

# INVARIANTS OF QUARTIC PLANE CURVES AS AUTOMORPHIC FORMS

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ABSTRACT. We identify the algebra of regular functions on the space of quartic polynomials in three complex variables invariant under  $SL(3, \mathbb{C})$  with an algebra of meromorphic automorphic forms on the complex 6-ball. We also discuss the underlying geometry.

*To Igor Dolgachev, for his 60th birthday.*

## 1. MOTIVATION AND GOAL

One of the most beautiful mathematical gems of the 19th century is, at least to my taste (but I expect Igor to agree), the theorem that says that the algebra of invariants of plane cubics is the algebra of modular forms. To be precise, consider the vector space of cubic homogeneous forms in three complex variables, which as a  $SL(3, \mathbb{C})$ -representation may also be thought of as the third symmetric power of the dual of the defining representation  $\mathbb{C}^3$ . Since  $SL(3, \mathbb{C})$  acts through its simple quotient  $PGL(3, \mathbb{C})$ , we prefer to regard this a representation of the latter. Then the theorem I am referring to says that the algebra of  $PGL(3, \mathbb{C})$ -invariant polynomials on that space is as a graded  $\mathbb{C}$ -algebra isomorphic to the algebra of  $SL(2, \mathbb{Z})$ -modular forms (recall that this is the polynomial algebra generated by the Eisenstein series  $E_4$  and  $E_6$ ). This theorem is in a sense the optimal algebraic form of the geometric property which says that a genus one curve is completely determined by its periods. Yet there is subtlety here which deserves to be explicated. Denote the vector space of homogeneous cubic forms on  $\mathbb{C}^3$  by  $\mathcal{K}$  and let  $\mathcal{K}^\circ \subset \mathcal{K}$  be the open subset of  $F \in \mathcal{K}$  whose zero set  $C(F)$  in  $\mathbb{P}^2$  defines a smooth cubic curve (hence of genus one). Then any  $F \in \mathcal{K}^\circ$  also determines a distinguished holomorphic differential  $\omega(F)$  on  $C(F)$ , which perhaps is best described as an iterated residue:

$$\omega(F) = \text{Res}_{C(F)} \text{Res}_{Z(F)} \frac{dZ_0 \wedge dZ_1 \wedge dZ_2}{F}.$$

Here  $Z(F)$  denotes the zero set of  $F$  in  $\mathbb{C}^3$  (the affine cone over  $C(F)$ ) and  $C(F)$  is regarded as the locus where  $Z(F)$  meets the hyperplane at infinity  $\mathbb{P}^2$ . It is easy to see that this double residue is indeed a nonzero holomorphic differential on  $C(F)$ . The lattice of periods of  $\omega(F)$  (a lattice in  $\mathbb{C}$ ) only depends on the  $PGL(3, \mathbb{C})$ -orbit of  $F$ .

Now recall that the space of lattices in  $\mathbb{C}$  is naturally the  $SL(2, \mathbb{Z})$ -orbit space of the space  $\text{Iso}^+(\mathbb{R}^2, \mathbb{C})$  of oriented  $\mathbb{R}$ -isomorphisms  $\mathbb{R}^2 \rightarrow \mathbb{C}$  (where  $\sigma \in SL(2, \mathbb{Z})$  takes  $\zeta \in \text{Iso}^+(\mathbb{R}^2, \mathbb{C})$  to  $\zeta\sigma^{-1}$ ). So we have defined a map

$$PGL(3, \mathbb{C}) \backslash \mathcal{K}^\circ \rightarrow SL(2, \mathbb{Z}) \backslash \text{Iso}^+(\mathbb{R}^2, \mathbb{C}).$$

It is easy to see that this map is in fact an isomorphism of (affine) algebraic varieties (as for its injectivity, observe that the  $PGL(3, \mathbb{C})$ -stabilizer of a nonsingular cubic plane curve is the group of automorphisms of that curve which preserve its degree three polarization).

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2000 *Mathematics Subject Classification*. Primary: 11D25, 32N15.

*Key words and phrases*. Quartic curve, ball quotient, meromorphic automorphic form.

Both sides come with a natural  $\mathbb{C}^\times$ -action:  $\lambda \in \mathbb{C}^\times$  acts on  $\mathcal{K}^\circ$  by multiplication with  $\lambda^{-1}$  and acts on  $\text{Iso}^+(\mathbb{R}^2, \mathbb{C})$  by composing with multiplication by  $\lambda$  on  $\mathbb{C}$ . These actions descend to the orbit spaces and make the isomorphism above  $\mathbb{C}^\times$ -equivariant. (These actions are no longer effective, for  $-1 \in \mathbb{C}^\times$  acts as the identity.) The graded algebra  $\mathbb{C}[\mathcal{K}]^{SL(3, \mathbb{C})}$  can be understood as an algebra of regular functions on the left hand side and the graded algebra of  $SL(2, \mathbb{Z})$ -modular forms  $\mathbb{C}[E_4, E_6]$  can be understood as an algebra of regular functions on right hand side. In either case the grading comes from the  $\mathbb{C}^\times$ -action we just described. The cited theorem says that the displayed map identifies these graded  $\mathbb{C}$ -algebras. This has a geometrical consequence which goes somewhat beyond the observed isomorphism: the proj construction on the left is interpreted by Geometric Invariant Theory:  $\text{Proj}(\mathbb{C}[\mathcal{K}]^{\text{PGL}(3, \mathbb{C})})$  adds to  $\text{PGL}(3, \mathbb{C}) \backslash \mathbb{P}(\mathcal{K}^\circ)$  a singleton which is represented by the unique closed strictly semistable  $\text{PGL}(3, \mathbb{C})$ -orbit in  $\mathbb{P}(\mathcal{K})$ , namely, the orbit of three nonconcurrent lines in  $\mathbb{P}^2$ . The proj construction on the right compactifies the modular curve in question, that is, the  $j$ -line  $SL(2, \mathbb{Z}) \backslash \mathbb{P} \text{Iso}^+(\mathbb{R}^2, \mathbb{C})$ , in the standard manner: it is the simplest instance of a Baily-Borel compactification.

To me this theorem serves as a model for the theory of period maps. It seems to tell us that whenever we know such a map to be an open embedding, then we should try to express that fact on a (richer) algebraic level, amounting to the identification of two graded algebras of invariants, one with respect to a reductive algebraic group, the other with respect to a discrete group. The ensuing identification of their proj's should give us then an additional piece of geometric information, which in the end is a sophisticated way to understand the period map's boundary behavior. This is most likely to happen when that map takes values in a ball quotient or a locally symmetric variety of type IV. If the period map is an isomorphism, then one hopes for an exact analogue of the theorem above. For example, Allcock-Carlsen-Toledo [2] have essentially established this for the case of cubic surfaces, where  $\mathcal{K}$  is replaced by the space of homogenous cubic forms in 4 variables and the algebra of modular forms by the algebra of automorphic forms on a 4-ball with respect to an arithmetic group.

But if the period map is not surjective, then some modifications on the automorphic side are in order and it is precisely for this purpose that I developed the geometric theory of meromorphic automorphic forms in [5] and [6]. What I want to do here is to illustrate that theory in the (other) case that logically comes after our guiding example, namely that of quartic plane curves. To be precise, let  $\mathcal{Q}$  stands for the vector space of *quartic* (rather than cubic) homogeneous forms in three complex variables and regard this space as a representation of  $SL(3, \mathbb{C})$ , then we shall interpret the algebra of invariants  $\mathbb{C}[\mathcal{Q}]^{SL(3, \mathbb{C})}$  (or rather  $\mathbb{C}[\mathcal{Q}]^{\mu_4 \times SL(3, \mathbb{C})}$  with  $\mu_4$  acting on  $\mathcal{Q}$  by scalar multiplication) as an algebra of (meromorphic) automorphic forms on the complex 6-ball. This will be an algebra isomorphism which rescales the degrees by a factor three. We shall, of course, also interpret the proj construction on either side.

This example is not an isolated one. For instance, Allcock [1] has determined the semistable cubic threefolds and his results suggest that the situation is very much like the case of quartic curves.

It is pleasure to dedicate this paper to my longtime friend Igor Dolgachev. Igor and I share a passion for our field (which is not confined to algebraic geometry) and we have a similar mathematical taste, but I only wish I had his extensive knowledge of the classical literature of our subject. I learned a lot from him.

## 2. THE BAILY-BOREL COMPACTIFICATION OF A BALL QUOTIENT

Let  $V$  be complex vector space  $V$  of finite dimension, endowed with a nondegenerate Hermitian form  $h : V \times V \rightarrow \mathbb{C}$  of signature  $(1, n)$ . So we can find a coordinate system  $(z_0, \dots, z_n)$  for  $V$  on which  $h$  takes the standard form:  $h(z, z) = |z_0|^2 - |z_1|^2 - \dots - |z_n|^2$ . This shows that the unitary group  $U(V)$  of  $(V, h)$  is isomorphic to  $U(1, n)$ . Let us denote by  $V_+$  the open subset of  $z \in V$  with  $h(z, z) > 0$ . This set is  $\mathbb{C}^\times$ -invariant and hence defines an open subset  $\mathbb{P}(V_+)$  of  $\mathbb{P}(V)$ . In terms of the above coordinates,  $\mathbb{P}(V_+)$  is defined by  $\sum_{i=1}^n |z_i/z_0|^2 < 1$ , which shows that  $\mathbb{P}(V_+)$  is biholomorphic to the complex  $n$ -ball. The group  $U(V)$  acts properly and transitively on it; in fact, the stabilizer of a point is a maximal compact subgroup  $U(V)$ , so that  $\mathbb{P}(V_+)$  can be understood as the symmetric space of  $U(V)$ .

Suppose we are also given a discrete subgroup  $\Gamma \subset U(V)$  of finite *covolume* (which means that  $U(V)/\Gamma$  has finite  $U(V)$ -invariant volume). Since  $\Gamma$  is discrete, it acts properly discontinuously on  $\mathbb{P}(V_+)$  (and hence also on  $V_+$ ) so that the formation of the orbit spaces  $\Gamma \backslash \mathbb{P}(V_+)$  and  $\Gamma \backslash V_+$  takes place in the complex-analytic category (the orbit spaces will be normal). However, the Baily-Borel theory tells us that these spaces have the richer structure of quasi-projective variety. In order to state a precise result, we recall that a  $\Gamma$ -automorphic form of degree  $d \in \mathbb{Z}$  is in the present setting a  $\Gamma$ -invariant holomorphic function  $f : V_+ \rightarrow \mathbb{C}$  which is homogeneous of degree  $-d$ :  $f(\lambda z) = \lambda^{-d} f(z)$  and which in case  $n = 1$  also obeys a growth condition which we do not bother to specify. If we denote the space of such forms by  $A_d^\Gamma$ , then it is clear that their direct sum  $A_\bullet^\Gamma$  is a graded  $\mathbb{C}$ -algebra. The Baily-Borel theory has the following to say about it:

**Theorem 2.1.** *The graded  $\mathbb{C}$ -algebra  $A_\bullet^\Gamma$  has finitely many generators of positive degree. This algebra separates the  $\Gamma$ -orbits in  $V_+$  so that we have injective complex-analytic maps*

$$\Gamma \backslash V_+ \rightarrow \text{Spec}(A_\bullet^\Gamma), \quad \Gamma \backslash \mathbb{P}(V_+) \rightarrow \text{Proj}(A_\bullet^\Gamma).$$

*The images of these morphisms are Zariski open-dense so that  $\Gamma \backslash V_+$  resp.  $\Gamma \backslash \mathbb{P}(V_+)$  acquires the structure of a quasi-affine resp. quasi-projective complex variety. Moreover, we have natural (so-called Baily-Borel) extensions  $V_+ \subset V_+^*$  and  $\mathbb{P}(V_+) \subset \mathbb{P}(V_+^*)$  as topological spaces with  $\Gamma$ -action (which we describe below) such that these morphisms extend to homeomorphisms*

$$\Gamma \backslash V_+^* \cong \text{Spec}(A_\bullet^\Gamma), \quad \Gamma \backslash \mathbb{P}(V_+^*) \cong \text{Proj}(A_\bullet^\Gamma),$$

*if we endow the targets with their Hausdorff topology.*

In order to describe the extensions mentioned in this theorem, we introduce the following notion. Let  $H \subset V$  be a linear hyperplane. If  $H$  is nondegenerate, then the  $U(V)$ -stabilizer  $U(V)_H$  is simply  $U(H) \times U(H^\perp)$ . Suppose now that  $H$  is degenerate. Then  $H^\perp \subset H$ ,  $H/H^\perp$  is negative definite and the evident homomorphism  $U(V)_H \rightarrow U(H/H^\perp) \times GL(H^\perp)$  is surjective with kernel a unipotent group (the unipotent radical of  $U(V)_H$ ). This unipotent group is in fact a Heisenberg group: an extension of the complex vector group  $(H/H^\perp) \otimes \overline{H^\perp}$  by a real vector group of dimension one.

**Definition 2.2.** We call a linear hyperplane  $H \subset V$  that is not negative definite  $\Gamma$ -rational if its  $\Gamma$ -stabilizer  $\Gamma_H$  maps to a subgroup of finite covolume in  $U(H)$ .

(Since the  $U(V)$ -stabilizer of a negative definite hyperplane is compact, this notion would be for such hyperplanes without much interest.) Let  $\mathcal{I}$  denote the collection of isotropic lines  $I \subset V$  for which  $I^\perp$  is  $\Gamma$ -rational. The  $\Gamma$ -rationality amounts here to requiring that  $\Gamma_I$  meets the unipotent radical of  $U(V)_I$  in a cocompact subgroup. According to

the reduction theory for such forms,  $\Gamma$  has only finitely many orbits in the set  $\mathcal{I}$ . Now we can describe the extensions mentioned in Theorem 2.1 as a set:

$$V_+^* = V_+ \sqcup \left( \coprod_{I \in \mathcal{I}} V_+/I^\perp \right) \sqcup (V_+/V), \quad \mathbb{P}(V_+^*) = \mathbb{P}(V_+) \sqcup \coprod_{I \in \mathcal{I}} \mathbb{P}(V_+/I^\perp),$$

where  $V_+/I^\perp$  simply denotes the image of  $V_+$  in  $V/I^\perp$  and likewise in other cases. This notation may appear unnecessarily complicated, since  $V_+/I^\perp = V/I^\perp - \{0\}$  is just a copy of  $\mathbb{C}^\times$  and both  $V_+/V$  and  $\mathbb{P}(V_+/I^\perp)$  are even singletons. Yet it is convenient notation, not just because we need to be able to distinguish singletons by name, but also because it is typical for the general situation that the extension is obtained by adding quotients of the very space we are extending. Still we may observe that  $\mathbb{P}(V_+^*) - \mathbb{P}(V_+)$  can be identified with the discrete set  $\mathcal{I}$  so that  $\Gamma \backslash \mathbb{P}(V_+^*) - \Gamma \backslash \mathbb{P}(V_+)$  is identified with the finite set  $\Gamma \backslash \mathcal{I}$ .

We will not define the topology of these extensions; suffices to say here that this topology induces the given one on each piece, and that the elements of both  $\Gamma$  and  $\mathbb{C}^\times$  (acting by scalar multiplication) act as homeomorphisms. The space  $V_+^*$  has the structure of a cone with the singleton  $V_+/V$  as vertex and  $\mathbb{P}(V_+^*)$  as its base.

### 3. $\Gamma$ -ARRANGEMENTS AND ASSOCIATED COMPACTIFICATIONS

We keep the situation of the previous section, but now assume that  $n \geq 2$ .

**$\Gamma$ -arrangements.** Suppose that  $H \subset V$  is a  $\Gamma$ -rational hyperplane with  $H^\perp$  a negative definite line. So  $H$  has signature  $(1, n-1)$ ,  $H_+ = H \cap V_+$  and  $\Gamma_H$  maps to a subgroup of  $U(H)$  of cofinite volume. That subgroup will be discrete and so the preceding applies: we find a quasi-affine variety  $\Gamma_H \backslash H_+$  with its Baily-Borel extension  $\Gamma_H \backslash H_+^*$  (a normal affine variety). The natural map  $\Gamma_H \backslash H_+ \rightarrow \Gamma \backslash V_+$  is evidently complex-analytic, but since  $\Gamma$ -automorphic forms restrict to  $\Gamma_H$ -automorphic forms it is in fact an algebraic morphism. For the same reason, this map has a Baily-Borel extension  $\text{Spec}(A_{\Gamma_H}^\bullet) \rightarrow \text{Spec}(A_\Gamma^\bullet)$  in the algebraic category. The underlying map  $\Gamma_H \backslash H_+^* \rightarrow \Gamma \backslash V_+^*$  is defined in an evident manner. In particular, the preimage of the vertex is the vertex and so this Baily-Borel extension is finite. It is also birational onto its image and hence this map is a normalization of that image. Since the preimage of  $\Gamma \backslash V_+$  in  $\Gamma_H \backslash H_+^*$  is  $\Gamma_H \backslash H_+$ , it follows that  $\Gamma_H \backslash H_+ \rightarrow \Gamma \backslash V_+$  is a finite morphism as well. The preimage in  $V_+$  of the image of  $\Gamma_H \backslash H_+ \rightarrow \Gamma \backslash V_+$  is of course the union of the  $\Gamma$ -translates of  $H_+$ . It follows that these translates form a collection of hyperplane sections of  $V_+$  that is locally finite. This remains true if instead of a single  $H$  we take a finite number of them. This leads up to the following

**Definition 3.1.** A  $\Gamma$ -arrangement is a collection  $\mathcal{H}$  of  $\Gamma$ -rational hyperplanes with negative definite orthogonal complement, which, when viewed as a subset of the appropriate Grassmannian, is a union of finitely many  $\Gamma$ -orbits.

For the remainder of this section we fix a  $\Gamma$ -arrangement  $\mathcal{H}$ .

**Meromorphic automorphic forms.** The preceding discussion shows that the collection  $(H_+)_{H \in \mathcal{H}}$  is locally finite on  $V_+$  so that its union  $D_{\mathcal{H}} := \cup_{H \in \mathcal{H}} H_+$  is closed in  $V_+$ . Moreover,  $D_{\mathcal{H}}$  is the preimage of a hypersurface in  $\Gamma \backslash V_+$ . We shall denote  $V_+ - D_{\mathcal{H}}$  simply by  $V_{\mathcal{H}}$ . Any open subset of  $V$  of this form will be referred to as a  $\Gamma$ -arrangement complement. It is clear that  $\Gamma \backslash V_{\mathcal{H}}$  is the complement of a hypersurface in  $\Gamma \backslash V_+$ . It is our goal to state a generalization of Theorem 2.1 for  $\Gamma \backslash V_{\mathcal{H}}$ .

For this purpose we introduce an algebra of meromorphic  $\Gamma$ -automorphic forms: for any integer  $d$ , we denote by  $A_{\mathcal{H},d}^\Gamma$  the space of  $\Gamma$ -invariant holomorphic functions homogeneous of degree  $d$  on  $V_{\mathcal{H}}$  that are meromorphic on  $V_+$  and have a pole of order at most  $d$  along the hyperplane sections  $H_+, H \in \mathcal{H}$ . It is clear that  $A_{\mathcal{H},\bullet}^\Gamma$  is an algebra of holomorphic functions on  $\Gamma \backslash V_{\mathcal{H}}$  which contains  $A_{\bullet}^\Gamma$  as a subalgebra. In particular, this algebra separates the points of  $\Gamma \backslash V_{\mathcal{H}}$ . We now quote from [5]:

**Theorem 3.2.** *The graded  $\mathbb{C}$ -algebra  $A_{\mathcal{H},\bullet}^\Gamma$  has finitely many generators of positive degree and the resulting complex-analytic injections*

$$\Gamma \backslash V_{\mathcal{H}} \rightarrow \text{Spec}(A_{\mathcal{H},\bullet}^\Gamma), \quad \Gamma \backslash \mathbb{P}(V_{\mathcal{H}}) \rightarrow \text{Proj}(A_{\mathcal{H},\bullet}^\Gamma)$$

are open embeddings onto Zariski open-dense subsets so that  $\Gamma \backslash V_{\mathcal{H}}$  resp.  $\Gamma \backslash \mathbb{P}(V_{\mathcal{H}})$  acquires the structure of a quasi-affine resp. quasi-projective complex variety. Moreover, we have natural topological extensions  $V_{\mathcal{H}} \subset V_{\mathcal{H}}^*$  and  $\mathbb{P}(V_{\mathcal{H}}) \subset \mathbb{P}(V_{\mathcal{H}}^*)$  as  $\Gamma$ -spaces (described below) such that these embeddings extend to homeomorphisms

$$\Gamma \backslash V_{\mathcal{H}}^* \cong \text{Spec}(A_{\bullet}^\Gamma), \quad \Gamma \backslash \mathbb{P}(V_{\mathcal{H}}^*) \cong \text{Proj}(A_{\bullet}^\Gamma)$$

when the targets are endowed with their Hausdorff topology. These topological extensions come as  $\Gamma$ -equivariant stratified spaces which result in partitions of  $\text{Spec}(A_{\mathcal{H},\bullet}^\Gamma)$  and  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma)$  by subvarieties.

**Small modification of the Baily-Borel extension.** The  $\Gamma$ -extensions in the above theorem share a number of properties with the Baily-Borel and the toric extensions: the boundary material that we add comes partitioned into pieces ('strata'), where each piece is given as a topological quotient of the extended space. Before we describe the extension in question, we first discuss a small modification of the Baily-Borel extension.

Given  $I \in \mathcal{I}$ , we denote by  $\mathcal{H}_I$  the collection of  $H \in \mathcal{H}$  containing the isotropic line  $I$  and by  $I^{\mathcal{H}}$  the intersection of  $I^\perp$  with all the members of  $\mathcal{H}_I$ . So  $I \subset I^{\mathcal{H}} \subset I^\perp$ . Our rationality assumptions imply that  $\Gamma_I$  has finitely many orbits in  $\mathcal{H}_I$ . The small modification of the Baily-Borel extension amounts to replacing in its definition  $I^\perp$  by  $I^{\mathcal{H}}$ :

$$\mathbb{P}(V_+^{\mathcal{H}}) := \mathbb{P}(V_+) \sqcup \coprod_{I \in \mathcal{I}} \mathbb{P}(V_+ / I^{\mathcal{H}})$$

This extension comes with a natural  $\Gamma$ -invariant topology which makes the evident map  $\mathbb{P}(V_+^{\mathcal{H}}) \rightarrow \mathbb{P}(V_+^*)$  continuous. Its  $\Gamma$ -orbit spaces come with the structure of a normal complex-analytic space so that the induced map  $\Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}}) \rightarrow \Gamma \backslash \mathbb{P}(V_+^*)$  is an analytic morphism. This is in fact a modification: it is proper and an isomorphism over the open-dense stratum. The lemma below describes the exceptional fibers and also helps us to understand the behavior of the collection  $\mathcal{H}_I$  near  $I$ .

**Lemma 3.3.** *The collection  $\mathcal{H}_I$  induces on  $I^\perp$  a finite arrangement (which we denote  $\mathcal{H}_I|I^\perp$ ). Any proper intersection  $L \neq I^\perp$  of members of  $\mathcal{H}_I|I^\perp$  is also an intersection of members of  $\mathcal{H}_I$  only; moreover, there exists an intersection  $\tilde{L}$  of members of  $\mathcal{H}_I$  which meets  $V_+$  and is such that  $L = I \cap \tilde{L}$ .*

*The fiber of  $\Gamma \backslash V_+^{\mathcal{H}} \rightarrow \Gamma \backslash V_+^*$  over a point of the  $\mathbb{C}^\times$ -orbit defined by  $I \in \mathcal{I}$  is the quotient of an affine space over  $I^\perp / I^{\mathcal{H}}$  by a crystallographic group.*

*Proof.* We observe  $V_+ / I_{\mathcal{H}} = V / I_{\mathcal{H}} - I^\perp / I_{\mathcal{H}}$  is the complement of a linear hyperplane and so its projectivization  $\mathbb{P}(V_+ / I_{\mathcal{H}}) = \mathbb{P}(V / I_{\mathcal{H}}) - \mathbb{P}(I^\perp / I_{\mathcal{H}})$  is in a natural manner an affine space. Let us denote this affine space simply by  $A$ . The group  $\Gamma_I$  acts on  $A$  through a

complex crystallographic group (and so the orbit space of that action is a compact complex-analytic variety (in fact, a quotient of an abelian variety by a finite group). The exceptional fibers over the image of  $V_+/I^\perp$  are all isomorphic to  $\Gamma_I \backslash A$  and so the last assertion of the lemma follows.

Every  $H \in \mathcal{H}_I$  determines a hyperplane  $A_H$  of  $A$  and this defines a bijection between  $\mathcal{H}_I$  and a collection of affine hyperplanes of  $A$ , a collection we shall denote by  $\mathcal{H}|A$ . By construction, the common intersection of the collection  $\mathcal{H}|A$  is empty. Notice that for  $H_1, H_2 \in \mathcal{H}_I$ , we have  $H_1 \cap I = H_2 \cap I$  if and only if  $A_{H_1}$  and  $A_{H_2}$  are parallel.

The group  $\Gamma_I$  has finitely many orbits in  $\mathcal{H}_I$  and hence also in  $\mathcal{H}_I|I^\perp$  and  $\mathcal{H}_I|A$ . The assertions of the lemma now follow easily. The finiteness of  $\mathcal{H}_I|I^\perp$  is a consequence of the fact that  $\Gamma_I$  acts in  $I^\perp/I$  through a finite group. This is equivalent to  $\mathcal{H}|A$  decomposing into finitely many equivalence classes for the relation of parallelism. If  $L \neq I^\perp$  is as in the lemma, then the collection  $\mathcal{H}_L$  of  $H \in \mathcal{H}$  containing  $L$  corresponds to a nonempty finite union of such equivalence classes; a minimal nonempty intersection of these is an affine subspace  $B$  of  $A$  which has  $L$  as translation space; the common intersection  $\tilde{L}$  of the corresponding subset of  $\mathcal{H}_L$  will have the property that  $\tilde{L}$  meets  $V_+$  and  $\tilde{L} \cap I = L$ . Finally, if  $B'$  is another such minimal nonempty intersection distinct from  $B$  so that  $B \cap B' = \emptyset$ , then the collection of  $H \in \mathcal{H}_L$  with  $A_H \supset B$  or  $A_H \supset B'$  has  $L$  as its common intersection.  $\square$

*Remark 3.4.* The modification  $\Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}}) \rightarrow \Gamma \backslash \mathbb{P}(V_+^*)$  has a simple algebro-geometric meaning: for every  $H \in \mathcal{H}$ , the evident map  $\Gamma_H \backslash \mathbb{P}(H_+) \rightarrow \Gamma \backslash \mathbb{P}(V_+)$  is a finite morphism which extends to the Baily-Borel compactifications:  $\Gamma_H \backslash \mathbb{P}(H_+^*) \rightarrow \Gamma \backslash \mathbb{P}(V_+^*)$ . The latter is also finite with image a hypersurface (in fact, it is a normalization of this image), but that hypersurface need not support a  $\mathbb{Q}$ -Cartier divisor. According to Lemma 5.2 of [5], the morphism  $\Gamma_H \backslash \mathbb{P}(H_+^*) \rightarrow \Gamma \backslash \mathbb{P}(V_+^*)$  lifts naturally to a morphism  $\Gamma_H \backslash \mathbb{P}(H_+^*) \rightarrow \Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}})$  whose image *does* support a  $\mathbb{Q}$ -Cartier divisor. The modification  $\Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}}) \rightarrow \Gamma \backslash \mathbb{P}(V_+^*)$  is the smallest one for which the images of all the  $\Gamma_H \backslash \mathbb{P}(H_+)$  extend to hypersurfaces which are  $\mathbb{Q}$ -Cartier divisors.

**Baily-Borel extension of an arithmetic arrangement complement.** Let  $\mathcal{L}_+(\mathcal{H})$  denote the collection of all the linear subspaces that arise as an intersection of members of  $\mathcal{H}$  and meet  $V_+$  (this includes  $V$  as the empty intersection) and denote by  $\mathcal{L}_+(\mathcal{H}, \mathcal{I})$  the set of of pairs  $(L, I) \in \mathcal{L}_+(\mathcal{H}) \times \mathcal{I}$  with  $L \supset I$ . Then the extensions appearing in Theorem 3.2 are

$$V_{\mathcal{H}}^* = V_{\mathcal{H}} \sqcup \left( \prod_{(L, I) \in \mathcal{L}_+(\mathcal{H}, \mathcal{I})} V_{\mathcal{H}} / (L \cap I^\perp) \right) \sqcup \prod_{L \in \mathcal{L}_+(\mathcal{H})} (V_{\mathcal{H}} / L),$$

$$\mathbb{P}(V_{\mathcal{H}}^*) = \mathbb{P}(V_{\mathcal{H}}) \sqcup \left( \prod_{(L, I) \in \mathcal{L}_+(\mathcal{H}, \mathcal{I})} \mathbb{P}(V_{\mathcal{H}} / (L \cap I^\perp)) \right) \sqcup \prod_{V \neq L \in \mathcal{L}_+(\mathcal{H})} \mathbb{P}(V_{\mathcal{H}} / L),$$

with a topology enjoying similar properties as in the Baily-Borel case. Notice, incidentally, that we recover the latter if  $\mathcal{H}$  is empty.

**Structure of the strata.** A valuable piece of information contained in Theorem 3.2 is the stratified structure it exhibits in the boundary  $\text{Proj}(A_{\mathcal{H}, \bullet}^\Gamma) - \Gamma \backslash \mathbb{P}(V_+)$ . Any stratum is clearly of the form  $\Gamma_L \backslash \mathbb{P}(V_{\mathcal{H}} / L)$  with  $L \in \mathcal{L}_+(\mathcal{H})$  or  $\Gamma_{L, I} \backslash \mathbb{P}(V_{\mathcal{H}} / L \cap I^\perp)$  with  $(L, I) \in \mathcal{L}_+(\mathcal{H}, \mathcal{I})$ .

Let us see what we get in the first case. Denote by  $\mathcal{H}_L$  the collection of members of  $\mathcal{H}$  which contain  $L$  and by  $(V/L)_{\mathcal{H}}$  the complement of the union of the members of  $\mathcal{H}_L$

in  $V/L$ . Then  $V/L \cong L^\perp$  is negative definite,  $\mathcal{H}_L$  is finite and  $V_{\mathcal{H}}/L = (V/L)_{\mathcal{H}}$ . So  $\mathbb{P}(V_{\mathcal{H}}/L) = \mathbb{P}(V/L)_{\mathcal{H}}$  is a projective arrangement complement. The stabilizer  $\Gamma_L$  acts on  $L^\perp$  through a finite group and hence the stratum  $\Gamma_L \backslash \mathbb{P}(V_{\mathcal{H}}/L)$  of  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma)$  is well-understood. In particular, its codimension is the dimension of  $L$ .

In the second case, we only note that the maximal stratum associated to  $I$  is the  $\Gamma_I$ -orbit space of the affine arrangement complement  $A_{\mathcal{H}} := A - \cup_{H \in \mathcal{H}_I} A_H$ , where  $A$  is the affine space and  $A_H$  the affine hyperplane that we encountered in the proof of Lemma 3.3. (So this is an open subset of a finite quotient of an abelian variety). This discussion has an interesting corollary:

**Corollary 3.5.** *Suppose that  $\mathcal{H} \neq \emptyset$ . Then the codimension of the boundary that  $\Gamma \backslash V_{\mathcal{H}}$  has in  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma)$  is the minimal dimension of a member of  $\mathcal{L}_+(\mathcal{H}) \cup \{I^{\mathcal{H}}\}_{I \in \mathcal{I}}$ . In particular, if every one-dimensional intersection of members of  $\mathcal{H}$  is negative definite, then the boundary of  $\Gamma \backslash \mathbb{P}(V_+)$  in  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma)$  is of codimension  $> 1$  and the meromorphicity requirement in the definition of  $A_{\mathcal{H},\bullet}^\Gamma$  is superfluous:  $A_{\mathcal{H},\bullet}^\Gamma$  is simply the algebra of  $\Gamma$ -invariant holomorphic functions on  $V_{\mathcal{H}}$ .*

*Proof.* The first statement follows immediately from Theorem 3.2 and Lemma 3.3.

Under the assumption that every one-dimensional intersection of members of  $\mathcal{H}$  is negative definite, the boundary of our compactification is of codimension  $\geq 2$ . A meromorphic  $\Gamma$ -invariant holomorphic functions on  $V_{\mathcal{H}}$  which is homogeneous of degree  $d$  defines a holomorphic function on  $\text{Spec}(A_{\mathcal{H},\bullet}^\Gamma) - \Gamma \backslash V_+$ . Since  $\text{Spec}(A_{\mathcal{H},\bullet}^\Gamma)$  is normal, such a function extends to all of  $\text{Spec}(A_{\mathcal{H},\bullet}^\Gamma)$ . The homogeneity assumption implies that this extension is regular, i.e., lies in  $A_{\mathcal{H},d}^\Gamma$ .  $\square$

**Comparison of two compactifications.** There is in general no natural map  $\mathbb{P}(V_{\mathcal{H}}^*) \rightarrow \mathbb{P}(V_+^*)$  and hence no natural morphism  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma) \rightarrow \text{Proj}(A_\bullet^\Gamma)$ . As both  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma)$  and  $\text{Proj}(A_\bullet^\Gamma)$  are projective compactifications of the same variety, we can only say that we have a birational map between them. In [5] we explicitly described the graph of the birational map  $\text{Proj}(A_{\mathcal{H},\bullet}^\Gamma) \dashrightarrow \text{Proj}(A_\bullet^\Gamma)$  and for its topological  $\Gamma$ -equivariant counterpart  $\mathbb{P}(V_{\mathcal{H}}^*) \dashrightarrow \mathbb{P}(V_+^*)$ . We shall not recall this, but it is worth mentioning an interesting special case, which is relevant for the example that we will discuss below.

**Proposition 3.6.** *If  $\mathcal{H}$  has the property that any two distinct members of  $\mathcal{H}$  do not intersect each other in  $V_+$ , then*

- (i) *we have a natural map  $\pi : \mathbb{P}(V_+^{\mathcal{H}}) \rightarrow \mathbb{P}(V_{\mathcal{H}}^*)$  which is continuous  $\Gamma$ -equivariant and such that the resulting map  $\Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}}) \rightarrow \Gamma \backslash \mathbb{P}(V_{\mathcal{H}}^*)$  is a morphism,*
- (ii) *the dimension of the boundaries  $\Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}}) - \Gamma \backslash \mathbb{P}(V_+)$  and  $\Gamma \backslash \mathbb{P}(V_{\mathcal{H}}^*) - \Gamma \backslash \mathbb{P}(V_{\mathcal{H}})$  are at most one,*
- (iii) *the images of the natural morphisms  $\Gamma_H \backslash \mathbb{P}(H_+) \rightarrow \Gamma \backslash \mathbb{P}(V_+^{\mathcal{H}})$  are disjoint if we let  $H$  run over a system of representatives of the  $\Gamma$ -orbits in  $\mathcal{H}$ , and  $\pi$  contracts each of these images.*

*In particular,  $\mathbb{P}(V_+^{\mathcal{H}})$  appears as the normalization of the graph of the map  $\Gamma \backslash \mathbb{P}(V_+^*) \dashrightarrow \Gamma \backslash \mathbb{P}(V_{\mathcal{H}}^*)$ .*

This is a special case of Theorem 5.7 of [5]. We confine ourselves here to describing the map  $\pi$ . Under the hypothesis of the proposition, the boundary strata of  $\mathbb{P}(V_{\mathcal{H}}^*)$  consist of  $\mathbb{P}(V_{\mathcal{H}}/H)$  (a singleton),  $H \in \mathcal{H}$  and  $\mathbb{P}(V_{\mathcal{H}}/I^{\mathcal{I}})$  (a singleton in case  $\mathcal{H}_I$  is empty and an affine line minus a discrete set otherwise),  $I \in \mathcal{I}$ . Now  $\pi$  sends  $\mathbb{P}(H_+)$  to the singleton  $\mathbb{P}(V_{\mathcal{H}}/H)$ ; if  $I \in \mathcal{I}$ , then it is identity on  $\mathbb{P}(V_{\mathcal{H}}/I^{\mathcal{I}})$ , and if  $H \in \mathcal{H}_I$ , then the image of  $\mathbb{P}(H_+)$  in  $\mathbb{P}(V_+/I^{\mathcal{I}})$  (a singleton) maps to the singleton  $\mathbb{P}(V_{\mathcal{H}}/H)$ .

## 4. THE MODULI SPACE PLANE QUARTICS

We illustrate the preceding (and especially Proposition 3.6) with the case of quartic plane curves.

**Geometric invariant theory of quartics.** Fix a complex vector space  $W$  of dimension three. In what follows a central role is played by the space of homogeneous quartic forms on  $W$ ,  $\text{Sym}^4(W^*)$ , and so we prefer a briefer name: we abbreviate that space simply by  $\mathcal{Q}$ . The fundamental theorem of geometric invariant theory says that each fiber of the natural map  $\mathcal{Q} \rightarrow \text{Spec } \mathbb{C}[\mathcal{Q}]^{\text{SL}(W)}$  contains a unique closed  $\text{SL}(W)$ -orbit which lies in the closure of all other orbits in that fiber. In other words,  $\text{Spec } \mathbb{C}[\mathcal{Q}]^{\text{SL}(W)}$  is the orbit space in the separated category,  $\text{SL}(W) \backslash \backslash \mathcal{Q}$ . The preimage of 0 is called the *unstable locus* and has the origin of  $\mathcal{Q}$  as its unique closed orbit. The complement of the unstable locus is by definition the *semistable locus*  $\mathcal{Q}^{\text{ss}}$ . The *stable locus*  $\mathcal{Q}^s \subset \mathcal{Q}^{\text{ss}}$  is the set of  $F \in \mathcal{Q}$  for which the orbit map  $\text{SL}(W) \rightarrow \mathcal{Q}, g \mapsto gF$ , is proper (so that the orbit of  $F$  is closed in  $\mathcal{Q}$ ). Following Mumford these are precisely the  $F$  which define a reduced quartic curve with only ordinary double points or (ordinary) cusps. The closed orbits in the *strictly semistable locus*  $\mathcal{Q}^{\text{ss}} - \mathcal{Q}^s$  are the  $F \in \mathcal{Q}$  of the form  $(Z_1 Z_2 - Z_0^2)(s Z_1 Z_2 - t Z_0^2)$  with  $s \neq 0$ , where  $Z_0, Z_1, Z_2$  is a coordinate system for  $W$ . So these define quartics that are the union  $C' \cup C''$  of two conics with  $C'$  smooth,  $C''$  not a double line and for which either  $C'$  and  $C''$  meet in two points of multiplicity two, or  $C' = C''$ .

The degrees appearing in  $\text{Spec } \mathbb{C}[\mathcal{Q}]^{\text{SL}(W)}$  are divisible by 3 because the center  $\mu_3$  of  $\text{SL}(W)$  acts effectively on  $\mathcal{Q}$ .

We denote by  $\mathcal{Q}^\circ \subset \mathcal{Q}^s$  the set of  $F \in \mathcal{Q}$  for which  $C(F)$  is smooth.

**Associated K3 surface.** The following construction follows S. Kondō. Fix  $F \in \mathcal{Q}^s$  and denote by  $C(F) \subset \mathbb{P}(W)$  the curve defined by  $F$ . The equation  $T^4 = F$  defines an affine cone  $Z(F)$  in  $W \oplus \mathbb{C}$ , whose base at infinity is a quartic surface  $S(F) \subset \mathbb{P}(W \oplus \mathbb{C})$ . The projection  $S(F) \rightarrow \mathbb{P}(W)$  from  $[0 : 1] \in \mathbb{P}(W \oplus \mathbb{C})$  has as its fibers the  $\mu_4$ -orbits and  $S(F)$  as discriminant. In this way  $S(F)$  is a  $\mu_4$ -covering of  $\mathbb{P}(W)$  with total ramification along  $C(F)$ . Since the singularities of  $C(F)$  are ordinary double points or ordinary cusps, those of  $S(F)$  are DuVal singularities of type  $A_3$  (local-analytic equation  $w^4 = x^2 + y^2$ ) or  $E_6$  (local-analytic equation  $w^4 = x^3 + y^2$ ) and hence, if  $\tilde{S}(F) \rightarrow S(F)$  resolves these in the standard (minimal) manner, then  $\tilde{S}(F)$  is a nonsingular K3 surface. This resolution is unique up to unique isomorphism and so the  $\mu_4$ -action on  $S(F)$  lifts to  $\tilde{S}$ . The hyperplane class  $\eta(F) \in H^2(\tilde{S}(F), \mathbb{Z})$  is a semipolarization of  $\tilde{S}(F)$  invariant under  $\mu_4$  and of degree 4: we have  $\eta(F) \cdot \eta(F) = 4$  (the dot denotes the intersection form on the cohomology).

We now fix a generator  $\mu \in \wedge^3 W^*$  and view this generator as a translation invariant 3-form on  $W$ . Then the surface  $\tilde{S}(F)$  comes with a holomorphic differential defined by

$$\omega(F) := \text{Res}_{S(F)} \left( \frac{\mu}{T^3} \Big|_{Z(F)} \right).$$

Since  $\mu_4$  acts on the last factor  $\mathbb{C}$  by the tautological character, it acts on the  $T$ -coordinate by  $\bar{\chi}$  and hence on  $\omega(F)$  by  $\bar{\chi}^{-3} = \bar{\chi}$ . We often regard  $\omega(F)$  as a cohomology class in  $H^2(\tilde{S}(F), \mathbb{C})_{\bar{\chi}}$ . Since  $\omega(F) \wedge \bar{\omega}(F)$  is everywhere positive (for the complex orientation), we have  $\omega(F) \cdot \bar{\omega}(F) > 0$ . It is clear that  $\tilde{S}(F)$  is a K3-surface. The pull-back of the hyperplane class defines a semipolarization  $\eta(F)$  of degree four for  $\tilde{S}(F)$  relative to which the surface is *nonhyperelliptic*:  $\tilde{S}(F)$  has no elliptic fibration such that the semipolarization is of degree 1 or 2 on the fibers. (Otherwise these fibers would map with degree 2 to their image or get contracted.)

If we carry out this construction universally, then we find a hypersurface  $\mathcal{Z} \subset \mathbb{Q}^s \times (W \oplus \mathbb{C})$  (defined by the equation  $F = T^4$ ), the projection  $\mathcal{Z} \rightarrow \mathbb{Q}^s \times W$  is a  $\mu_4$ -cover which ramifies (totally) over the zero set  $\mathcal{C} \subset \mathbb{Q}^s \times W$  of  $F$  and at infinity we get  $\mathcal{S} \subset \mathbb{Q}^s \times \mathbb{P}(W \oplus \mathbb{C})$  as a  $\mu_4$ -cover of  $\mathbb{Q}^s \times \mathbb{P}(W)$ . If we let  $\mathrm{GL}(W)$  act as the identity on the last factor  $\mathbb{C}$ , then this action preserves  $\mathcal{Z}$  and its defining equation. We also get a section  $\omega$  of the relative dualizing sheaf of  $\mathcal{S}$ . This section transforms under  $\mu_4$  according to the character  $\det : \mathrm{GL}(W) \rightarrow \mathbb{C}^\times$ .

**Proposition 4.1.** *Let  $F \in \mathbb{Q}^s$ , denote by  $a_1$  resp.  $a_2$  the number of ordinary double points resp. ordinary cusps of  $C$  and put  $d := a_1 + 2a_2$ . Then  $\mu_4$  acts on  $H^2(S(F), \mathbb{C})$  with character  $1 + (7 - d)(\chi + \chi^2 + \chi^3)$  and on  $H^2(\tilde{S}(F), \mathbb{C})$  with character*

$$1 + 7(\chi + \chi^2 + \chi^3) + a_1(3 - \chi - \chi^2 - \chi^3) + a_2(4 - 2\chi - 2\chi^3).$$

*Proof.* The fixed point set of  $\mu_4$  in  $S = S(F)$  is  $C = C(F)$ , whereas the action of  $\mu_4$  on  $S - C$  is free. We can now invoke the Lefschetz theorem which says that the (alternating) character of  $\mu_4$  on the total cohomology of  $S$  takes on  $\xi \in \mu_4$  the value  $e(S^\xi)$  (the Euler characteristic of the fixed point set of  $\xi$ ). The latter is  $e(S)$  for  $\xi = 1$  and  $e(C)$  otherwise. In the presence of  $a_1$  ordinary double points and  $a_2$  cusps, the Euler characteristic of  $C$  resp.  $S$  is  $-4 + d$  resp.  $24 - 3d$ . It then easily follows that the character on the total cohomology is  $3 + (7 - d)(\chi + \chi^2 + \chi^3)$ . So the character on  $H^2(S, \mathbb{C})$  is as asserted.

We use this to compute the character of  $\mu_4$  on  $H^2(\tilde{S}, \mathbb{C})$ : The difference between the two is accounted for by  $H^2$  of the exceptional set. A double point resp. a cusp yields a DuVal curve  $D$  of type  $A_3$  resp.  $E_6$ . Its reduced cohomology only lives in dimension 2 and has as basis the fundamental classes of the irreducible components of  $D$ . A generator of  $\mu_4$  leaves in the first case each irreducible component invariant (so we get character 3) and induces in the second case the only non trivial symmetry of an  $E_6$ -diagram (so we get character  $4 + 2\chi^2$ ). Hence the character on  $H^2(\tilde{S}, \mathbb{C})$  is  $1 + (7 - d)(\chi + \chi^2 + \chi^3) + a_1 3 + a_2(4 + 2\chi^2) = 1 + 7(\chi + \chi^2 + \chi^3) + a_1(3 - \chi - \chi^2 - \chi^3) + a_2(4 - 2\chi - 2\chi^3)$ .  $\square$

It is clear that the orbit space  $S'(F) := \mu_2 \backslash S(F)$  is the degree two cover of  $\mathbb{P}^2$  which ramifies along  $C(F)$ . So  $S'(F)$  is a Del Pezzo surface of degree 2 which is allowed to have DuVal singularities of type  $A_1$  and  $A_2$ .

In Proposition 4.1,  $H^2(S(F), \mathbb{C})^{\mu_4}$  is spanned by the hyperplane class and  $\sqrt{-1} \in \mu_4$  acts on  $S(F)$  with Lefschetz number  $-4 + d$ . So the following proposition provides a converse in case the latter is  $\leq 0$ .

**Proposition 4.2.** *Let  $(S, \eta)$  be a polarized K3 surface (DuVal singularities allowed) of degree 4 with  $\mu_4$ -action  $\rho : \mu_4 \rightarrow \mathrm{Aut}(S, \eta)$  and let  $\omega$  a generator  $\omega$  of the dualizing sheaf of  $S$  such that*

- (i)  $H^2(S, \mathbb{C})^{\mu_4}$  is spanned by  $\eta$ ,
- (ii) the Lefschetz number of  $\rho(\sqrt{-1})$  is  $\leq 0$  and
- (iii)  $\omega \in H^2(S, \mathbb{C})_{\bar{\chi}}$ .
- (iv) is  $\tilde{S} \rightarrow S$  is the minimal resolution, then for every  $\varepsilon \in \mathrm{Pic}(\tilde{S})$  with  $\varepsilon \cdot \varepsilon = 0$  we have  $|\varepsilon \cdot \eta| > 2$ .

*Then there is an embedding of  $S$  in  $\mathbb{P}^3$  such that  $\eta$  is the hyperplane class, the image has an equation of the form  $F(Z_0, Z_1, Z_2) = Z_0^4$ , with  $F$  defining a stable quartic plane curve (i.e.,  $F \in \mathbb{Q}^s$ ), the  $\mu_4$ -action on  $S$  is the restriction of its diagonal action on  $\mathbb{P}^3$  with characters  $(1, 1, 1, \chi)$  and  $\omega$  is the residue of  $F^{-1}dZ_0 \wedge dZ_1 \wedge dZ_2$ . Moreover the  $\mathrm{SL}(3, \mathbb{C})$ -orbit of  $F$  is unique.*

*Proof.* The last assumption says that  $(S, \eta)$  is nonhyperelliptic, i.e., that the degree of every elliptic curve on a minimal resolution  $\tilde{S}$  of  $S$  is  $\geq 3$ . According to Mayer [7] and St-Donat [8], the nonhyperellipticity of  $\tilde{S}$  implies that the polarization defines up to a projective transformation an embedding  $S \subset \mathbb{P}^3$ .

Let  $S^{\mu_4} \subset S$  denote the fixed point set of the  $\mu_4$ -action. The Lefschetz formula says that the Euler characteristic of  $S^{\mu_4}$  is the Lefschetz number of  $\sqrt{-1} \in \mu_4$ , which is  $\leq 0$ . Since  $S^{\mu_4}$  is nonempty, it follows that  $S^{\mu_4}$  contains an irreducible curve  $C$  of positive genus. That curve spans a subspace of  $\mathbb{P}^3$  on which  $\mu_4$  acts trivially and that subspace is not a line. So  $\mu_4$  acts trivially on a plane in  $\mathbb{P}^3$ . It follows that we can represent the  $\mu_4$ -action on  $\mathbb{C}^4$  by a diagonal action of type  $(1, 1, 1, \chi^{\pm 1})$ . Then  $S$  will have an equation of the form  $F(Z_0, Z_1, Z_2) = cT^4$  for some  $c \in \mathbb{C}$ . Since  $S$  has only DuVal singularities, we cannot have  $c = 0$ . We rescale the coordinates in such a manner that  $\omega$  is the residue as above. It is now also clear that the  $\mu_4$ -action is of type  $(1, 1, 1, \chi)$ . At the same time we see that  $F$  defines a stable quartic curve. The uniqueness of the  $\mathrm{SL}(3, \mathbb{C})$ -orbit of  $F$  is left to the reader.  $\square$

**The period map.** We briefly review the period map, again essentially following Kondō [4]. Fix an even unimodular lattice  $\Lambda$  of signature  $(3, 19)$  and a  $\eta \in \Lambda$  with  $\eta \cdot \eta = 4$ . As is well-known, such a pair  $(\Lambda, \eta)$  is unique up to isometry. Consider the collection of  $\mu_4$ -actions  $\rho : \mu_4 \rightarrow \mathrm{O}(\Lambda)_\eta$  on  $\Lambda$  fixing  $\eta$  for which

- (i)  $\eta$  spans  $\Lambda^{\mu_4}$ ,
- (ii) the sublattice  $\Lambda^{\mu_2}$  is nondegenerate of signature  $(1, 7)$ , and
- (iii) if  $\varepsilon \in \Lambda^{\mu_2}$  is such that  $\varepsilon \cdot \varepsilon = 0$ , then  $\varepsilon \cdot \eta \neq 2$ .

Notice that (i) and (ii) imply that  $\mu_4$  has character  $1 + 7(\chi + \chi^2 + \chi^3)$  on  $\Lambda$ . The group  $\mathrm{O}(\Lambda)_\eta$  acts on this set (by composition). One can either invoke the surjectivity of the period map and the above discussion or use more intrinsically the theory of lattices to see that this action is transitive.

We therefore fix one such  $\rho$  and we write simply  $a'$  for  $\rho(\sqrt{-1})a$ . We let  $\Lambda_\pm$  denote the set of  $a \in \Lambda$  with  $a'' = \pm a$  (so  $\Lambda_+ = \Lambda^{\mu_2}$ ). According to Kondō,  $(\Lambda_+, \eta)$  is naturally isomorphic to the Picard lattice of a Del Pezzo surface of degree two with its anticanonical class, scaled by a factor two:  $\Lambda_+$  admits an orthogonal basis  $e_0, \dots, e_7$  with  $e_0 \cdot e_0 = 2$ ,  $e_i \cdot e_i = -2$  for  $i = 1, \dots, 7$  and such that  $\eta = 3e_0 - (e_1 + \dots + e_7)$ .

Notice that  $V := (\Lambda \otimes \mathbb{C})_{\bar{\chi}}$  is a 7-dimensional complex vector space. If  $u, v \in V$ , then  $u \cdot v = u' \cdot v' = -u \cdot v$  and so  $V$  is totally isotropic relative to the  $\mathbb{C}$ -bilinear extension of the form on  $\Lambda$ . We have  $\bar{V} = (\Lambda \otimes \mathbb{C})_{\chi}$  and  $V \oplus \bar{V}$  is real of signature  $(2, 12)$ . So the Hermitian extension of the bilinear form to  $\Lambda \otimes \mathbb{C}$  has signature  $(1, 6)$  on  $V$ . Denote half that Hermitian form by  $h$  and let  $V_{\mathcal{O}}$  be the set of  $v \in V$  with  $\frac{1}{2}(v + \bar{v}) \in \Lambda_-$ . So if for instance  $\alpha \in \Lambda_-$  is such that  $\alpha \cdot \alpha = -2$ , then  $\alpha \cdot \alpha' = 0$ ,  $v := \alpha + \sqrt{-1}\alpha' \in V_{\mathcal{O}}$  and we have  $h(v, v) = \frac{1}{2}(\alpha \cdot \alpha + \alpha' \cdot \alpha') = -2$ .

Clearly,  $V_{\mathcal{O}}$  is a Hermitian module over the Gaussian integers  $\mathcal{O} := \mathbb{Z}[\sqrt{-1}]$ . According to Heckman  $V_{\mathcal{O}}$  admits a  $\mathcal{O}$ -basis  $\{v_i\}_{i=1}^7$  which at the same time enumerates the vertices of a  $E_7$ -graph such that  $h(v_i, v_j)$  equals  $-2$  when  $i = j$ ,  $0$  when  $v_i$  and  $v_j$  are not connected and  $1 + \mathrm{sign}(j - i)\sqrt{-1}$  if  $v_i$  and  $v_j$  are connected. In particular, the pair  $(V, h)$  is naturally defined over  $\mathbb{Q}(\sqrt{-1})$ . Denote by  $\tilde{\Gamma} \subset \mathrm{O}(\Lambda)$  the group of  $\mu_4$ -automorphisms of  $(\Lambda, \eta)$  and by  $\Gamma$  its image in the unitary group of  $(\Lambda, \eta)$ . Both groups are arithmetic and contain  $\mu_4$  as their center (which acts on  $V$  as group of scalars with character  $\bar{\chi}$ ). This implies that  $\Gamma$  splits as a direct product  $\mu_4 \times \Gamma_1$ : since  $V_{\mathcal{O}}$  is a free  $\mathcal{O}$ -module of rank 7, the kernel  $\Gamma_1$  of the determinant homomorphism  $\det_{\mathcal{O}} : \Gamma \rightarrow \mathcal{O}^\times = \mu_4$  supplements  $\mu_4 \subset \Gamma$ .

Given  $F \in \mathcal{Q}^\circ$ , then a choice of an equivariant isometry  $H^2(S(F); \mathbb{Z}) \cong \Lambda$  which takes  $\eta(F)$  to  $\eta$  will take  $\omega(F)$  to a point of  $V_+$ . The latter's  $\Gamma$ -orbit does not change if we pick another such isometry and hence we get a well-defined element of  $\Gamma \backslash V_+$ . The resulting map

$$P : \mathcal{Q}^\circ \rightarrow \Gamma \backslash V_+$$

is analytic and constant on the  $SL(W)$ -orbits; in fact, for  $g \in GL(W)$ , we have  $P(gF) = \det(g)P(F)$ . Since  $\lambda \in \mathbb{C}^\times \subset GL(W)$  takes  $F$  to  $\lambda^{-4}F$  and  $P(F)$  to  $\lambda^3P(F)$ , it follows that  $P$  is homogenous of degree  $-3/4$  relative to scalar multiplication in  $\mathcal{Q}$  and  $V_+$ . (There is no contradiction here: if we descend the  $\mathbb{C}^\times$ -action on  $V_+$  given by scalar multiplication to  $\Gamma \backslash V_+$ , then it is no longer faithful: the kernel is  $\Gamma \cap \mathbb{C}^\times = \mu_4$ .)

The map  $P$  extends across  $\mathbb{P}(\mathcal{Q}^s)$  (for then  $S(F)$  only acquires DuVal singularities) and yields an analytic map  $\mathbb{P}(\mathcal{Q}^s) \rightarrow \Gamma \backslash \mathbb{P}(V_+)$ . The Torelli theorem for  $K3$ -surfaces implies that this map is an open embedding. But it fails to be surjective: if  $H$  is a hyperplane of  $V$  of signature  $(1, 5)$  that is orthogonal to some  $\varepsilon \in \Lambda$  with  $\varepsilon \cdot \varepsilon = 0$  and  $\varepsilon \cdot \eta = 2$ , then  $\mathbb{P}(H \cap V_+)$  parametrizes hyperelliptic  $K3$ -surfaces and hence its image in  $\Gamma \backslash \mathbb{P}(V_+)$  is disjoint with the image of the above period map. The following lemma describes the situation in a more precise manner:

**Lemma 4.3.** *Let  $\varepsilon \in \Lambda$  be such that  $\varepsilon \cdot \varepsilon = 0$ ,  $\varepsilon \cdot \eta = 2$  and the span of  $\eta$  and the  $\mu_4$ -orbit of  $\varepsilon$  is of hyperbolic signature. Then we have the following eigenspace decomposition in  $\Lambda \otimes \mathbb{C}$ :*

$$2\varepsilon = \eta + \alpha + \beta = \eta + \frac{1}{2}(\alpha - \sqrt{-1}\alpha') + \frac{1}{2}(\alpha + \sqrt{-1}\alpha') + \beta,$$

with  $\alpha, \beta \in L$  such that  $\alpha'' = -\alpha$ ,  $\beta' = -\beta$  and  $\alpha \cdot \alpha = \beta \cdot \beta = -2$ . In particular,  $\varepsilon$  is primitive and the orthogonal projection of  $4\varepsilon$  in  $V$ ,  $v := \alpha + \sqrt{-1}\alpha'$ , lies in  $V_{\mathcal{O}}$  and satisfies  $h(v, v) = -2$ . Moreover,  $\mathcal{O}v$  is an orthogonal direct summand of  $V_{\mathcal{O}}$ .

We first prove:

**Lemma 4.4.** *Let  $L \subset \Lambda$  be a  $\mu_4$ -invariant sublattice of hyperbolic signature containing  $\eta$  and two distinct isotropic vectors  $\varepsilon_1, \varepsilon_2$  with  $\varepsilon_i \cdot \eta = 2$ . Then  $\varepsilon_1 \cdot \varepsilon_2 = 1$ . In particular, each  $\varepsilon_i$  is primitive.*

*Proof.* Since  $\mu_4$  fixes  $\eta$ , it will also preserve each connected component of  $\{x \in L \otimes \mathbb{R} - \{0\} : x \cdot x \geq 0\}$ . This implies that  $\varepsilon_1 \cdot \varepsilon_2 > 0$ . Now  $a_i := \eta - 2\varepsilon_i$  is perpendicular to  $\eta$  and we have  $a_i \cdot a_i = -4$  and  $a_1 \cdot a_2 = -4 + 4\varepsilon_1 \cdot \varepsilon_2$ . Since  $a_1$  and  $a_2$  span a negative definite lattice, it follows that  $\varepsilon_1 \cdot \varepsilon_2 = 1$ .  $\square$

*Proof of Lemma 4.3.* We put  $\alpha := \varepsilon - \varepsilon''$  and  $\beta := \varepsilon + \varepsilon'' - \eta$  and so that we have the orthogonal decomposition  $2\varepsilon = \alpha + \beta + \eta$  with  $\alpha'' = -\alpha$  and  $\beta' = -\beta$ . If we apply the preceding lemma to the pairs  $\varepsilon, \varepsilon'$  and  $\varepsilon, \varepsilon''$ , we find that  $\alpha \cdot \alpha = \beta \cdot \beta = -2$ . Clearly  $\alpha + \sqrt{-1}\alpha'$  is an eigenvector of our automorphism with eigenvalue  $-\sqrt{-1}$ .

To prove the last assertion, let  $z \in \Lambda_-$ . Then we have  $\alpha \cdot z = (2\varepsilon - \eta - \beta) \cdot z = 2\varepsilon \cdot z \in 2\mathbb{Z}$  and likewise  $\alpha' \cdot z \in 2\mathbb{Z}$ . Hence  $z$  can be written as  $z_1 + \frac{1}{2}(\alpha \cdot z)\alpha + \frac{1}{2}(\alpha' \cdot z)\alpha'$  with  $z_1 \in \Lambda_-$  perpendicular to  $\alpha$  and  $\alpha'$ . This proves that  $\mathbb{Z}\alpha + \mathbb{Z}\alpha'$  is an orthogonal direct summand of  $\Lambda_-$ . Hence  $\mathcal{O}v$  is an orthogonal direct summand of  $V_{\mathcal{O}}$ .  $\square$

It is clear that in this situation the orthogonal complement  $H$  of  $v$  in  $V$  is also the intersection of  $V$  with the orthogonal complement of  $\varepsilon$  in  $\Lambda_{\mathbb{C}}$ . Lemma 4.3 tells us that  $V_{\mathcal{O}}$  is the orthogonal direct sum of  $H \cap V_{\mathcal{O}}$  and  $\mathcal{O}v$ , in particular,  $H$  is defined over  $\mathbb{Q}(\sqrt{-1})$ . Kondō shows that the underlying  $\mathbb{Z}$ -lattice, the orthogonal complement of  $\mathbb{Z}\alpha + \mathbb{Z}\alpha'$ , is isometric to  $U(2) \perp U(2) \perp D_8(-1)$ , where  $U$  denotes the hyperbolic plane,  $D_8$  the

root lattice of type  $D_8$ , and the number between parenthesis indicates the scaling factor of the form. The signature of  $H$  is  $(1, 5)$  and so  $H$  is  $\Gamma$ -rational. We shall refer to a  $\varepsilon$  as in this lemma as a *hyperelliptic vector* and we call the associated  $H \subset V$  a *hyperelliptic hyperplane*. We denote the collection of the latter by  $\mathcal{H}_h$ . This is clearly a  $\Gamma$ -arrangement. In particular, we have defined  $V_{\mathcal{H}_h}^*$ . We verify that this arrangement satisfies the hypotheses of Proposition 3.6.

**Lemma 4.5.** *Two distinct members of  $\mathcal{H}_h$  do not meet inside  $V_+$ .*

*Proof.* Suppose we have two distinct hyperelliptic hyperplanes  $H_1, H_2$  which meet inside  $V_+$ . This means that the corresponding elliptic vectors  $\varepsilon_1, \varepsilon_2$  will be contained in a  $\mu_4$ -invariant sublattice  $L \subset \Lambda$  of hyperbolic signature. According to Lemma we then have  $\varepsilon_1^{(k)} \cdot \varepsilon_2^{(l)} = 1$  for all  $k, l$ . It follows that  $\alpha_i = \varepsilon_i - \varepsilon_i''$  satisfy  $\alpha_1^{(k)} \cdot \alpha_2^{(l)} = 0$  so that by Lemma 4.4 we have an orthogonal  $\mu_4$ -invariant decomposition

$$\Lambda_- = (\mathbb{Z}\alpha_1 \perp \mathbb{Z}\alpha_1') \perp (\mathbb{Z}\alpha_2 \perp \mathbb{Z}\alpha_2') \perp K$$

This implies that  $M := U(2) \perp U(2) \perp D_8(-1)$  has an orthogonal direct summand (of rank one) spanned by a vector  $\alpha$  with  $\alpha \cdot \alpha = -2$ . This, in turn, implies that the discriminant quadratic form of  $M$ ,  $M^*/M \rightarrow \mathbb{Q}/\mathbb{Z}$  (the reduction of  $x \in M^* \mapsto \frac{1}{2}x \cdot x$ ), represents  $-\frac{1}{4} \in \mathbb{Q}/\mathbb{Z}$  (namely its value on  $\frac{1}{2}\alpha$ ). But a straightforward calculation shows that this is not the case. This proves the first assertion.  $\square$

The following statements are proved by Kondō or are implicit in his discussion [4] (a more detailed discussion can be found in the thesis by M. Artebani [3]):

- (i)  $\Gamma$  acts transitively on the collection  $\mathcal{I}$  of degenerate hyperplanes defined over  $\mathbb{Q}(\sqrt{-1})$  and
- (ii)  $\tilde{\Gamma}$  acts transitively on the collection of hyperelliptic vectors and hence  $\Gamma$  acts transitively on the collection  $\mathcal{H}_h$  of hyperelliptic hyperplanes.

According to Corollary 3.5, the algebra of  $\Gamma$ -invariant holomorphic functions on  $V_{\mathcal{H}_h}$  is a  $\mathbb{C}$ -graded algebra admitting a finite set of homogeneous generators of positive degree. Since  $\mu_4$  acts faithfully as a group of scalars on  $V$ ,  $A_{\mathcal{H}_h, k}^\Gamma$  is zero when  $k$  is not a multiple of 4. It follows from the two properties above that the projective compactification  $\mathbb{P}(\Gamma \backslash V_{\mathcal{H}_h}) \subset \mathbb{P}(\Gamma \backslash V_{\mathcal{H}_h}^*) = \text{Proj}(A_{\mathcal{H}_h}^\Gamma)$  adds to  $\Gamma \backslash \mathbb{P}(V_{\mathcal{H}_h})$  just two strata: a singleton  $\{\infty_h\}$  (corresponding to a member of  $\mathcal{H}_h$ ) and an affine curve  $C_h$  (corresponding to some  $I^{\mathcal{H}_h}$ ). The closure of  $C_h$  is the union of these two:  $\overline{C_h} = C_h \cup \{\infty_h\}$ .

**Theorem 4.6.** *Let  $\mu_4$  acts on  $\mathcal{Q}$  by scalar multiplication. Then the period map defines an isomorphism  $(\mu_4 \times \text{SL}(W)) \backslash \mathcal{Q}^s \rightarrow \Gamma \backslash V_{\mathcal{H}_h}$  and induces an isomorphism of  $\mathbb{C}$ -algebras*

$$A_{\mathcal{H}_h}^\Gamma \xrightarrow{\cong} \mathbb{C}[\mathcal{Q}]^{\mu_4 \times \text{SL}(W)}$$

*which multiplies degrees by 3 and gives rise to an isomorphism  $(\mu_4 \times \text{SL}(W)) \backslash \mathcal{Q} \cong \Gamma \backslash V_{\mathcal{H}_h}^*$ . The points  $\infty_h$  corresponds to the strictly semistable orbit defined by  $(Z_1 Z_2 - Z_0^2)^2$  (double conic) and the curve  $C_h$  corresponds to the curve of strictly semistable orbits defined by  $(Z_1 Z_2 - Z_0^2)(Z_1 Z_2 - t Z_0^2)$  with  $t \neq 0$ .*

*Proof.* The Torelli theorem for  $K3$ -surfaces implies that  $\text{SL}(W) \backslash \mathcal{Q}^s \rightarrow \Gamma \backslash V_{\mathcal{H}_h}$  is an isomorphism after projectivization. We noticed that it is homogeneous of degree  $-3$  relative to the  $\mathbb{C}^\times$ -actions on  $W$  and  $V_+$ . In  $\Gamma \backslash V_+$  the  $\mathbb{C}^\times$ -action is not effective, but has kernel  $\mathbb{C}^\times \cap \Gamma = \mu_4$ . It follows that this map drops to an isomorphism  $(\mu_4 \times \text{SL}(W)) \backslash \mathcal{Q}^s \rightarrow \Gamma \backslash V_{\mathcal{H}_h}$ .

Since  $\mathcal{Q} - \mathcal{Q}^s$  is of codimension  $> 1$  in  $\mathcal{Q} - \mathcal{Q}^s$ ,  $\mathbb{C}[\mathcal{Q}]^{\mu_4 \times \mathrm{SL}(W)}$  is the algebra of regular functions on  $(\mu_4 \times \mathrm{SL}(W)) \backslash \mathcal{Q}^s$ . Similarly, since  $\Gamma \backslash V_{\mathcal{H}_h}^* - \Gamma \backslash V_{\mathcal{H}_h}$  is of codimension  $> 1$  in  $\Gamma \backslash V_{\mathcal{H}_h}^*$ ,  $A_{\mathcal{H}_h}^\Gamma$  is the algebra of regular functions on  $\Gamma \backslash V_{\mathcal{H}_h}$ . It follows that the isomorphism  $(\mu_4 \times \mathrm{SL}(W)) \backslash \mathcal{Q}^s \cong \Gamma \backslash V_{\mathcal{H}_h}$  induces an isomorphism  $A_{\mathcal{H}_h}^\Gamma \cong \mathbb{C}[\mathcal{Q}]^{\mu_4 \times \mathrm{SL}(W)}$  which multiplies the degrees by 3.  $\square$

*Question 4.7.* It seems likely that the same statements hold for  $\mathrm{SL}(W)$  in relation to  $\Gamma_1$ , so that for instance  $A_{\mathcal{H}_h}^{\Gamma_1}$  gets identified with  $\mathbb{C}[\mathcal{Q}]^{\mathrm{SL}(W)}$ . This raises the following question: given  $F \in \mathcal{Q}^\circ$ , does the naturally defined  $\mathcal{O}$ -lattice inside  $H^2(S(F); \mathbb{C})_{\bar{\chi}}$  (of rank 7) have a canonical generator for its top exterior power?

*Remark 4.8.* It is clear that the Baily-Borel compactification  $\Gamma \backslash \mathbb{P}(V_+^*)$  of  $\Gamma \backslash \mathbb{P}(V_+)$  has a unique cusp and so is a one-point compactification. According to Proposition 3.6, the small modification  $\Gamma \backslash \mathbb{P}(V^{7t})$  replaces the cusp by a curve and there is a natural contraction  $\Gamma \backslash \mathbb{P}(V^{7t}) \rightarrow \Gamma \backslash \mathbb{P}(V_{\mathcal{H}_h}^*)$  whose exceptional divisor is the image of a natural morphism  $\Gamma_H \backslash \mathbb{P}(H_+^*) \rightarrow \Gamma \backslash \mathbb{P}(V^{\mathcal{H}_h})$  (for any  $H \in \mathcal{H}_h$ ). The intersection of this exceptional divisor with  $\Gamma \backslash \mathbb{P}(V_+)$  parametrizes the hyperelliptic curves of genus three.

The small blow-up has a counterpart in the geometric invariant theory of quartic curves: Artebani [3] shows in her thesis that it is a GIT quotient of a blow-up of the orbit of the double conic in  $\mathbb{P}(\mathcal{Q}^{\mathrm{ss}})$  and that we thus obtain a compactification of the moduli space of genus three curves.

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