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Ordinary Calabi-Yau-3 Crystals

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Abstract. We show that crystals with the properties of crystalline cohomology of ordinary Calabi-Yau threefolds in characteristic $p > 0$, exhibit a remarkable similarity with the well known structure on the cohomology of complex Calabi-Yau threefolds near a boundary point of the moduli space with maximal unipotent local monodromy. In particular, there are canonical coordinates and an analogue of the prepotential of the Yukawa coupling. Moreover in Formulas (2.25) and (2.29) we show p-adic analogues of the integrality properties for the canonical coordinates and the prepotential of the Yukawa coupling, which have been observed in the examples of Mirror Symmetry.

Introduction

Calabi-Yau manifolds of dimensions 1 and 2, i.e. elliptic curves and K3-surfaces, have a long and successful tradition in geometry and number theory. In the 1980's, in connection with developments in string theory, Calabi-Yau manifolds of dimension 3 moved to the forefront. Emphasis has been on their complex and symplectic geometry, and in particular on the variation of the Hodge structure for Calabi-Yau threefolds near the so-called large complex structure limit. This is the first of a series of papers in which we want to describe certain aspects of the arithmetic geometry of families of ordinary Calabi-Yau threefolds and analogies with complex Calabi-Yau threefolds near the large complex structure limit. More specifically, this paper discusses the associated crystals, i.e. modules with an integrable connection, like the Gauss-Manin connection on the cohomology of a family of Calabi-Yau threefolds. Crystals which originate from geometry come with a Hodge filtration, which is not preserved by the connection but instead satisfies Griffiths transversality. Crystals which originate from geometry in characteristic p additionally carry an action of Frobenius operators behaving in an appropriate way with respect to the connection and the Hodge filtration: they are divisible Hodge F-crystals. For so-called divisible *ordinary* Hodge F-crystals the space of flat sections decomposes into eigenspaces for the Frobenius operators. The position of the Hodge filtration

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with respect to this eigenspace decomposition is the source for the canonical coordinates and the Yukawa coupling: see Theorems 1.8, 2.2, 2.4. The ordinariness requirement implies, exactly as the requirement of maximally unipotent local monodromy in the traditional analysis of complex Calabi-Yau threefolds near the large complex structure limit [8], that there is a *filtration which is opposite to the Hodge filtration and is invariant under the Gauss-Manin connection and whose associated graded is of Hodge-Tate type* (cf. [4] prop.1.3.6, [5] §6).

The theory of ordinary Hodge F-crystals has its origins in Dwork's work on the variation of the zeta function in a family of varieties over a finite field [7] and in the work of Serre and Tate on the formal moduli of deformations of an ordinary elliptic curve over a field of positive characteristic. In [4] Deligne and Illusie present a general theory of ordinary Hodge F-crystals and apply it to investigate the formal moduli of ordinary abelian varieties and ordinary K3-surfaces. The general theory works equally well for ordinary CY3 crystals, i.e. crystals of the kind that arises as the crystalline cohomology of ordinary Calabi-Yau threefolds in characteristic $p > 0$. Thus, exactly as in [4], there are canonical coordinates on the formal moduli space. In Section 1 we briefly recall the general theory of [4]. In Section 2 we take the same algebraic path as Bryant, Griffiths, Morrison and Deligne [2, 6, 8, 5] in their analysis of the variation of the Hodge structure of complex Calabi-Yau threefolds near a maximally degenerate ($\approx$ maximally unipotent local monodromy $\approx$ large complex structure) boundary point of the moduli space. In particular we find for ordinary CY3 crystals an exact analog of the cubic form of Bryant-Griffiths ($\approx$ Yukawa coupling). Interestingly, besides the standard characterization (2.21) of the prepotential of this cubic form we find a characterization (2.28) by means of the action of the canonical lift of Frobenius. The latter characterization remains valid if there are no parameters, i.e. if $n = 0$ in (1.1), while in that case the former description is vacuous. Also for the canonical coordinates the formalisms in the ordinary and the large complex structure situations match very well, but the match is not perfect: in the ordinary case the canonical coordinates take the value 1 at the origin of the deformation space (cf. (2.22)), whereas the canonical coordinates at the large complex structure limit point vanish.

The standard algorithms (cf. [3, 8]) for computing canonical coordinates and Yukawa coupling for complex Calabi-Yau threefolds near the "maximally unipotent local monodromy boundary point" use special solutions of the Picard-Fuchs equations associated with the family of CY3's and a nowhere vanishing global 3-form. In Remark 3.4 we point out that in the ordinary case one has basically the same relation between canonical coordinates, Yukawa coupling and special solutions of the Picard-Fuchs equations (although explicit computations seem here out of reach).

The present paper is a survey of structures none which – except maybe (2.28) and (2.29) – is new by itself, but which appear in the literature in different contexts. So we hope that there is an inspiring and cross-fertilizing effect from putting them into one (con)text. For instance, being alerted that for the deformation theory of ordinary Calabi-Yau threefolds the cubic form of Bryant-Griffiths (Yukawa coupling) should be relevant, one may wonder how that shows up in connection with the Hodge-Tate decomposition of p-adic étale cohomology as in [1] p. 109.

The original motivation for our work was to find a general method of proof for the integrality conjectures in Mirror Symmetry by reducing them to known results or easy to check conditions on the crystalline cohomology for families of Calabi-Yau threefolds in positive characteristics: It has been observed in many examples in the literature on Mirror Symmetry for Calabi-Yau threefolds, that the coefficients in appropriate expansions of the canonical coordinates and of the prepotential of the Yukawa coupling are integers. In the famous example of the quintics in $\mathbb{P}^4$ (see [3, 8]) one has for instance

$$q = t + 770\,t^2 + 1014275\,t^3 + 1703916750\,t^4 + 3286569025625\,t^5 + \dots$$

where $t = \psi^{-5}$ is the parameter and q is the canonical coordinate of the family

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - \psi X_0 X_1 X_2 X_3 X_4 = 0\,;$$

the Yukawa coupling Y is given by

$$Y = 5 \,+\, 5 \sum_{n \geq 1} b_n\, n^3\, \frac{q^n}{1-q^n}\,,$$

$b_1 = 575\,,\; b_2 = 121850\,,\; b_3 = 63441275\,,\; b_4 = 48493506000\,,\; b_5 = 45861177777525\,.$

The prepotential Z is defined (up to terms of order ≤ 2 in $\log q$) by $Y = \left(q\frac{d}{dq}\right)^3 Z$ and has an expansion

$$Z(q) = \frac{5}{6}\log^3 q + 5 \sum_{n \geq 1} b_n \mathrm{Li}_3(q^n)$$

where $\mathrm{Li}_3(x) = \sum_{j \geq 1} \frac{x^j}{j^3}$ is the trilogarithm function. According to the Mirror Symmetry Conjecture [3, 8] the number $5b_n$ is the number of rational curves of degree n on a generic quintic hypersurface in $\mathbb{P}^4$. For the Integrality Conjecture one is interested in the less astonishing, but still highly non-trivial fact that these numbers are integers. Now one may note that the numbers $5b_n$ are integers if and only if for every prime number p

$$Z(q) - p^{-3} Z(q^p) \in \mathbb{Z}_p[[q]] \tag{0.1}$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.

In this paper we show that p-adic analogues of the integrality conjectures, in particular an analogue of (0.1), hold true for ordinary CY3 crystals; see Formulas (2.25) and (2.29) [1]. Throughout this paper the prime p is fixed and the constructions involve choices which may depend on p. For the integrality conjectures for families of Calabi-Yau threefolds, like the above quintics, one must however deal with all primes. Thus one is confronted with the challenge to 'glue' or to 'synchronize' those choices for the various primes p. In [10] we will describe the beginnings of such a synchronized set-up for "ordinariness". The present paper starts at the fairly abstract heights of F-crystals and wants to find out from there whether the path of ordinariness leads to interesting vistas. In [10] we will start at a less abstract more geometric level.

1 Ordinary Hodge F-crystals

We recall the theory of ordinary Hodge F-crystals; the general reference is [4]. We work over a perfect field k of characteristic $p > 0$ and explicitly keep track of which results require passing to the algebraic closure $\overline{k}$ of k. Let W be the ring of Witt vectors of k and K the field of fractions of W. So W is a local ring with residue field k; its field of fractions K has characterisitic 0; its maximal ideal is

[1] For some examples it has been proved in [9] that the expansion coefficients of the mirror map (i.e. the coordinate transformation to canonical coordinates) are really integers, in $\mathbb{Z}$.

pW and W is complete and separated for the p-adic topology: $W = \varprojlim_m W/p^mW$. Moreover we need the rings of formal power series

$$A_0 = k[[t_1, \ldots, t_n]] \,, \quad A = W[[t_1, \ldots, t_n]]. \tag{1.1}$$

A *crystal over* A_0 is, by definition, a finitely generated free A-module H together with a connection

$$\nabla : \, H \longrightarrow \Omega^1_{A/W} \otimes_A H$$

which is integrable and p-adically topologically nilpotent; this means that if we set $D_i = (\frac{d}{dt_i} \otimes 1) \circ \nabla : H \to H$, with the derivation $\frac{d}{dt_i}$ viewed as a linear map $\Omega^1_{A/W} \to A$, then $D_i D_j = D_j D_i$ and $\lim_{m \to \infty} D_i^m = 0$ in the p-adic topology on $\mathrm{End}_W(H)$.

On the characteristic p rings k and A_0 one has the *Frobenius endomorphism* σ raising elements to their p-th power: $\sigma(x) = x^p$. On the perfect field k this is an automorphism and it lifts canonically to an automorphism on W, also denoted by σ. There are many different *lifts of Frobenius on* A, i.e. ring endomorphisms ϕ of A which restrict to σ on W and reduce modulo p to σ on A_0. If H is an A-module and $\phi : A \to A$ is a lift of Frobenius we write

$$\phi^* H = A_\phi \otimes_A H$$

where A_ϕ is A viewed as an A-A-bimodule with left structure via the identity map $id : A \to A$ and right structure via $\phi : A \to A$. The natural map

$$\phi^* : H \to \phi^* H \,, \qquad \phi^*(h) = 1 \otimes h$$

is ϕ-linear, i.e. $\phi^*(a_1 h_1 + a_2 h_2) = \phi(a_1)\phi^*(h_1) + \phi(a_2)\phi^*(h_2)$.

Now let (H, ∇) be a crystal as above and let $\phi, \psi : A \to A$ be two lifts of Frobenius. Since ϕ and ψ are equal modulo p the connection ∇ provides, via a kind of Taylor expansion, a canonical isomorphism of A-modules

$$\chi(\phi, \psi) : \phi^* H \xrightarrow{\simeq} \psi^* H;$$

more precisely, the map $\chi(\phi, \psi)\phi^* : H \to \psi^* H$ is given by the formula

$$\chi(\phi, \psi)\phi^*(x) = \sum_{m_1, \ldots, m_n \geq 0} \prod_{j=1}^n \frac{p^{m_j}}{m_j!} \left(\frac{\phi(t_j) - \psi(t_j)}{p} \right)^{m_j} \psi^*(D_1^{m_1} \circ \ldots \circ D_n^{m_n}(x)).$$

Note that for $m \geq 1$ the rational number $\frac{p^m}{m!}$ has p-adic valuation $> m - \frac{m}{p-1} \geq 0$.

Definition 1.1 One says that a crystal (H, ∇) is an *F-crystal over* A_0 if for every lift of Frobenius $\phi : A \to A$ there is given a homomorphism of A-modules

$$F(\phi) : \, \phi^* H \to H \tag{1.2}$$

which is horizontal for the connection ∇, i.e. the square

$$\begin{array}{ccc} H & \xrightarrow{\nabla} & \Omega^1_{A/W} \otimes H \\ {\scriptstyle F(\phi)\phi^*} \downarrow & & \downarrow {\scriptstyle \phi \otimes F(\phi)\phi^*} \\ H & \xrightarrow{\nabla} & \Omega^1_{A/W} \otimes H \end{array} \tag{1.3}$$

is commutative, and such that for every pair of lifts of Frobenius $\phi, \psi : A \to A$

$$F(\psi) \circ \chi(\phi, \psi) = F(\phi). \tag{1.4}$$

Moreover $F(\phi) \otimes \mathbb{Q}_p : \phi^* H \otimes \mathbb{Q}_p \to H \otimes \mathbb{Q}_p$ should be an isomorphism.
If for one, and hence every, lift of Frobenius $\phi : A \to A$ the homomorphism $F(\phi)$ in (1.2) is an isomorphism one says that H is a *unit F-crystal*.

Combining Formula (1.4) with the Taylor expansion formula for $\chi(\phi, \psi)$ one gets

$$F(\phi)\phi^*(x) = \sum_{m_1,\ldots,m_n \geq 0} \prod_{j=1}^{n} \frac{p^{m_j}}{m_j!} \left(\frac{\phi(t_j) - \psi(t_j)}{p} \right)^{m_j} F(\psi)\psi^*(D_1^{m_1} \circ \ldots \circ D_n^{m_n}(x)). \tag{1.5}$$

The horizontality relation for $F(\phi)$ and ∇ (1.3) makes it possible to solve the problem of finding ∇-flat sections (i.e. essentially solving differential equations) by finding $F(\phi)\phi^*$-fixed sections (i.e. essentially solving polynomial equations). Indeed, one has the following result.

Lemma 1.2 *Suppose $e \in H$ is such that $F(\phi)\phi^* e = e$ for some lift of Frobenius $\phi : A \to A$. Then $\nabla e = 0$ and $F(\psi)\psi^* e = e$ holds for every lift of Frobenius ψ.*

Proof By definition there is for every $a \in A$ an element $a_1 \in A$ such that $\phi(a) = a^p + pa_1$. From this one gets by induction for every $a \in A$ and $m \geq 1$ elements $a_1, \ldots, a_m \in A$ such that

$$\phi^m(a) = a^{p^m} + pa_1^{p^{m-1}} + \ldots + p^m a_m$$

and hence $\phi^m(bda) := \phi^m(b)d(\phi^m(a)) \in p^m \Omega^1_{A/W}$ for every 1-form bda.
If $F(\phi)\phi^* e = e$, then (1.3) yields for every $m \geq 1$

$$\nabla e = (\phi^m \otimes (F(\phi)\phi^*)^m) \circ \nabla e \in p^m \Omega^1_{A/W} \otimes H.$$

Taking $m \to \infty$ we see $\nabla e = 0$. The equality $F(\psi)\psi^* e = e$ for any lift of Frobenius ψ now follows from (1.5). □

Let H be an F-crystal[2] and $H_0 = H \otimes_A A_0$. For a lift of Frobenius $\phi : A \to A$ and an integer i we set

$$\begin{aligned} \mathrm{Fil}_i H_0 &= \{x \in H_0 \,|\, \exists y \in H \text{ lifting } x \text{ s.t. } p^i y \in \mathrm{Im} F(\phi)\}, \\ \mathrm{Fil}^i H_0 &= \{x \in H_0 \,|\, \exists y \in H \text{ lifting } x \text{ s.t. } F(\phi)\phi^* y \in p^i H\}. \end{aligned}$$

These are A_0-submodules of H_0. One can show that they are independent of the chosen lift of Frobenius ϕ; see [4] §1.3. Trivially, $\mathrm{Fil}_i H_0 = 0$ for $i \leq -1$ and $\mathrm{Fil}^i H_0 = H_0$ for $i \leq 0$. Moreover, it can be shown that there is an integer N such that $\mathrm{Fil}_N H_0 = H_0$ and $\mathrm{Fil}^{N+1} H_0 = 0$. One then says that H *is of level* $\leq N$. Thus, one finds the filtrations

$$0 = \mathrm{Fil}_{-1} H_0 \subset \mathrm{Fil}_0 H_0 \subset \mathrm{Fil}_1 H_0 \subset \ \ldots \ \subset \mathrm{Fil}_{N-1} H_0 \subset \mathrm{Fil}_N H_0 = H_0 \tag{1.6}$$
$$H_0 = \mathrm{Fil}^0 H_0 \supset \mathrm{Fil}^1 H_0 \supset \mathrm{Fil}^2 H_0 \supset \ \ldots \ \supset \mathrm{Fil}^N H_0 \supset \mathrm{Fil}^{N+1} H_0 = 0 \tag{1.7}$$

(1.6) is called the *conjugate filtration* and (1.7) is called the *Hodge filtration*. The connection ∇ on H induces a connection ∇_0 on H_0. The conjugate filtration is horizontal for ∇_0 and the Hodge filtration satisfies the *Griffiths transversality condition*, i.e. for all i:

$$\nabla_0 \mathrm{Fil}_i H_0 \subset \Omega^1_{A_0/k} \otimes \mathrm{Fil}_i H_0 \qquad \text{and} \qquad \nabla_0 \mathrm{Fil}^i H_0 \subset \Omega^1_{A_0/k} \otimes \mathrm{Fil}^{i-1} H_0.$$

[2]For simplicity, we do not explicitly mention the connection ∇ and the maps $F(\phi)$.

Let $\mathrm{gr}_\bullet H_0$ and $\mathrm{gr}^\bullet H_0$ denote the graded modules associated with the conjugate and Hodge filtrations respectively. If $\mathrm{gr}_\bullet H_0$ is a free A_0-module, then so is $\mathrm{gr}^\bullet H_0$ and for every i the modules $\mathrm{gr}_i H_0 = \mathrm{Fil}_i H_0/\mathrm{Fil}_{i-1} H_0$ and $\mathrm{gr}^i H_0 = \mathrm{Fil}^i H_0/\mathrm{Fil}^{i+1} H_0$ are canonically isomorphic. The rank of the free A_0-module $\mathrm{gr}_i H_0 \simeq \mathrm{gr}^i H_0$ is called *the i-th Hodge number of H*.

Definition 1.3 A *Hodge F-crystal* over A_0 is an F-crystal H over A_0 equipped with a filtration by free A-submodules

$$H = \mathrm{Fil}^0 H \supset \mathrm{Fil}^1 H \supset \ldots \supset \mathrm{Fil}^i H \supset \mathrm{Fil}^{i+1} H \supset \ldots \tag{1.8}$$

(called the Hodge filtration on H) which lifts the Hodge filtration from H_0 and satisfies Griffiths transversality, i.e. for every i

$$\mathrm{Fil}^i H \otimes_A A_0 = \mathrm{Fil}^i H_0 \qquad \text{and} \qquad \nabla \mathrm{Fil}^i H \subset \Omega^1_{A/W} \otimes_A \mathrm{Fil}^{i-1} H. \tag{1.9}$$

Definition 1.4 An F-crystal H over A_0 is said to be *ordinary* if $\mathrm{gr}_\bullet H_0$ is a free A_0-module and the conjugate filtration and Hodge filtration on H_0 are opposite, i.e.

$$H_0 = \mathrm{Fil}_i H_0 \oplus \mathrm{Fil}^{i+1} H_0 \qquad \text{for every } i.$$

Proposition 1.5 ([4] prop.1.3.2) *Let H be an F-crystal over A_0 such that $\mathrm{gr}_\bullet H_0$ is a free A_0-module. Then H is ordinary if and only if there is a filtration by sub-F-crystals*

$$0 = U_{-1} \subset U_0 \subset U_1 \subset \ldots \subset U_i \subset U_{i+1} \subset \ldots \tag{1.10}$$

such that for every i

$$U_i \otimes_A A_0 = \mathrm{Fil}_i H_0 \quad \textit{and} \quad U_i/U_{i-1} \simeq V_i(-i)$$

where V_i is a unit F-crystal and $(-i)$ is Tate twist, i.e. $V_i(-i)$ is the same A-module with connection as V_i, but for every lift of Frobenius ϕ the map $F(\phi)\phi^*$ on $V_i(-i)$ is $p^i F(\phi)\phi^*$ on V_i. The filtration (1.10) is unique. □

Proposition 1.6 ([4] prop.1.3.6) *For an ordinary Hodge F-crystal H the filtrations $U_\bullet$ and $\mathrm{Fil}^\bullet H$ are opposite, i.e. for every i*

$$H = U_i \oplus \mathrm{Fil}^{i+1} H.$$

As a consequence one has a decomposition

$$H = \oplus_i H^i, \qquad H^i = U_i \cap \mathrm{Fil}^i H.$$

□

The above proposition gives a first result on the structure of ordinary Hodge F-crystals. Stronger results (Theorem 1.8 below) can be obtained over the algebraic closure $\overline{k}$ of the base field k. Let $\overline{W}$ the ring of Witt vectors of $\overline{k}$ and

$$\overline{A}_0 = \overline{k}[[t_1, \ldots, t_n]], \qquad \overline{A} = \overline{W}[[t_1, \ldots, t_n]], \qquad K^{\mathrm{u}} = \overline{W}[\tfrac{1}{p}].$$

One can base change an F-crystal H over A_0 to an F-crystal $\overline{H}$ over $\overline{A}_0$ as follows. By tensoring with $\overline{A}$ one gets the free $\overline{A}$-module with connection

$$\overline{H} = \overline{A} \otimes_A H, \qquad \nabla : \overline{H} \to \Omega^1_{\overline{A}/\overline{W}} \otimes_{\overline{A}} \overline{H}.$$

If $\phi : \overline{A} \to \overline{A}$ is a lift of Frobenius such that $\phi(A) \subset A$, then the A-linear map $F(\phi) : \phi^* H \to H$ gives the $\overline{A}$-linear map $F(\phi) : \phi^* \overline{H} \to \overline{H}$. If $\psi : \overline{A} \to \overline{A}$ is an arbitrary lift of Frobenius, the connection ∇ provides an isomorphism $\chi(\psi, \phi) : \psi^* \overline{H} \xrightarrow{\simeq} \phi^* \overline{H}$ and we can define $F(\psi) = F(\phi) \circ \chi(\psi, \phi) : \psi^* \overline{H} \to \overline{H}$.

Definition 1.7 Let $\overline{H}$ be a Hodge F-crystal over $\overline{A}_0$ with Hodge filtration $\{\mathrm{Fil}^i\overline{H}\}_{0\le i\le N}$. One says that $\overline{H}$ is *divisible* if for some lift of Frobenius ϕ

$$F(\phi)\phi^*\mathrm{Fil}^i\overline{H} \subset p^i\overline{H} \qquad \text{for} \quad i=0,1,\ldots,N. \tag{1.11}$$

If $p > N$ and (1.11) holds for one lift of Frobenius, then it holds for every lift of Frobenius (see [7] §5.0).

Théorème 1.4.2 of [4] describes the structure of ordinary Hodge F-crystals of level ≤ 1. The following theorem generalizes that result to higher levels.

Theorem 1.8 *Let $\overline{H}$ be a divisible ordinary Hodge F-crystal over $\overline{A}_0$ of level $\le N < p$ with filtrations $U_\bullet$ and $\mathrm{Fil}^\bullet H$ as in (1.10) and (1.8) respectively. Then*

$$\overline{H} = \bigoplus_{i=0}^{N} \overline{H}^i\,, \qquad \overline{H}^i = \overline{U}_i \cap \mathrm{Fil}^i\overline{H}$$

and there is a basis $\{e_{im}\}_{0\le i\le N, 1\le m\le h^i}$ of $\overline{H}$ with $e_{im}\in\overline{H}^i$ for all i,m and there is a matrix $\mathcal{T}$ with entries in $K^{\mathrm{u}}[[t_1,\ldots,t_n]]$, such that

- *$\mathcal{T} = (\tau_{ij})_{0\le i,j\le N}$ with τ_{ij} an $h^i\times h^j$-matrix, $\tau_{ij}=0$ for $j<i$, τ_{ii} =identity-matrix, the matrix $\tau_{ij}(0)$ has entries in $p^{j-i}\overline{W}$.*
- *the matrix of the connection ∇ with respect to the basis $\{e_{im}\}$ is*
$$\mathcal{T}^{-1}\cdot d\mathcal{T} \tag{1.12}$$
- *for every lift of Frobenius ψ the matrix of the map $F(\psi)\psi^*$ with respect to the basis $\{e_{im}\}$ is*
$$\mathcal{T}^{-1}\cdot P\cdot \psi(\mathcal{T}) \tag{1.13}$$
where P is the diagonal matrix
$$P = \mathrm{diag}(1,\ldots,1,p,\ldots,p,\ldots,p^j,\ldots,p^j,\ldots,p^N,\ldots,p^N) \tag{1.14}$$
with the entry p^j repeated h^j-times.

Proof For $i=0,\ldots,N$ there is an isomorphism of $\overline{A}$-modules $\overline{H}^i \simeq \overline{U}_i/\overline{U}_{i-1} = \overline{V}_i(-i)$ with $\overline{V}_i$ a unit F-crystal. According to [4] prop.1.2.2 there is an $\overline{A}$-basis $f_{i1},\ldots,f_{ih^i}$ of $\overline{V}_i$ such that

$$\nabla f_m = 0\,, \qquad F(\psi)\psi^* f_m = f_m \qquad \text{for} \quad m=1,\ldots,h^i \tag{1.15}$$

for every lift of Frobenius $\psi:\overline{A}\to\overline{A}$. Lift the basis $f_{i1},\ldots,f_{ih^i}$ of $\overline{V}_i$ to a basis $e_{i1},\ldots,e_{ih^i}$ of $\overline{H}^i$. This gives the basis $\{e_{im}\}_{0\le i\le N,1\le m\le h^i}$ for $\overline{H}$.

Equations (1.15) and Griffiths transversality show that the connection matrix with respect to this basis has the following shape

$$M_\nabla = \begin{pmatrix} 0 & \eta_{01} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \eta_{12} & 0 & \ldots & 0 \\ & & \ddots & \ddots & & \\ 0 & \ldots & \ldots & 0 & \eta_{N-2,N-1} & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & \eta_{N-1,N} \\ 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \tag{1.16}$$

with $\eta_{i,i+1}$ a matrix of size $h^i\times h^{i+1}$ with entries in $\Omega^1_{\overline{A}/\overline{W}}$. Integrability of the connection implies for all i

$$d\eta_{i,i+1} = 0 \qquad \text{and} \qquad \eta_{i,i+1}\eta_{i+1,i+2} = 0. \tag{1.17}$$

The entries of $\eta_{i,i+1}$ are closed 1-forms and hence exact, by the Poincaré lemma. So there is a matrix $\tau_{i,i+1}$ of size $h^i \times h^{i+1}$ with entries in $K^{\mathrm{u}}[[t_1,\dots,t_n]]$ such that

$$\eta_{i,i+1} = d\tau_{i,i+1}. \tag{1.18}$$

Using the equations in (1.17) and induction on $j-i$ one sees that for $i+2 \le j \le N$ there exist matrices τ_{ij} of size $h^i \times h^j$ with entries in $K^{\mathrm{u}}[[t_1,\dots,t_n]]$ such that

$$\tau_{i,j-1}\eta_{j-1,j} = d\tau_{ij}. \tag{1.19}$$

For $j < i$ we set $\tau_{ij} = h^i \times h^j$-zero-matrix and for $j = i$ we set $\tau_{ii} = h^i \times h^i$-identity-matrix. We collect the matrices τ_{ij} into one (block structured) matrix

$$\mathcal{T} = (\tau_{ij})_{0\le i,j\le N}.$$

Then (1.18) and (1.19) are equivalent with

$$\mathcal{T} M_\nabla = d\mathcal{T}. \tag{1.20}$$

We fix the constants of integration by taking the constant term $\mathcal{T}(0)$ of $\mathcal{T}$ as in (1.22) below. Take the lift of Frobenius $\phi : \overline{A} \to \overline{A}$ given by

$$\phi(\sum a_{m_1,\dots,m_n} t_1^{m_1}\cdots t_n^{m_n}) = \sum \sigma(a_{m_1,\dots,m_n}) t_1^{pm_1}\cdots t_n^{pm_n} \tag{1.21}$$

with σ the standard Frobenius map on $\overline{W}$. Equations (1.15) and the assumption on divisibility show

$$F(\phi)\phi^* e_{im} - p^i e_{im} \in p^i \overline{U}_{i-1}.$$

Therefore the matrix M_ϕ of the ϕ-linear map $F(\phi)\phi^* : \overline{H} \to \overline{H}$ has the shape

$$M_\phi = LP$$

with P as in (1.14) and L an uppertriangular matrix with entries in $\overline{A}$ and 1's along the diagonal. Let $L(0)$ denote the constant term of L. After these preparations we define $\mathcal{T}$ to be the unique matrix which satisfies the differential equation (1.20) and the initial condition

$$\mathcal{T}(0) = \prod_{m=1}^{\infty} \sigma^{-m}(P^{-m} L(0) P^m); \tag{1.22}$$

this product (with the factors ordered from right to left for increasing m) converges since the matrix $P^{-m}L(0)P^m - I$ has entries in $p^m\overline{W}$.

The fact that $F(\phi)$ is horizontal for the connection ∇ (see diagram (1.3)) is expressed by the matrix equation $M_\phi \phi(M_\nabla) = M_\nabla M_\phi + dM_\phi$. Multiplying this matrix equation by $\mathcal{T}$ and using Equation (1.20) gives

$$\mathcal{T} M_\phi \phi(\mathcal{T})^{-1} d\phi(\mathcal{T}) = d(P^{-1}\mathcal{T} M_\phi)$$

and hence $\mathcal{T} M_\phi \phi(\mathcal{T})^{-1}$ is constant. Since $\phi(\mathcal{T})(0) = P^{-1}\mathcal{T}(0)M_\phi(0)$ by (1.22) we conclude:

$$M_\phi = \mathcal{T}^{-1}\, P\, \phi(\mathcal{T}).$$

This shows that (1.13) holds for the particular lift of Frobenius ϕ. In order to check it for an arbitrary lift of Frobenius we first define the new basis $\{\varepsilon_{im}\}_{0\le i\le N, 1\le m\le h^i}$ for $\overline{H}\otimes K^{\mathrm{u}}[[t_1,\dots,t_n]]$ so that $\mathcal{T}^{-1}$ is the matrix whose columns give the coordinates of the new basis vectors with respect to the old basis $\{e_{im}\}_{0\le i\le N, 1\le m\le h^i}$:

$$\{\varepsilon_{im}\} = \{e_{im}\}\mathcal{T}^{-1}. \tag{1.23}$$

The triangular shape of the matrix $\mathcal{T}$ implies

$$\varepsilon_{im} \in \overline{U}^i \otimes K^{\mathrm{u}}[[t_1, \ldots, t_n]] \qquad \text{for all} \quad i, m. \tag{1.24}$$

The connection matrix with respect to the new basis is $d(\mathcal{T}^{-1}) + M_\nabla \mathcal{T}^{-1} = 0$. So

$$\nabla \varepsilon_{im} = 0 \qquad \text{for all} \quad i, m. \tag{1.25}$$

The matrix of $F(\phi)\phi^*$ with respect to the new basis is $\mathcal{T} M_\phi \phi(\mathcal{T})^{-1} = P$. So $F(\phi)\phi^* \varepsilon_{im} = p^i \varepsilon_{im}$ for all i, m. This holds for the special lift of Frobenius ϕ (1.21). If ψ is any lift of Frobenius (1.5) and (1.25) show $F(\psi)\psi^* \varepsilon_{im} = F(\phi)\phi^* \varepsilon_{im}$. Thus for any lift of Frobenius ψ we have

$$F(\psi)\psi^* \varepsilon_{im} = p^i \varepsilon_{im} \qquad \text{for all} \quad i, m. \tag{1.26}$$

The matrix M_ψ of $F(\psi)\psi^*$ with respect to the basis $\{e_{im}\}$ is therefore

$$M_\psi = \mathcal{T}^{-1} \cdot P \cdot \psi(\mathcal{T}),$$

as claimed in (1.13). To finish the proof of Theorem 1.8 we note that (1.22) implies that the matrix $\tau_{ij}(0)$ has entries in $p^{j-i}\overline{W}$ for $j > i$. □

2 Ordinary CY3-crystals over an algebraically closed field

We now specialize the general result of Theorem 2.2 to crystals of the type that appears as the third crystalline cohomology of an ordinary Calabi-Yau threefold over the algebraically closed field $\overline{k}$. More precisely, we want to see what happens, if we combine the hypotheses in Theorem 2.2 with the hypotheses for the variation of Hodge structure of Calabi-Yau threefolds, which are explicitly stated in [2] §2. So *from now on* $p > 3$ *and* $\overline{H}$ *is a divisible ordinary Hodge* F*-crystal over* $\overline{A}_0$ *of level* 3 with filtrations

$$\begin{array}{l} 0 \subset \overline{U}_0 \subset \overline{U}_1 \subset \overline{U}_2 \subset \overline{U}_3 = \overline{H} \\ \overline{H} = \mathrm{Fil}^0\overline{H} \supset \mathrm{Fil}^1\overline{H} \supset \mathrm{Fil}^2\overline{H} \supset \mathrm{Fil}^3\overline{H} \supset 0. \end{array}$$

such that in the decomposition $\overline{H} = \overline{H}^0 \oplus \overline{H}^1 \oplus \overline{H}^2 \oplus \overline{H}^3$, $\quad \overline{H}^i = \overline{U}_i \cap \mathrm{Fil}^i\overline{H}$,

$$\mathrm{rank}\,\overline{H}^0 = \mathrm{rank}\,\overline{H}^3 = 1 \qquad \text{and} \qquad \mathrm{rank}\,\overline{H}^1 = \mathrm{rank}\,\overline{H}^2 = h; \tag{2.1}$$

$\overline{H}^i$ would be denoted as $H^{i,3-i}$ in the Hodge theoretic setting of [2].
Moreover we assume that there is given a non-degenerate alternating bilinear form

$$\langle\ ,\ \rangle \,:\, \overline{H} \times \overline{H} \to \overline{A}$$

such that for all $x, y \in \overline{H}$ and for every lift of Frobenius $\phi : \overline{A} \to \overline{A}$

$$\begin{array}{rcl} \langle \nabla x, y \rangle + \langle x, \nabla y \rangle &=& d\langle x, y \rangle \\ \langle F(\phi)\phi^*(x), F(\phi)\phi^*(y) \rangle &=& p^3 \phi(\langle x, y \rangle). \end{array} \tag{2.2}$$

We also require the Riemann bilinear relations:

$$(\mathrm{Fil}^3\overline{H})^\perp = \mathrm{Fil}^1\overline{H}\,, \qquad (\mathrm{Fil}^2\overline{H})^\perp = \mathrm{Fil}^2\overline{H}\,. \tag{2.3}$$

Definition 2.1 An *ordinary CY3 crystal* over $\overline{A}_0$ is a divisible ordinary Hodge F-crystal of level 3 over $\overline{A}_0$ equipped with a non-degenerate alternating bilinear form $\langle\ ,\ \rangle$ such that (2.1), (2.2), (2.3) are satisfied.

In view of the Griffiths tranversality condition (1.9) the connection ∇ induces an $\overline{A}$-linear map

$$T_{\overline{A}/\overline{W}} \to \mathrm{Hom}(\overline{H}^3, \overline{H}^2) \tag{2.4}$$

where $T_{\overline{A}/\overline{W}} = \mathrm{Hom}(\Omega^1_{\overline{A}/\overline{W}}, \overline{A})$.

Theorem 2.2 *Let $p > 3$ and let $\overline{H}$ be an ordinary CY3 crystal over $\overline{A}_0$. Then there is a basis $\{e_i\}_{i=0,\ldots,2h+1}$, such that*

$$e_0 \in \overline{H}^0, \qquad e_1, \ldots, e_h \in \overline{H}^1, \qquad e_{h+1}, \ldots, e_{2h} \in \overline{H}^2, \qquad e_{2h+1} \in \overline{H}^3.$$

and such that the Gramm matrix of the form $\langle\ ,\ \rangle$ with respect to this basis is

$$(\langle e_i, e_j\rangle)_{0\le i,j\le 2h+1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & I_h & 0 \\ 0 & -I_h & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.5}$$

with $I_h = h \times h$-identity-matrix. Moreover there is a matrix $\mathcal{T}$ such that the matrix of the connection ∇ with respect to the basis $e_0, \ldots, e_{2h+1}$ is

$$\mathcal{T}^{-1} \cdot d\mathcal{T} \tag{2.6}$$

and such that for every lift of Frobenius ψ the matrix of the map $F(\psi)\psi^*$ with respect to the basis $e_0, \ldots, e_{2h+1}$ is

$$\mathcal{T}^{-1} \cdot P \cdot \psi(\mathcal{T}) \tag{2.7}$$

with P the (block) diagonal matrix $diag(1, pI_h, p^2 I_h, p^3)$.

This matrix $\mathcal{T}$ factors as [3]

$$\mathcal{T} = \begin{pmatrix} 1 & \tau_{23}^\star & 0 & 0 \\ 0 & I_h & 0 & 0 \\ 0 & 0 & I_h & \tau_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\tau_{13}^\star & Z \\ 0 & I_h & \tau_{12} & \tau_{13} \\ 0 & 0 & I_h & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.8}$$

where Z, τ_{23}, τ_{13}, τ_{12} are matrices with entries in $K^{\mathrm{u}}[[t_1, \ldots, t_n]]$ of respective sizes 1×1, $h \times 1$, $h \times 1$, $h \times h$.

Assume in addition to the preceding assumptions that the map in (2.4) is an isomorphism, and hence $n = h$. Let $\tau_1, \ldots, \tau_h \in K^{\mathrm{u}}[[t_1, \ldots, t_h]]$ denote the components of the $h \times 1$-matrix τ_{23}:

$$\tau_{23} = (\tau_1, \ldots, \tau_h)^\star. \tag{2.9}$$

Then the matrix

$$\left(\frac{\partial \tau_i}{\partial t_j}\right)_{1\le i,j\le h} \tag{2.10}$$

is invertible over $K^{\mathrm{u}}[[t_1, \ldots, t_h]]$ and

$$\tau_{13} \;=\; -\frac{1}{2}\left(\frac{\partial Z}{\partial \tau_1}, \ldots, \frac{\partial Z}{\partial \tau_h}\right)^\star \tag{2.11}$$

$$\tau_{12} \;=\; -\frac{1}{2}\left(\frac{\partial^2 Z}{\partial \tau_i \partial \tau_j}\right)_{1\le i,j\le h}. \tag{2.12}$$

[3] $.^\star$ means matrix transpose

Proof Take any $\overline{A}$-basis of $\overline{H}$ and matrix for which the statements in Theorem 1.8 hold and denote these as $e'_0, e'_1, \ldots, e'_h, e'_{h+1}, \ldots, e'_{2h}, e'_{2h+1}$ and $\mathcal{T}'$ respectively. So

$$e'_0 \in \overline{H}^0, \qquad e'_1, \ldots, e'_h \in \overline{H}^1, \qquad e'_{h+1}, \ldots, e'_{2h} \in \overline{H}^2, \qquad e'_{2h+1} \in \overline{H}^3.$$

Also take the basis $\varepsilon'_0, \varepsilon'_1, \ldots, \varepsilon'_h, \varepsilon'_{h+1}, \ldots, \varepsilon'_{2h}, \varepsilon'_{2h+1}$ as in (1.24), related to the previous basis by the transformation with the matrix $\mathcal{T}'$; see (1.23).

Then $\nabla \varepsilon'_i = \nabla \varepsilon'_j = 0$ and hence $d\langle \varepsilon'_i, \varepsilon'_j\rangle = \langle \nabla\varepsilon'_i, \varepsilon'_j\rangle + \langle \varepsilon'_i, \nabla\varepsilon'_j\rangle = 0$ for all i, j. This implies $\langle \varepsilon'_i, \varepsilon'_j\rangle \in \overline{W}$ for all i, j. Moreover we have $F(\phi)\phi^*(\varepsilon'_i) = p^r \varepsilon'_i$ with $r = \lceil \frac{i}{h} \rceil :=$ smallest integer $\geq \frac{i}{h}$, and $\langle F(\phi)\phi^*(\varepsilon'_i), F(\phi)\phi^*(\varepsilon'_j)\rangle = p^3\sigma(\langle \varepsilon'_i, \varepsilon'_j\rangle)$. This implies $\sigma(\langle \varepsilon'_i, \varepsilon'_j\rangle) = \langle \varepsilon'_i, \varepsilon'_j\rangle$ and hence $\langle \varepsilon'_i, \varepsilon'_j\rangle \in \mathbb{Z}_p$ if $\lceil \frac{i}{h} \rceil + \lceil \frac{j}{h} \rceil = 3$, whereas $\langle \varepsilon'_i, \varepsilon'_j\rangle = 0$ if $\lceil \frac{i}{h} \rceil + \lceil \frac{j}{h} \rceil \neq 3$. Now take the (block structured) matrix with entries in $\mathbb{Z}_p$:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I_h & 0 & 0 \\ 0 & 0 & (\langle \varepsilon'_i, \varepsilon'_{j+h}\rangle)_{1\leq i,j\leq h} & 0 \\ 0 & 0 & 0 & -\langle \varepsilon'_0, \varepsilon'_{2h+1}\rangle \end{pmatrix}$$

This matrix is invertible because the form $\langle\, , \,\rangle$ is non-degenerate. We use it for the basis transformations

$$\begin{aligned}(e_0, e_1, \ldots, e_h, e_{h+1}, \ldots, e_{2h}, e_{2h+1}) &:= (e'_0, e'_1, \ldots, e'_h, e'_{h+1}, \ldots, e'_{2h}, e'_{2h+1})M^{-1} \\ (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_h, \varepsilon_{h+1}, \ldots, \varepsilon_{2h}, \varepsilon_{2h+1}) &:= (\varepsilon'_0, \varepsilon'_1, \ldots, \varepsilon'_h, \varepsilon'_{h+1}, \ldots, \varepsilon'_{2h}, \varepsilon'_{2h+1})M^{-1}\end{aligned}$$

For the new basis $\varepsilon_0, \ldots, \varepsilon_{2h+1}$, the (block structured) Gramm matrix is

$$(\langle \varepsilon_i, \varepsilon_j\rangle)_{0\leq i,j\leq 2h+1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & I_h & 0 \\ 0 & -I_h & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

We set

$$\mathcal{T} = M\mathcal{T}'M^{-1},$$

so that we have the basis relation

$$(e_0, e_1, \ldots, e_h, e_{h+1}, \ldots, e_{2h}, e_{2h+1}) = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_h, \varepsilon_{h+1}, \ldots, \varepsilon_{2h}, \varepsilon_{2h+1})\mathcal{T}.$$

The block structure of the matrix $\mathcal{T}$ is

$$\mathcal{T} = \begin{pmatrix} 1 & \tau_{01} & \tau_{02} & \tau_{03} \\ 0 & I_h & \tau_{12} & \tau_{13} \\ 0 & 0 & I_h & \tau_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

The statements contained in formulas (1.25), (1.26), (1.12), (1.13) are still valid with respect to the new bases.

The Gramm matrix of the form $\langle\, , \,\rangle$ with respect to the basis $e_0, \ldots, e_{2h+1}$ is

$$(\langle e_i, e_j\rangle)_{0\leq i,j\leq 2h+1} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & I_h & \tau_{23} - \tau^\star_{01} \\ 0 & -I_h & -\tau_{12} + \tau^\star_{12} & -\tau_{13} + \tau^\star_{12}\tau_{23} - \tau^\star_{02} \\ 1 & \tau_{01} - \tau^\star_{23} & \tau_{02} - \tau^\star_{23}\tau_{12} + \tau^\star_{13} & 0 \end{pmatrix} \tag{2.15}$$

The Riemann bilinear relations (2.3) boil down to

$$\langle e_i, e_{2h+1}\rangle = 0 \quad \text{for} \;\; i \geq 1\,, \qquad \langle e_i, e_j\rangle = 0 \quad \text{for} \;\; i,j \geq h+1.$$

In view of (2.15) this means firstly that the Gramm matrix is indeed as in (2.5) and secondly that

$$\tau_{23} = \tau_{01}^{\star}\,, \qquad \tau_{12} = \tau_{12}^{\star}\,, \qquad \tau_{02} = \tau_{23}^{\star}\tau_{12} - \tau_{13}^{\star}. \tag{2.16}$$

The latter equalities are equivalent with the factorization of $\mathcal{T}$ as in (2.8), with

$$Z = \tau_{03} - \tau_{23}^{\star}\tau_{13}. \tag{2.17}$$

From Equations (1.18), (1.19), (2.16) one gets immediately

$$d\tau_{13} = \tau_{12}d\tau_{23} \tag{2.18}$$

and with some straightforward extra calculation

$$dZ = d(\tau_{03} - \tau_{13}^{\star}\tau_{23}) = -2\tau_{13}^{\star}d\tau_{23}. \tag{2.19}$$

Now assume that the map in (2.4) is an isomorphism. The partial derivations $\frac{\partial}{\partial t_j}$ $(j = 1, \ldots, h)$ constitute a natural basis for $T_{\overline{A}/\overline{W}}$. A natural basis for $\mathrm{Hom}(\overline{H}^3, \overline{H}^2)$ consists of the maps $e_{2h+1} \mapsto e_{h+i}$ $(i = 1, \ldots, h)$. With respect to these bases the map (2.4) is given by the matrix

$$\left(\frac{\partial \tau_i}{\partial t_j}\right)_{1 \leq i,j \leq h}. \tag{2.20}$$

This matrix is therefore invertible. Equation (2.11) is an obvious restatement of (2.19), while Equation (2.12) follows from (2.18) and (2.11). □

In the preceding proof we have essentially followed the same algebraic path as Bryant and Griffiths in their analysis of the variation of the Hodge structure of (complex) Calabi-Yau threefolds [2, 6]. Formulas (2.17), (2.11), (2.12) are familiar from the theory of Bryant and Griffiths. The following computation shows that the equally familiar formula for the *cubic form of Bryant and Griffiths* (also known as *Yukawa coupling*) holds in the present situation as well:

Corollary 2.3 *In the situation of Theorem 2.2 one has*

$$\langle e_{2h+1}, \nabla\left(\frac{\partial^3}{\partial\tau_i\partial\tau_j\partial\tau_k}\right) e_{2h+1}\rangle \;=\; -\frac{1}{2}\frac{\partial^3 Z}{\partial\tau_i\partial\tau_j\partial\tau_k}. \tag{2.21}$$

here $\nabla\left(\frac{\partial^3}{\partial\tau_i\partial\tau_j\partial\tau_k}\right)$ *is shorthand for* $\left(\frac{\partial}{\partial\tau_i}\otimes 1\right)\circ\nabla\circ\left(\frac{\partial}{\partial\tau_j}\otimes 1\right)\circ\nabla\circ\left(\frac{\partial}{\partial\tau_k}\otimes 1\right)\circ\nabla.$

Proof

$$\langle e_{2h+1}, \nabla\left(\frac{\partial^3}{\partial\tau_i\partial\tau_j\partial\tau_k}\right) e_{2h+1}\rangle \;=\; \frac{\partial^3\tau_{03}}{\partial\tau_i\partial\tau_j\partial\tau_k} - \tau_{23}^{\star}\frac{\partial^3\tau_{13}}{\partial\tau_i\partial\tau_j\partial\tau_k} \;=$$

$$=\; \frac{\partial^3 Z}{\partial\tau_i\partial\tau_j\partial\tau_k} - \frac{1}{2}\frac{\partial^2}{\partial\tau_j\partial\tau_k}\frac{\partial Z}{\partial\tau_i} - \frac{1}{2}\frac{\partial^2}{\partial\tau_i\partial\tau_k}\frac{\partial Z}{\partial\tau_j} - \frac{1}{2}\frac{\partial^2}{\partial\tau_i\partial\tau_j}\frac{\partial Z}{\partial\tau_k}.$$

□

All hypotheses and notations of Theorem 2.2 remain in force. The divisibility hypothesis (1.11) and Formula (2.7) imply that for every lift of Frobenius ψ the matrix $\mathcal{T}^{-1} \cdot P \cdot \psi(\mathcal{T}) \cdot P^{-1}$ has entries in $\overline{A}$. This means in particular

$$p^{-1}\psi(\tau_i) - \tau_i \in \overline{A} \qquad \text{for} \quad i = 1, \ldots, h.$$

From Theorem 1.8 we also know that $\tau_i(0) \in p\overline{W}$ for $i = 1, \ldots, h$. By a lemma of Dwork this implies (see [4] cor.1.4.5) that, if we set

$$q_i = exp(\tau_i) \qquad \text{for} \quad i = 1, \ldots, h,$$

then q_i is well defined and

$$q_i \in \overline{A} \qquad \text{and} \qquad q_i(0) - 1 \in p\overline{W}. \tag{2.22}$$

[4] cor.1.4.7 now yields that $p, q_1 - 1, \ldots, q_h - 1$ is a regular system of parameters for the ring $\overline{A}$ i.e.

$$\overline{A} = \overline{W}[[t_1, \ldots, t_h]] \;=\; \overline{W}[[q_1 - 1, \ldots, q_h - 1]].$$

The elements $q_1, \ldots, q_h$ are called the *canonical coordinates*. There is an associated *canonical lift of Frobenius*

$$\varphi : \overline{A} \to \overline{A}\,, \qquad \varphi|_{\overline{W}} = \sigma\,, \qquad \varphi(q_i) = q_i^p \quad \text{for } i = 1, \ldots, h.$$

For every lift of Frobenius ψ the matrix of the map $F(\psi)\psi^*$ with respect to the basis $e_0, \ldots, e_{2h+1}$ is given by Formula (2.7). A straightforward computation makes the entries of this matrix explicit. One then sees that exactly for the canonical lift of Frobenius φ this matrix takes the following elegant form:
For the canonical lift of Frobenius φ the matrix of $F(\varphi)\varphi^*$ with respect to the basis $e_0, \ldots, e_{2h+1}$ is

$$\begin{pmatrix} 1 & 0 & -p^{-2}\varphi(\tau_{13}^\star) + \tau_{13}^\star & p^{-3}\varphi(Z) - Z \\ 0 & I_h & p^{-1}\varphi(\tau_{12}) - \tau_{12} & p^{-2}\varphi(\tau_{13}) - \tau_{13} \\ 0 & 0 & I_h & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & pI_h & 0 & 0 \\ 0 & 0 & p^2 I_h & 0 \\ 0 & 0 & 0 & p^3 \end{pmatrix} \tag{2.23}$$

This formula generalizes [4] (1.4.7.1) to the CY3 case.

We summarize the preceding discussion in

Theorem 2.4 *All hypotheses and notations of Theorem 2.2 remain in force. Define*

$$q_i = exp(\tau_i) \qquad \textit{for} \quad i = 1, \ldots, h. \tag{2.24}$$

Then

$$q_i \in \overline{A} \qquad \textit{and} \qquad q_i(0) - 1 \in p\overline{W}. \tag{2.25}$$

and $p, q_1 - 1, \ldots, q_h - 1$ is a regular system of parameters for the ring $\overline{A}$ i.e.

$$\overline{A} = \overline{W}[[t_1, \ldots, t_h]] \;=\; \overline{W}[[q_1 - 1, \ldots, q_h - 1]]. \tag{2.26}$$

The elements $q_1, \ldots, q_h$ are called the canonical coordinates. *There is an associated* canonical lift of Frobenius

$$\varphi : \overline{A} \to \overline{A}\,, \qquad \varphi|_{\overline{W}} = \sigma\,, \qquad \varphi(q_i) = q_i^p \quad \textit{for } i = 1, \ldots, h. \tag{2.27}$$

Moreover

$$\langle e_{2h+1}, F(\varphi)\varphi^*(e_{2h+1})\rangle = \varphi(Z) - p^3 Z \tag{2.28}$$

and

$$p^{-3}\varphi(Z) - Z \in \overline{A} = \overline{W}[[t_1, \dots, t_h]]. \qquad (2.29)$$

Proof (2.28) and (2.29) follow directly from (2.23) and the divisibility hypothesis (1.11). □

3 Ordinary CY3-crystals over a perfect field

Having a good hold on the structures when the base field is algebraically closed we now turn to a base field which is just a *perfect field* k. So:
Let $p > 3$ *and let* H *be a divisible ordinary Hodge* F*-crystal of level* 3 *over* A_0 *equipped with a non-degenerate alternating bilinear form* $\langle\, ,\, \rangle$ *such that (2.1), (2.2), (2.3) hold and such that (2.4) is an isomorphism*, everything with H in place of $\overline{H}$.

Let $\overline{H}$ be the Hodge F-crystal over $\overline{A}_0$ obtained from H by base change from k to $\overline{k}$. Then all results of Section 2 hold for $\overline{H}$. We use the notations of Section 2 for referring to matters of $\overline{H}$.

Fix a non zero element ω_0 in $\mathrm{Fil}^3 H = H^3$. Since the $\overline{A}$-module $\mathrm{Fil}^3\overline{H} = \overline{H}^3$ has rank 1 with basis e_{2h+1}, there is an invertible element $f \in \overline{A}$ such that

$$\omega_0 = f e_{2h+1}. \qquad (3.1)$$

Lemma 3.1 *The element* f *in (3.1) has the form*

$$f = a\tilde{f}^{-1} \qquad \text{with} \quad a \in \overline{W},\ \tilde{f} \in A,\ \tilde{f}(0) = 1. \qquad (3.2)$$

Proof Let $\hat{\omega} \in V_3$ and $\hat{e}_{2h+1} \in \overline{V}_3$ denote the respective images of ω_0 and e_{2h+1} under the isomorphisms $\mathrm{Fil}^3 H \xrightarrow{\simeq} U_3/U_2 = V_3$, $\mathrm{Fil}^3\overline{H} \xrightarrow{\simeq} \overline{U}_3/\overline{U}_2 = \overline{V}_3$. Take the lift of Frobenius $\phi : A \to A$ such that $\phi(t_i) = t_i^p$ for $i = 1, \dots, h$. Then in the unit F-crystal V_3

$$F(\phi)\phi^*(\hat{\omega}) = cg\hat{\omega}$$

with $c \in W$, $g \in A$, $g(0) = 1$. The product $\tilde{f} = \prod_{i=0}^{\infty} \phi^i(g)$ converges in A, because $\phi^i(g) \equiv 1 \bmod (t_1^{p^i}, \dots, t_h^{p^i})$. Moreover there is an element $b \in \overline{W}$ such that $c = \sigma(b)b^{-1}$. Thus

$$F(\phi)\phi^*(b^{-1}\tilde{f}\hat{\omega}) = \sigma(b^{-1})\phi(\tilde{f})cg\hat{\omega} = b^{-1}\tilde{f}\hat{\omega}.$$

On the other hand, (2.7) implies $F(\phi)\phi^* e_{2h+1} = p^3 e_{2h+1}$ and hence in the (untwisted) unit F-crystal $\overline{V}_3$ we have

$$F(\phi)\phi^*\hat{e}_{2h+1} = \hat{e}_{2h+1}.$$

Since $\hat{\omega} = f\hat{e}_{2h+1}$ we see that $\phi(b^{-1}\tilde{f}f) = b^{-1}\tilde{f}f$ and hence $b^{-1}\tilde{f}f \in \mathbb{Z}_p$. Thus $f = a\tilde{f}^{-1}$ with $a \in \overline{W}$. □

We set for $j = 1, \dots, h$

$$\omega_j := fD_j\left(\frac{1}{f}\omega_0\right) \qquad (3.3)$$

where $D_j := (\frac{d}{dt_j} \otimes 1) \circ \nabla$. Lemma 3.1 shows $\omega_j = \tilde{f}^{-1} D_j\left(\tilde{f}\omega_0\right)$ and hence $\omega_j \in H$. On the other hand, $\nabla e_{2h+1} = \sum_{i=1}^{h} d\tau_i \otimes e_{h+i}$ implies

$$\omega_j = fD_j(e_{2h+1}) = f\frac{\partial\tau_1}{\partial t_j}e_{h+1} + \ldots + f\frac{\partial\tau_h}{\partial t_j}e_{2h}. \qquad (3.4)$$

From (2.20) we know that $\left(\frac{\partial \tau_i}{\partial t_j}\right)_{1\le i,j\le h}$ is precisely the matrix of the isomorphism $T_{\overline{A}/W} \to \mathrm{Hom}(\overline{H}^3, \overline{H}^2)$. Since $\{e_{h+1},\dots,e_{2h}\}$ is an $\overline{A}$-basis of $\overline{H}^2$, we now see that $\{\omega_1,\dots,\omega_h\}$ is an A-basis for H^2.

Next we set for $i = 1,\dots,h$

$$\check{\omega}_i = f^{-1}\frac{\partial t_i}{\partial \tau_1}e_1 + \ldots + f^{-1}\frac{\partial t_i}{\partial \tau_h}e_h. \tag{3.5}$$

Then

$$\langle \check{\omega}_i, \omega_j\rangle \;=\; \sum_{m=1}^{h} \frac{\partial t_i}{\partial \tau_m}\frac{\partial \tau_m}{\partial t_j} \;=\; \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases} . \tag{3.6}$$

The form $\langle\, ,\,\rangle$ induces a duality between H^1 and H^2. This fact together with (3.6) shows that $\check{\omega}_1,\dots,\check{\omega}_h$ which a priori are in $\overline{H}^1$, lie in fact in H^1 and constitute the basis of H^1 dual to the basis $\omega_1,\dots,\omega_h$ of H^2.

Finally we set

$$\check{\omega}_0 = f^{-1}e_0. \tag{3.7}$$

Then $\check{\omega}_0 \in \overline{H}^0$ and $\langle\check{\omega}_0,\omega_0\rangle = -1$. So $\check{\omega}_0$ lies in fact in H^0 and gives a basis dual to the basis ω_0 of H^3.

We summarize the preceding discussion in the following proposition.

Proposition 3.2 *Starting from $\omega_0 \in \mathrm{Fil}^3 H = H^3$ define $f \in \overline{A}$ by (3.1), define $\omega_1,\dots,\omega_h \in H^2$ by (3.3) and define $\check{\omega}_0 \in H^0$ and $\check{\omega}_1,\dots,\check{\omega}_h \in H^1$ by (3.7) resp. (3.5). Write*

$$\check{\omega}_{2h+1} = \omega_0\,, \qquad \check{\omega}_{h+i} = \omega_i \quad \textit{for } i = 1,\dots,h.$$

Then $\{\check{\omega}_0,\dots,\check{\omega}_{2h+1}\}$ is a basis for H such that

$$\check{\omega}_0 \in H^0\,, \quad \check{\omega}_1,\dots,\check{\omega}_h \in H^1\,, \quad \check{\omega}_{h+1},\dots,\check{\omega}_{2h} \in H^2\,, \quad \check{\omega}_{2h+1} \in H^3.$$

The matrix $\mathcal{S}$ in the basis transformation

$$(\check{\omega}_0,\dots,\check{\omega}_{2h+1}) = (e_0,\dots,e_{2h+1})\mathcal{S}$$

(i.e. the j-th column gives the coordinates of $\check{\omega}_j$ with respect to $e_0,\dots,e_{2h+1}$) is

$$\mathcal{S} = \begin{pmatrix} f^{-1} & 0 & 0 & 0 \\ 0 & f^{-1}\left(\frac{\partial t_j}{\partial \tau_i}\right) & 0 & 0 \\ 0 & 0 & f\left(\frac{\partial \tau_i}{\partial t_j}\right) & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \tag{3.8}$$

□

Recall that in the proof of Theorem 2.2 we constructed a basis $\{\varepsilon_0,\dots,\varepsilon_{2h+1}\}$ from the basis transformation

$$(e_0,\dots,e_{2h+1}) = (\varepsilon_0,\dots,\varepsilon_{2h+1})\mathcal{T}.$$

The relation with the basis $\{\check{\omega}_0,\dots,\check{\omega}_{2h+1}\}$ is therefore

$$(\check{\omega}_0,\dots,\check{\omega}_{2h+1}) = (\varepsilon_0,\dots,\varepsilon_{2h+1})\mathcal{T}\mathcal{S}.$$

The virtues of $\varepsilon_0,\dots,\varepsilon_{2h+1}$ are

$$\nabla\varepsilon_i = 0 \qquad \text{for} \quad i = 0,\dots,2h+1$$

and for every lift of Frobenius $\phi : A \to A$

$$F(\phi)\phi^*\varepsilon_0 = \varepsilon_0\,, \quad F(\phi)\phi^*\varepsilon_{2h+1} = p^3\varepsilon_{2h+1}$$
$$F(\phi)\phi^*\varepsilon_i = p\varepsilon_i\,, \quad F(\phi)\phi^*\varepsilon_{h+i} = p^2\varepsilon_{h+i} \qquad \text{for} \quad i = 1, \ldots, h.$$

Thus one finds the following analogue of formulas (2.6) and (2.7)

Corollary 3.3 *Let the hypotheses and notations be as in Proposition 3.2. Then the matrix of the connection ∇ with respect to the basis $\{\breve{\omega}_0, \ldots, \breve{\omega}_{2h+1}\}$ is*

$$(\mathcal{TS})^{-1} \cdot d(\mathcal{TS}) \tag{3.9}$$

and for every lift of Frobenius $\phi : A \to A$ the matrix of the map $F(\phi)\phi^*$ with respect to the basis $\{\breve{\omega}_0, \ldots, \breve{\omega}_{2h+1}\}$ is

$$(\mathcal{TS})^{-1} \cdot P \cdot \phi(\mathcal{TS}) \tag{3.10}$$

with matrices $\mathcal{T}$ and P as in Theorem 2.2 and $\mathcal{S}$ as in Proposition 3.2. □

Remark 3.4 Up to now we have discussed all structures in terms of a connection, to be thought of as the Gauss-Manin connection of a family of Calabi-Yau threefolds. In the literature descriptions of canonical coordinates and Yukawa coupling for complex Calabi-Yau threefolds near the large complex structure limit usually start from the Picard-Fuchs equations and their solutions. We want to point out how such a description can also be seen in our discusssion of ordinary CY3 crystals. The hypotheses and notations are as before in this section.

We start with a non zero element ω_0 in $\mathrm{Fil}^3 H$. In the geometric situation $\mathrm{Fil}^3 H$ would be $H^{3,0}$ and thus ω_0 is the analogue of a nowhere vanishing global 3-form. The connection induces an action of differential operators (in $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_h}$) on H. The differential operators which annihilate ω_0 are called *the Picard-Fuchs operators*. We expand ω_0 with respect to the basis $\{\varepsilon_0, \ldots, \varepsilon_{2h+1}\}$:

$$\omega_0 = f_0\varepsilon_0 + \ldots + f_{2h+1}\varepsilon_{2h+1}.$$

Since $\nabla\varepsilon_i = 0$ for all i, all f_i are annihilated by the Picard-Fuchs operators, i.e. are solutions of the Picard-Fuchs equations. This set of solutions is split into subsets $\{f_0\}\,,\ \{f_1, \ldots, f_h\}\,,\ \{f_{h+1}, \ldots, f_{2h}\}\,,\ \{f_{2h+1}\}$ according to the action of Frobenius on the basis vectors $\varepsilon_0, \ldots, \varepsilon_{2h+1}$. In the complex setting near the large complex structure point such a splitting of a basis of the solution space of the Picard-Fuchs equations is made according to the monodromy action, i.e. according to the degree of the logarithmic terms in the solution.

Since $\breve{\omega}_{2h+1} = \omega_0$ and $(\breve{\omega}_0, \ldots, \breve{\omega}_{2h+1}) = (\varepsilon_0, \ldots, \varepsilon_{2h+1})\mathcal{TS}$ the column vector $(f_0, \ldots, f_{2h+1})^\star$ is the last column of the matrix $\mathcal{TS}$. Thus $f = f_{2h+1}$ and $\left(\frac{f_0}{f}, \ldots, \frac{f_{2h+1}}{f}\right)$ is the last column of the matrix $\mathcal{T}$. This is analogous to the step of dividing the solutions of the Picard-Fuchs equations by the unique solution which is holomorphic and has value 1 at the large complex structure point.

From here on the algorithms for computing the canonical coordinates and the prepotential of the Yukawa coupling are identical for the ordinary case and for the large complex structure case.

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