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Yang-Mills Instantons in the Large-N Limit and the AdS/CFT Correspondence

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We examine a certain 16-fermion correlator in $\mathcal{N}=4$ supersymmetric $SU(N)$ gauge theory in 4 dimensions. Generalizing recent $SU(2)$ results of Bianchi, Green, Kovacs and Rossi, we calculate the exact N-dependence of the effective 16-fermion vertex at the 1-instanton level, and find precise agreement in the large-N limit with the prediction of the type IIB superstring on $\mathrm{AdS}_5\times S^5$. This suggests that the string theory prediction for the 1-instanton amplitude considered here is not corrected by higher-order terms in the α' expansion.

In interesting recent work by Maldacena [1] (see also [2] for relevant previous work by other authors), the large-N limit of $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) has been related to the low-energy behavior of type IIB superstrings on $\mathrm{AdS}_5\times S^5$. In the conjectured correspondence, the gauge coupling g and vacuum angle θ of the four-dimensional theory are given by

$$g \;=\; \sqrt{4\pi g_{st}} \;=\; \sqrt{4\pi e^{\phi}}\;, \qquad \theta \;=\; 2\pi c^{(0)} \tag{1}$$

Here g_{st} is the string coupling while $c^{(0)}$ is the expectation value of the Ramond-Ramond scalar of IIB string theory. Also N appears explicitly, through the relation

$$\frac{L^2}{\alpha'} \;=\; \sqrt{g^2\,N} \tag{2}$$

where L is the radius of both the AdS_5 and S^5 factors of the background. One striking consequence of these identifications is that the action of a Yang-Mills instanton in the gauge theory is equated to that of a D-instanton in the string theory. The relation between these two seemingly different types of instantons has been investigated further by Bianchi, Green, Kovacs and Rossi (BGKR) [3]. These authors compared the leading semiclassical contribution of a single Yang-Mills instanton in the $\mathcal{N}=4$ theory with gauge group $SU(2)$ with that of a D-instanton in the IIB theory on $\mathrm{AdS}_5\times S^5$. Specifically, the gauge theory is at its conformal point where all the Higgs VEVs vanish and the instanton is an exact solution of the field equations. BGKR found an interesting correspondence between the moduli-space and zero modes of the two classical configurations and between the resulting contributions to various correlation functions in their respective theories. A particularly attractive aspect of the correspondence is that the scale size of the Yang-Mills instanton is mapped onto the radial position of the D-instanton in AdS_5. This is consistent with the interpretation of Maldacena's conjecture in which the four-dimensional gauge theory lives on the boundary of AdS_5 [4,5].

An obvious puzzle about the results of Ref. [3] is that such an agreement between weakly-coupled gauge theory and the low-energy effective field theory of the IIB string is found for gauge group $SU(N)$ with $N=2$. In contrast, Maldacena's conjecture only predicts such an agreement in the large-N limit.[1] In this letter we generalize to arbitrary

[1] More accurately, as explained below, the low-energy IIB prediction should only hold when g^2N is large. On the other hand, to justify using semiclassical methods we must also have g^2 small. These conditions together certainly require N to be large.

N the $SU(2)$ calculation of the 1-instanton contribution to a sixteen fermion correlator considered in [3]. In the $SU(N)$ SYM theory a single instanton has a total of $8N$ adjoint fermion zero modes and, for $N = 2$, the resulting Grassmann integrations are saturated by the sixteen fermion insertions of this correlator. For $N > 2$ there are additional fermion zero modes which must be lifted in order to obtain a non-zero result. Thus the main technical challenge in generalizing the calculation of BGKR is to account correctly for this lifting. As in several other cases in three [6] and four [7] dimensions, this can be accomplished by determining the Grassmann quadrilinear term in the instanton action. Our final result for the instanton contribution has a complicated algebraic dependence on N. However, in the large-N limit, the dependence on both g^2 and N is exactly that extracted from the superstring by BGKR, and earlier by Banks and Green [8], on the basis of Maldacena's conjecture.

It is important to emphasize that, although we are working in the large-N limit, the agreement we have found is still somewhat mysterious. Maldacena's conjecture relates the ratio of the string length-scale, α', and the radius of curvature of the spacetime, L, to the gauge theory parameters via Eq. (2). This means that the α' expansion of the IIB theory, on which the prediction of Refs. [3] and [8] relies, is only valid in the regime of large g^2N. In particular, it is not obvious that stringy corrections to the low-energy IIB effective action (i.e., higher orders in the α' expansion) can be neglected when comparing to our semiclassical calculation. A similar situation arises for the calculation of an eight-fermion correlator in the three-dimensional theory with sixteen supercharges [9,6], where *weak*-coupling multi-instanton calculations agree exactly with the predictions based on the M(atrix) model of M-theory [9,10] which, strictly speaking, only apply in a *strong*-coupling limit. As emphasized by Banks and Green, in general such an agreement would only be expected if the relevant correlator is constrained by a supersymmetric nonrenormalization theorem. Very recently [11], exactly such a non-renormalization theorem has been proved for eight-fermion terms in the effective action of the three-dimensional theory. Our present results suggest that a similar nonrenormalization theorem should be at work in the four-dimensional context. Specifically, it appears that the prediction for the 16 fermion correlator extracted from the low-energy IIB action in [8,3] is not modified by higher-order stringy corrections. On the other hand, the exact N-dependence of our one-instanton result given below includes an infinite series of $1/N$ corrections which should correspond to quantum corrections on the IIB side.

We first briefly review the type IIB superstring prediction, closely following [3]. At leading order beyond the Einstein-Hilbert term in the derivative expansion, the IIB effective action is expected to contain a totally antisymmetric 16-dilatino effective vertex of the form [12,8]

$$(\alpha')^{-1} \int d^{10}X \sqrt{\det g}\, e^{-\phi/2} f_{16}(\tau,\bar\tau)\, \Lambda^{16} \;+\; \text{H.c.} \tag{3}$$

Here Λ is a complex chiral $SO(9,1)$ spinor, and f_{16} is a certain weight $(12,-12)$ modular form under $SL(2,\mathbb{Z})$. In particular f_{16} has the following weak-coupling expansion:

$$f_{16} \;=\; a_0\, e^{-3\phi/2} + a_1\, e^{\phi/2} + \sum_{k=1}^{\infty} \mathcal{G}_k\, e^{\phi/2} \;, \tag{4}$$

where

$$\mathcal{G}_k \;=\; \Big(\sum_{n|k} \frac{1}{n^2}\Big) \big(ke^{-\phi}\big)^{25/2} \exp\big(-2\pi k(e^{-\phi} + ic^{(0)})\big) \tag{5}$$

neglecting perturbative corrections; the summation in (5) runs over the positive integral divisors of k. Notice that with the conjectured correspondence (1) to the couplings of 4D SYM theory, the expansion (4) has the structure of a semiclassical expansion: the first two terms correspond to the tree and one-loop pieces, respectively (a_0 and a_1 are numerical constants), while the sum on k is interpretable as a sum on Yang-Mills instanton number. In the IIB theory, these terms, which are non-perturbative in the string coupling come from D-instantons.

From this effective vertex one can construct Green's functions for 16 fermions $\Lambda(x_i)$, $1 \le i \le 16$, which live on the boundary of AdS$_5$:

$$\langle\, \Lambda(x_1)\cdots\Lambda(x_{16})\rangle \;\sim\; (\alpha')^{-1}\, e^{-\phi/2} f_{16}\, t_{16} \int \frac{d^4x_0\, d\rho}{\rho^5} \prod_{i=1}^{16} K^F_{7/2}(x_0,\rho;x_i,0) \tag{6}$$

suppressing spinor indices. Here $K^F_{7/2}$ is the bulk-to-boundary propagator for a spin-1/2 Dirac fermion of mass $m = -3/2L$ and scaling dimension $\Delta = 7/2$ [13,4,5,14]:

$$K^F_{7/2}(x_0,\rho;x,0) \;=\; K_4(x_0,\rho;x,0)\,\big(\rho^{1/2}\gamma_5 + \rho^{-1/2}(x_0-x)_n\gamma^n\big) \tag{7}$$

with

$$K_4(x_0,\rho;x,0) \;=\; \frac{\rho^4}{\big(\rho^2 + (x-x_0)^2\big)^4} \tag{8}$$

In these expressions the x_i are 4-dimensional space-time coordinates for the boundary of AdS$_5$ while ρ is the fifth, radial, coordinate; we suppress the coordinates on S^5 as the propagator does not depend on them (save through an overall multiplicative factor which we drop). The quantity t_{16} in Eq. (6) is (in the notation of BGKR) a 16-index antisymmetric invariant tensor which enforces Fermi statistics and ensures, *inter alia*, that precisely 8 factors of $\rho^{1/2}\gamma_5$ and 8 factors of $\rho^{-1/2}\gamma^n$ are picked out in the product over $K^F_{7/2}$.

According to Maldacena's conjecture, the correlator (6) in the IIB theory should correspond to a certain 16-fermion correlator in 4D large-N SYM. The correspondence of operators in the two pictures was established in Refs. [4,5,15,16]. For present purposes, the fermion operator in the 4D SYM picture with the right transformation properties is the gauge invariant composite operator

$$\Lambda^A_\alpha \;=\; \sigma^{mn}{}_\alpha{}^\beta\,\mathrm{Tr}_N(v_{mn}\,\lambda^A_\beta) \tag{9}$$

which is a spin-1/2 fermionic Noether current associated with a particular superconformal transformation. Here v_{mn} is the $SU(N)$ gauge field strength while the λ^A_β are the Weyl gauginos, with the index $A = 1,2,3,4$ labeling the four supersymmetries. The numerical tensor σ^{mn} projects out the self-dual component of the field strength,[2] so that only instantons rather than anti-instantons can contribute.

In Ref. [3], BGKR explicitly compared the form of the first term in the D-instanton expansion of the 16 dilatino correlator (6) with the 1-instanton contribution to a correlator in $\mathcal{N}=4$ $SU(2)$ Yang-Mills theory with 16 insertions of the operator (9). These authors noted that the two correlators agree up to an overall normalization. In particular the integration measure and integrand appearing in Eq. (6) exactly match their counterparts in the gauge theory calculation. As reviewed below, technically this identification is possible because the expression (8) is proportional to the 1-instanton action density:

$$\mathrm{tr}_N(v_{mn})^2\Big|_{\text{1-inst}} \;=\; \frac{96}{g^2}\,K_4\ . \tag{10}$$

As mentioned earlier, for gauge group $SU(2)$ the single $\mathcal{N}=4$ superinstanton contains precisely 16 adjoint fermion zero modes: a supersymmetric plus a superconformal zero mode, each of which is a Weyl 2-spinor, times four supersymmetries. A nonvanishing 16-fermion correlator is therefore obtained by saturating each of the fermion insertions with a distinct such zero mode. In what follows we generalize the calculation of BGKR to $SU(N)$.

[2] We use Wess and Bagger conventions throughout [17].

Since for any value of N the relation (10) still works, the functional similarity between the 16-fermion correlators in the two pictures noted by BGKR continues to hold. However, by correctly accounting for the $8N-16$ lifted fermion modes, we will also extract the overall multiplicative constant C_N which determines the strength of the effective 16-fermion SYM vertex. As shown below, in the large-N limit, C_N scales like $\sqrt{N}$. We identify this behavior with the factor of $1/\alpha'$ in front of the IIB effective vertex (3), using the dictionary (2). The extraction of this overall constant as a function of N is our chief result.

In order to calculate the required 16-fermion correlator in 4D SYM theory, one needs to understand (i) the collective coordinate integration measure, (ii) the instanton action S_{inst}, and (iii) the form of the fermionic insertions (9). Let us discuss each of these, in turn:

(i) The measure. For general topological number k, the collective coordinate integration measure in the $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric cases was derived in Refs. [18-20].[3] The form of these measures is uniquely fixed by supersymmetry, the index theorem, renormalization group decoupling, and cluster decomposition. The collective coordinates used in these expressions are those of the ADHM multi-instanton [21,22], suitably supersymmetrized as explained in Refs. [23,24,20]. In particular the bosonic and adjoint fermionic collective coordinates are encoded in ADHM matrices a and $\mathcal{M}^A$, respectively, where the index A runs over the independent supersymmetries. For gauge group $SU(N)$ and topological number k, a is an $(N+2k)\times 2k$ complex-valued matrix, while $\mathcal{M}^A$ is an $(N+2k)\times k$ matrix of complex Grassmann numbers (see Ref. [20] for a review). The elements of a and $\mathcal{M}^A$ are subject to polynomial constraints and gauge-like invariances which reduce the number of independent collective coordinates to the number required by the index theorem. In the 1-instanton sector these matrices have the simple canonical form [20]:

$$a \;=\; \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \rho & 0 \\ 0 & \rho \\ \multicolumn{2}{c}{-x_0^m\sigma_m} \end{pmatrix} \;, \quad \mathcal{M}^A \;=\; \begin{pmatrix} \mu_1^A \\ \vdots \\ \mu_{N-2}^A \\ 4i\rho\bar\eta^{A1} \\ 4i\rho\bar\eta^{A2} \\ 4\xi_1^A \\ 4\xi_2^A \end{pmatrix} \tag{11}$$

[3] The one specific case of the $\mathcal{N}=4$ k-instanton measure for general $SU(N)$ is not explicitly given in these references, but may be written down by inspection, by generalizing the $SU(2)$ expression (7) of Ref. [19] to $SU(N)$ by the methods of Ref. [20]. When $k=1$, this measure reduces to Eq. (13) below.

We have chosen familiar notation whereby $\rho \in \mathbb{R}$ and $x_0^m \in \mathbb{R}^4$ denote the size and position of the instanton, and ξ^A_α and $\bar\eta^{A\dot\alpha}$ are the supersymmetric and superconformal fermion zero modes, respectively. Equation (11) assumes the canonical 'North pole' embedding of the instanton within $SU(N)$; more generally there is a manifold of equivalent instantons obtained by acting on (11) by group generators Ω in the coset space

$$\Omega \;\in\; \frac{U(N)}{U(N-2)\times U(1)} \tag{12}$$

The complex Grassmann coordinates μ^A_i in Eq. (11) (which do not carry a Weyl spinor index) may be thought of as the superpartners of the coset embedding parameters (12). Together, ξ^A_α, $\bar\eta^A_{\dot\alpha}$, μ^A_i and $\bar\mu^A_i$ constitute $8N$ fermionic collective coordinates, as needed.

In terms of these 1-instanton variables, the correctly normalized $\mathcal{N}=4$ collective coordinate integration measure may be easily deduced from the early literature [25-27]:[4]

$$\begin{aligned}&\frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!}\,\frac{1}{g^{4N}}\int d^4x_0\int\frac{d\rho}{\rho^5}(\mu_0\rho)^{4N}\\ &\times\int\prod_{A=1}^4 d^2\xi^A\,\big(\frac{g^2}{16\pi^2\mu_0}\big)^4\int\prod_{A=1}^4 d^2\bar\eta^A\,\big(\frac{g^2}{32\pi^2\rho^2\mu_0}\big)^4\int\prod_{A=1}^4\prod_{i=1}^{N-2} d\mu^A_i d\bar\mu^A_i\,\big(\frac{g^2}{2\pi^2\mu_0}\big)^{4(N-2)}\end{aligned} \tag{13}$$

As usual in instanton calculations [25], $g = g(\mu_0)$ is evaluated in the Pauli-Villars scheme; since the $\mathcal{N}=4$ model is a finite theory, the subtraction scale μ_0 should, and by inspection does, cancel out of Eq. (13). The overall constant in front (see Eq. (33) of [26]) comes from the volume of the coset space (12); this factor presupposes that the measure will be used to integrate only gauge singlets, as we shall be doing (since all adjoint Higgs VEVs, which pick out special directions in the color space, will be set to zero in the present paper).

(ii) The instanton action. For all instanton numbers k, the $\mathcal{N}=4$ instanton action was derived in Ref. [7], for the gauge group $SU(2)$. Using the methods of Ref. [20], that expression immediately generalizes to $SU(N)$ for arbitrary N. In particular, in the

[4] The normalization factors of ξ^A and $\bar\eta^A$, $g^2/16\pi^2\mu_0$ and $g^2/32\pi^2\rho^2\mu_0$ respectively, used in this measure, agree with Refs. [28,23] but disagree, by factors of two, with Ref. [29] and much of the subsequent literature; see Ref. [28] for a discussion. In the ADHM language, the relative values of the ξ^A, $\bar\eta^A$ and μ^A normalizations used in this measure follow from Eq. (11) together with the fact that the inner product of $\mathcal{M}^A$ matrices is proportional to $\mathrm{Tr}\bar{\mathcal{M}}^A(\mathcal{P}_\infty+1)\mathcal{M}^A$ where the $(N+2k)\times(N+2k)$ diagonal matrix $\mathcal{P}_\infty+1$ has 2's in the first N diagonal entries and 1's in the remaining $2k$ diagonal entries [22,23].

conformal case where all adjoint Higgs VEVs are set to zero, the instanton action has the form

$$S_{\rm inst} \;=\; \frac{8\pi^2 k}{g^2} + S_{\rm quad} \tag{14}$$

Here $S_{\rm quad}$ is a particular fermion quadrilinear term, with one fermion collective coordinate chosen from each of the four gaugino sectors $A = 1, 2, 3, 4$. In the 1-instanton sector this term collapses to[5]

$$S_{\rm quad} \;=\; \frac{\pi^2}{2\rho^2 g^2}\, \epsilon_{ABCD}\, \Lambda_f^N(\mathcal{M}^A, \mathcal{M}^B)\Lambda_f^N(\mathcal{M}^C, \mathcal{M}^D) \tag{15}$$

where

$$\Lambda_f^N(\mathcal{M}^A, \mathcal{M}^B) \;=\; \frac{1}{2\sqrt{2}} \sum_{i=1}^{N-2} \big(\bar\mu_i^A \mu_i^B - \bar\mu_i^B \mu_i^A\big) \tag{16}$$

As we explained in Ref. [7], for all k, in the absence of VEVs, $S_{\rm quad}$ is a supersymmetric invariant quantity that lifts all the fermion zero modes except for the 16 supersymmetric and superconformal modes. For $k = 1$ this latter property is obvious from Eq. (16), which explicitly depends only on the collective coordinates μ_i^A and $\bar\mu_i^A$ and not on ξ_α^A or $\bar\eta_{\dot\alpha}^A$.

(iii) The fermion insertions. As stated earlier, the 16 explicit fermion insertions (9) are needed to saturate the 16 global supersymmetric and superconformal zero modes, which are not otherwise lifted by the action (15). Accordingly we substitute for the gaugino λ_β^A in Eq. (9):

$$\lambda_\beta^A(x) \;=\; -\big(\xi^{A\gamma} - \bar\eta_{\dot\gamma}^A \bar\sigma_m^{\dot\gamma\gamma}\cdot(x^m - x_0^m)\big)\sigma_{\gamma\beta}^{kl}\, v_{kl}(x - x_0) + \cdots \tag{17}$$

as follows from Eqs. (4.3a) and (A.5) of [23]; the dots stand for admixtures of the remaining fermion modes which we can neglect (since these are saturated by $S_{\rm inst}$). The field strengths in Eqs. (9) and (17) are to be evaluated on the instanton. With Eq. (10) together with the identity

$${\rm tr}_N\, v_{mn} v_{kl} \;=\; \frac{1}{3}\,\mathcal{P}^{\rm SD}_{mn,kl}\,{\rm tr}_N (v_{pq})^2\ , \tag{18}$$

where $\mathcal{P}^{\rm SD}_{mn,kl}$ is the projector onto self-dual antisymmetric tensors, Eqs. (9) and (17) reduce to

$$\begin{aligned}\Lambda^{A\gamma}(x) \;&=\; -\big(\xi^{A\gamma} - \bar\eta_{\dot\gamma}^A \bar\sigma_m^{\dot\gamma\gamma}\cdot(x^m - x_0^m)\big){\rm tr}_N\big(v_{pq}(x - x_0)\big)^2 \\ &=\; -\frac{96}{g^2}\,\big(\xi^{A\gamma} - \bar\eta_{\dot\gamma}^A \bar\sigma_m^{\dot\gamma\gamma}\cdot(x^m - x_0^m)\big)K_4(x_0, \rho; x, 0)\end{aligned} \tag{19}$$

[5] See Eqs. (14), (4) and (6) of [7], and Eqs. (3.20), (8.6) and (8.7) of [20].

Putting this all together, we now insert 16 copies of the composite fermion (19), at 16 spacetime points x_i, times $\exp(-S_{\rm inst})$, into the 1-instanton collective coordinate measure (13). Apart from the integrations over the lifted fermion modes $\{\mu_i^A,\bar\mu_i^A\}$ and the N-dependent constant in front of the measure, the resulting expression is identical (up to some corrected factors of 2) to the $SU(2)$ expression examined by BGKR. Our task is precisely to evaluate the N-dependent effect of the lifted modes. On a formal level, for general N, and general topological number k, the integral of $\exp(-S_{\rm quad})$ over the lifted fermion modes is given by the Gauss-Bonnet-Chern theorem, and equals the Euler character of a certain quotient space formed from the charge-k $SU(N)$ ADHM moduli space [6,30]. However, since much less is known about such spaces than about the multi-monopole spaces which govern analogous instanton calculations in 3D [6], here we will adopt an alternative, direct calculational approach. Specializing to $k=1$, we define I_N to be the (unnormalized) contribution of the lifted modes to the correlator:

$$I_N \;=\; \int\prod_{A=1}^{4}\prod_{i=1}^{N-2}d\mu_i^A d\bar\mu_i^A\,e^{-S_{\rm quad}} \tag{20}$$

By explicit computation we find

$$I_3 \;=\; \frac{3\pi^4}{\rho^4 g^4} \tag{21}$$

and we also define $I_2=1$.

To evaluate I_N for general N, it is helpful to rewrite the quadrilinear term $S_{\rm quad}$ as a quadratic form. To this end we introduce six independent auxiliary bosonic variables $\chi_{AB}=-\chi_{BA}$, and substitute into Eq. (20) the integral representation

$$e^{-S_{\rm quad}} \;=\; \frac{i\rho^6 g^6}{2^9\pi^6}\int\prod_{1\le A'<B'\le 4}d\chi_{A'B'}\;\exp\big(\frac{\rho^2 g^2}{32\pi^2}\epsilon_{ABCD}\chi_{AB}\chi_{CD}+\frac12\,\chi_{AB}\Lambda_f^N(\mathcal{M}^A,\mathcal{M}^B)\big) \tag{22}$$

where an appropriate analytic continuation of the integration contours is understood. Our strategy is to perform only the integrations over μ^A_{N-2} and $\bar\mu^A_{N-2}$ in Eq. (20), and thereby relate I_N to I_{N-1}. Accordingly we break out these terms from Λ_f^N :

$$\Lambda_f^N(\mathcal{M}^A,\mathcal{M}^B) \;=\; \Lambda_f^{N-1}(\mathcal{M}^A,\mathcal{M}^B) \;+\; \frac{1}{2\sqrt2}\big(\bar\mu^A_{N-2}\mu^B_{N-2}-\bar\mu^B_{N-2}\mu^A_{N-2}\big) \tag{23}$$

The $\{\mu^A_{N-2},\bar\mu^A_{N-2}\}$ integration in Eqs. (20) and (22) brings down a factor of $\frac{1}{64}\det\chi_{AB}$. Next we exploit the fact that the determinant of an even-dimensional antisymmetric matrix is a perfect square:

$$\det\chi_{AB} \;=\; \big(\frac18\,\epsilon_{ABCD}\chi_{AB}\chi_{CD}\big)^2 \tag{24}$$

Since the right-hand side of Eq. (24) is proportional to the square of the first term in Eq. (22), the result of the $\{\mu^A_{N-2}, \bar\mu^A_{N-2}\}$ integration can be rewritten as a parametric second derivative relating I_N to I_{N-1} :

$$I_N \;=\; \frac{1}{4}\,\pi^4\rho^6 g^6\,\frac{\partial^2}{\partial(\rho^2g^2)^2}\,\big((\rho^2g^2)^{-3}I_{N-1}\,\big) \tag{25}$$

(The insertion of $(\rho^2g^2)^{-3}$ under the parentheses forces the derivatives to act on the exponent of Eq. (22), and not on the prefactor.) This recursion relation, combined with the initial condition (21), gives finally

$$I_N \;=\; \frac{1}{2}\,(2N-2)!\,\big(\frac{\pi^2}{2\rho^2g^2}\big)^{2N-4}\;. \tag{26}$$

Combining Eqs. (13), (19) and (26), we therefore find for the 1-instanton contribution to the 16-fermion correlator in $\mathcal{N}=4$ SYM theory:

$$\begin{aligned}\langle\Lambda^1_{\alpha_1}(x_1)\cdots\Lambda^4_{\alpha_{16}}(x_{16})\rangle \;=\;& C_N\int\frac{d^4x_0d\rho}{\rho^5}\int\prod_{A=1}^4 d^2\xi^A\,d^2\bar\eta^A\\ &\times\,\big(\xi^1_{\alpha_1}-\bar\eta^1_{\dot\gamma}\bar\sigma^{\dot\gamma}_{m\alpha_1}\cdot(x_1^m-x_0^m)\big)K_4(x_0,\rho;x_1,0)\\ &\times\cdots\times\,\big(\xi^4_{\alpha_{16}}-\bar\eta^4_{\dot\gamma}\bar\sigma^{\dot\gamma}_{m\alpha_{16}}\cdot(x_{16}^m-x_0^m)\big)K_4(x_0,\rho;x_{16},0)\end{aligned} \tag{27}$$

where the overall constant C_N is given by

$$C_N \;=\; g^{-24}\,\frac{(2N-2)!}{(N-1)!(N-2)!}\,2^{-2N+57}\,3^{16}\,\pi^{-10}\;. \tag{28}$$

Taking the $N\to\infty$ limit using Stirling's formula gives

$$C_N \;\longrightarrow\; g^{-24}\,\sqrt{N}\,2^{55}\,3^{16}\,\pi^{-21/2}\;. \tag{29}$$

This is in agreement with the IIB prediction, which can be read off Eqs. (1)-(6):

$$(\alpha')^{-1}\,e^{-\phi/2}\,f_{16}\Big|_{\text{1-inst}}\;\sim\; g^{-24}\,\sqrt{N}\;, \tag{30}$$

up to an overall numerical constant. We reiterate that the structural agreement between the IIB and SYM integrals (6) and (27), and between the powers of g in Eqs. (28) and (30), was already noted in the $SU(2)$ analysis of BGKR; the new ingredient here is the nontrivial agreement in the N-dependence between Eqs. (29) and (30).

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