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Neighboring powers

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ABSTRACT

In this article we discuss how close different powers of integers can be to each other. In addition we study pairs of powers of polynomials with rational coefficients which have differences of small positive degree.

1. Introduction

Let m and n be coprime integers with $n > m \geqslant 2$. It follows from the *abc* conjecture, see e.g. [13], that for each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for any positive integers x, y with $x^m \neq y^n$ we have

$$\left|x^n - y^m\right| > c(\varepsilon)X^{1-1/m-1/n-\varepsilon}, \quad X = \max\left(x^n, y^m\right). \tag{1}$$

For a proof of this implication, see Section 2. In the case $m = 2$, $n = 3$ this inequality is known as Hall's (modified) conjecture. Unconditionally a weaker result applies. It follows from work of Sprindzuk [11, 12] and Schmidt [9], which refined earlier work of Baker [1], that there is a positive number $\delta = \delta(m, n)$, which is effectively computable in terms of m and n such that $|x^m - y^n| \gg (\log X)^{\delta}$.

We believe that (1) is optimal in the following sense.

Conjecture 1.1. *Let m, n be coprime integers with $n > m \geqslant 2$. Then, for any $c > 0$ there exist infinitely many positive integers x, y such that*

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$$0 < \left|x^n - y^m\right| < cX^{1-1/m-1/n},$$

where $X = \max(x^n, y^m)$.

For the case $m = 2$, $n = 3$ this conjecture contradicts Hall's original conjecture which predicted a lower bound of the form $cX^{1/6}$ with $c > 0$. It is generally believed that Hall's original conjecture is not true. In Section 2 we give a heuristic argument for the validity of Conjecture 1.1.

Given m, n we define the number $\theta(m,n)$ as the infimum over all θ such that $0 < |x^n - y^m| < X^\theta$, $X = \max(x^n, y^m)$ has infinitely many solutions in positive integers x, y. In view of the two conjectures above we believe that $\theta(m,n) = 1 - 1/n - 1/m$. Only the trivial lower bound 0 is known and any improvement seems very hard to achieve. In this paper we focus our attention on upper bounds. We define the number $\gamma(m,n)$ by $\theta(m,n) = 1 - (1/m) - (1/n) + \gamma(m,n)/mn$ and find upper bounds for $\gamma(m,n)$. Our conjecture concerning $\theta(m,n)$ is now equivalent to

Conjecture 1.2. *For any coprime integers m, n with $n > m \geqslant 2$ we have $\gamma(m,n) = 0$.*

For any real number x let $[x]$ denote the greatest integer less than or equal to x.

Theorem 1.3. *Let n and m be coprime integers with $n > m \geqslant 2$. We have*

$$\gamma(m,n) \leqslant \min\bigl\{n - m[n/m], m - \bigl(n/\bigl([n/m] + 1\bigr)\bigr)\bigr\}.$$

If n and m are both odd integers and n is slightly larger than an odd multiple of m we are able to improve on Theorem 1.3.

Theorem 1.4. *Let n and m be coprime odd integers with $n > m \geqslant 2$. Write $n = (2q+1)m + w$ with $q \geqslant 0$ and $0 < |w| < m$. Then*

$$\gamma(m,n) \leqslant \frac{n(|w| - 1) - m}{n - 1}.$$

One can check that Theorem 1.4 yields an improvement on Theorem 1.3 precisely when w is positive and $m > (2q+3)(w-1)$.

Here is a table of values for the upper bounds for $\gamma(m,n)$ given by Theorems 1.3 and 1.4. We appealed to Theorem 1.4 only for the entries $(5,7)$, $(7,9)$ and $(9,11)$.

n \ m	2	3	4	5	6	7	8	9	10
3	1/2								
4	∗	1							
5	1/3	1/2	1						
6	∗	∗	∗	1					
7	1/4	2/3	1/2	1/3	1				
8	∗	1/3	∗	1	∗	1			
9	1/5	∗	1	1/2	∗	1/4	1		
10	∗	1/2	∗	∗	∗	2	∗	1	
11	1/6	1/4	1/3	1	1/2	3/2	5/2	1/5	1
12	∗	∗	∗	1	∗	1	∗	∗	∗
13	1/7	2/5	3/4	2/3	1	1/2	3/2	5/2	3
14	∗	1/5	∗	1/3	∗	∗	∗	2	∗
15	1/8	∗	1/4	∗	∗	1	1/2	∗	∗
16	∗	1/3	∗	1	∗	5/3	∗	1	∗
17	1/9	1/6	3/5	3/4	1/3	4/3	1	1/2	3/2

The proofs of Theorems 1.3–1.7 depend upon polynomial constructions and as a consequence for each pair (m,n) there are positive numbers $c_1 = c_1(m,n)$, $c_2 = c_2(m,n)$ and $\lambda = \lambda(m,n)$ such that for each positive integer N larger than c_2 there are at least N^λ pairs (x^n, y^m) with $1 \leqslant x^n \leqslant N$ for which

$$0 < \left|x^n - y^m\right| < c_1 X^\theta. \tag{2}$$

Birch, Chowla, Hall and Schinzel [2], see Section 6.3, showed that $\gamma(2,3) \leqslant 1/5$ by a polynomial construction. In 1982 Danilov [4] proved that $\gamma(2,3) \leqslant 0$. Danilov's proof yielded a much thinner set of pairs (x^3, y^2) for which (2) holds with $\theta = 1/6$. The pairs exhibited by Danilov correspond to solutions of a Pell's equation and because of this their counting function grows as a constant times $\log N$.

We are able to improve on Theorems 1.3 and 1.4 for a sparse but infinite set of pairs (m,n).

Theorem 1.5. *Let n and m be coprime positive integers with $n > m \geqslant 2$. Suppose that*

$$\frac{6}{5} < \frac{n}{m} < \frac{3}{2}$$

and that there are positive integers u and v for which either $(m,n) = (6v^2 - u^2, 9v^2 - 2u^2)$ or $(m,n) = (2v^2 - 3u^2, 3v^2 - 6u^2)$ then

$$\gamma(m,n) \leqslant m - \frac{2n}{3}.$$

Theorem 1.5 yields an improvement of Theorem 1.3 whenever it is applicable. We are also able to improve on Theorem 1.3 when the conditions of the next result apply.

Theorem 1.6. *Let n and m be coprime positive integers with $n > m \geqslant 2$. Suppose that there is a rational number a such that $(x,y) = (a, n/m)$ is a point on the curve*

$$E: \quad x^3 + 3(y-3)x^2 + (y-3)(y-4)x + \frac{(y-3)(y-4)(y-5)}{15} = 0.$$

Then

$$\gamma(m,n) \leqslant n - \frac{5m}{2} \quad \text{if } \frac{5}{2} < \frac{n}{m} < \frac{21}{8}$$

and

$$\gamma(m,n) \leqslant m - \frac{2n}{5} \quad \text{if } \frac{15}{7} < \frac{n}{m} < \frac{5}{2}.$$

The curve E together with a rational point, such as $(0,3)$, is an elliptic curve. It has Weierstrass form

$$Y^2 = X^3 - 2475X - 5850. \tag{3}$$

The set of real points (X,Y) on (3) consists of two connected components, one of which is unbounded. Since $(235, 3520)$, a non-torsion rational point, is in the latter component the rational solutions are dense in that component. Thus one may check that there are infinitely many coprime pairs of positive integers (m,n) for which $5/2 < n/m < 21/8$ and for which $15/7 < n/m < 5/2$. However there are only two pairs (m,n) with $m < 10000$ for which n/m is in the above ranges. They are $(23, 59)$ and $(7991, 19980)$ and we have $\gamma(23,59) \leqslant 3/2$ and $\gamma(7991, 19980) \leqslant 5/2$.

Theorem 1.7. $\gamma(11, 28) \leqslant 1/2$.

Our strategy for proving Theorems 1.3–1.7 is the same as that employed by Birch, Chowla, Hall and Schinzel [2]. That is, we look for polynomials f and g with rational coefficients and degrees mk and nk respectively for which $f^n - g^m$ is a non-zero polynomial of small degree and then we specialize to produce an m-th power of an integer and an n-th power of an integer which are close. Davenport [5] has shown that there are limitations on how small the degree of $f^n - g^m$ can be. He proved that if f and g are non-constant polynomials in $\mathbb{C}[x]$ then either $f^n = g^m$ or

$$\deg\bigl(f^n - g^m\bigr) \geqslant kmn - km - kn + 1. \tag{4}$$

We shall call pairs (f^n, g^m) of polynomials with $f, g \in \mathbb{Q}[x]$ for which equality holds in (4) Davenport pairs. These pairs are interesting in their own right and the last part of the paper is devoted to finding Davenport pairs.

In the final section we include some numerical tables of pairs of powers that are very close. Observe that in the tables with $m = 2$ there are some examples with pairs of x-values that are extremely close. Elaboration of this observation gave us the following theorem.

Theorem 1.8. *Let m be an integer with $m \geqslant 2$. Let r be a positive even integer and put $n = mr - 1$. Then there exist positive integers x and y for which*

$$0 < \bigl|x^n - y^m\bigr| < \frac{2m^2}{r+1} X^{1-1/m-1/n},$$

where $X = \max(x^n, y^m)$.

An immediate consequence of the above result is the following theorem.

Theorem 1.9. *Let m be an integer $\geqslant 2$. Then, for any $c > 0$ the inequality*

$$0 < \bigl|x^n - y^m\bigr| < cX^{1-1/m-1/n},$$

where $X = \max(x^n, y^m)$, has an infinite number of positive integer solutions x, y, n with $n > m$.

Note that this theorem goes some way towards Conjecture 1.1 with the big difference that now n is not kept fixed.

We would like to thank Wouter van der Bilt who performed a number of calculations for us. In particular, his work led us to Theorems 1.5 and 1.8.

2. Justification of the conjectures

2.1. The abc conjecture implies inequality (1)

The abc-conjecture states that for any $\varepsilon > 0$ there exists $c'(\varepsilon) > 0$ such that for any positive integers a, b, c satisfying $a + b = c$ and $\gcd(a, b, c) = 1$ we have

$$c^{1-\varepsilon} < c'(\varepsilon) Q(abc),$$

where for any positive integer N, $Q(N)$ denotes the greatest square free factor of N or the radical of N and so is the product of all distinct prime divisors of N. Put $k = x^n - y^m$, assume it is positive (otherwise we take $k = y^m - x^n$) and define $d = \gcd(x^m, y^n, k)$. Apply the abc-conjecture to $a = k/d$, $b = y^m/d$, $c = x^n/d$. We find that

$$\bigl(x^n/d\bigr)^{1-\varepsilon} < c'(\varepsilon) Q\bigl(x^n y^m k/d\bigr) \leqslant c'(\varepsilon) xyk/d.$$

Note that $X = x^n$ and $x \leqslant X^{1/n}$, $y \leqslant X^{1/m}$. Thus

$$X^{1-\varepsilon}/d \leqslant (X/d)^{1-\varepsilon} < c'(\varepsilon)X^{1/m+1/n}k/d.$$

Multiply on both sides by $dX^{-1/m-1/n}$ to obtain our inequality with $c(\varepsilon) = 1/c'(\varepsilon)$.

2.2. Heuristics for Conjecture 1.1

We base our conjecture on the following easy statement from probability theory.

Proposition 2.1. *For each positive number c there exists a positive number $\rho(c)$ such that if $I \subset \mathbb{R}$ is a closed interval of length L and S and T are two disjoint finite sets which are distributed over I with uniform probability distribution, then the probability that there exist $s \in S$, $t \in T$ with $|s - t| < cL/(|S| \cdot |T|)$ is at least $\rho(c)$.*

Let I be the interval $[N, 2N]$ where N is a positive integer. We choose S to be the set of m-th powers which are not mn-th powers and we let T be the set of n-th powers, the mn-th powers excluded. The numbers $|S|$ and $|T|$ are proportional to respectively $N^{1/m}$ and $N^{1/n}$. Our working assumption is that n-th powers and m-th powers behave as if they are uniformly distributed when n and m are coprime. Then, for any $c > 0$ there exists $\rho(c) > 0$ such that the probability for an m-th power and an n-th power to have distance $< cN/N^{1/m+1/n}$ is at least $\rho(c)$. There are infinitely many disjoint intervals of the form $[N, 2N]$ so the expected number of m-th and n-th powers of size X, whose distance is positive and at most $cX^{1-1/m-1/n}$ is infinite.

3. Proof of Theorem 1.3

Our approach to construct very close pairs of m-th and n-th powers is via polynomials. More particularly, it rests on the following lemma.

Lemma 3.1. *Let m, n be coprime integers with $n > m \geqslant 2$ and let k be a positive integer. Suppose there exist polynomials $f, g, h \in \mathbb{Q}[x]$ of degrees km, kn, D respectively with $0 < D < kmn$ such that*

$$f^n - g^m = h.$$

Put $D = kmn - km - kn + \delta$. Then $\theta(m,n) \leqslant (1 - 1/m - 1/n) + \delta/kmn$ and $\gamma(m,n) \leqslant \delta/k$.

Proof. Let N be the common denominator of the coefficients of the polynomials f, g. Then $(N^m f(x))^n - (N^n g(x))^m = N^{mn}h(x)$ is an identity between polynomials with integer coefficients. We construct close m-th and n-th powers by substitution of x by an integer a. Since there exist positive constants a_0, b_0 and c_0 such that

$$\bigl(N^m f(a)\bigr)^n/a^{kmn} \to a_0, \qquad \bigl(N^n g(a)\bigr)^m/a^{kmn} \to b_0, \qquad N^{mn}h(a)/a^D \to c_0$$

as $a \to \infty$, our estimate for $\theta(m,n)$ follows. □

To prove Theorem 1.3 we use the following polynomial constructions.

Lemma 3.2. *Let m, n be coprime integers with $n > m \geqslant 2$ and let $s = [n/m]$. Let*

$$B_1(x) = \sum_{j=0}^{s} \binom{n/m}{j} x^{n-jm}.$$

Then $(x^m + 1)^n - B_1(x)^m$ is a polynomial of degree $mn - m(s+1)$.

Lemma 3.3. *Let m, n be coprime integers with $n > m \geqslant 2$ and let $s = [n/m] + 1$. Let*

$$B_2(x) = \sum_{j=0}^{s} \binom{n/m}{j} x^{sn-jn}.$$

Then $(x^{ms} + x^{ms-n})^n - B_2(x)^m$ is a polynomial of degree $mns - n(s+1)$.

Proof of Lemmas 3.2 and 3.3. Consider the following Taylor expansion in t,

$$(1+t)^{n/m} = \sum_{j=0}^{s} \binom{n/m}{j} t^j + O\left(t^{(s+1)}\right). \tag{5}$$

Replace t by $1/x^m$, raise both sides to the power m, and multiply on both sides by x^{mn}. We obtain

$$\left(x^m + 1\right)^n = B_1(x)^m + O\left(x^{mn-m(s+1)}\right).$$

A more careful analysis of the constant in the O-term shows that the degree is precisely $mn - m(s+1)$ and Lemma 3.2 follows.

For the proof of Lemma 3.3 we replace t by $1/x^n$ in (5), raise both sides to the power m and multiply on both sides by x^{mns}. We obtain

$$\left(x^{ms} + x^{ms-n}\right)^n = B_2(x)^m + O\left(x^{mns-n(s+1)}\right).$$

A more careful analysis of the constant in the O-term shows that the degree is precisely $mns - n(s+1)$ and so Lemma 3.3 holds. □

The proof of Theorem 1.3 now goes as follows. Application of Lemma 3.1 with $k = 1$ and Lemma 3.2 gives us $\gamma(m,n) \leqslant mn - m([n/m]+1) - mn + m + n = n - [n/m]m$. Application of Lemma 3.1 with $k = s$ and Lemma 3.3 gives us $\gamma(m,n) \leqslant mn - n(1 + 1/([n/m]+1)) - mn + m + n = m - (n/([n/m]+1))$.

4. A refinement of Theorem 1.3

Put

$$\theta = \frac{x + \sqrt{x^2 - 4}}{2}$$

and notice that

$$\theta^{-1} = \frac{x - \sqrt{x^2 - 4}}{2}.$$

Define

$$T_n(x) = \theta^n + \theta^{-n}.$$

By putting $x = 2\cos\phi$ one sees that $\theta = e^{i\phi}$ and $T_n(2\cos\phi) = 2\cos n\phi$. Up to the appearance of the factor 2 this is the definition of a Chebyshev polynomial of the first kind.

Let n and m be coprime integers with $n > m \geqslant 2$ and let q and w be the integers with $q \geqslant 0$ and $0 < |w| < m$ for which $n = (2q+1)m + w$. Notice that

$$\left(\theta^m+\theta^{-m}\right)^{n/m}=\sum_{r=0}^{q}\binom{n/m}{2r}\theta^{n-2rm}+O\left(\theta^{-m+w}\right)$$

and so

$$\left(T_m(x)\right)^{n/m}=\sum_{r=0}^{q}\binom{n/m}{2r}T_{n-2rm}(x)+O\left(\theta^{-m+|w|}\right),$$

hence

$$\left(T_m(x)\right)^{n}=\left(\sum_{r=0}^{q}\binom{n/m}{2r}T_{n-2rm}(x)\right)^m+O\left(x^{n(m-1)-m+|w|}\right). \tag{6}$$

The identity (6) together with Lemma 3.1 does not yield an improvement of Theorem 1.3. However, the following observation will yield improvement in some cases, as formulated in Theorem 1.4.

If n is odd then all non-zero coefficients of T_n are associated with odd powers of x while if n is even all non-zero coefficients of T_n are associated with even powers of n. Suppose that m,n are odd and let

$$M=\frac{m-1}{2},\qquad N=\frac{n-1}{2}.$$

Write

$$P_M(x)=T_m(\sqrt{x})/\sqrt{x},\qquad Q_N(x)=\sum_{r=0}^{q}\binom{n/m}{2r}T_{n-2rm}(\sqrt{x})/\sqrt{x}.$$

The newly defined polynomials P_M,Q_N have degrees M,N respectively and we obtain

$$\left(P_M(x)\right)^n x^{(n-m)/2}=\left(Q_N(x)\right)^m+O\left(x^{n(m-1)/2-m+|w|/2}\right).$$

Replace x by x^n in the identity. We get

$$\left(P_M\left(x^n\right)x^{(n-m)/2}\right)^n=\left(Q_N\left(x^n\right)\right)^m+O\left(x^{n^2(m-1)/2-mn+|w|n/2}\right).$$

The degree of $P_M(x^n)x^{(n-m)/2}$ equals $nM+(n-m)/2=mN$, the degree of $Q_N(x^n)$ equals nN. The remainder term has order

$$n^2(m-1)/2-mn+|w|n/2=mnN-mN-nN+\left(|w|-1\right)n/2-m/2.$$

We can now apply Lemma 3.1 with $k=N$ to conclude that

$$\gamma(m,n)\leqslant\frac{(|w|-1)n/2-m/2}{N}=\frac{(|w|-1)n-m}{n-1}.$$

5. Further improvements on Theorem 1.3

In this section we generalize the constructions employed in the proof of Lemmas 3.2 and 3.3. Instead of $(1+t)^{n/m}$ we consider

$$\left(1+t+at^2\right)^{n/m} = 1 + b_1 t + b_2 t^2 + \cdots + b_k t^k + b_{k+1} t^{k+1} + \cdots.$$

Now assume that we can find $a \in \mathbb{Q}$ such that $b_{k+1} = 0$. We get

$$\left(1+t+at^2\right)^{n/m} = 1 + b_1 t + b_2 t^2 + \cdots + b_k t^k + O\left(t^{k+2}\right). \tag{7}$$

We consider two possibilities for construction, depending on whether $2n - km \geqslant 0$ or $2n - km \leqslant 0$.

In the first case, when $2n - km \geqslant 0$, we replace t by $1/x^m$, raise everything to the power m and multiply by x^{2mn}. This yields

$$\left(x^{2m} + x^m + a\right)^n = \left(x^{2n} + b_1 x^{2n-m} + \cdots + b_k x^{2n-km}\right)^m + O\left(x^{2mn-(k+2)m}\right).$$

Using Lemma 3.1 we get the bound $\gamma(m,n) \leqslant n - mk/2$. This is an improvement over the bound given by Theorem 1.3 if and only if k is odd and $k/2 < n/m < ((k+1)(k+2))/(2(k+3))$.

In the second case, when $2n - km \leqslant 0$, we replace t by $1/x^n$, raise everything to the power m and multiply by x^{kmn}. We obtain

$$\left(x^{km} + x^{km-n} + ax^{km-2n}\right)^n = \left(x^{kn} + b_1 x^{kn-n} + \cdots + b_k\right)^m + O\left(x^{kmn-(k+2)n}\right).$$

Using Lemma 3.1 we get the bound $\gamma(m,n) \leqslant m - 2n/k$. This is an improvement over Theorem 1.3 if and only if k is odd and $(k(k+1))/(2(k+2)) < n/m < k/2$. Everything we said, of course, relies on our success in finding a rational number a for which $b_{k+1} = 0$.

We now see if we are successful in finding such a for increasing values of k. When $k = 1$ we arrive at the polynomials found in Lemmas 3.2 and 3.3. When k is even we have seen that we cannot expect any improvement over Theorem 1.3.

5.1. The case $k = 3$ and the proof of Theorem 1.5

A brief calculation shows that $b_4 = 0$ implies

$$12a^2m^2 - 24am^2 + 12amn + 6m^2 - 5mn + n^2 = 0. \tag{8}$$

This is quadratic in a, so a is rational if and only if the discriminant

$$48m^2(3m-2n)(2m-n)$$

is a square. Since $\gcd(m,n) = 1$ there are four possibilities:

1. $2n - 3m = u^2$ and $n - 2m = 3v^2$ for some coprime positive integers u, v. Hence $m = -6v^2 + u^2$ and $n = -9v^2 + 2u^2$.
2. $2n - 3m = 3u^2$ and $n - 2m = v^2$ for some coprime positive integers u, v. Hence $m = -2v^2 + 3u^2$ and $n = -3v^2 + 6u^2$.
3. $3m - 2n = u^2$ and $2m - n = 3v^2$ for some coprime positive integers u, v. Hence $m = 6v^2 - u^2$ and $n = 9v^2 - 2u^2$.
4. $3m - 2n = 3u^2$ and $2m - n = v^2$ for some coprime positive integers u, v. Hence $m = 2v^2 - 3u^2$ and $n = 3v^2 - 6u^2$.

In the first two cases we have $2n-3m \geqslant 0$ and so we obtain an improvement of Theorem 1.3 when $3/2 < n/m < 5/3$. However, one can check that this does not occur. In the next two cases $2n-3m < 0$ and we obtain an improvement of Theorem 1.3 when $6/5 < n/m < 3/2$. In case 3 we see that this is equivalent to the condition $2u < 3v$ while in case 4 it is equivalent to the condition $2u < v$. Further in both cases, by our earlier remarks, $\gamma(m,n) \leqslant m - 2n/3$ and so Theorem 1.5 follows.

5.2. The case $k = 5$ and the proof of Theorem 1.6

The equation $b_6 = 0$ is equivalent to

$$15a^3 + 45\left(\frac{n}{m} - 3\right)a^3 + 15\left(\frac{n}{m} - 3\right)\left(\frac{n}{m} - 4\right)a + \left(\frac{n}{m} - 3\right)\left(\frac{n}{m} - 4\right)\left(\frac{n}{m} - 5\right) = 0.$$

We have an improvement of Theorem 1.3 if $5/2 < n/m < 21/8$ in which case $\gamma(m,n) \leqslant n - 5m/2$ or if $15/7 < n/m < 5/2$ in which case $\gamma(m,n) \leqslant m - 2n/5$. Theorem 1.6 now follows.

5.3. The proof of Theorem 1.7

One may check that

$$\left(\left(x^{22} + 11x^{11} + 22\right)^{28}, \left(x^{56} + 28x^{45} + 294x^{34} + 1428x^{23} + 3213x^{12} + 2856x\right)^{11}\right) \tag{9}$$

is a Davenport pair. It now follows from Lemma 3.1 with $\delta = 1$ and $k = 2$ that $\gamma(11, 28) \leqslant 1/2$.

6. The bounds of Davenport and of Mason and Stothers

The inequality of Davenport is a consequence of the Mason–Stothers inequality. For any non-zero polynomial p in $\mathbb{C}[x]$ we let $N_0(p)$ denote the number of distinct roots of p. Mason [8] and Stothers [14] proved that if p and q are coprime polynomials in $\mathbb{C}[x]$ which are not both constant then

$$\max\{\deg p, \deg q\} \leqslant N_0\big(pq(p-q)\big) - 1. \tag{10}$$

Here is a statement where we drop the condition of coprimality of p, q.

Proposition 6.1. *Let p, q be non-zero polynomials in $\mathbb{C}[x]$ whose ratio p/q is non-constant. Then*

$$\max\big(\deg(p), \deg(q)\big) \leqslant N_0(p) + N_0(q) + \deg(p-q) - 1.$$

Moreover, if equality holds then p and q are coprime and $p - q$ is square-free.

Proof. Let $d = \gcd(p, q)$ and apply (10) to $p/d, q/d$. We obtain

$$\begin{aligned}\max\left(\deg\left(\frac{p}{d}\right), \deg\left(\frac{q}{d}\right)\right) &\leqslant N_0\left(\left(\frac{p}{d}\right)\left(\frac{q}{d}\right)\frac{p-q}{d}\right) - 1 \\ &\leqslant N_0(p) + N_0(q) - N_0(d) + \deg\left(\frac{(p-q)}{d}\right) - 1.\end{aligned}$$

Adding $\deg(d)$ on both sides gives us the inequality

$$\max\big(\deg(p), \deg(q)\big) \leqslant N_0(p) + N_0(q) - N_0(d) + \deg(p-q) - 1$$

from which our desired inequality follows. Furthermore, when we are in a case when equality holds, we necessarily have that $N_0(d) = 0$, hence p, q are coprime. In that case application of (10) gives us

$$\max\bigl(\deg(p), \deg(q)\bigr) \leqslant N_0\bigl(pq(p-q)\bigr) - 1 \leqslant N_0(p) + N_0(q) + \deg\bigl((p-q)\bigr) - 1$$

and the equality implies that $N_0(p-q) = \deg(p-q)$, in other words, $p-q$ is square-free. □

When we apply this proposition to $p = f^n$, $q = g^m$ we find, assuming that $\deg(f) = mk$, $\deg(g) = nk$,

$$mnk \leqslant N_0(f) + N_0(g) + \deg\bigl(f^n - g^m\bigr) - 1 \leqslant mk + nk + \deg\bigl(f^n - g^m\bigr) - 1. \tag{11}$$

This immediately implies inequality (4). Furthermore, our proposition implies that for a Davenport pair (f^n, g^m) we have that $\gcd(f, g) = 1$ and $f^n - g^m$ is a square-free polynomial. In addition inequality (11) shows that we have $nk = N_0(f)$, $mk = N_0(g)$ in the case of a Davenport pair. Hence f, g are square-free.

7. Construction of Davenport pairs

7.1. Infinite families of Davenport pairs

In this subsection we list the Davenport pairs (f^n, g^m) with f and g in $\mathbb{Q}[x]$ which we have found. Suppose that the degree of f is km and the degree of g is kn with k a positive integer. We consider the pairs $(f(x)^n, g(x)^m)$ and $(c^{km} f(ax+b)^n, (c^{kn} g(ax+b))^m)$ with $a, b, c \in \overline{\mathbb{Q}}$, $ac \neq 0$ as equivalent. Accordingly we may choose a representative (f^n, g^m) from each equivalence class with f a monic polynomial of degree km having 0 as the coefficient of degree $km-1$ and with the next non-zero coefficient an integer of smallest absolute value which, when possible, is taken to be positive. We shall refer to this as the normalized form for a representative.

We have found seven infinite families of Davenport pairs as well as a number of sporadic examples. Two of the families arise from the Taylor series expansion of $(1+t)^{n/m}$, three from the Taylor series expansion of $(1+t+at^2)^{n/m}$ and the two other infinite families are connected with the Chebyshev polynomials.

Notice that if $n - sm = 1$ for some positive integer s, or equivalently that $m \mid n-1$ then by Lemma 3.2

$$\bigl((x^m+1)^n, B_1(x)^m\bigr) \tag{12}$$

is a Davenport pair. Further if $m \mid n+1$ then the polynomials in Lemma 3.3 form a Davenport pair. However these polynomials belong to a larger family. To see this raise both sides of (5) to the power m, substitute $t = 1/x^d$ and multiply by x^{rmn}. We obtain

$$\bigl(x^{rm} + x^{rm-d}\bigr)^n = \left(x^{rn} + \cdots + \binom{n/m}{s} x^{rn-ds}\right)^m + m\binom{n/m}{s+1} x^{mnr-d(s+1)} + O\bigl(x^{mnr-d(s+2)}\bigr).$$

Here d, r and s are positive integers for which $rm - d \geqslant 0$ and $rn - ds \geqslant 0$. They form a Davenport pair whenever $rm + rn - d(s+1) = 1$. Thus either $rm - d = 0$ and $rn - ds = 1$ in which case $r = 1$ and we recover the pairs (12) or $rm - d = 1$ and $rn - ds = 0$. In the latter case d divides n and m divides $d+1$. Therefore we have the following result.

Proposition 7.1. *Let n and m be coprime positive integers with $n > m \geqslant 2$. Suppose that there is a positive divisor d of n such that m divides $d+1$. Put $s = (d+1)/m$ and*

$$B_3(x) = \sum_{j=0}^{ns/d} \binom{n/m}{j} x^{ns-jd}.$$

Then $(x^{d+1}+x)^n - B_3^m$ *has degree* $mns - ms - ns + 1$.

It follows from Proposition 7.1 that if $d > 0$, $d \mid n$ and $m \mid d+1$ then

$$\left(\left(x^{d+1}+x\right)^n, B_3^m(x)\right) \tag{13}$$

is a Davenport pair.

Let us now raise both sides of (7) to the power m, substitute $t = 1/x^d$ and multiply by x^{rmn}. We obtain $(x^{rm} + x^{rm-d} + ax^{rm-2d})^n = (x^{rn} + \cdots + b_k x^{rn-kd})^m + O(x^{rmn-d(k+2)})$. In order to have a polynomial identity we need $rm - 2d$ and $rn - kd$ to be non-negative. Here we are assuming that there is a rational number a for which $b_{k+1} = 0$. We have a Davenport pair precisely when $rm + rn - d(k+2) = 1$. Therefore there are two possibilities:

i) $rm - 2d = 0$ and $rn - kd = 1$
ii) $rm - 2d = 1$ and $rn - kd = 0$.

In case i) we see that r divides 2 so r is 1 or 2. If r is 1 then $m = 2d$ and $n = kd + 1$ while if $r = 2$ then $m = d$ and $n = (kd+1)/2$. Observe that if $k = 2$ then $r = 1$ and $m = 2d$, $n = 2d + 1$. The condition $b_3 = 0$ holds when $a = 1/3 - n/6m$ and so

$$\left(\left(x^{2d} + x^d + \frac{2d-1}{12d}\right)^{2d+1}, \left(x^{2d+1} + \frac{2d+1}{2d}x^{d+1} + \frac{2d^2+3d+1}{12d^2}x\right)^{2d}\right) \tag{14}$$

is a Davenport pair for d a positive integer. We did not find new families for $k = 1, 3, 4, 5$. However when $k = 5$ we found two interesting examples. For the first example we take $d = 2$ and $r = 1$ so that $n = 11$ and $m = 4$. This yields the Davenport pair

$$\left(\left(x^4 + x^2 + \frac{1}{8}\right)^{11}, \left(x^{11} + \frac{11}{4}x^9 + \frac{11}{4}x^7 + \frac{77}{64}x^5 + \frac{231}{1024}x^3 + \frac{77}{4096}x\right)^4\right)$$

which is equivalent to a Davenport pair from the infinite family given by (17). For the second example we take $d = 11$ and $r = 2$ so that $n = 28$ and $m = 11$. This yields the Davenport pair

$$\left(\left(x^{22} + x^{11} + \frac{2}{11}\right)^{28}, \left(x^{56} + \frac{28}{11}x^{45} + \frac{294}{121}x^{34} + \frac{1428}{1331}x^{23} + \frac{3213}{14641}x^{12} + \frac{2856}{161051}x\right)^{11}\right)$$

which is equivalent to the Davenport pair (9).

In case ii) r divides k with $(r,d) = 1$ and r odd. If $r = 1$ then $m = 2d + 1$ and $n = kd$ with k at least 3. If $k = 3$ then $3m - 2n = 3$ and on examining the four cases from Section 5.1 we find that in case 4 we have $u = 1$. Thus $m = 2v^2 - 3$ and $n = 3v^2 - 6$ and so, by (8), a is either $\frac{v(v-1)}{2(2v^2-3)}$ or $\frac{v(v+1)}{2(2v^2-3)}$. Therefore we have, for e in $\{1, -1\}$,

$$\left(f_e^{3v^2-6}, g_e^{2v^2-3}\right) \tag{15}$$

is a Davenport pair with

$$f_e(x) = x^{2v^2-3} + x^{v^2-1} + \frac{v(v+e)}{2(2v^2-3)}x$$

and

$$g_e(x) = x^{3v^2-6} + \frac{3v^2-6}{2v^2-3}x^{2v^2-4} + \frac{3(v^2-2)(2v^2+ev-3)}{2(2v^2-3)^2}x^{v^2-2} + \frac{(v^2-2)v(v^2-3)(2v+3e)}{2(2v^2-3)^3}$$

for $v = 2, 3, \ldots$. Next if $r = 3$ and $k = 3$ then $m = (2d+1)/3$ and $n = d$ so $3m - 2n = 1$. On again examining the four cases from Section 4.1 we see that the only possibility is case 3 where $u = 1$. Thus $m = 6v^2 - 1$ and $n = 9v^2 - 2$. By (8) a is either $\frac{v(3v-1)}{2(6v^2-1)}$ or $\frac{v(3v+1)}{2(6v^2-1)}$. Therefore we have, for e in $\{1, -1\}$,

$$\left(f_e^{9v^2-2}, g_e^{6v^2-1}\right) \tag{16}$$

is a Davenport pair with

$$f_e(x) = x^{18v^2-3} + x^{9v^2-1} + \frac{v(3v+e)}{2(6v^2-1)}x$$

and

$$g_e(x) = x^{27v^2-6} + \frac{9v^2-2}{6v^2-1}x^{18v^2-4} + \frac{(9v^2-2)(6v^2+ev-1)}{2(6v^2-1)^2}x^{9v^2-2} + \frac{(9v^2-2)v(3v^2-1)(2v+e)}{2(6v^2-1)^3}$$

for $v = 1, 2, \ldots$.

We remark that the representatives given for the equivalence classes of Davenport pairs for the families (12), (13), (14), (15) and (16) are in normalized form with the exception of the case $d = 1$ in (13) and the case $d = 1$ in (14). Further we may take f_e and g_e in non-normalized form to be

$$f_e(x) = x^{2v^2-3} + \left(2v^2-3\right)x^{v^2-1} + \frac{v(v+e)(2v^2-3)}{2}x$$

and

$$g_e(x) = x^{3v^2-6} + \left(3v^2-6\right)x^{2v^2-4} + \frac{3(v^2-2)(2v^2+ev-3)}{2}x^{v^2-2} + \frac{(v^2-2)v(v^2-3)(2v+3e)}{2}$$

in (15) and to be

$$f_e(x) = x^{18v^2-3} + \left(6v^2-1\right)x^{9v^2-1} + \frac{v(3v+e)(6v^2-1)}{2}x$$

and

$$g_e(x) = x^{27v^2-6} + \left(9v^2-2\right)x^{18v^2-4} + \frac{(9v^2-2)(6v^2+ev-1)}{2}x^{9v^2-2} + \frac{(9v^2-2)v(3v^2-1)(2v+e)}{2}$$

in (16).

7.2. The Chebyshev families

Let m, q and e be integers with $m \geqslant 2$, $q \geqslant 0$ and e from $\{1, -1\}$. Put $n = (2q+1)m + e$. Then, by (6),

$$\left(\left(T_m(x)\right)^n, \left(\sum_{r=0}^{q} \binom{n/m}{2r} T_{n-2rm}(x)\right)^m\right) \tag{17}$$

is a Davenport pair.

Further by taking m odd and $n = m+2$ in Section 4 we see that

$$\left(f_m^{m+2}, g_{m+2}^m\right) \tag{18}$$

is a Davenport pair where

$$f_m(x) = x^{-m/2} T_m\left(x^{(m+2)/2}\right)$$

and

$$g_{m+2}(x) = x^{-m+2/2} T_{m+2}\left(x^{(m+2)/2}\right).$$

The first few polynomials $T_m(x)$ are

$$T_2(x) = x^2 - 2,$$
$$T_3(x) = x^2 - 3x,$$
$$T_4(x) = x^4 - 4x^2 + 2,$$
$$T_5(x) = x^5 - 5x^3 + 5x,$$
$$T_6(x) = x^6 - 6x^4 + 9x^2 - 2,$$
$$T_7(x) = x^7 - 7x^5 + 14x^3 - 7x$$

and in normalized form they are $t_m(x)$ where

$$t_2(x) = x^2 + 1,$$
$$t_3(x) = x^3 + x,$$
$$t_4(x) = x^4 + x^2 + \frac{1}{8},$$

$$t_5(x) = x^5 + x^3 + \frac{x}{5},$$
$$t_6(x) = x^6 + x^4 + \frac{x^2}{4} + \frac{1}{108},$$
$$t_7(x) = x^7 + x^5 + \frac{2}{7}x^3 + \frac{1}{49}x.$$

7.3. Davenport pairs with $(m,n) = (2,3)$

The case when $(m,n) = (2,3)$ has been studied intensively. It is readily checked that when k is $1,2$ or 3 there is only one equivalence class of solutions with coefficients in $\mathbb{Q}$. When $k=1$, $f = x^2+1$, $y = x^3 + (3/2)x$ is a representative solution, when $k=2$ we may take

$$f = x^4 + x^2 + \frac{1}{4}, \qquad g = x^6 + \frac{3}{2}x^4 + \frac{3}{4}x^2 + \frac{1}{8}$$

and when $k=3$ we may take

$$f = x^6 + x^4 + \frac{5}{8}x^2 + \frac{3}{32}, \qquad g = x^9 + \frac{3}{2}x^7 + \frac{21}{16}x^5 + \frac{35}{64}x^3 + \frac{63}{512}x.$$

In [7] Hall found an example with $k=4$. Normalized in the usual manner the example is

$$f = x^8 + 21x^6 + 22x^5 + \frac{1183}{8}x^4 + 423x^3 + \frac{6721}{16}x^2 + \frac{13679}{8}x + \frac{268777}{256},$$
$$g = x^{12} + \frac{63}{2}x^{10} + 33x^9 + \frac{6195}{16}x^8 + 981x^7 + \frac{42339}{16}x^6 + \frac{78783}{8}x^5$$
$$+ \frac{3758439}{256}x^4 + \frac{632675}{16}x^3 + \frac{32269011}{512}x^2 + \frac{13826697}{256}x + \frac{280013653}{4096},$$

and it follows, for example, from work of Zannier, see p. 126 of [15], that there is only one equivalence class with coefficients in $\mathbb{Q}$ with $(n,m,k) = (3,2,4)$. Zannier relates equivalence classes of solutions over $\mathbb{C}$ to certain regular trees. His approach is connected with the dessins d'enfants of Grothendieck, see e.g. [10]. Similarly one can show that there are at most two equivalence classes of solutions over $\mathbb{Q}$ for $k=5$. Birch, Chowla, Hall and Schinzel [2] found one example

$$f = x^{10} + x^7 + \frac{5}{12}x^4 + \frac{1}{18}x,$$
$$g = x^{15} + \frac{3}{2}x^{12} + x^9 + \frac{1}{3}x^6 + \frac{5}{96}x^3 + \frac{1}{576}.$$

Further Elkies [6] found the example

$$f = x^{10} - 2x^9 + 33x^8 - 12x^7 + 378x^6 + 336x^5 + 2862x^4 + 2652x^3 + 14397x^2 + 9922x + 18553,$$
$$g = x^{15} - 3x^{14} + 51x^{13} - 67x^{12} + 969x^{11} + 33x^{10} + 10963x^9 + 9729x^8 + 96507x^7 + 108631x^6$$
$$+ 580785x^5 + 700503x^4 + 2102099x^3 + 1877667x^2 + 3904161x + 1164691,$$

which we have not normalized in the usual manner.

Therefore the complete list of Davenport pairs with $(n,m) = (3,2)$ and k at most 5 is known. It is not known if there exist any with k larger than 5. This question was posed already in [2].

7.4. Some sporadic Davenport pairs

In this subsection we collect some examples of Davenport pairs with $(m, n) \neq (2, 3)$ which do not belong to one of the seven infinite families given by (12), (13), (14), (15), (16), (17) and (18).

We remark that if (n, m, k) is specified with $n > m$ and (f^n, g^m) is a Davenport pair then the coefficients of g are determined once the coefficients of f are known.

Since the pairs are determined by f when n, m and k are given we have only listed f. We have found the pairs by means of the Groebner package in MAPLE.

(m, n, k)	f
$(2, 5, 2)$	$x^4 + 6x^2 + 64x - 55$
$(3, 5, 1)$	$x^3 + x + \frac{1}{3}$
$(3, 7, 1)$	$x^3 + 2x + \frac{2}{3}$
$(3, 8, 1)$	$x^3 + 3x + 3$
$(3, 10, 1)$	$x^3 + 6x + 6$
$(4, 5, 1)$	$x^4 - 2x^2 + 2x + \frac{3}{2}$
$(5, 6, 1)$	$x^5 + x^3 + \frac{3}{5}x$
$(5, 9, 1)$	$x^5 + 2x^3 + \frac{4}{5}x + \frac{4}{25}$
$(5, 11, 1)$	$x^5 + 3x^3 + \frac{9}{5}x + \frac{9}{25}$
$(5, 14, 1)$	$x^5 + 6x^3 + \frac{36}{5}x + \frac{108}{25}$
$(5, 16, 1)$	$x^5 + x^3 + \frac{x}{5} + \frac{1}{25}$
$(11, 28, 2)$	$x^{22} + x^{11} + \frac{2}{11}$

8. Numerical results and calculations

8.1. Proof of Theorem 1.8

Proof. Let r be a positive even integer and put $n = mr - 1$. Let t be a real number with $-1/4 < t < 0$. By Taylor's theorem

$$(1+t)^{n/m} = \sum_{j=0}^{r} \binom{n/m}{j} t^j + \Delta t^{r+1}, \tag{19}$$

where Δ is the $(r+1)$-th derivative of $(1+t)^{n/m}$ evaluated at some point ξ between t and 0. Thus

$$\Delta = \binom{n/m}{r+1} (1+\xi)^{-1/m-1}. \tag{20}$$

We have

$$-m\binom{n/m}{r+1} = \frac{1}{r+1}\binom{n/m}{r}.$$

Since $-1/4 < \xi < 0$ and $m \geqslant 2$

$$|m\Delta| \leqslant \frac{1}{r+1}\binom{n/m}{r}\left(\frac{4}{3}\right)^{3/2}. \tag{21}$$

Put

$$Q(m,n,t)=\sum_{j=0}^{r}\binom{n/m}{j}t^j.$$

If a and b are real numbers with $a>b>0$ and $a=b+\delta$ then for any positive integer m

$$a^m-b^m\leqslant m\delta a^{m-1}. \tag{22}$$

Since t and Δ are negative and r is even we see from (19) that $(1+t)^{n/m}>Q(m,n,t)$. Further, since $-1/4<t<0$, $m\geqslant 2$, $r\geqslant 2$ and $\binom{n/m}{r}<1$, we see from (19) and (21) that $Q(m,n,t)>0$. We now apply (22) with $a=(1+t)^{n/m}$ and $b=Q(m,n,t)$ to conclude that

$$\left|(1+t)^n-Q(m,n,t)^m\right|\leqslant m\Delta t^{r+1}(1+t)^{n-n/m}. \tag{23}$$

Suppose that A is a positive integer and B is a negative integer with $-1/4<B/A^m<0$. Put $t=B/A^m$ and then by (21) and (23) we see that

$$\begin{aligned}&\left|\left(A^m+B\right)^n-Q(m,n,A,B)^m\right|\\&\quad\leqslant\frac{1}{r+1}\binom{n/m}{r}\left(\frac{4}{3}\right)^{3/2}\left(1+\frac{B}{A^m}\right)\frac{|B|^{r+1}}{A}\left(A^m+B\right)^{n(1-1/m-1/n)},\end{aligned} \tag{24}$$

where

$$Q(m,n,A,B)=\sum_{j=0}^{r-1}\binom{n/m}{j}A^{n-jm}B^j+\binom{n/m}{r}\frac{B^r}{A}.$$

By, for example, Lemma 4.1 of [3], $m^{2j}\binom{n/m}{j}$ is an integer for $j=0,\dots,r$. Thus if we take $B=-m^2$ and $A=\binom{n/m}{r}m^{2r}$ then $Q(m,n,A,B)$ is an integer. Further $B/A^m<0$ and $|B/A^m|=m^2/\left(\binom{n/m}{r}m^{2r}\right)^m$. We have

$$\frac{n/m}{r-1}\cdot\frac{n/m-1}{r-2}\cdot\frac{n/m-(r-2)}{1}>1.$$

Hence

$$\binom{n/m}{r}>\frac{n/m-(r-1)}{r}=\frac{1-1/m}{r}.$$

Thus

$$\left|\frac{B}{A^m}\right|<\left(\frac{r}{1-1/m}\right)^m\frac{1}{m^{2rm-2}}\leqslant\left(\frac{2r}{m^{2r}}\right)^m m^2\leqslant\left(\frac{1}{4}\right)^m m^2\leqslant\frac{1}{4}$$

as required. Furthermore $|B|^r/A = \binom{n/m}{r}^{-1}$ and $0 < (1 + B/A^m) < 1$ so by (24),

$$\left|\left(A^m + B\right)^n - Q(m,n,A,B)^m\right| \leqslant \frac{2m^2}{r+1}\left(A^m + B\right)^{n(1-1/m-1/n)}.$$

We take $x = A^m + B$ and $y = Q(m,n,A,B)$ and our result follows. □

A remarkable consequence of the above construction is the following. We constructed the close pair of powers $(A^m + B)^n$ and $Q(m,n,A,B)^m$. Changing B into $-B$ does not affect the estimates substantially and we find that $(A^m - B)^n$, $Q(m,n,A,-B)^m$ is a close pair as well with almost the same quality of approximation. Similarly we may remove the requirement that r is even at a small cost to our estimates. In the table for $m = 2$, $n = 5$ we see for example the x-values $135^2 \pm 6$, in the table for $m = 2$, $n = 7$ the x-values $70^2 \pm 4$, in the table for $m = 2$, $n = 9$ the x-values $252^2 \pm 4$ and in the table for $m = 2$, $n = 11$ we observe the occurrence of $x = 924^2 \pm 4$.

8.2. The tables

The following tables display all $x \in \mathbb{Z}$ that are part of a pair $(x, y) \in \mathbb{Z}^2_{>0}$ with $0 < |x^n - y^m| < x^{n(1-1/m-1/nn)}$ and $0 < x \leqslant 4000000$, $m < n \leqslant 12$ and $\gcd(m,n) = 1$. The second entry in each row is the quality $q(x,m,n)$ of nearness defined by $x^{n(1-1/m-1/n)}/|x^n - y^n|$. The y's are omitted because they can be easily calculated from the corresponding x's ($y = \mathrm{round}(x^{n/m})$). These examples were found using PARI/GP.

$m=2$, $n=3$	
2	1.41
5234	4.26
8158	3.76
93844	1.03
367806	2.93
421351	1.05
720114	3.77
939787	3.16

$m=2$, $n=5$	
5	1.02
8	3.23
23	4.24
27	3.79
55	21.5
73	1.69
76	11.0
377	21.5
396	1.01
432	1.09
18219	1.33
18231	1.33
747343	2.27
748635	1.09

$m=2$, $n=7$	
12	1.29
93	1.11
239	3.42
4896	1.25
4904	1.25
6546	2.44
7806	1.69
9104	51.9
20542	1.54
35962	3.30
43783	2.23
96569	3.12
616400	4.05
635331	36.9
842163	1.78

$m=2$, $n=9$	
892	1.10
1110	1.65
1498	2.38
1827	8.94
3657	9.69
9249	2.03
10637	1.08
27590	9.41
63500	1.50
63508	1.50
248461	1.29
300221	1.15
357450	1.18
1317619	1.88

$m=2$, $n=11$	
3	1.49
21	1.10
145	1.09
1005	1.72
1746	24.2
5559	3.79
29005	36.7
34320	1.55
76053	1.12
146402	24.6
154269	1.00
553624	1.27
853772	1.75
853780	1.75
1841222	1.64
2582634	1.22
3051972	145.

$m=3$, $n=4$	
15	3.26
42	1.15
71	2.42
168	1.34
9172	4.55
15844	3.56
542482	1.03
548554	1.15

$m=3,\ n=5$	
4	1.06
23	1.01
122	1.02
199	4.65
408	1.34
4995	2.71
7320	1.34
44217	1.53
177682	3.24
394826	9.23
1706886	1.58
1738064	34.2

$m=3,\ n=7$	
2	4.23
3	5.62
32	3.94
33	2.07
34	1.12
88	1.04
442	1.77
498	5.04
942	1.03
2266144	73.5

$m=3,\ n=8$	
97	2.46
37840	1.07
199652	4.95
2905727	10.2

$m=3,\ n=10$	
2	2.12
3	1.87
48	5.76
73	1.19
436	1.03
23494	1.57
37381	2.30
621706	3.65
781913	1.13
2351612	1.62

$m=3,\ n=11$	
82	2.52
858	4.58
28439	1.06
166378	2.50
174879	1.70
977170	1.10
1330997	5.00
1595395	1.40
3393037	2.04

$m=4,\ n=5$	
3	1.58
53	1.63
7702	1.10
10836	5.27
11944	4.09
338295	1.33
422295	1.20
857745	1.68

$m=4,\ n=7$	
6	21.4
13	8.64
21	1.52
59	1.06
3053	3.95
7075	1.04
8509	6.02
168511	3.08
1413693	1.82

$m=4,\ n=9$	
12	1.47
127	1.12
3137	1.15
208870	1.09
298574	2.01

$m=4,\ n=11$	
6	3.56
28	1.20
402	16.3
892	1.63
1065	1.32
2818	1.04
15197	1.10
314820	11.0

$m=5,\ n=6$	
41031	1.15
60840	1.20
994895	1.52

$m=5,\ n=7$	
4	1.39
93	1.29
389	1.04
1184258	1.09

$m=5,\ n=8$	
2	3.25
947971	9.59

$m=5,\ n=9$	
39	2.81
233817	1.48
878236	1.07
1853987	2.77
3845948	9.87

$m=5,\ n=11$	
8	4.27
26	1.85
92	1.06
59613	1.02
483168	7.52
838882	1.05

$m=5,\ n=12$	
3	1.99
61	1.54
2889	16.4
7295	2.66
22434	1.88

$m=6,\ n=7$	
4	1.07
337	1.15
817	1.02
45858	1.15
923498	2.76
1097437	3.37
2028879	1.10

$m=6,\ n=11$	
19	1.62
95	1.74
184	1.52
238	15.7
1938094	3.23

$m=7,\ n=8$	
23	1.78
37502	1.35
73319	1.91
232450	4.30

$m=7$, $n=9$	
6	2.16
206	2.18
351	1.68
34477	1.34
403982	1.21
576482	1.52

$m=7$, $n=10$	
45	1.39
155	1.33
8116	47.3
61834	2.39
1812959	11.3

$m=7$, $n=11$	
2	2.48
109	11.4
105936	24.4
438963	4.38
944988	1.25

$m=7$, $n=12$	
11	1.08
39	1.54
163	2.13
876	1.89
259632	4.85
310504	1.06
1521835	4.92

$m=8$, $n=9$	
426	2.83
9807	1.66
84332	1.91

$m=8$, $n=11$	
177	1.51
11266	1.21
115871	4.98

$m=9$, $n=10$	
5	1.05
44	1.01
133	4.86
3550208	1.07

$m=9$, $n=11$	
55	1.27
706	10.8

$m=10$, $n=11$	
20598	1.63
125496	2.50
681143	1.05
803178	1.12

$m=11$, $n=12$	
65	113.
70	1.83
473	70.3
692	2.16
290599	2.12

8.3. *Examples with high quality*

The tables in the previous subsection show that the quality q doesn't get very big in general. To find examples with very high quality, it is better to restrict the search to small x with high exponents. The following table contains all examples with quality over 1000 with $x \leqslant 100$ and $2 \leqslant m < n \leqslant 650$.

q	x	n	m
37704	26	361	2
29736	7	303	154
16317	48	525	344
15234	7	510	431
9038	3	120	17
5442	42	131	4
4640	13	482	129
4248	7	457	154
3784	67	470	17
3013	9	60	17
3013	3	137	17
2578	39	399	43
2451	7	434	11
2176	10	431	255

q	x	n	m
2124	46	308	303
2006	7	71	32
1728	15	374	5
1670	89	384	265
1537	13	560	69
1450	26	363	2
1410	6	211	4
1219	10	569	6
1202	6	595	463
1166	69	360	29
1009	18	466	37
1005	11	373	26
1004	27	40	17
1004	3	154	17

The quality of the pair $(3^{120}, 2333^{17})$ is so high, that $(9^{60}, 2333^{17})$ and $(27^{40}, 2333^{17})$ still have quality over 1000. Also, if a pair (x^n, y^m) has quality q, then one straightforwardly checks that $(x^{m+n}, (xy)^m)$ has quality $\frac{q}{x}$.

References

[1] A. Baker, Bounds for the solutions of the hyperelliptic equation, Proc. Cambridge Philos. Soc. 65 (1969) 439–444.

[2] B.J. Birch, S. Chowla, M. Hall Jr., A. Schinzel, On the difference $x^3 - y^2$, Norske Vidensk. Selsk. Forh. 38 (1965) 65–69.

[3] G.V. Chudnovsky, On the method of Thue–Siegel, Ann. of Math. 117 (1983) 325–382.

[4] L.V. Danilov, The Diophantine equation $x^3 - y^2 = k$ and Hall's conjecture, Math. Notes Acad. Sci. USSR 32 (1982) 617–618.

[5] H. Davenport, On $f^3(t) - g^2(t)$, Norske Vidensk. Selsk. Forh. 38 (1965) 86–87.

[6] Noam D. Elkies, Rational points near curves and small non-zero $|x^3 - y^2|$ via lattice reduction, in: Wieb Bosma (Ed.), Algorithmic Number Theory, in: Lecture Notes in Comput. Sci., vol. 1838, Springer-Verlag, 2000, pp. 33–63.

[7] M. Hall Jr., The Diophantine equation $x^3 - y^2 = k$, in: A. Atkin, B. Birch (Eds.), Computers in Number Theory, Academic Press, 1971, pp. 173–198.

[8] R.C. Mason, Diophantine Equations Over Function Fields, London Math. Soc. Lecture Note Ser., vol. 96, Cambridge University Press, 1984.

[9] W.M. Schmidt, Integer points on curves of genus 1, Compos. Math. 81 (1992) 33–59.

[10] G.B. Shabat, V.A. Voevodsky, Drawing curves over number fields, in: P. Cartier, L. Illusie, N.M. Katz, G. Laumon, Y. Manin, K.A. Ribet (Eds.), The Grothendieck Festschrift, vol. III, in: Progr. Math., vol. 88, Birkhäuser, 1990, pp. 199–227.

[11] V.G. Sprindzuk, Square-free divisors of polynomials, and class numbers of algebraic number fields, Acta Arith. 24 (1973) 143–149 (in Russian).

[12] V.G. Sprindzuk, The arithmetic structure of integer polynomials and class numbers, Tr. Mat. Inst. Steklova 143 (1977) 153–174 (in Russian).

[13] C.L. Stewart, Kunrui Yu, On the *abc* conjecture, II, Duke Math. J. 108 (2001) 169–181.

[14] W.W. Stothers, Polynomial identities and Hauptmoduln, Quart. J. Math. Oxford 32 (1981) 349–370.

[15] U. Zannier, On Davenport's bound for the degree of $f^3 - g^2$ and Riemann's existence theorem, Acta Arith. 71 (1995) 107–137.