Lambda^2)} \prod_i \lambda_i^{-\frac{1}{2}}. \tag{15}$$

The formulas (11)–(15) imply that

$$\exp F(-T(\Lambda)) = \mathrm{const}\,.\, \det(\lambda_i^{j-1})^{-1} \det(z_{j-1}(\lambda_i)), \tag{16}$$

where

$$z_k(\lambda) = c_k \lambda^{\frac{1}{2}} \exp(\frac{1}{3}\lambda^3) a^{(k)}(2^{-\frac{2}{3}}\lambda^2), \quad \mathrm{Re}(\lambda) > 0, \tag{17}$$

with c_k a nonzero constant that we are still free to choose. Notice that $z_0(\lambda)$ is proportional to $f(2^{-1}\lambda^3)$. In particular, its asymptotic expansion at ∞ is in $\mathbf{C}[[\lambda^{-3}]]$ and has nonzero constant term. We choose c_0 such that its constant term equals 1 and we set $z(\lambda) := z_0(\lambda)$. Let us now identify a simple recursive relationship between the functions z_k. The pull-back of d/dx under $x = c\lambda^2$ is the vector field $(2c\lambda)^{-1} d/d\lambda$. From this we readily deduce that the Airy equation amounts to the property that $a(2^{-\frac{2}{3}}\lambda^2)$ is annihilated by the operator $(\lambda^{-1}d/d\lambda)^2 - \lambda^2$. So if we put

$$D := \lambda^{\frac{1}{2}} \exp(\tfrac{1}{3}\lambda^3)(-\lambda^{-1}\frac{d}{d\lambda})\lambda^{-\frac{1}{2}} \exp(-\tfrac{1}{3}\lambda^3) = -\lambda^{-1}\frac{d}{d\lambda} + \tfrac{1}{2}\lambda^{-2} + \lambda,$$

then $D(z_k)$ is proportional to z_{k+1} and $D^2(z) = \lambda^2 z$. This allows us to take $z_{2k}(\lambda) := \lambda^{2k} z(\lambda)$ and $z_{2k+1}(\lambda) := \lambda^{2k}(Dz)(\lambda)$. Notice that the asymptotic expansion of z_k at ∞ is of the form λ^k plus lower powers of λ. Since F has no constant term, a comparison of the constant terms in both sides of (16) reveals that the constant in (16) must be 1. In particular we have for $N = 1$:

$$z(\lambda) = \exp F(-(\frac{\lambda^{-k}}{k})_k) = (\frac{\lambda}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} dy\, \exp(-\frac{\lambda y^2}{2} + \frac{\sqrt{-1}y^3}{6}), \quad \mathrm{Re}(\lambda) > 0. \tag{18}$$

This proves most of

Theorem 10 *The function* $\exp F$ *is the* τ*-function for the* $\mathbf{C}[\lambda^2]$*-submodule of* $\mathbf{C}((\lambda^{-1}))$ *generated by* z *and* Dz*, where* z *is given by* (18) *and*

$$D = -\lambda^{-1}\frac{d}{d\lambda} + \frac{1}{2}\lambda^{-2} + \lambda.$$

The corresponding KdV hierarchy has initial value $\partial^2 + 2x$*.*

For the last property we first observe that $\langle \tau_0 \cdots \tau_0 \rangle$ is equal to 1 if there are three factors (the contribution from the three-pointed Riemann sphere) and zero else. This implies that $F(T_1, 0, 0, \ldots) = \frac{1}{6}T_1^3$. Now apply formula (9).

If we put $D_k := \frac{1}{2}\lambda^{2k+2}D$, $k \geq -1$, then it is easily verified that $[D_k, D_l] = (l-k)D_{k+l}$. So their linear span is a Lie algebra $\mathcal{L}$ isomorphic to the Lie algebra of algebraic vector fields of the affine line. This Lie algebra acts on the Sato Grassmannian, and on the space of τ functions. This action can be made explicit by means of the *boson-fermion* correspondence (see [12]). It turns out that this transforms $\mathcal{L}$ into the linear span of the L_k's. The fact that each element of $\mathcal{L}$ maps V properly into itself (with zero determinant), implies that the L_k's must kill its τ-function $\exp(F)$, which proves theorem 3. (This argument is due to Kac–Schwartz [13].)

The method described in this section for producing τ-functions has been discovered by Kontsevich. Theorem 10 as stated here can be found in Itzykson–Zuber [11].

7. FURTHER RESULTS AND CONJECTURES

After Kontsevich had proved theorem 2, Witten conjectured that the derivatives of $\exp(F)$ should admit a similar description. That conjecture has been recently proved by Di Franceso–Itzykson–Zuber and reads as follows:

Theorem 11 [5] *There is a* $\mathbf{Q}$*-linear isomorphism*

$$P \in \mathbf{Q}[\partial/\partial T_1, \partial/\partial T_3, \partial/\partial T_5 \ldots] \mapsto Q_P \in \mathbf{Q}[T_1, T_3, T_5, \ldots]$$

such that

$$P(\exp(F(T_k(-\Lambda)_k)) = \langle Q_P(\operatorname{tr} X, \operatorname{tr} X^2, \ldots) \exp \operatorname{tr}(\frac{\sqrt{-1}X^3}{6}))\rangle_\Lambda$$

and the degree of Q_P *is the order of* P.

Their proof is algebraic.

Witten [24],[25] has also proposed a generalization of his conjecture which relates Gelfand–Dikii hierarchies of order $p \geq 2$ with intersection theory on certain Galois coverings of Deligne–Mumford compactifications.

Start with integers $k_1, \ldots k_n \in \{0, 1, \ldots, p-2\}$ and an integer $g \geq 0$ such that $2g - 2 - \sum_i k_i$ is divisible by p. Let $(\Sigma; x_1, \ldots, x_n)$ be a pointed curve of genus g. Then the line bundle $\omega_\Sigma(-\sum k_i(x_i))$ is divisible in $Pic(\Sigma)$ by p. Up to isomorphism there are p^{2g} pth roots of this line bundle and they are simply transitively permuted by an action of $H_1(\Sigma;\mathbf{Z}/p)$. Let $\mathcal{T}$ be one such root. If $\mathcal{T}$ has no nonzero sections (which is often the case), then by Riemann–Roch, $\tilde{V} := H^0(\Sigma, \mathcal{T}^* \otimes \omega)$ will be a vector space of dimension

$$d := \frac{p-2}{p}(g-1) + \sum_i \frac{k_i}{p}.$$

If we only fix the isomorphism class of $\mathcal{T}$, then $\tilde{V}$ is unique up to scalar multiplication by a p-th root of unity, and so the orbit space $V := \tilde{V}/\mu_p$ is unique up to unique isomorphism. We shall regard V as a vector space in the orbifold sense.

We now vary $(\Sigma; x_1, \ldots, x_n; \mathcal{T})$. Let $\mathcal{M}'$ denote the moduli space of n-pointed genus g curves $(\Sigma; x_1, \ldots, x_n)$ equipped with an isomorphism class of a pth root of $\omega(-\sum k_i(x_i))$. This is a covering over $\mathcal{M}_g^n$ of degree p^{2g}. The orbivector spaces V define an orbivector bundle $\mathcal{V}$ of rank d over a Zariski open part of $\mathcal{M}'$. Let $\overline{\mathcal{M}}'$ be the normalization of $\mathcal{M}'$ over $\overline{\mathcal{M}}_g^n$. Index theory suggests a way of extending its euler class to a class defined on $\overline{\mathcal{M}}'$ [25]. Let us denote the direct image of this class on $\overline{\mathcal{M}}_g^n$ by

$$e(p; k_1, \ldots, k_n) \in H^{2d}(\overline{\mathcal{M}}_g^n; \mathbf{Q}).$$

By bringing these euler classes into the game we produce many more characteristic numbers: put

$$\langle \tau_{d_1+k_1/p} \cdots \tau_{d_n+k_n/p} \rangle_{p,g} := p^{-g} \int_{\overline{\mathcal{M}}_g^n} \tau_1^{d_1} \cdots \tau_n^{d_n} e(p; k_1, \ldots, k_n),$$

and form the generating function

$$\sum_{g,n} \frac{1}{n!} \sum \langle \tau_{d_1+k_1/p} \cdots \tau_{d_n+k_n/p} \rangle_{p,g} t_{d_1+k_1/p} \cdots t_{d_n+k_n/p}.$$

We pass to the variables T_i by means of the substitution

$$t_{d+k/p} = (k+1)(p+k+1)(2p+k+1)\cdots(dp+k+1)T_{dp+k+1}.$$

The resulting expression F_p is independent of T_m if m is a multiple of p. It can be verified that $F_2 = F$, and so the truth of the following conjecture will generalize theorem 1:

Conjecture (Witten) The expansion $\exp(F_p)$ is the τ-function of the Gelfand–Dikii hierarchy whose initial value is $\partial^p + px$.

This means that if we define inductively $u_k(x+T_1, T_2, T_3 \ldots)$ by formula (10) ($k = 2, \ldots p$), then the corresponding operator $L(\mathbf{T})$ satisfies the Gelfand–Dikii equations (8). Witten has shown that F_p satisfies a string equation. He also analyzed the situation for $g = 0$. Kontsevich noticed that one can write down the matricial τ-function associated to the operator $\partial^p + px$ in a form generalizing theorem 2. In addition, one can specify differential operators analogous to the L_n's, that annihilate this τ-function, see [3].

Penner [18] reinterpreted the Harer–Zagier calculation of the orbifold euler characteristic of $\mathcal{M}_g^n$ in terms of a matrix model. Although this matrix model is not of the type considered here, it may be thought of as the matrix model corresponding to the case $p = -1$. (Kontsevich gives in an appendix to [16] a short proof of this formula.)

Let us finally mention that the expression F_p is related to the Lie algebra A_{p-1} (so that the original conjecture is related to A_1). One expects that for every simple Lie algebra there is a Witten expansion whose exponential is a τ-function of the KP-hierarchy.

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