ON PERIOD MAPS THAT ARE OPEN EMBEDDINGS

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Abstract. For certain complex projective manifolds (such as K3 surfaces and their higher dimensional analogues, the complex symplectic projective manifolds) the period map takes values in a locally symmetric variety of type IV. It is often an open embedding and in such cases it has been observed that the image is the complement of a locally symmetric divisor. We explain that phenomenon and get our hands on the complementary divisor in terms of geometric data.

Introduction

The period map assigns to a complex nonsingular projective variety the isomorphism type of the polarized Hodge structure on its primitive cohomology in some fixed degree; it therefore goes from a moduli space which parametrizes varieties to a moduli space which parametrizes polarized Hodge structures. The latter is always a locally homogeneous complex manifold which comes with an invariant metric. This manifold need not be locally symmetric, but when it is, then the period map is often a local isomorphism. Prime examples are (besides the somewhat tautological case of polarized abelian varieties) projective K3 surfaces and more generally, projective complex symplectic manifolds: the period map then takes values a locally symmetric variety of type IV or a locally symmetric subvariety thereof such as a ball quotient. In many of these cases, the period map can be proved to be even an open immersion and we then find ourselves immediately wondering what the complement of the image might be. It has been observed that this is almost always a locally symmetric arrangement complement, that is, the complement of a finite union of locally symmetric hypersurfaces and this was the main reason for one of us to develop a compactification technique (that generalizes the Baily-Borel theory) for such complements.

In this paper we approach the issue from the other, geometric, side, by offering (among other things) an *explanation* for the said observation. Our main result in this direction is Corollary 2.2 which gives sufficient conditions for the image of a period map to be a locally symmetric arrangement complement once it is known to be an open embedding. The point is that in many cases of interest these conditions are known to be satisfied or are verified with relative ease. This is particularly so with K3 surfaces and we illustrate that with the Kulikov models and a case that was already analyzed by Kōndo [4], namely the quartic surfaces that arise as a cyclic cover of the projective plane along a quartic curve. But our chief motivation is to apply this to cases where the image of the period map has not yet been

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established, such as cubic fourfolds (whose period map takes values in a locally symmetric variety of type IV of dimension 20), and more specifically, to those that arise as a cyclic cover of $\mathbb{P}^4$ along a cubic hypersurface (whose period map takes values in a locally complex hyperbolic variety of dimension 10). This last case will be the subject of a subsequent paper.

It is worth noting that our main result also provides a criterion for the surjectivity of the period map; this refers of course to the case when it implies that the locally symmetric divisor is empty. What we like about this criterion is that it is *nonintrusive*: it deals with degenerations as we encounter them in nature, while leaving them untouched and, as Lemma 2.4 will testify, does not ask us to subject the degenerating family to some kind of artificial processing (of which is unpredictable how and when it will terminate).

The title of this a paper is explained by the fact that almost every significant result in this paper has as a hypothesis that some period map is an open embedding. This can be hard to verify, even if we know the period map to be a local isomorphism. Nevertheless the technique developed here sometimes allows us to derive this hypothesis from much weaker assumptions (see Remarks 1.11 and 3.5).

Finally a few details about the way this paper is organized. The first section is about polarized variations of Hodge structure 'of type IV' defined on the generic point of an irreducible variety. It begins with a simple Lemma (namely 1.2), which contains one of the basic ideas on which this paper is built. We develop this idea with the help of our notion of *boundary extension* of a polarized variation of Hodge structure, leading up to the main results of this section, Propositions 1.10 and 1.12. Such a boundary extension yields rather precise information on the image of the classifying map of certain variations of polarized Hodge structure (also the injectivity of the classifynig map is addressed, here). Section 2 brings the discussion to the geometric stage, specifically, to the setting of geometric invariant theory. It contains our main result Corollary 2.2. One can see this corollary at work in the next section, where it is being applied to (families of) K3 surfaces. We here also make the connection with the classical theory of Kulikov models and we illustrate the power of the theory by applying it to the (well-understood) case of the moduli space of quartic curves. In the appendix we show that the Baily-Borel type compactifactions of locally symmetric arrangement complements come with a natural boundary extension. In the appendix we show that the natural compactification of a locally symmetric arrangement complement comes with a natural boundary extension.

In what follows a Hodge structure is always assumed to be defined over $\mathbb{Q}$ and if the Hodge structure is polarized, then this also applies to the polarizing form. This allows us to identify a polarized Hodge structure of weight zero with its dual. A variation of Hodge structure is not only supposed to have an underlying local system of $\mathbb{Q}$-vector spaces, but the latter is also assumed to contains some unspecified (flat) $\mathbb{Z}$-sublattice.

If $\mathbb{H}$ is a variation of Hodge structure over a complex manifold M, then we denote by $\mathcal{F}^\bullet(\mathbb{H})$ the Hodge flag on $\mathcal{O}_M^{\mathrm{an}} \otimes_{\mathbb{C}_M} \mathbb{H}$. We use the traditional superscript * as the generic way to indicate the dual of an object, where of course the meaning depends on the category (which we usually do not mention since it is clear from the context). So $\mathbb{H}^*$ is the $\mathbb{C}_M$-dual of $\mathbb{H}$ as a variation of Hodge structure, but $\mathcal{F}^p(\mathbb{H})^*$ is the $\mathcal{O}_M^{\rm an}$-dual of $\mathcal{F}^p(\mathbb{H})$.

1. A limit theorem for certain period maps

A local discussion. The following notion will be used throughout.

Definition 1.1. A Hodge structure of even weight $2k$ is said to be *of type IV* if is polarized and its Hodge number $h^{k+i,k-i}$ is 1 for $|i| = 1$ and 0 for $|i| > 1$.

This implies that its quadratic form has signature $(h^{k,k}, 2)$ or $(2, h^{k,k})$ according to whether k is even or odd. Notice that tensoring with the Tate structure $\mathbb{C}(k)$ turns a weight $2k$ Hodge structure of type IV into a weight 0 Hodge structure of that type.

If we are only given a vector space H with $\mathbb{Q}$-structure endowed with a $\mathbb{Q}$-quadratic form of signature $(m, 2)$, then to make it a weight zero Hodge structure H of type IV is to choose a complex line $F^1 \subset H$ with the property that F^1 is isotropic for the quadratic form and negative definite for the hermitian form: if $\alpha \in F^1$ is a generator, then we want $\alpha \cdot \alpha = 0$ and $\alpha \cdot \bar{\alpha} < 0$. The subset of H defined by these two conditions has two connected components, interchanged by complex conjugation, one of which we will denote by H_+. The projectivization $\mathbb{P}(H_+)$ is the symmetric space for the orthogonal group of $H(\mathbb{R})$ and is known as a symmetric domain of type IV. This explains our terminology, since $\mathbb{P}(H_+)$ classifies Hodge structures of that type.

We shall also consider situations where the group μ_l of lth roots of unity with $l \geq 3$ acts on H in such a manner that the eigenspace H_χ for the tautological character $\chi : \mu_l \subset \mathbb{C}^\times$, has hyperbolic signature $(m, 1)$ (relative to the hermitian form). We observe that H_χ is isotropic for the quadratic form, for given $\alpha, \beta \in H_\chi$ and $\zeta \in \mu_l$ of order l, then $\alpha \cdot \beta = \zeta\alpha \cdot \zeta\beta = \zeta^2 \alpha \cdot \beta$ and hence $\alpha \cdot \beta = 0$. The open subset $H_{+,\chi}$ of H_χ defined by $\alpha \cdot \bar{\alpha} < 0$ has as its projectivization $\mathbb{P}(H_{\chi,+})$ a complex ball in $\mathbb{P}(H_\chi)$. This ball is the symmetric space of the unitary group of the Hermitian form on H_χ and is apparently also the classifying space for Hodge structures of type IV with a certain μ_l-symmetry. Since $H_{\chi,+}$ is connected, it is contained in exactly one component H_+ as above.

We now state and prove a simple lemma and discuss its consequences.

Lemma 1.2. *Let be given a normal complex analytic variety B, a smooth Zariski open-dense subset $j : B^\circ \subset B$ and a variation of weight zero polarized Hodge structure $\mathbb{H}$ of of type IV over B°. Suppose further given a point $o \in B - B^\circ$, a subspace $V \subset H^0(B^\circ, \mathbb{H})$ (the space of flat sections of $\mathbb{H}$) and a generating section $\alpha \in H^0(B^\circ, \mathcal{F}^1(\mathbb{H}))$ such that*

(i) *for every $v \in V$, the function $s \in B^\circ \mapsto v \cdot \alpha(s)$ extends holomorphically to B and*
(ii) *$-\alpha(s) \cdot \bar{\alpha}(s) \to \infty$ as $s \in B^\circ \to o$.*

Then any limit of lines $F^1(s)$ for $s \to o$ lies in $V^\perp$ in the following sense: if $U \subset B^\circ$ is simply connected with o in its closure and $p \in U$ is some base point, then

$$P : s \in U \mapsto [F^1(s) \subset \mathbb{H}(s) \cong \mathbb{H}(p)] \in \mathbb{P}(\mathbb{H}(p)),$$

has the property that any accumulation point of $P(s)$ for $s \to o$ lies in the orthogonal complement of V in $\mathbb{H}(s)$.

Proof. If $k := \dim V$, then choose a basis $e_1, \dots, e_N$ of $\mathbb{H}|_U$ such that $e_{k+1}, \dots, e_N$ are perpendicular to V. So if we write $\alpha(s) = \sum_{i=1}^N f_i(s)e_i$, with each f_i holomorphic on U, then the first assumption implies that $f_1, \dots, f_k$ extend holomorphically across o. The second assumption tells us that $\sum_r |f_r| \to \infty$ as $s \to o$. This implies that any accumulation point of $[f_(s) : \cdots : f_N(s)]$ for $s \to o$ has its first k coordinates zero. □

Remark 1.3. Condition (i) essentially comes down to requiring that the image of

$$V \subset (j_*\mathbb{H})_o \cong (j_*\mathbb{H}^*)_o \subset (j_*(\mathcal{O}^{\mathrm{an}}_{B,o} \otimes \mathbb{H}^*))_o \to (j_*\mathcal{F}^1(\mathbb{H})^*)_o$$

be contained in an principal $\mathcal{O}^{\mathrm{an}}_{B,o}$-submodule: $v \in V$ sends α to $v \cdot \alpha \in \mathcal{O}^{\mathrm{an}}_{B,o}$ and so the $\mathcal{O}^{\mathrm{an}}_{B,o}$-module spanned by the image of V in $(j_*\mathcal{F}^1(\mathbb{H})^*)_o$ can be identified with the ideal spanned by such functions in $\mathcal{O}^{\mathrm{an}}_{B,o}$. Conversely, if the image of the above map is contained in a principal $\mathcal{O}^{\mathrm{an}}_{B,o}$-submodule of $(j_*\mathcal{F}^1(\mathbb{H})^*)_o$, generated by α, say, then α satisfies (i) on a neighborhood of o in B. Notice that if the image of $\cdot\alpha$ in V^* (under evalution in o) is nonzero, then V itself generates a pricipal $\mathcal{O}^{\mathrm{an}}_{B,o}$-submodule of $(j_*\mathcal{F}^1(\mathbb{H})^*)_o$.

Condition (ii) says that on the given principal submodule (which is generated by $\alpha \mapsto 1$), the Hodge norm goes to zero at o (so that on the norm on its dual goes to infinity).

Types of degeneration. We return to the situation of Lemma 1.2. Let $p \in B^\circ$ be fixed base point and let us write H for the vector space underlying $\mathbb{H}(p)$ while retaining its $\mathbb{Q}$-structure and the polarizing quadratic form. The latter has signature $(\dim H - 2, 2)$. We recall that the set of $\alpha \in H$ with $\alpha \cdot \alpha = 0$, $\alpha \cdot \bar\alpha < 0$ has two connected components, one of which, denoted H_+, contains a generator of $F^1(p)$. Its projectivization $\mathbb{P}(H_+)$ is a symmetric domain for the orthogonal group of $H(\mathbb{R})$; it is also the domain for a classifying map of $\mathbb{H}$, for we can think of P as taking values in $\mathbb{P}(H_+)$. This is also a good occasion to recall that the boundary of $\mathbb{P}(H_+)$ in $\mathbb{P}(H)$ decomposes naturally into *boundary components*: a boundary component is given by a nontrivial isotropic subspace $J \subset H$ defined over $\mathbb{R}$ (so $\dim J = \{1, 2\}$): the corresponding boundary component is the $\mathbb{P}(J)$-interior of $\mathbb{P}(J) \cap \mathbb{P}(H_+)^-$. So if $\dim J = 1$, it is the singleton $\mathbb{P}(J)$ and if $\dim J = 2$ we get an open half space on $\mathbb{P}(J)$. The only incidence relations between these boundary components come from inclusions: if the closure of the boundary component attached to J meets the one associated to J', then $J' \subset J$.

Assume now that V is defined over $\mathbb{R}$. We are given that there exists a sequence $(\alpha_i)_i$ in H_+ with the lines $(\mathbb{C}\alpha_i)_i$ converging to some line $F_\infty \subset H$. According to Lemma 1.2 we have $F_\infty \perp V$ so that $[F_\infty] \in \mathbb{P}(V^\perp) \cap \mathbb{P}(H_+)^-$. The following is clear.

Lemma 1.4. *Let $V_0 \subset V$ denote the nilspace of the quadratic form. If the image of V in $(j_*\mathbb{H})_o$ is defined over $\mathbb{R}$, then we are in one of the following three cases:*

1. $V_0 = 0$. *Then V is positive definite, $V^\perp$ has signature $(\dim V^\perp - 2, 2)$ and $\mathbb{P}(H_+) \cap \mathbb{P}(V^\perp)$ is a nonempty (totally geodesically embedded) symmetric subdomain of $\mathbb{P}(H_+)$.*
2. $\dim V_0 = 2$. *Then V is positive semidefinite, $V^\perp$ is negative semidefinite and $\mathbb{P}(V^\perp) \cap \mathbb{P}(H_+)^- = \mathbb{P}(V_0) \cap \mathbb{P}(H_+)^-$.*
3. $\dim V_0 = 1$ *and $\mathbb{P}(V^\perp) \cap \mathbb{P}(H_+)^-$ is the union of the boundary components of $\mathbb{P}(H_+)$ that have $\mathbb{P}(V_0)$ in their closure.*

So in the last two cases, $\mathbb{P}(V^\perp)$ does not meet $\mathbb{P}(H_+)$, but does meet its boundary. We would like to be able to say that in case (1) $[F_\infty] \in \mathbb{P}(H_+)$, that in the other two cases $[F_\infty]$ lies in the boundary component defined by V_0 and that V is also positive semidefinite in case (3). We shall see that we come close to fulfilling these wishes if we assume:

(i) the subspace $V \subset (j_*\mathbb{H})_o$ is defined over $\mathbb{Q}$ and
(ii) the image of α in V^* under evaluation in o is nonzero so that it spans a line $F \subset V^*$.

But then the discussion is no longer elementary, as we need to invoke the mixed Hodge theory of one-parameter degenerations. For this purpose we make a base change over the open unit disk $\Delta \subset \mathbb{C}$, which sends 0 to o and Δ^* to B°. We assume here simply that H is the fiber of a base point in the image of Δ^*. Let $T : H \to H$ denote the monodromy operator of the family over Δ. It is known that some positive power T^k is unipotent. Since we can arrive at this situation by a finite base change, we assume that this is already the case.

Now $T - 1$ is nilpotent (we shall see that in the present case its third power is zero) and hence $N := \log T = -\sum_{k\geq 1} \frac{1}{k}(1-T)^k$ is a finite sum and nilpotent also. Notice that N will be a rational element of the Lie algebra of the orthogonal group of H. If N is not the zero map, then there exist linearly independent $\mathbb{Q}$-vectors e, u in H with $e \cdot e = e \cdot u = 0$ such that

$$N(a) = (a \cdot e)u - (a \cdot u)e.$$

Hence T lies canonically in the natural one-parameter subgroup (take $w = 1$)

$$\exp(wN)(a) = a + w(a \cdot e)u - w(a \cdot u)e - \tfrac{1}{2}w^2(u \cdot u)(a \cdot e)e, \quad w \in \mathbb{C}.$$

We have three cases:

(I) $N = 0$,
(II) $N \neq 0 = N^2$, which means that $(u \cdot u) = 0$ and
(III) $N^2 \neq 0 = N^3$, which means that $(u \cdot u) \neq 0$.

Let J denote the span of e and u and J_0 the nilspace of J (so J_0 equals J resp. $\mathbb{C}e$ in case II resp. case III). Notice that $V \subset \mathrm{Ker}(N) = J^\perp$ and that V^* is a quotient of $\mathrm{Coker}(N) = H/J$.

We kill the monodromy by counteracting it as follows. A universal cover $\widetilde{\Delta^*} \to \Delta^*$ of Δ^* can be taken to be the upper half plane with coordinate w so that the covering projection is given by $s = \exp(2\pi\sqrt{-1}w)$ and $w \mapsto w + 1$ generates the covering group. The variation of Hodge structure over Δ^* is given by a holomorphic map $P : \widetilde{\Delta^*} \to \mathbb{P}(H_+)$ with $P(w + 1) = TP(w)$. Then $\exp(-w \log N)P(w)$ only depends on $s = \exp(2\pi\sqrt{-1}w)$ and so we get a holomorphic map $\phi : \Delta^* \to \mathbb{P}(H)$. Schmid [10] proved that the latter extends holomorphically across $0 \in \Delta$. The line $F_{\lim}$ defines the Hodge filtration of a mixed Hodge structure $H_{\lim}$ on H whose weight filtration is the Jacobson-Morozov filtration $W_\bullet$ defined by N. This makes N a morphism of mixed Hodge structures $N : H_{\lim} \to H_{\lim}(-1)$. The pure weight subquotients are polarized with the help of N. It follows that H/J has a natural mixed Hodge structure with $\pi_J(F_{\lim})$ defining the Hodge filtration. In particular $\pi_J(F_{\lim})$ is nontrivial. Since N is the zero map in H/J, it follows that if take limits in H/J instead, we find that

$$\lim_{\mathrm{Im}(w)\to\infty} \pi_J(F_w) = \pi_J(F_{\lim}).$$

where $\pi_J : H \to H/J$ is the projection. Notice that if the image of the lefthand side under the projection $H/J \to V^*$ is nonzero, then it must equal F. So if in addition V^* has a mixed Hodge structure for which $H/J \to V^*$ is a MHS-morphism, then its Hodge filtration is given by F.

Let us now go through the three cases.

(I) $H_{\rm lim}$ is pure of weight zero and the period map factors through an analytic map $\Delta \to \mathbb{P}(H_+)$ which takes in 0 the value $[F_{\rm lim}]$. So in this case V is positive definite and is a Hodge substructure of $H_{\rm lim}$ which is pure of bidegree $(0,0)$. In particular, the line $F \subset V^*$ has no Hodge theoretic significance.

(II) Here $0 = W_{-2} \subset W_{-1} = J \subset J^\perp = W_0 \subset W_1 = H$ and $F_{\rm lim}$ projects nontrivially in $H/J^\perp \cong J^*$; the latter has a Hodge structure of weight 1 and $J^\perp/J$ which is pure of bidegree $(0,0)$. The line $F \subset V^*$ is the image of $F_{\rm lim}$ in V^*, but we cannot conclude that V^* thus acquires a Hodge structure for which $H/J \to V^*$ is a MHS-morphism unless we know that $V_0 = J$ (a priori, $0 \neq V_0 \subset J$); in that case $(V/V_0)^*$ is pure of bidegree $(0,0)$ and V_0^* is of weight 1.

(III) Then $0 = W_{-3} \subset W_{-2} = W_{-1} = J_0 \subset J_0^\perp = W_0 = W_1 \subset W_2 = H$ and $F_{\rm lim}$ projects isomorphically onto $H/J_0^\perp \cong J_0^*$. The latter is isomorphic to $\mathbb{C}(-1)$ and polarized by the form $(N^2a \cdot b)$. Since we have $(N^2a\cdot a) = -(u\cdot u)(a\cdot e)^2$ it follows that $(u \cdot u) < 0$. So J is negative semidefinite and $J^\perp$ is positive semidefinite (and hence so is V). In particular, $V_0 = J_0$. Thus V^* acquires a Hodge structure with $(V/V_0)^*$ pure of bidegree $(0,0)$ and V_0^* of bidegree $(1,1)$.

We sum up our findings and use the occasion to make a definition:

Proposition-Definition 1.5. *Assume that in the situation of Lemma 1.2, $V \subset (j_*\mathbb{H})_o$ is defined over $\mathbb{Q}$ and that the image of α in V^* under evaluation in o spans a nonzero line $F \subset V^*$. Then V is positive semidefinite and V^* has a mixed Hodge structure characterized as follows: $(V/V_0)^*$ is of bidegree $(0,0)$, V_0^* has pure weight equal to its dimension and if $V_0 \neq 0$, then $F = F^1(V^*)$. It has the property that if we make a base change over an analytic curve $\Delta \to B$ whose special point goes to o and whose generic point to B°, then the natural maps $V \to H_{\rm lim}$ and $H_{\rm lim} \to V^*$ are morphisms of mixed Hodge structure, unless $\dim V_0 = 1$ and the base change is of type II.*

If this last case never occurs, we say that $V \subset (j_*\mathbb{H})_o$ is a mixed Hodge *subspace.*

It is obvious that $(j_*\mathbb{H})_o$ itself is a mixed Hodge subspace.

A global version. Suppose now that we are given a irreducible normal variety S and a variation of polarized Hodge structure $\mathbb{H}$ of type IV weight zero over a Zariski open-dense subset $j : S^\circ \subset S$. Let $S^f \subset S$ denote the set of $s \in S$ where $j_*\mathbb{H}$ has finite monodromy. This is a Zariski open subset which contains S°. We put $S_\infty := S - S^f \subset S$. We choose a base point $p \in S^\circ$ and let H and H_+ have the same meaning as before. We have a monodromy representation $\pi_1(S^\circ, p) \to \mathrm{O}(H)$, whose image is the *monodromy group* Γ of $\mathbb{H}$. It preserves H_+ and defines an unramified Γ-covering $\widetilde{S}^\circ \to S^\circ$ on which is defined the classifying map $P : \widetilde{S}^\circ \to \mathbb{P}(H_+)$. This map is Γ-equivariant. The Γ-covering $\widetilde{S}^\circ \to S^\circ$ extends canonically as a ramified Γ-covering $\widetilde{S}^f \to S^f$ with normal total space. We know that the classifying map then extends as a complex-analytic Γ-equivariant map $P : \widetilde{S}^f \to \mathbb{P}(H_+)$.

Recall that the Baily-Borel theory asserts among other things that in case Γ is arithmetic, $\Gamma\backslash\mathbb{P}(H_+)$ admits a natural projective completion $\Gamma\backslash\mathbb{P}(\widehat{H}_+)$ whose boundary is Zariski closed (so that $\Gamma\backslash\mathbb{P}(H_+)$ is in a natural manner a quasiprojective variety). Under mild assumptions, we are in that situation:

Lemma 1.6. *If S is complete, $\mathbb{H}$ has regular singularities along S_∞ and $P : \widetilde{S}^f \to \mathbb{P}(H_+)$ is an open map, then Γ is arithmetic and P descends to an open morphism $S^f \to \Gamma\backslash\mathbb{P}(H_+)$ in the quasiprojective category.*

Proof. To prove that Γ is arithmetic, choose an arithmetic $\Gamma' \supset \Gamma$ (since Γ stabilizes a lattice, we can take for Γ' the orthogonal group of that lattice). Then P determines an analytic map $S^f \to \Gamma'\backslash\mathbb{P}(H_+)$. Domain and range have algebraic compactifications (namely S and the Baily-Borel compactification $\Gamma'\backslash\mathbb{P}(\widehat{H}_+)$) and since the singularities of this map have no essential singularities, its graph in $S \times \Gamma'\backslash\mathbb{P}(\widehat{H}_+)$ has algebraic closure. This implies that $S^f \to \Gamma'\backslash\mathbb{P}(H_+)$ has finite (positive) degree. Since that map factors through $\Gamma'\backslash\mathbb{P}(H_+)$, it follows that $[\Gamma : \Gamma']$ is finite. This proves that Γ is arithmetic also and that P descends to an open morphism $S^f \to \Gamma\backslash\mathbb{P}(H_+)$ in the quasiprojective category. □

Suppose now the variation of polarized Hodge structure $\mathbb{H}$ comes with a (fiberwise) faithful action of μ_l so that $\mathbb{H}_\chi$ has hyperbolic signature and hence the classifying map induces a morphism taking values in $\mathbb{P}(H_{\chi,+})$. We then have the following counterpart of the above lemma (whose proof is similar to the case treated and therefore omitted):

Proposition 1.7. *If S is complete, $\mathbb{H}$ has regular singularities along S_∞ and $P : \widetilde{S}^f \to \mathbb{P}(H_{\chi,+})$ is an open map, then Γ is arithmetic and P descends to an open morphism $S^f \to \Gamma\backslash\mathbb{P}(H_{\chi,+})$ in the quasiprojective category.*

We ready now ready to introduce the notions that are central to this paper.

Definition 1.8. We call a constructible subsheaf $\mathbb{V}$ of $j_*\mathbb{H}$ a *boundary extension* for $(S, \mathbb{H})$ if it is defined over $\mathbb{Q}$, its quotient sheaf has support on S_∞ and the image of the sheaf map

$$\mathbb{V} \subset j_*\mathbb{H} \cong j_*\mathbb{H}^* \subset j_*(\mathcal{O}_S \otimes \mathbb{H}^*) \to j_*\mathcal{F}^1(\mathbb{H})^*$$

generates an invertible $\mathcal{O}_S$-submodule (whose restriction to S° will be $\mathcal{O}_{S^\circ} \otimes \mathcal{F}^1\mathbb{H})^*$) such that its (degenerating) norm on defined by the polarization of $\mathbb{H}$ tends to zero along S_∞. If in addition the stalks of $\mathbb{V}$ define mixed Hodge subspaces of the stalks of $j_*\mathbb{H}$ (in the sense of Proposition-definition 1.5) then we say that $\mathbb{V}$ is the boundary extension of is of *Hodge type*. If $j_*\mathbb{H}$ is a boundary extension, then we say that $\mathbb{H}$ is *tight* on S.

Clearly, a tight boundary extension is of Hodge type. We shall denote the $\mathcal{O}_S$-dual of the image of $\mathbb{V}$ in $j_*\mathcal{F}^1(\mathbb{H})^*$ by $\mathcal{F}$. It is an extension of $\mathcal{F}^1(\mathbb{H})$ cross S as a line bundle and has the property that the norm on $\mathcal{F}$ relative to the polarization of $\mathbb{H}$ tends to infinity along S_∞. So this reproduces the situation of Proposition-definition 1.5 stalkwise. We shall refer to $\mathcal{F}$ as the *Hodge bundle* of the boundary extension.

Remark 1.9. It can be shown that boundary Hodge extension carries in a natural manner the structure of a polarized Hodge modules in the sense of M. Saito [9]. In particular, if we are given a boundary extension $\mathbb{V}$ of $(S, \mathbb{H})$ as in Definition 1.8, then we can stratify S into smooth subvarieries such that the restriction of $\mathbb{V}$ to every stratum is a variation of polarized mixed Hodge structure.

Let $\mathbb{V} \subset j_*\mathbb{H}$ be a boundary extension for $(S, \mathbb{H})$. According to Lemma 1.2, for every $s \in S - S^f$, $\mathbb{V}_s$ determines a Γ-orbit in the Grassmannian $\mathrm{Gr}(H)$ of H. We denote by $\mathcal{K}_s \subset \mathrm{Gr}(H)$ the collection of orthogonal complements of these subspace (which is also a Γ-orbit). It has a type (1, 2 or 3) according to the distinction made in Lemma 1.4. If we stratify S_∞ into connected strata in such a manner that $\mathbb{V}$ is locally constant on each stratum, then $s \mapsto \mathcal{K}_s$ is constant on strata and so $\mathcal{K} := \cup_{s \in S - S^f} \mathcal{K}_s$ is a finite union of Γ-orbits in $\mathrm{Gr}(H)$. We decompose according to type:

$$\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3.$$

The members of $\mathcal{K}_2 \cup \mathcal{K}_3$ do not meet H_+, whereas the collection of linear sections $\{\mathbb{P}(K_+)\}_{K \in \mathcal{K}_1}$ is locally finite on $\mathbb{P}(H^+)$, because Γ preserves a lattice in H (see [6]). So if we put

$$H_+^\circ := H_+ - \cup_{K \in \mathcal{K}_1} K_+, \quad \mathbb{P}(H_+^\circ) := \mathbb{P}(H_+) - \cup_{K \in \mathcal{K}_1} \mathbb{P}(K_+),$$

then H_+° is open in H_+ and $\mathbb{P}(H_+^\circ)$ is open in $\mathbb{P}(H_+)$. It is known that for every linear subspace $K \in H$ of signature $(n, 2)$, the image of $\mathbb{P}(K_+)$ in $\Gamma\backslash\mathbb{P}(H_+)$ is a closed subvariety (see [6] for a proof). So $\Gamma\backslash\mathbb{P}(H_+^\circ)$ is Zariski open in $\Gamma\backslash\mathbb{P}(H_+)$.

Proposition 1.10. *Suppose that S is complete, $\mathbb{H}$ has regular singularities along S_∞ and $P : \widetilde{S}_f \to \mathbb{P}(H_+)$ is an open map. Let also be given a boundary extension of $\mathbb{H}$ across S with associated collection of type 1 subspaces $\mathcal{K}_1 \subset \mathrm{Gr}(H)$. Put $H_+^\circ := H_+ - \cup_{K \in \mathcal{K}_1} K_+$ and denote by $H_+^\circ \subset H_+^\diamond \subset H_+$ the hyperplane arrangement complement of the collection of* hyperplanes *$K \in \mathcal{K}_1$ for which the image of P is disjoint with $\mathbb{P}(K_+)$ (so that P maps to $\mathbb{P}(H_+^\diamond)$). Then $P(\widetilde{S}_f)$ contains $\mathbb{P}(H_+^\circ)$ and $\mathbb{P}(H_+^\diamond) - P(\widetilde{S}_f)$ is everywhere of codim ≥ 2.*

If moreover S_∞ is everywhere of codim ≥ 2 in S, $\mathcal{F}$ is ample and P is injective, then the $\mathbb{C}$-algebra of meromorphic automorphic forms $\oplus_{d \geq 0} H^0(\mathbb{P}(H_+^\diamond), \mathcal{L}^{\otimes d})^\Gamma$ is finitely generated, S can be identified with its Proj *and $H_+^\circ = P(\widetilde{S}^f) = H_+^\diamond$.*

Proof. Under these hypotheses Lemma 1.6 applies so that the monodromy group Γ is arithmetic and $\Gamma\backslash P : S^f \to \Gamma\backslash\mathbb{P}(H_+)$ is an open morphism of varieties. This implies that $\Gamma\backslash P$ has Zariski open image and that the image of P is open-dense in $\mathbb{P}(H_+)$.

We first prove that $P(\widetilde{S}^f) \supset \mathbb{P}(H_+^\circ)$. Let $\alpha \in \mathbb{P}(H_+^\circ)$. Since α is in the closure of the image of P, there exists a sequence $(\tilde{s}_i \in \widetilde{S}^f)_i$ for which $(\alpha_i := P(\tilde{s}_i))_i$ converges to $\alpha \in \mathbb{P}(H_+^\circ)$. Pass to a subsequence so that the image sequence $(s_i \in S^f)_i$ converges to some $s \in S$. We cannot have $s \in S_\infty$, for then we must have $\alpha \in \mathbb{P}(K)$ for some $K \in \mathcal{K}_s$. So $s \in S^f$. Choose $\tilde{s} \in \widetilde{S}^f$ over s. It is clear that there exist $\gamma_i \in \Gamma$ such that $(\gamma_i \tilde{s}_i)_i$ converges to $\tilde{s}$. So the sequences $(\gamma_i \alpha_i = P(\gamma_i \tilde{s}_i))_i$ and $(\alpha_i)_i$ converge in $\mathbb{P}(H_+)$ to $P(\tilde{s})$ and α respectively. From the the fact that Γ acts properly discontinuously on $\mathbb{P}(H_+)$, it follows that a subsequence of $(\gamma_i)_i$ is stationary, say equal to γ, so that $\alpha = P(\gamma^{-1}\tilde{s}) \in P(\widetilde{S}^f)$.

If $K \in \mathcal{K}_1$ is such that the image of P meets $\mathbb{P}(K_+)$, then P meets $\mathbb{P}(K_+)$ in a nonempty open subset and hence the same is true for the image of the induced open embedding $\Gamma\backslash P : \widetilde{S}^f \to \Gamma\backslash\mathbb{P}(H_+)$ with regard to the image of $\mathbb{P}(K_+)$. In this last case, it does not matter whether we use the Hausdorff topology or the Zariski topology. This implies that $\mathbb{P}(H_+^\diamond) - P(\widetilde{S}^f)$ has codim ≥ 2 in $\mathbb{P}(H_+^\diamond)$ everywhere.

The line bundle $\mathcal{F}$ is the pull-back of the automorphic line bundle $\mathcal{L}$ on $\Gamma\backslash\mathbb{P}(H_+)$. So if S_∞ is everywhere of codim ≥ 2 in S, then for every integer $d \geq 0$ we have an injection

$$H^0(\mathbb{P}(H_+^\diamond), \mathcal{L}^{\otimes d})^\Gamma \subset H^0(S^f, \mathcal{F}^{\otimes d}) \cong H^0(S, \mathcal{F}^{\otimes d}).$$

If in addition P injective and $\mathcal{F}$ is ample, then the displayed injection is an isomorphism and $S = \mathrm{Proj}\left(\oplus_{d\geq 0} H^0(S, \mathcal{F}^{\otimes d})\right)$ is a projective completion of $\Gamma\backslash\mathbb{P}(H_+^\diamond)$. Hence $\Gamma\backslash P$ will map S^f isomorphically onto $\Gamma\backslash\mathbb{P}(H_+^\diamond)$ and $\mathbb{P}(H_+^\circ) = \mathbb{P}(H_+^\diamond)$. □

Remark 1.11. The requirement that P be injective is (under the assumption that P is open) the conjunction of two conditions: that P be a local isomorphism and that P be of degree one. While it is often not difficult to verify the former, the latter is in general much harder to settle. Sometimes the injectivity can be established along the way if all the assumptions of Proposition 1.10 are known to be fulfilled (including the ampleness of $\mathcal{F}$ and the codimension condition for S_∞) except that instead of P being injective, we only know it to be a local isomorphism. To explain what we mean, let us regard $\Gamma\backslash P$ as a rational map between two projective varieties: from S to some compactification of $\Gamma\backslash\mathbb{P}(H_+^\diamond)$ (for instance the Baily Borel compactification of $\Gamma\backslash\mathbb{P}(H_+)$ or the one we discuss below). If we are lucky enough to find a boundary point in the latter compactification such that the rational map is regular over a Hausdorff (or formal) neighborhood of that point and is there an isomorphism, then we may conclude P that is injective so that the last assertion of Proposition 1.10 still holds.

A natural completion of a variety of the form $\Gamma\backslash H_+^\diamond$ was constructed in the two papers [6]. This completion $\Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ)$ has the property that the automorphic line bundle on $\Gamma\backslash H_+^\diamond$ extends over it as an ample line bundle (in the orbifold sense, of course), so that the completion is in fact projective. We recall its construction in the appendix and also show that it comes with a natural boundary extension of the tautological variation of type IV Hodge structure over $\Gamma\backslash H_+^\diamond$. Following Theorem 4.4 it is tight over this compactification in case the boundary of the completion is everywhere of codim ≥ 2. The question comes up whether we get in the situation of the previous proposition the same projective completion with the same boundary extension. Presumably the answer is always yes. Here we prove this to be so under some additional hypotheses.

Proposition 1.12. *Assume we are in the situation of Proposition 1.10: S is complete, $\mathbb{H}$ has regular singularities on S, $\mathcal{F}$ is ample on S and S_∞ everywhere of codim ≥ 2 in S and P an open embedding. If in addition $\dim S \geq 3$ and the hermitian form on H takes a positive value on every two-dimensional intersection of hyperplanes in $\mathcal{K}_1$, then the isomorphism $S^f \cong \Gamma\backslash\mathbb{P}(H_+^\circ)$ (asserted by Proposition 1.10) extends to an isomorphism of S onto the natural compactification $\Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ)$ of $\Gamma\backslash\mathbb{P}(H_+^\circ)$. This isomorphism underlies an isomorphism of polarized variations of Hodge structure with boundary extension: the boundary extension of $\mathbb{H}$ across S is tight over S.*

Proof. The additional assumption and the fact that $\dim H = 2 + \dim\mathbb{P}(H_+) = 2 + \dim S \geq 5$ imply that the boundary of the compactification of $\Gamma\backslash\mathbb{P}(H_+^\circ) \subset \Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ)$ is of codim ≥ 2 everywhere. Since the automorphic line bundle $\mathcal{L}$ on $\Gamma\backslash\mathbb{P}(H_+^\circ)$ extends as an ample line bundle over the completion $\Gamma\backslash\mathbb{P}(H_+^\circ) \subset \Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ)$, it follows that

$$H^0(\mathbb{P}(H_+^\circ), \mathcal{L}^{\otimes d})^\Gamma = H^0(\mathbb{P}(\widehat{H}_+^\circ), \mathcal{L}^{\otimes d})^\Gamma$$

and so

$$\Gamma\backslash\mathbb{P}(\widehat{H}^\circ_+) = \operatorname{Proj}\left(\oplus_{d\geq 0} H^0(\mathbb{P}(H^\circ_+), \mathcal{L}^{\otimes d})^\Gamma\right).$$

This shows that P induces an isomorphism $S \cong \Gamma\backslash\mathbb{P}(\widehat{H}^\circ_+)$. The assertion concerning the boundary extension follows from Theorem 4.4. □

If $\mathbb{H}$ and its boundary extension over S come with a (fiberwise) faithful action of μ_l so that $\mathbb{H}_\chi$ has hyperbolic signature. Then we have the following counterpart of Propositions 1.10 and 1.12 (proofs are omitted since they are similar to the case treated):

Proposition 1.13. *Suppose S is complete, $\mathbb{H}$ has regular singularities along S_∞ and $P : \widetilde{S}^f \to \mathbb{P}(H_{\chi,+})$ is an open map. Let $H^\circ_{\chi,+} \subset H^\diamond_{\chi,+} \subset H_{\chi,+}$ denote the arrangement complement of the collection of* hyperplanes *K_χ, $K \in \mathcal{K}_1$, for which the image of P is disjoint with $\mathbb{P}(K_{\chi,+})$ (so that P maps to $\mathbb{P}(H^\diamond_{\chi,+})$). Then $P(\widetilde{S}_f)$ contains $\mathbb{P}(H^\circ_{\chi,+})$ and has a complement in $\mathbb{P}(H^\diamond_{\chi,+})$ everywhere of codim ≥ 2.*

If moreover P is injective, S_∞ is everywhere of codim ≥ 2 in S and $\mathcal{F}$ is ample, then in fact $H^\circ_{\chi,+} = P(\widetilde{S}^f) = H^\diamond_{\chi,+}$.

If in addition the hermitian form takes a positive value on every two-dimensional intersection of the hyperplanes of the form K_χ, $K \in \mathcal{K}_1$, then the isomorphism $S^f \cong \Gamma\backslash\mathbb{P}(H^\circ_{\chi,+})$ extends to an isomorphism of S onto the natural compactification $\Gamma\backslash\mathbb{P}(\widehat{H}^\circ_{\chi,+})$ of $\Gamma\backslash\mathbb{P}(H^\circ_{\chi,+})$ which underlies an isomorphism of polarized variations of Hodge structure with boundary extension. In particular, the boundary extension is the direct image of the dual of $\mathbb{H}$ on S.

2. The geometric context

We begin with a definition:

Definition 2.1. Let $f : \mathcal{X} \to S$ be a proper family of pure relative dimension m which is smooth over an open-dense connected $S^\circ \subset S$ and let $0 < k \leq m$ be an integer such that the fibers over S° have their cohomology in degree $2k$ of type IV. Denoting by $f_{\mathrm{reg}} : (\mathcal{X}/S)_{\mathrm{reg}} \to S$ the restriction of f to the part where f is smooth, then a *geometric Hodge bundle for f in degree* $2k$ is a line bundle $\mathcal{F}$ over S together with an embedding of quasi-coherent sheaves $u : \mathcal{F} \to (R^{2k} f^*_{\mathrm{reg}} f^{-1}_{\mathrm{reg}}\mathcal{O}_S)(k)$ such that the induced fiber maps $u(s) : \mathcal{F}(s) \to H^{2k}(X_{s,\mathrm{reg}}, \mathbb{C})(k)$ enjoy the following two properties:

(i) for every $s \in S$, $u(s) : \mathcal{F}(s) \to H^{2k}(X_{s,\mathrm{reg}}, \mathbb{C})(k)$ is injective and
(ii) when $s \in S^\circ$, the image of $u(s)$ is $H^{k+1,k-1}(X_s, \mathbb{C})(k)$ (so that u identifies $\mathcal{F}|S^\circ$ with $R^{k-1} f_*\Omega^{k+1}_{\mathcal{X}/S}|S^\circ$).

If f is projective (so that the primitive part $\mathbb{H}$ of $R^{2k} f_* f^{-1}\mathcal{O}_S(k)|S^\circ$ is a polarized variation of Hodge structure), then we say that such a geometric Hodge bundle has *a proper norm* if

(iii) the Hodge norm stays bounded on $\mathcal{F}$ at s, precisely when $R^{2k} f_*\mathbb{Q}_\mathcal{X}|S^\circ$ has finite monodromy at s.

The justification of this definition is that a geometric Hodge bundle $\mathcal{F}$ with proper norm determines a boundary extension $\mathbb{V}_f \subset j_*\mathbb{H}$ as follows. For every

$s \in S$, there exists a neighborhood B of s in S and a retraction $r : X_B \to X_s$ which is C^∞-trivial over the smooth part $X_{s,\mathrm{reg}}$ of X_s so that we get a B-embedding

$$\iota^s : X_{s,\mathrm{reg}} \times B \to (X_B/B)_{\mathrm{reg}}$$

in the C^∞-category. This embedding is unique up to B-isotopy. In particular, we have a well-defined map $\iota^s_{s'*} : H_{2k}(X_{s,\mathrm{reg}}, \mathbb{C}) \to H_{2k}(X_{s',\mathrm{reg}}, \mathbb{C})$ for every $s' \in B$. This defines a constructible sheaf $\tilde{\mathbb{V}}_f$ on S whose stalk at s is $H_{2k}(X_{s,\mathrm{reg}}, \mathbb{C})(-k)$. (By means of fiberwise Poincaré duality (or rather Verdier duality), this sheaf can also be identified with $R^{2(m-k)}(f_{\mathrm{reg}})_!\mathbb{C}(m-k)$.) The homomorphism $\mathbb{H}^* \subset j^*\tilde{\mathbb{V}}_f$ dualizes to a sheaf homomorphism $\tilde{\mathbb{V}}_f \to j_*\mathbb{H}^* \cong j_*\mathbb{H}$ and we let $\mathbb{V}_f$ be the image of the latter. So the stalk $\mathbb{V}_{f,s}$ is the image of $H_{2k}(X_{s,\mathrm{reg}}, \mathbb{C})(-k)$ in $\mathbb{H}(s')^* \cong \mathbb{H}(s')$, where $s' \in S^\circ$ is close to s. In order that $\mathbb{V}_f$ defines a boundary extension, we need that the image of

$$\tilde{\mathbb{V}}_f \to j_* \operatorname{Hom}_{\mathcal{O}_{S^\circ}}(j^* R^k f_* \Omega^k_{X/S}(k), \mathcal{O}_{S^\circ})$$

generates a line bundle. This is the case: that line bundle can be identified with the dual of $\mathcal{F}$. The assumption that $\mathcal{F}$ has proper norm ensures that $\mathbb{V}_f$ defines a boundary extension.

We now prepare for a geometric counterpart of the discussion in Section 1. If $\mathbb{H}$ is a variation of a polarized Hodge structure $\mathbb{H}$ over a quasi-projective base, then according to Deligne [2], $\mathbb{H}$ is semisimple as a local system and its isotypical decomposition is one of variation of a polarized Hodge structures. In particular, the part invariant under monodromy is a polarized Hodge substructure and so is its orthogonal complement. We refer to the latter as the *transcendental part* of $\mathbb{H}$.

Part of the preceding will be summed up by Corollary 2.2 below. We state it in such a manner that it includes the ball quotient case, but our formulation is dictated by the applications that we have in mind, rather than by any desire to optimize for generality.

Let be given

(a) a projective family $f : \mathcal{X} \to S$ of pure relative dimension m with smooth general fiber and normal projective base,
(b) an action on f of a group of the form $G \times \mu_l$ where G is (algebraic and) semisimple and μ_l is the group of lth roots of unity ($l = 1, 2, \dots$),
(c) a $G \times \mu_l$-equivariant ample line bundle $\mathcal{F}$ over S (so that are defined the stable locus S^{st} and the semistable locus S^{ss} relative to the G-action on $\mathcal{F}$)
(d) and an integer $0 < k \le m$

with the following properties:

(i) the transcendental part H of a general fiber of $R^{2k} f_* \mathbb{C}_{\mathcal{X}}(k)$ is of type IV,
(ii) μ_l preserves the fibers of f and acts on the generic stalk of $R^{k-1} f_* \Omega^{k+1}_{\mathcal{X}/S}(k)$ with tautological character $\chi : \mu_l \subset \mathbb{C}^\times$,
(iii) if $H_\chi \subset H$ denotes the eigenspace of the tautological character $\chi : \mu_l \subset \mathbb{C}^\times$, then the open subset $S^f \subset S$ where the local system $\mathbb{H}_\chi$ defined by H_χ has finite monodromy is contained in the stable locus S^{st},
(iv) $G\backslash\backslash(S^{\mathrm{ss}} - S^f)$ has codim ≥ 2 everywhere in $G\backslash\backslash S^{\mathrm{ss}}$,
(v) $\mathcal{F}|S^{\mathrm{ss}}$ can be given the structure of a geometric Hodge bundle for $f_{S^{\mathrm{ss}}}$ in degree $2k$ such that it has proper norm,

Then a period map $\widetilde{S}^f \to \mathbb{P}(H_{\chi,+})$ is defined on the monodromy covering $\widetilde{S}^f$ of S^f. Since G is semisimple, the G action on S^f lifts to an action of $\widetilde{S}^f$, perhaps after passing to a connected covering $\tilde{G}$ of G such that the period map factors through $\tilde{G}\backslash\widetilde{S}^f$. We finally require:

(vi) (Torelli property) the map $\tilde{G}\backslash\widetilde{S}^f \to \mathbb{P}(H_{\chi,+})$ through which the period map factors is an open embedding.

Corollary 2.2. *Under these assumptions the monodromy group* Γ *of* $\mathbb{H}_\chi$ *is arithmetic in its unitary resp. orthogonal group. The collection hyperplanes of* $\mathbb{H}_\chi$ *that appear as the kernel of a member of the natural* Γ*-orbit of maps* $H_\chi \to H^{2k}(X_{s,\mathrm{reg}},\mathbb{C})_\chi$*, where* $s \in S^{\mathrm{ss}} - S^f$ *has a closed orbit in* S^{ss}*, and which meet* $H_{\chi,+}$ *(that is, are of type I) make up a* Γ*-arrangement whose associated arrangement complement* $\mathbb{P}(H^\circ_{\chi,+}) \subset \mathbb{P}(H_{\chi,+})$ *is the exact image of the period map so that there results an isomorphism* $G\backslash S^f \cong \Gamma\backslash\mathbb{P}(H^\circ_{\chi,+})$*. If* $\dim S \geq 3$ *(or* ≥ 2 *in case* $l \geq 3$*), then the associated boundary extension of* $\mathbb{H}$ *over* S^{ss} *is tight.*

Proof. It is known that a variation of polarized Hodge structure of geometric origin has regular singularities. Since $G\backslash\backslash S^{\mathrm{ss}} = \mathrm{Proj}\left(\oplus_{d\geq 0} H^0\left(S,\mathcal{L}^{\otimes d}\right)\right)^\Gamma$ is a projective variety which carries the G-quotient of $\mathcal{L}$ as an ample line bundle in the orbifold sense, the hypotheses of Propositions 1.10 and 1.13 are fulfilled after this passage to G-quotients and the corollary follows. ☐

The conditions (i)-(iv) and (vi) of Corollary 2.2 are often easily checked in practice or are known to hold—it is usually the verification of (v) that requires work. This motivates the following definition.

Definition 2.3. Let $X \subset \mathbb{P}^N$ be a projective variety of pure dimension m and k a positive integer $\leq m$ such that X admits a projective smoothing in $\mathbb{P}^N$ whose general fiber has cohomology in degree $2k$ of type IV. We shall call a one dimensional subspace $F \subset H^{2k}(X_{\mathrm{reg}},\mathbb{C})$ a *residual Hodge line* if every such smoothing admits a geometric Hodge bundle in degree $2k$ which has F as closed fiber (this F will be unique, of course). If in addition, the geometric Hodge bundle has proper norm, then we say that X is a *boundary variety in dimension* $2k$ and call (X,F) a *boundary pair in degree* $2k$.

A smoothing of such a boundary variety will have a type. Mixed Hodge theory allows us to read off this type from the pair (X,F) without any reference to a smoothing:

Lemma 2.4. *Let* (X,F) *be a boundary pair in degree* $2k$ *and let* $w(F)$ *be such that* F *embeds in the weight* $2k+w(F)$ *subquotient of* $H^{2k}(X_{\mathrm{reg}},\mathbb{C})$*. Then* $w(F) \in \{0,1,2\}$ *and the type of any smoothing is* $w(F)+1$*.*

Proof. If $f : (\mathcal{X},X) \to (\Delta,0)$ is a projective smoothing of any projective variety X over the unit disk, then we have a natural map $H^0(\Delta^*,R^\bullet f_*\mathbb{Q}_\mathcal{X}) \to H^\bullet(X_{\mathrm{reg}},\mathbb{Q})$, and hence also a map $H^0(\widetilde{\Delta^*},R^\bullet f_*\mathbb{Q}_\mathcal{X}) \to H^\bullet(X_{\mathrm{reg}},\mathbb{Q})$. It is known that if we give $H^0(\widetilde{\Delta^*},R^\bullet f_*\mathbb{Q}_\mathcal{X})$ the limiting mixed Hodge structure and $H^\bullet(X_{\mathrm{reg}},\mathbb{Q})$ the usual mixed Hodge structure, then this becomes a MHS-morphism. This fact, combined with the discussion following Lemma 1.4 leading up to the three cases I, II and III immediately yields the assertion. ☐

3. Examples

If in the setting of Corollary 2.2 the general fiber is a K3-surface (and $k = 1$), then a logical choice for $\mathcal{F}$ is $f_*\omega_{\mathcal{X}/S}$ and (v) then boils down to the statement that for $s \in S^{\mathrm{ss}}$, a generator of $H^0(X_s, \omega_{X_s})$ defines a nonzero class in $H^2(X_{s,\mathrm{reg}}, \mathbb{C})$ and does not lift as a regular form to a resolution of X_s. However, this last condition tends to be always fulfilled:

Proposition 3.1. *Let X be a projective surface with $H^1(X, \mathcal{O}_X) = 0$ and with trivial dualizing sheaf ω_X. Then every smoothing of X is a K3 surface. A generator of ω_X is square integrable on X precisely when X is a K3 surface with only rational double point singularities (so that X is smoothable with finite local monodromy group).*

Proof. A smoothing of X will be a surface with vanishing irregularity and trivial dualizing sheaf and hence is a K3 surface.

Assume now that a generator of ω_X is square integrable on X. We first show that X has isolated singularities. If not, then a general hyperplane section has a Gorenstein curve singularity with the property that a generator of its dualizing module is regular on the normalization. But this is impossible, since the quotient of the two modules has dimension equal to the Serre invariant, which vanishes only in the absence of a singularity. Since X has only isolated singularities, then these must be normal in view of the fact that X has no embedded components and $H^1(X, \mathcal{O}_X) = 0$. According to Laufer [8], the square integrability of α is then equivalent to X having only rational double points as singularities. Then X has a minimal resolution $\tilde{X}$ with $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ and trivial dualizing sheaf. This implies that $\tilde{X}$ is a K3 surface. □

If X is a smoothable projective surface as in the previous proposition (so with vanishing irregularity and trivial dualizing sheaf) and a generator of ω_X is *not* square integrable and has *nonzero* image in $H^2(X_{\mathrm{reg}}, \mathbb{C})$, then it is clear from the definitions that X is a boundary surface. With the help of Lemma 2.4 we can specify the type. Let us first do this for some special cases.

Proposition 3.2. *A smoothable projective surface with vanishing irregularity and trivial dualizing sheaf which has a simple-elliptic resp. cusp singularity is a boundary surface of Hodge type 2 resp. 3.*

Proof. Let X be such a surface and let α be a generating section of ω_X and let X have at p a simple-elliptic or a cusp singularity.

If p is simple elliptic, then the exceptional curve C of the minimal resolution of the germ X_p is a smooth genus one curve C and α has a simple pole of order one along C with residue a generator of the dualizing sheaf of C. It follows that we have an embedding of Hodge structures $H^1(C, \mathbb{Q})(-1) \subset H^2(X_{\mathrm{reg}}, \mathbb{Q})$ whose image subsists in the coholomology of a limiting mixed Hodge structure of a smoothing. This implies that X is of type 2.

If p is a cusp singularity, then the exceptional curve C of the minimal resolution of the germ X_p is a is a cycle of rational curves and it is still true that α has a simple pole of order one along C with residue a generator of the dualizing sheaf of C. The latter will have a nonzero residue at a singular point of C. It follows that we have an embedding $(\mathbb{Q})(-2) \subset H^2(X_{\mathrm{reg}}, \mathbb{Q})$ whose image subsists in the coholomology of a limiting mixed Hodge structure of a smoothing. Hence X is of type 3. □

By essentially the same argument we find:

Proposition 3.3. *Let X be a smoothable projective surface with with vanishing irregularity and trivial dualizing sheaf. Suppose that for some resolution $\tilde X$ of X there exists a connected component C of $\tilde X - X_{\rm reg}$ that is smooth of genus one or a cycle of rational curves with only rational double points, and has the property that a generator of ω_X has a pole of exact order one along C. Then X is a boundary surface of Hodge type type 2 resp. 3 if C is smooth resp. a rational cyle.*

The customary approach to studying the period map of a degenerating family of K3 surfaces over the unit disk is to pass to a standard situation, called a *Kulikov model*. This involves a combination of a base change and a blowing up over the central fiber (whose effect on the period map is merely a base change). The model in question has then the property that it has a smooth total space, a trivial dualizing sheaf and a reduced normal crossing divisor as central fiber. But this process is highly nonunique and arriving at such a model tends to be nontrivial task. The interest of our approach lies in the fact that we leave such degenerations untouched and deal with them in the way they come: neither a central modification, nor usually a base change are needed to extract useful information about the limiting behavior of the period map. Let us nevertheless see how our method deals with the Kulikov models. These models fall into three classes according to the type of the central normal crossing surface X:

(i) X is a smooth K3 surface.
(ii) X is a chain of (at least two) smooth surfaces (the dual graph is an interval) whose double curves are mutually isomorphic smooth genus one curves. The surfaces at the end are rational and the surfaces in between are elliptic ruled surfaces.
(iii) the normalization of X consists of smooth rational surfaces and the incidence complex of these surfaces is a triangulated two-sphere. The double curve meets every component in an anticanonical cycle.

In case (i) there is nothing to do and Proposition 3.3 is applicable to the other two cases: a resolution of X is given by normalization. In case (ii) a member at the end of a chain in a rational surface Y which meets the rest of the chain along a smooth genus one curve C and we have $\mathcal{O}_Y \cong \omega_X \otimes \mathcal{O}_Y = \omega_Y(C)$, so that X is a boundary surface of Hodge type 2. In case (iii), let Y be any connected component of the normalization of X. Then Y is rational and the preimage $C \subset Y$ of $X_{\rm sing}$ is a cycle of rational curves. So it is a boundary surface and according to Proposition 3.3, X is a boundary surface of Hodge type 3.

A worked example: the moduli space of quartic curves. The result we are going to discuss is essentially due to Kōndo ([4], see also [7])–the point of what follows is merely to show how effectively it is reproduced by our method. Let $G \subset \mathrm{SL}(4,\mathbb{C})$ be the stabilizer of the decomposition $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}$. This group is clearly isogenous to $\mathrm{SL}(3,\mathbb{C}) \times \mathbb{C}^\times$. It acts on the space of quartic forms in four complex variables (Z_0, Z_1, Z_2, Z_3) of the shape $F(Z_0, Z_1, Z_2, Z_3) = f(Z_0, Z_1, Z_2) + \lambda Z_3^4$. Let us denote the projectivization of that space by S. This projective space supports a quartic surface $\mathcal{X} \subset \mathbb{P}^3_S$ with μ_4-action. It is clear that the open subset defined by $f \neq 0 \neq \lambda$ parametrizes quartic surfaces in $\mathbb{P}^3$ that are μ_4-covers of quartic curves in $\mathbb{P}^2$. The G-semistable surfaces all lie in this open subset. The surface $X_s \subset \mathbb{P}^3$ is G-(semi)stable if and only if the corresponding quartic plane curve $C_s \subset \mathbb{P}^2$ is so relative to the quotient $\mathrm{SL}(3,\mathbb{C})$ of G. Following Mumford, $s \in S^{\mathrm{st}}$ precisely when C_s has only ordinary double points or (ordinary) cusps (this means that X_s has singularities of type A_3 or E_6) and $s \in S^{\mathrm{ss}} - S^{\mathrm{st}}$ has a closed orbit in $S^{\mathrm{ss}} - S^{\mathrm{st}}$ precisely when C_s is a union of two conics C', C'' with C' smooth, C'' not a double line and for which either C' and C'' meet in two points of multiplicity two, or $C' = C''$. So if $C' \neq C''$, then either C'' is nonsingular and C' and C'' have common tangents where they meet or C'' is the union of two distinct tangents of C'. These are represented by the one-parameter family $(Z_1Z_2 - Z_0^2)(Z_1Z_2 - tZ_0^2)$, with $t \in \mathbb{C}$ (but notice that we get the same orbit in S for t and t^{-1}). This shows that $S^f = S^{\mathrm{st}}$.

If $C = C_s$ is smooth, then $X = X_s$ is a polarized K3 surface of degree 4 with μ_4-action. According to [4] the eigenspace $H^2(X,\mathbb{C})_\chi$ is hyperbolic of dim 7 and defined over $\mathbb{Q}(\sqrt{-1})$.

There is a G-equivariant isomorphism of $f_*\omega_{\mathcal{X}/S}|X^{\mathrm{ss}} \cong \mathcal{O}_{S^{\mathrm{ss}}}(1)$ defined as follows: to every $F \in \mathbb{C}[Z_0, Z_1, Z_2, Z_3]$ as above defining a G-semistable surface X, we associate the generator

$$\alpha_F := \mathrm{Res}_X \, \mathrm{Res}_{\mathbb{P}^3} \frac{dZ_0 \wedge dZ_1 \wedge dZ_2 \wedge dZ_3}{F}.$$

of ω_X. Since F appears here with degree -1, we get the asserted line bundle isomorphism. It is evidently G-invariant. So if we take $\mathcal{F} = \mathcal{O}_S(1)$, then have all the data in place for the verification of the properties of Corollary 2.2.

Proposition 3.4. *All the conditions of Corollary 2.2 are satisfied for $\mathcal{F} = \mathcal{O}_S(1)$ and hence so is its conclusion: we get an isomorphism $G\backslash\backslash S^{\mathrm{ss}} \cong \Gamma\backslash\mathbb{P}(\widehat{H}^\circ_{\chi,+})$. Here H_χ is a vector space of dim 7 endowed with a hermitian form of hyperbolic signature defined over $\mathbb{Q}(\sqrt{-1})$ and $H_{\chi,+}$ is the arrangement complement and the Γ-arrangement defining $H_{\chi,+}$ consists of a single Γ-orbit of hyperplanes of H_χ (these are perpendicular to a sublattice of $H_\mathbb{Z}$ of type $2I(-2)$ on which μ_4 acts faithfully). We have $S^f = S^{\mathrm{st}}$ and $S^{\mathrm{ss}} - S^f$ has two strata: one stratum parametrizes unions of distinct conics, one of which is nonsingular and having exactly two points in common and is of Hodge type 2, whereas the other is the orbit double nonsingular conics and is of type 1.*

Proof. Conditions (i) and (ii) of Corollary 2.2 are clearly satisfied. We have already seen that $S^f = S^{\mathrm{st}}$, so that also (iii) holds. We found that the orbit space $G\backslash\backslash(S^{\mathrm{ss}} - S^f)$ is of dimension one, and hence (iv) is satisfied. Property (vi) is a consequence of the Torelli theorem for K3-surfaces. So it remains to verify (v). We first check that $\mathcal{F}$ indeed defines a geometric Hodge bundle over $S^{\mathrm{ss}} - S^f$. A semicontinuity argument shows that it is enough to do this in the closed orbits of G. We verify this property and the remaining (iv) at a point of such an orbit at the same time. We therefore assume that $C = C_s$ is the union of two conics C' and C'' as described above.

The case $C' = C''$. So C is a nonsingular conic with multiplicity 2. Since $X = X_s \subset \mathbb{P}^3$ is a μ_4-cover over $\mathbb{P}^2$ totally ramified along $C \subset \mathbb{P}^2$, it consists of two copies S', S'' of $\mathbb{P}^1 \times \mathbb{P}^1$ joined along their diagonal $D \subset \mathbb{P}^1 \times \mathbb{P}^1$. A generating section α of ω_X is obtained as follows. If z is the affine coordinate on $\mathbb{P}^1$, then $\zeta := (z'-z'')^{-2}dz'\wedge dz''$ extends as a regular 2-form on $\mathbb{P}^1\times\mathbb{P}^1 - D$ with a pole of order 2 along D. Then let α be on S' equal to this form and on S'' minus this form.

It is clear that ζ (and hence α) is not square integrable. We next verify that its cohomology class in $H^2(\mathbb{P}^1\times\mathbb{P}^1 - D,\mathbb{C})$ is nonzero (so that the cohomology class of α in $H^2(X_{\rm reg},\mathbb{C})$ is nonzero, also.) This we verify by evaluating the integral of ζ over the integral generator of $H_2(\mathbb{P}^1\times\mathbb{P}^1-D,\mathbb{C})$. A generator can be represented as follows: the algebraic cycle $[\mathbb{P}^1]\otimes 1 - 1\otimes[\mathbb{P}^1]$ on $\mathbb{P}^1\times\mathbb{P}^1$ is homologous to a (nonalgebraic) cycle which avoids D. The homology takes place on a small neighborhood of $(0,0)$: we let Γ_ϵ be the sum of the holed Riemann spheres $(\mathbb{P}^1-\Delta_\epsilon)\times\{0\}$ and $\{0\}\times(\mathbb{P}^1-\Delta_\epsilon)$ with opposite orientation (where Δ_ϵ is the open ϵ-ball centered at $0\in\mathbb{P}^1$) with and the tube T_ϵ of $(z',z'')\in\mathbb{C}\times\mathbb{C}\subset\mathbb{P}^1\times\mathbb{P}^1$ with $z'-z''=e^{\sqrt{-1}\theta}$ and $z'+z''=te^{\sqrt{-1}\theta}$, $-1\le t\le 1$. Then up to sign, we have

$$\int_{\Gamma_\epsilon}\zeta=\int_{T_\epsilon}\frac{dz'\wedge dz''}{(z'-z'')^2}=$$
$$=\int_{T_\epsilon}\frac{\frac{1}{2}d(e^{\sqrt{-1}\theta})\wedge d(te^{\sqrt{-1}\theta})}{e^{2\sqrt{-1}\theta}}=\int_{-1}^{1}\int_0^{2\pi}\tfrac{1}{2}\sqrt{-1}t d\phi\wedge dt=2\pi\sqrt{-1}.$$

Notice that the selfintersection number of Γ_ϵ is that of $[\mathbb{P}^1]\otimes 1-1\otimes[\mathbb{P}^1]$ and hence equal to -2. Let γ be the corresponding class in $H_2(\mathbb{P}^1\times\mathbb{P}^1-D)$ lifted to $H_2(X_{\rm reg})$. If $g\in\mu_4$ is a generator, then $g^2\gamma=-\gamma$ and $H_2(X_{\rm reg})$ is freely generated by γ and $g\gamma$. Thus, $H_2(X_{\rm reg})$ is a μ_4-module isomorphic to the Gauss lattice $\mathbb{Z}[\sqrt{-1}]$ endowed with the quadratic form $-2\|z\|^2$. This lattice has no even overlattices and so a copy of it gets *primitively* and μ_4-equivariantly embedded in the primitive homology of a nearby smooth quartic surface of the above type. This lattice is negative definite and hence defines a hyperplane section of $\mathbb{P}(H_{\chi,+})$; we are in the type 1 case (compare [4]).

The case $C'\neq C''$. Then the associated μ_4-cover X has simple-elliptic singularities of degree 2 at the two points of $C'\cap C''$. We can therefore invoke Proposition 3.2 to conclude that property (v) holds. Let us nevertheless do this in some detail. The two conics generate a pencil of which the generic member completely decomposes in X. If we resolve the simple-elliptic singularities minimally, we get a surface $\tilde X$ with μ_4-action that is obtained as follows. Let E be a smooth genus one curve with μ_4-action isomorphic to $\mathbb{C}/\mathbb{Z}[\sqrt{-1}]$ with its obvious μ_4-action. This action has two distinct fixed points which we denote by p',p''. Now let μ_4 act on $\mathbb{P}^1$ via $\mathbb{C}^\times\subset\mathbb{P}^1$ and consider $\mathbb{P}^1\times E$ with the diagonal μ_4-action. Blow up $(0,p')$ and $(0,p'')$ in $\mathbb{P}^1\times E$ and then blow down the strict transforms of $\mathbb{P}^1\times\{p'\}$ and $\mathbb{P}^1\times\{p''\}$. The result is a is a $\mathbb{P}^1$-bundle $\bar X\to E$ with μ_4-action. The fixed point set of the action is the union of the fibers over p' and p''. The 'zero section' has self-intersection -2 and the 'section at infinity' has self-intersection $+2$. Now choose $q\in E$.

If q is not fixed by the μ_4-action, then we let $\tilde X\to\bar X$ be the blow up the μ_4-orbit of (∞,q). The sections E_0,E_∞ of $\tilde X\to E$ at zero and infinity have both self-intersection -2 so that their contraction $\tilde X\to X$ yields a surface X with two simple-elliptic singularities. The latter inherits a μ_4-action whose orbit space is isomorphic to $\mathbb{P}^2$. The fiber over p' and p'' map to C' and C'' and the fiber over q maps to the union of the common tangents of C' and C''. We have that $X_{\rm reg} \cong \tilde X - E_0 - E_\infty$ and a straightforward calculation shows that we have an exact sequence

$$0 \to \mathbb{C}(-1) \to H^2(X_{\rm reg}, \mathbb{C})_\chi \to H^1(E_0, \mathbb{C})(-1) \to\ 0,$$

in particular, $\dim H^2(X_{\rm reg}, \mathbb{C})_\chi = 2$. If q is a fixed point, say $q = p''$, then we let $\tilde X \to \bar X$ be obtained by blowing up four times over q (always on the strict transform of $\{\infty\} \times E$). This produces over q a string of three (-2)-curves that can be contracted to A_3-singularity. The morphism $\tilde X \to X$ contracts also the sections E_0 and E_∞. Then $H^2(X_{\rm reg}, \mathbb{C})_\chi = H^2(\tilde X - E_0 - E_\infty, \mathbb{C})_\chi \cong H^1(E_0, \mathbb{C})(-1)$, hence has dim 1. In either case, a generating section of ω_X is obtained as follows: if α_E is a nowhere zero regular differential α_E on E, then $z^{-1}dz \otimes \alpha_E$ defines a rational differential on $\tilde X$ with divisor $-E_0 - E_\infty$ which is the pull-back of a unique generating section α of ω_X. By Proposition 3.2 we are in the type 2 case. □

Remark 3.5. The use of the Torelli theorem for K3 surfaces to establish property (v) of Corollary 2.2 can be avoided if we use an argue as in Remark 1.11 by taking as compactification of $\Gamma\backslash\mathbb{P}(H_+^\circ)$ the natural one: over the point of a one-dimensional stratum of the latter we find the simple-elliptic locus. The period map is near that locus is sufficiently well understood [5] for proving that along this locus we have a local isomorphism for the Hausdorff topology.

4. Appendix: the boundary extension attached to an arrangement compactification

We show here that the arrangement compactifications introduced in [6] come with a natural boundary extension. The setting in which this construction is carried out is essentially the one that we are dealing with, for it needs the following input:

(i) a complex vector space H with $\mathbb{Q}$-structure of dimension $n+2 \geq 4$ endowed with a nondegenerate quadratic form, also defined over $\mathbb{Q}$ and of signature $(n, 2)$, and a connected component H_+ of the set of $\alpha \in H$ with $\alpha\cdot\alpha = 0$ and $\alpha \cdot \bar\alpha < 0$,

(ii) an arithmetic subgroup Γ of $\mathrm{O}(H)$ which stabilizes H_+ and

(iii) a Γ-invariant collection $\mathcal{H}$ of hyperplanes of H, all defined over $\mathbb{Q}$ and of signature $(n-1, 2)$ and consisting of finitely many Γ-equivalence classes.

We shall refer to these data as a Γ-*arrangement in* H_+ and we call

$$H_+^\circ := H_+ - \cup_{K\in\mathcal{H}} K_+ \quad \text{resp.} \quad \mathbb{P}(H_+^\circ) := \mathbb{P}(H_+) - \cup_{K\in\mathcal{H}} \mathbb{P}(K_+).$$

a Γ-*arrangement complement* in H_+ resp. $\mathbb{P}(H_+)$. Given our Γ-arrangement $\mathcal{H}$, we define an index set $\tilde\Sigma$ parametrizing such subspaces that is a union

$$\tilde\Sigma := \mathcal{K}_1 \cup \mathcal{K}_2 \cup (\cup_J \Sigma_J),$$

of which the first two members are self-indexing in the sense that these are collections of subspaces of H.

We let $\mathcal{K}_1$ be the collection of subspaces of H with positive definite orthogonal complement that arise as an intersection of members of $\mathcal{H}$ (this was denoted $\mathrm{PO}(\mathcal{H}|\mathbb{P}(H_+)$ in [6]). If $J \subset H$ is an isotropic plane defined over $\mathbb{Q}$, then denote by K_J the common intersection of $J^\perp$ and all the members of $\mathcal{H}$ which contain J (so $K_J = J^\perp$ if no member of $\mathcal{H}$ contains J). This will be our collection $\mathcal{K}_2$; it is clearly in bijective correpondence with the set isotropic planes defined over $\mathbb{Q}$.

Let now $J \subset H$ be an isotropic line defined over $\mathbb{Q}$. Then $J^\perp/J$ is nondegenerate defined over $\mathbb{Q}$ and has hyperbolic signature $(1,n)$. Let $e \in J(\mathbb{Q})$ be a generator. The map $p_e : H - J^\perp \to (J^\perp/J)(\mathbb{R})$, $\alpha \mapsto \operatorname{Im}\big((\alpha\cdot e)^{-1}\alpha\big)$ has the property that H_+ is the preimage of a component C_e of the quadratic cone in $J^\perp(\mathbb{R})/J(\mathbb{R})$ defined by $y\cdot y<0$. It is more intrinsic to consider instead $p_J : H - J^\perp \to (J\otimes J^\perp/J)(\mathbb{R})$ given by $\alpha \mapsto e\otimes \operatorname{Im}\big((\alpha\cdot e)^{-1}\alpha\big)$. So in the latter space there is a quadratic cone C_J such that $H_+ = p_J^{-1}(C_J)$. (This map factors through $\mathbb{P}(H_+)$ and gives rise to the latter's realization as a tube domain of the first kind.) Any member of $\mathcal{H}$ which contains J defines hyperplane section of $C_J \cong C_e$. These hyperplane sections are locally finite on C_J and decompose C_J into a locally polyhedral cones. Denote this collection of cones by Σ_J. For any $\sigma\in\Sigma_J$, denote by K_σ the subspace of $J^\perp$ defined by the complex-linear span of σ in $J^\perp/J$. So if no member of $\mathcal{H}$ contains J, then C_J is the unique member of Σ_J and we then have $K_{C_J}=J^\perp$. This is in general not an injectively indexed collection of subspaces, for it often happens that for distinct σ,σ' we have $K_\sigma = K_{\sigma'}$. Notice that Σ_J is in a natural manner a partially ordered set. The disjoint union $\mathcal{K}$ of the Σ_J, where J runs over all the $\mathbb{Q}$-isotropic lines in H, $\mathcal{K}_1$ and $\mathcal{K}_2$ is then also partially ordered by taking the inclusion relation between corresponding subspaces they define, except that on each Σ_J we replace this by the inclusion relation between cones.

For every linear subspace $K\subset H$ defined over $\mathbb{Q}$ for which $\mathbb{P}(K)$ meets the closure of $\mathbb{P}(H_+)$ (so that K is of type 1, 2 or 3), we denote by π_K the projection $H\to H/K$ and by $\mathbb{P}\pi_K : \mathbb{P}(H)-\mathbb{P}(K)\to\mathbb{P}(H/K)$ its projectivization. Consider the disjoint union

$$\widehat{H}_+^\circ := H_+^\circ \sqcup \coprod_{\sigma\in\tilde\Sigma} \pi_{K_\sigma}(H_+^\circ)$$

and its projectivization

$$\mathbb{P}(\widehat{H}_+^\circ) := \mathbb{P}(H_+^\circ) \sqcup \coprod_{\sigma\in\tilde\Sigma} \mathbb{P}\pi_{K_\sigma}\mathbb{P}(H_+^\circ).$$

In [6]-II, pp. 570 the latter is endowed with a Γ-invariant Hausdorff topology which induces the given topology on the parts. The former is not formally introduced there, but its definition is completely analogous for it is such that the obvious map $\widehat{H}_+^\circ \to \mathbb{P}(\widehat{H}_+^\circ)$ is the formation of a $\mathbb{C}^\times$-orbit space. Both have the property that the partial ordering prescribes the incidence relations. Notice that each of these spaces comes with a Γ-invariant *structure* sheaf of continuous complex valued functions, namely the functions that are holomorphic on every member of the partition. The Γ-orbit spaces $\mathbb{P}(\Gamma\backslash\widehat{H}_+^\circ)$ and $\Gamma\backslash\widehat{H}_+^\circ$ are Hausdorff and compact resp. locally compact and their structure sheaves make them normal analytic spaces for which the natural projection $\Gamma\backslash\widehat{H}_+^\circ \to \Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ)$ is a $\mathbb{C}^\times$-bundle in the orbifold sense. The associated orbifold line bundle (which we denote by $\mathcal{F}$) turns out to be ample, so that $\Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ)$ is projective.

We will need the following lemma, which almost captures the topology of $\widehat{H}_+^\circ$.

Lemma 4.1. *If a sequence $(\alpha_i \in H_+^\circ)_i$ converges to $\alpha_\infty \in \pi_K(H_+^\circ)$, with K a linear subspace indexed by $\tilde\Sigma$, then $\lim_{i\to\infty}\pi_K(\alpha_i)=\alpha_\infty$ and $\lim_{i\to\infty}\alpha_i\cdot\bar\alpha_i=-\infty$.*

This result is somewhat hidden in [6]-II, which makes it a bit hard to explicate. The construction of $\mathbb{P}(\widehat{H}_+^\circ)$ involves an extension of $\mathbb{P}(H_+)$ (which we will here denote by $\mathbb{P}(\widehat{H}_+)$, but which is there denoted $\mathbb{D}^{\Sigma(\mathcal{H})}$). We first notice that the linear subspaces which do not meet H_+ are indexed by $\Sigma := \mathcal{K}_2 \cup (\cup_J \Sigma_J)$, so that we can form

$$\widehat{H}_+ = H_+ \sqcup \coprod_{\sigma\in\Sigma} \pi_{K_\sigma}(H_+) \quad \text{and} \qquad \mathbb{P}(\widehat{H}_+) = \mathbb{P}(H_+) \sqcup \coprod_{\sigma\in\Sigma} \mathbb{P}\pi_{K_\sigma}\mathbb{P}(H_+).$$

The topologies are described in [6]-II, p. 566. In case $\mathcal{H}$ is empty, then $\widehat{H}_+$ is the Baily-Borel extension H_+^{bb} of H_+: the orbit space $\Gamma\backslash\mathbb{P}(H_+^{bb})$ is the Baily-Borel compactification of $\Gamma\backslash\mathbb{P}(H_+)$ and the $\mathbb{C}^\times$-bundle $H_+^{bb} \to \mathbb{P}(H_+^{bb})$ is associated to the basic automorphic line bundle: if $\mathcal{L}$ denotes the line bundle over $PP(H_+^{bb})$ obtained as a quotient of $\mathbb{C} \times H_+^{bb}$ by the $\mathbb{C}^\times$ action defined by $\lambda(z, \alpha) = (z\lambda^{-1}, \lambda\alpha)$, then the Γ-automorphic forms of degree l are the continuous Γ-equivariant sections of $\mathcal{L}^{\otimes l}$ that are holomorphic on strata.

There is an evident Γ-equivariant map $\widehat{H}_+ \to H_+^{bb}$. It is such that the resulting map $\Gamma\backslash\mathbb{P}(\widehat{H}_+) \to \Gamma\backslash\mathbb{P}(H_+^{bb})$ is a morphism and involves a modification of the Baily-Borel boundary. This modification has the property that the closure of the image of every $\mathbb{P}(K_+)$, $K \in \mathcal{H}$, is a $\mathbb{Q}$-Cartier divisor. In other words, it can locally be given by a single equation. Notice that the pull-back of the of the basic automorphic line bundle is the line bundle associated to the $\mathbb{C}^\times$-bundle $\widehat{H}_+ \to \mathbb{P}(\widehat{H}_+)$.

The relation between $\mathbb{P}(\widehat{H}_+)$ and $\mathbb{P}(\widehat{H}_+^\circ)$ is as follows: let $\widetilde{\mathbb{P}}(H_+)$ be the blow up of $\mathbb{P}(\widehat{H}_+)$ obtained by blowing up the closures of the linear sections $\mathbb{P}(K_+) \subset \mathbb{P}(H_+)$, $K \in \mathcal{K}_1$, in the order of increasing dimension (where we of course take their strict transforms). Then there is a blowdown $\widetilde{\mathbb{P}}(H_+) \to \mathbb{P}(\widehat{H}_+^\circ)$ obtained as follows: if $K \in \mathcal{K}_1$ is of codimension r, then the exceptional divisor in $\widetilde{\mathbb{P}}(H_+)$ associated to K has a product structure with one factor equal to a modified $\mathbb{P}^r$. We successively blow down onto these factors, starting with the factors of lowest dimension first with indeed $\mathbb{P}(\widehat{H}_+^\circ)$ as the final result. Let us denote by $\hat{\mathcal{L}}$ the line bundle over $\mathbb{P}(\widehat{H}_+^\circ)$ that is associated to $\widehat{H}_+ \to \mathbb{P}(\widehat{H}_+)$ (in the same way as $H_+^{bb} \to \mathbb{P}(H_+^{bb})$ is to $\mathcal{L}$). On the common blowup $\widetilde{\mathbb{P}}(H_+)$, the pull-back of $\hat{\mathcal{L}}$ is the pull-back of $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\sum_{K\in\mathcal{H}})$. This helps us to settle part of the proof of Lemma 4.1.

Proof of Lemma 4.1. Suppose that the K appearing in the lemma is a member of $\mathcal{K}_1$. We regard H_+° as an open subset of the total space of the pull-back of $\mathcal{L}(\mathcal{H})$ to $\widetilde{\mathbb{P}}(H_+)$. The assumption is that the sequence α_i converges in this total space to an element α_∞ that does not lie on the zero section. Notice that the sequence $([\alpha_i] \in \mathbb{P}(H_+))_i$ converges to some $[\alpha] \in \mathbb{P}(K_+)$. This means that if the sequence α_i is regarded as one which lies in the total space of $\mathcal{L}$, then there exist scalars $\lambda_i \in \mathbb{C}$ with $\lambda_i\alpha_i$ converging in H_+ to an element $\alpha \in H_+$ and $\lim_{i\to\infty} \lambda_i = 0$. So $\lim_{i\to\infty} |\lambda_i|^2\alpha_i \cdot \bar{\alpha}_i = \alpha \cdot \bar{\alpha} < 0$ and hence $\lim_{i\to\infty} \alpha_i \cdot \bar{\alpha}_i = -\infty$.

It remaining cases (when K is degenerate) involve a different type of argument and are settled in Lemma 4.2 below. □

Lemma 4.2. *If a sequence $(\alpha_i \in H_+)_i$ converges $\alpha_\infty \in \pi_K(H_+)$, with K a linear subspace indexed by Σ, then $\lim_{i\to\infty} \pi_K(\alpha_i) = \alpha_\infty$ and $\lim_{i\to\infty} \alpha_i \cdot \bar{\alpha}_i = -\infty$.*

Proof. Suppose first that $K = K_J$ for some isotropic plane J. Then $J(\mathbb{R})$ has a natural orientation characterized by the following property: if (e_0, e_1) is an oriented basis for $J(\mathbb{R})$, then for all $\alpha \in H^+$, we have $\mathrm{Im}((\alpha \cdot e_0)(\bar{\alpha} \cdot e_1)) > 0$. Given such a basis (e_0, e_1), we define a one parameter group of orthogonal transformations as follows: for $\tau \in \mathbb{C}$,

$$\psi_\tau : H \to H, \quad \psi_\tau(\alpha) = \alpha + \tau\left((\alpha \cdot e_0)e_1 - (\alpha \cdot e_1)e_0\right).$$

Notice that ψ_τ leaves H/J (and hence H/K) pointwise fixed. A small computation shows that ψ_τ preserves the quadratic form on $\mathbb{H}$ and that

$$\psi_\tau(\alpha) \cdot \overline{\psi_\tau(\alpha)} = \alpha \cdot \bar\alpha - 4\,\mathrm{Im}(\tau)\,\mathrm{Im}\left((\alpha \cdot e_0)(\bar\alpha \cdot e_1)\right).$$

In particular, ψ_τ preserves the domain $\mathbb{H}_+$ when $\mathrm{Im}(\tau) \ge 0$ and if $\alpha \in H_+$, then we have that $\lim_{\mathrm{Im}(\tau)\to\infty} \psi_\tau(\alpha)\cdot\overline{\psi_\tau(\alpha)} = -\infty$. For the topology on $\widehat{H}_+$ the convergence of $(\alpha_i \in H_+)_i$ to α_∞ means that there exist a sequence $(\gamma_i \in \Gamma)_i$ which leave K invariant and fix H/K pointwise and a sequence $(\eta_i \in \mathbb{R})_i$ converging to $+\infty$ such that $(\alpha'_i := \psi^{-1}_{\eta_i\sqrt{-1}}\gamma_i^{-1}\alpha_i)_i$ converges to some $\alpha'_\infty \in H_+$. Then $\alpha_\infty = \pi_K(\alpha'_\infty)$ and

$$\alpha_i \cdot \bar\alpha_i = \psi_{\eta_i\sqrt{-1}}\alpha'_i \cdot \overline{\psi_{\eta_i\sqrt{-1}}\alpha'_i} = \alpha'_i \cdot \bar\alpha'_i - 4\eta_i\,\mathrm{Im}\left((\alpha'_i \cdot e_0)(\alpha'_i \cdot e_1)\right),$$

which indeed tends to $-\infty$ as $i \to \infty$.

We next do the case when K has an associated isotropic line J: $K = K_\sigma$ for some $\sigma \in \Sigma_J$. Let $e \in J(\mathbb{Q})$ be a generator. For $f \in J^\perp$, we put

$$\psi_{e,f} : \alpha \in H \mapsto \alpha + (\alpha \cdot e)f - (\alpha \cdot f) - \tfrac{1}{2}(f \cdot f)(\alpha \cdot e)e \in H$$

This transformation respects the quadratic form on H and we have

$$\psi_{e,f}(\alpha) \cdot \overline{\psi_{e,f}(\alpha)} = \alpha \cdot \alpha + 4|\alpha \cdot e|^2\left(p_e(\alpha) \cdot \mathrm{Im}(f) + \tfrac{1}{2}\,\mathrm{Im}(f) \cdot \mathrm{Im}(f)\right).$$

So if f is such that $\mathrm{Im}(f) \in C_e$, then $\psi_{e,f}$ preserves H_+ and $\psi_{e,f}(\alpha)\cdot\overline{\psi_{e,f}(\alpha)} \to -\infty$ if $\mathrm{Im}(f)$ tends to infinity along a ray in C_e. For the topology on $\widehat{H}_+$ the convergence of $(\alpha_i \in H_+)_i$ to $\alpha_\infty \in \pi_K H_+$ means that there exist a sequence $(\gamma_i \in \Gamma)_i$ which leave σ invariant and fix H/K pointwise and a sequence $(y_i \in \sigma)_i$ converging to a point y_∞ at infinity in the relative interior of the projectivized σ such that $(\alpha'_i := \psi^{-1}_{e,y_i\sqrt{-1}}\gamma_i^{-1}\alpha_i)_i$ converges to some $\alpha'_\infty \in H_+$. Then $\alpha_\infty = \pi_K(\alpha'_\infty)$ and

$$\alpha_i \cdot \bar\alpha_i = \psi_{e,y_i\sqrt{-1}}\alpha'_i \cdot \overline{\psi_{e,y_i\sqrt{-1}}\alpha'_i} = \alpha'_i \cdot \bar\alpha'_i + 4|\alpha'_i \cdot e|^2\left(p_e(\alpha'_i) \cdot y_i + \tfrac{1}{2}y_i \cdot y_i\right)),$$

which tends to $-\infty$ as $i \to \infty$. $\square$

Let for $\sigma \in \tilde\Sigma$, Γ^σ stand for the group of $\gamma \in \Gamma$ that leave the stratum $\pi_{K_\sigma}(H_+)$ of $\widehat{H}_+$ pointwise fixed. This is also the group of $\gamma \in \Gamma$ which fix $K_\sigma^\perp$ pointwise and leave σ invariant. In certain cases these subgroups separate the strata:

Lemma 4.3. *If the form $\alpha \in H \mapsto \alpha\cdot\bar\alpha$ takes on every nonisotropic two-dimensional intersection of members of $\mathcal{H}$ a positive value, then for all $\sigma \in \tilde\Sigma$, $K_\sigma^\perp$ is the fixed point set of Γ^σ in H. In particular, $\pi_{K_\sigma}(H_+)$ is the fixed point set of Γ^σ in $\widehat{H}_+$.*

Proof. Denote by $G_{K^\perp}$ the group of orthogonal transformations of H which leave $K^\perp$ pointwise fixed. Suppose first that $K = K_\sigma$ is of type 1, so of signature $(\dim K - 2, 2)$. Then by definition Γ^σ is the group of $\gamma \in \Gamma$ which leave $K^\perp$ pointwise fixed. The latter is an arithmetic group in in the orthogonal group of K. Since $\dim K \ge 3$, this orthogonal group is not anisotropic. This implies [1] that it contains $\Gamma_{K^\perp}$ as a Zariski dense subgroup.

Next we do the case when K is of type 2. Then again Γ^σ is arithmetic in $G_{K^\perp}$. In particular, it meets the unipotent radical $U(G_{K^\perp})$ of the latter in an arithmetic subgroup. We observe that $U(G_{K^\perp})$ is the group of unipotent orthogonal transformations of H which leave $K^\perp$ pointwise fixed and has $K^\perp$ as its fixed point set in H. Hence this is also the fixed point set of Γ^σ.

Finally, we do the case when $\sigma \in \Sigma_J$, with J a $\mathbb{Q}$-isotropic line. So $K \supset J$ and K/J has hyperbolic signature. Choose a generator e of $J(\mathbb{Q})$ so that we have defined the distinguished quadratic cone $C_e \subset (J^\perp/J)(\mathbb{R})$ and Σ_J is a locally rational polyhedral decomposition of that cone. The span of σ is K/J. Our assumption implies that no member of Σ_J is of dimension one. So the one dimensional rays on the boundary of σ must all be all improper, that is, lie on $\mathbb{Q}$-isotropic lines in C_e. In particular, the collection $\mathcal{L}$ of such lines spans K/J. If $L \in \mathcal{L}$, then its preimage in $\tilde{L}$ in H is a $\mathbb{Q}$-isotropic plane which contains J. Let $e_L \in \tilde{L}(\mathbb{Q})$ be such that (e, e_L) is basis of $\tilde{L}$. Then the transformation

$$\psi_{e,e_L} : H \to H, \quad \psi_L(\alpha) = \alpha + (\alpha \cdot e)e_L - (\alpha \cdot e_L)e.$$

has $\tilde{L}^\perp$ as its fixed point set. It also acts trivially on $J^\perp/J$ and hence does so on K/J. Some power of ψ_{e,e_L} will lie in Γ and so upon replacing e_L be a positive multiple, we may assume that $\psi_{e,e_L} \in \Gamma$. It is clear that then $\psi_{e,e_L} \in \Gamma^\sigma$. Hence the fixed point set of Γ^σ equals $\cap_{L\in\mathcal{L}} \tilde{L}^\perp = (\sum_{L\in\mathcal{L}} \tilde{L})^\perp = K^\perp$. □

We abbreviate

$$M := \Gamma\backslash\mathbb{P}(H_+^\circ), \quad \widehat{M} := \Gamma\backslash\mathbb{P}(\widehat{H}_+^\circ), \quad M_\infty := \widehat{M} - M$$

and denote the inclusion $M \subset \widehat{M}$ by j. The variety $\widehat{M}$ is naturally stratified into orbifolds.

If Γ acts neatly on $\mathbb{P}(H_+^\circ)$ in the sense of Borel, then the strata of $\widehat{M}$ are smooth and $\mathcal{F}$ is a genuine line bundle. Since there is always a subgroup of Γ of finite index which is neat, we will assume that Γ already has that property. Then the trivial local system $\tilde{\mathbb{H}} := H_{\mathbb{P}(H_+^\circ)}$ over $\mathbb{P}(H_+^\circ)$ is in a tautological manner a polarized variation of Hodge structure of weight zero (whose classifying map is the identity). The group Γ acts on it so that we get a variation of Hodge structure $\mathbb{H}$ over M with the line bundle $j^*\mathcal{F} \subset \mathcal{O}_M \otimes \mathbb{H}$ defining its Hodge flag of level 1.

Theorem 4.4. *Assigning to $\sigma \in \tilde{\Sigma}$ the subspace $(H/K_\sigma)^*$ of H^* defines a subsheaf $\mathbb{V} \subset j_*\mathbb{H}$ that is a boundary extension of $\mathbb{H}$. The image of $\mathbb{V} \to j_*(\mathcal{F}^1(\mathbb{H})^*)$ spans a line bundle whose $\mathcal{O}_{\widehat{M}}$-dual can be identified with $\mathcal{F}$.*

If $\dim H \geq 5$ and the hermitian form takes on every two-dimensional intersection of members of $\mathcal{H}$ a positive value, then M_∞ is everywhere of codimension ≥ 2 in $\widehat{M}$ and $\mathbb{V} = j_*\mathbb{H}$.

Proof. A constructible subsheaf $\tilde{\mathbb{V}}$ of the direct image of $H^*_{\mathbb{P}(H_+^\circ)}$ on $\mathbb{P}(\widehat{H}_+^\circ)$ is defined by letting it on the stratum $\mathbb{P}\pi_{K_\sigma}\mathbb{P}(H_+^\circ)$ be constant equal to $(H/K_\sigma)^*$; this is indeed a sheaf because an incidence relation between strata implies the an inclusion relation of subspaces of H^*. Passage to the Γ-quotient then defines constructible subsheaf $\mathbb{V}$ of $j_*\mathbb{H}$, which is clearly defined over $\mathbb{Q}$. The assertion concerning $\mathcal{F}$ is essentially tautological since the restriction of $\mathcal{L}$ to a stratum associated to σ is the image of the $\mathbb{C}^\times$-bundle $\pi_{K_\sigma}(H_+^\circ) \to \mathbb{P}\pi_{K_\sigma}\mathbb{P}(H_+^\circ))$ under formation of the Γ-quotient. It remains to see that the norm on $\mathcal{F}^*$ vanishes on M_∞. This follows from Lemma 4.1.

Suppose now that no intersection of members of $\mathcal{H}$ is a plane on which the form is negative semidefinite. Clearly, the boundary $\widehat{M} - M$ is everywhere of codimension

≥ 2 precisely when there are no strata of dimension $n-1$. This is the case if and only if no member of $\tilde{\Sigma}$ is of dimension 2, or equivalently, if $\dim H \geq 5$ and no nonzero intersection of members of $\mathcal{H}$ is negative semidefinite. The equality $\mathbb{V} = j_*\mathbb{H}$ follows from Lemma 4.3. □

The preceding was also carried out in a ball quotient setting. Things then simplify considerably. This case may arise as a restriction of the case discussed above as follows. Suppose that H comes with a faithful action of the group μ_l of l-th roots of unity with $l \geq 3$, such that if $\chi : \mu_l \subset \mathbb{C}^\times$ is the tautological character, the hermitian form has on the χ-eigen space H_χ hyperbolic signature (i.e., $(1, m)$ for some $m \geq 0$), or equivalently, that the orthogonal complement of the $\mathbb{R}$-subspace $H_\chi + H_{\bar\chi}$ is positive definite.

We observed that H_χ is isotropic for the quadratic form so that intersection $H_{\chi,+} := H_\chi \cap H_+$ is simply the open set of $\alpha \in H_\chi$ with $\alpha \cdot \bar\alpha < 0$. This implies that $\mathbb{P}(H_{\chi,+})$ is a complex m-ball. Now let Γ' be an arithmetic subgroup in the centralizer of μ_l in the orthogonal group of H. Then Γ' acts properly on $\mathbb{P}(H_{\chi,+})$ and with finite covolume.

Let $\mathcal{H}'$ be a collection of μ_l-invariant hyperplanes of H_χ of hyperbolic signature that are obtained by intersecting H_χ with a hyperplane in H defined over $\mathbb{Q}$. We also assume that $\mathcal{H}'$ is a union of a finite number of Γ'-equivalence classes. The open subset $H^\circ_{\chi,+} \subset H_{\chi,+}$ is defined as usual, but the role of $\tilde{\Sigma}$ is now taken by a selfindexed collection $\mathcal{K}' = \mathcal{K}'_1 \cup \mathcal{K}'_2$ (the fact that there are no subspaces of type 3 is the reason that things simplify). Here $\mathcal{K}'_1$ is the collection of intersections of members of $\mathcal{H}'$ of hyperbolic signature and $\mathcal{K}'_2$ is bijectively labeled by the collection of the isotropic lines $J' \subset H_\chi$ for which $J = J' + \bar{J}'$ is defined over $\mathbb{Q}$ (this is then a $\mathbb{Q}$-isotropic plane): for each such J' we let $K_{J'}$ be the intersection of $J^\perp \cap H_\chi$ with the collection of members of $\mathcal{H}'$ which contain J'. From this point onwards one proceeds as in the case considered above. We obtain a projective compactification of $M' := \Gamma' \backslash H_{\chi,+}$, $M' \subset \widehat{M}'$ and an ample extension $\mathcal{F}'$ over $\widehat{M}'$ of the automorphic line bundle over M' and find:

Theorem 4.5. *There is a canonical boundary extension of Hodge type $\mathbb{V}' \subset j'_*\mathbb{H}_\chi$ such that the image of $\mathbb{V}' \to j'_*(\mathcal{F}^1(\mathbb{H})^*)$ spans a line bundle whose $\mathcal{O}_{\widehat{M}'}$-dual can be identified with $\mathcal{F}'$.*

If $\dim H_\chi \geq 3$ and every one-dimensional intersection of members of $\mathcal{H}'$ is positive for the hermitian form, then the boundary $M'_\infty := \widehat{M}' - M'$ is everywhere of codimension ≥ 2 in $\widehat{M}'$ and $\mathbb{V}' = j'_*\mathbb{H}_\chi$.

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