Multi-Instanton Calculus
and the AdS/CFT Correspondence
in $\mathcal{N}{=}4$ Superconformal Field Theory

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We present a self-contained study of ADHM multi-instantons in $SU(N)$ gauge theory, especially the novel interplay with supersymmetry and the large-N limit. We give both field- and string-theoretic derivations of the $\mathcal{N} = 4$ supersymmetric multi-instanton action and collective coordinate integration measure. As a central application, we focus on certain n-point functions G_n, $n = 16$, 8 or 4, in $\mathcal{N} = 4$ $SU(N)$ gauge theory at the conformal point (as well as on related higher-partial-wave correlators); these are correlators in which the 16 exact supersymmetric and superconformal fermion zero modes are saturated. In the large-N limit, for the first time in any 4-dimensional theory, we are able to evaluate all leading-order multi-instanton contributions exactly. We find compelling evidence for Maldacena's conjecture: (1) The large-N k-instanton collective coordinate space has the geometry of a single copy of $AdS_5 \times S^5$. (2) The integration measure on this space includes the partition function of 10-dimensional $\mathcal{N} = 1$ $SU(k)$ gauge theory dimensionally reduced to 0 dimensions, matching the description of D-instantons in Type IIB string theory. (3) In exact agreement with Type IIB string calculations, at the k-instanton level, $G_n = \sqrt{N} \, g^8 \, k^{n-7/2} e^{2\pi i k \tau} \sum_{d|k} d^{-2} \cdot F_n(x_1, \dots, x_n)$, where F_n is identical to a convolution of n bulk-to-boundary supergravity propagators.

Contents

I   Introduction

I.1   Overview of the paper

The $1/N$ expansion proposed by 't Hooft twenty-five years ago [1] continues to offer the tantalizing prospect of a tractable approximation scheme for QCD in which confinement is visible at leading order. Although this prospect is still a long way out of reach for QCD itself, the last year has seen significant progress in understanding the large-N limit in the more controlled context of non-abelian gauge theories with extended supersymmetry. In particular, Maldacena

has conjectured [2] that the large-N limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills with gauge group $SU(N)$, at the conformal point of vanishing Higgs VEVs, is dual to weakly coupled Type IIB superstring theory on an $AdS_5 \times S^5$ background. This provides a realization of the long-anticipated connection between the $1/N$ expansion, which is organized as a sum over Feynman diagrams of different topology, on the one hand, and string perturbation theory, which is a sum over string world-sheets of different topology, on the other. In particular, the gauge coupling g and vacuum angle θ of the four-dimensional theory are given in terms of the string parameters by

$$g \;=\; \sqrt{4\pi g_{st}} \;=\; \sqrt{4\pi e^\phi} \,, \qquad \theta \;=\; 2\pi c^{(0)} \,. \tag{1.1}$$

Here g_{st} is the string coupling while $c^{(0)}$ is the expectation value of the Ramond-Ramond scalar of Type IIB string theory. Also N appears explicitly, through the relation

$$\frac{L^2}{\alpha'} \;=\; \sqrt{g^2\,N} \tag{1.2}$$

where $(\alpha')^{-1}$ is the string tension and L is the radius of both the AdS_5 and S^5 factors of the background.

More generally, there is a precise correspondence between the finite mass string states in an $AdS_5 \times S^5$ background and the gauge-invariant composite operators on the Yang-Mills side, and hence, a proposed equivalence between all correlators in the two theories that are built from these operators [3–8]. In this paper we will focus on a class of such correlators which are known to receive contributions from all orders in D-instantons, on the superstring side, and from all orders in Yang-Mills instantons, on the gauge theory side. Moreover, on the superstring side, these contributions are known to sum to a specific modular form of the complexified coupling constants τ and $\bar\tau$, with

$$\tau \;=\; ie^{-\phi} + c^{(0)} \;\equiv\; \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \,. \tag{1.3}$$

Below we will calculate these contributions from first principles in the $\mathcal{N} = 4$ theory at large N. Specifically, for each topological number k, we will extract the leading semiclassical contribution to these correlators. This is the first time in a four-dimensional theory that the instanton series has been explicitly evaluated to all orders. Our calculation involves a novel interplay between supersymmetry, the large-N limit, and the multi-instanton formalism of Atiyah, Drinfeld, Hitchin and Manin (ADHM) [9]. For each k, we obtain perfect quantitative and qualitative agreement with the supergravity result, although, as described below, there is an outstanding puzzle about the differing regimes of validity of the supergravity and gauge theory calculations. Quantitatively, we precisely recover the kth Taylor coefficient of the predicted modular form. And qualitatively, we discover how the ten-dimensional $AdS_5 \times S^5$ space emerges, in a surprising way, from the four-dimensional picture: this space describes the geometry of the subspace of multi-instanton collective coordinates which dominates the path integral in the

large-N limit. Moreover, the integration measure on this space includes the partition function of ten-dimensional $\mathcal{N} = 1$ supersymmetric $SU(k)$ gauge theory dimensionally reduced to zero dimensions, matching the description of D-instantons in Type IIB string theory. In our view, this multi-faceted agreement constitutes compelling circumstantial evidence in favor of Maldacena's conjecture.

An abbreviated account of our calculation was presented in Ref. [10]. The present paper not only presents the calculational details of Ref. [10] in a self-contained manner, but several additional results as well, described below.

Let us comment further on both sides of the proposed correspondence. Unfortunately, little is known about the string theory on $AdS_5 \times S^5$, even at leading order in perturbation theory, due to the presence of background Ramond-Ramond fields. To make progress, it is necessary to focus on the regime where the radius of curvature of the background is large compared to the string length scale (i.e., small α') in addition to weak coupling (i.e., small g_{st}). In this case the IIB string theory on $AdS_5 \times S^5$ is well approximated by classical IIB supergravity in the same background. On the gauge theory side of the conjecture, per Eqs. (1.1)-(1.2), this regime corresponds to a large-N limit with $g^2 \ll 1$ but $g^2 N \gg 1$. Recall that the 't Hooft coupling $g^2 N$ is the effective expansion parameter of the large-N gauge theory [1]. Thus, via Maldacena's conjecture, classical supergravity yields predictions for the strong-coupling behavior of the four-dimensional theory. Stringy and quantum corrections to classical supergravity correspond to corrections in powers of $(g^2 N)^{-1}$ and g^2, respectively, on the gauge theory side.

On the four-dimensional side, the $\mathcal{N} = 4$ models constitute a particularly interesting class of quantum field theories in their own right. For any value of N and of the Higgs VEVs, they are finite theories, with vanishing chiral anomaly and β-function [11]. They are also the original setting for Olive-Montonen duality [12]; consequently, the spectrum of monopoles and dyons can in some cases be computed exactly [13]. Moreover, in the absence of such VEVs, the symmetry group of the theory is enlarged both at the classical and quantum levels: the model becomes a highly nontrivial superconformal field theory, in an interesting and ill-understood non-abelian Coulomb phase.

Yet apart from the well-studied constraints imposed by superconformal invariance for $n \leq 3$, little is known about n-point functions in this model for $n > 3$, beyond the regime $g^2 N \ll 1$ where they are amenable to standard perturbative and/or semiclassical analysis.[1] Maldacena's conjecture is of special interest to field theory, precisely because (at least for large N) it provides a quantitative means of extrapolating these correlators into a domain where perturbation theory fails. However this also raises an important problem: it is very hard to find quantitative tests of the conjecture against our existing knowledge of the $\mathcal{N} = 4$ theory. One way around this difficulty is to isolate a special class of correlators which are protected against quantum cor-

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[1]The fact that planar diagram corrections correspond to an expansion in $g^2 N$ is true in instanton backgrounds as well as around the perturbative vacuum.

rections by supersymmetry. The results of a weak coupling calculation can then legitimately be extrapolated to strong coupling and compared with the corresponding supergravity predictions.

The simplest examples of such supersymmetric nonrenormalization theorems involve two- and three-point functions of certain chiral primary operators [3, 4, 14–17]; these are highly constrained by the superconformal invariance of the $\mathcal{N} = 4$ theory and apparently are given exactly by their classical free field values. However, experience has shown that many supersymmetric theories contain a family of correlators which, though similarly protected, are far less trivial. Although the details vary from theory to theory, the characteristic (but not sufficient) property of these correlators is that they contain precisely the correct number of fermions to saturate the exact zero modes of an instanton. It is well-known that instanton contributions to the infra-red safe correlators of chiral superfields are protected against further quantum corrections [18–21]. Classic examples of exact results in $\mathcal{N} = 1$ supersymmetry include instanton derivation of the Affleck–Dine–Seiberg superpotential [22, 23] and of the NSVZ beta-function [24]. In theories with eight supercharges (both in $D = 4$ [19, 25, 26] and $D = 3$ [27, 28]), the quantities which are one-loop-exact in each multi-instanton sector are typically four-fermion correlators which come from the two-derivative/four-fermion terms in a low-energy action. For the three-dimensional theory with sixteen supercharges studied in [29, 30] the relevant correlator is related to an eight-fermion term in the effective action for which a non-renormalization theorem has recently been proved [31]. Other relevant examples involve the correlators of chiral operators which are the observables of a topological quantum field theory obtained by twisting the original supersymmetric gauge theory (see [32], in particular, for an application to the $\mathcal{N} = 4$ theory, which may be related to the calculation presented below).

In this paper we will study the analogous correlators in the $\mathcal{N} = 4$ superconformal theory, although, unlike the previous examples, it is not yet known to what extent these correlators are constrained by supersymmetry. In particular we will calculate the leading semiclassical contribution of ADHM multi-instantons to a sixteen-fermion correlator $G_{16}(x_1, \ldots, x_{16})$ in $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory at large N [33, 34] (see also the review [35]). The number of fermion insertions is dictated by the sixteen exact zero modes of the instanton which are protected by the supersymmetric and superconformal invariance of the $\mathcal{N} = 4$ theory with vanishing VEVs (turning on the VEVs reduces this number to eight). In addition, supersymmetry relates the fermionic correlator $G_{16}(x_1, \ldots, x_{16})$ to certain bosonic 8-point and 4-point functions $G_8(x_1, \ldots, x_8)$ and $G_4(x_1, \ldots, x_4)$ [34] in which the instanton zero modes are saturated, respectively, by fermion-bilinear and fermion-quadrilinear parts of bosonic fields.

By synthesizing Maldacena's conjecture with earlier results [36–42] about D-instanton contributions to the IIB effective action, Banks and Green [33] have obtained a closed-form supergravity prediction for the 4-point correlator of the stress-tensor operators in the $\mathcal{N} = 4$ superconformal Yang-Mills theory. Predictions for correlators G_n ($n = 16, 8$ or 4) of related

operators[2] were obtained in [34]. The G_n are expressed in terms of a non-holomorphic modular form $f_n(\tau, \bar{\tau})$, where the modular group $SL(2, \mathbb{Z})$ is the S-duality symmetry group of the Type IIB string theory. Suggestively, the f_n are amenable to a simultaneous Taylor expansion in g^2, $e^{2\pi i\tau}$ and $e^{-2\pi i\bar{\tau}}$. On the supergravity side, this amounts, respectively, to perturbative, D-instanton, and anti-D-instanton effects. The dictionary (1.1) then suggests that (anti-)D-instantons are mapped onto Yang-Mills (anti-)instantons by the correspondence. We stress once again that the supergravity prediction was derived by assuming large g^2N, while the semiclassical instanton calculation presented below is valid only for small g^2N. Despite this caveat, as stated earlier, for each topological number k we find precise accord between our weak-coupling results and the corresponding D-instanton terms in the Banks-Green prediction. Above, we asserted that this agreement for the G_n (which extends to the whole tower of Kaluza-Klein states of supergravity compactified on S^5, see Sec. VI.2) constitutes convincing evidence in favor of Maldacena's conjecture. In addition—though this is admittedly circular reasoning—it can be seen as equally compelling circumstantial evidence that a supersymmetric nonrenormalization theorem does, in fact, apply to these correlators too.

This paper is organized as follows. In the remainder of this introductory section, we outline the supergravity prediction for the G_n (Sec. I.2) [33, 34]. In particular, a Taylor expansion of the exact solution reveals a surprising and under-appreciated aspect of the D-instanton moduli space; namely that at any topological level k, it effectively contains only one copy—and not k copies—of $AdS_5 \times S^5$. This is contrary to what one usually expects from the physics of D-branes: the collective dynamics of a multi-charged configuration of D-branes is described by a non-abelian $U(k)$ Yang-Mills theory on the brane world-volume. The Coulomb branch of this gauge theory describes the freedom for the branes to separate from one another. However, it is more appropriate to interpret the D-instanton contribution to the correlation functions G_n, as being due to a charge k D-instanton "bound state". This is due to the surprising fact that the integrals over the coordinates, which one would ordinarily identify as the relative positions of the singly-charged D-objects, are actually convergent [37–39]. This fact will be seen to play a similarly crucial rôle on the Yang-Mills side of the story, as we elucidate in Sec. V. In Sec. I.3, we review the corresponding Yang-Mills calculation at the one-instanton level, and show how it matches the supergravity result, both in its space-time dependence [34], and in the overall coupling strength $(\alpha')^{-1} \sim \sqrt{N}$ [44]. Section I ends with a puzzle: naïvely, this agreement appears to be an accident of the one-instanton level, for only at this level does an instanton field-strength squared resemble a Euclidean supergravity bulk-to-boundary propagator. Indeed, even if a dilute instanton gas approximation were justified—and generally it is not—the k-instanton Yang-Mills calculation would naturally lead to k copies of AdS_5 (i.e., k independent instanton 4-positions and scale sizes) rather than just one copy as the supergravity side of the correspondence implies.

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[2]These operators as well as the stress-tensor operator correspond to different components of the Noether current superfield associated with the superconformal and chiral $SU(4)$ transformations of the $\mathcal{N} = 4$ theory written down in [34, 43].

Motivated in part by this puzzle, in Sec. II we first define what we mean precisely by the multi-instanton supermultiplet in the superconformal case and then give a thorough, self-contained review of multi-instanton calculus in supersymmetric $SU(N)$ gauge theory. Most of this review material is borrowed from Ref. [45] (written in collaboration with M. Slater) and is included herein for the reader's convenience. Sections II.1-II.4 are devoted to the ADHM construction of the general self-dual gauge field configuration $v_n(x)$ of arbitrary topological number k [9]. Subsequently we flesh out the remaining components of the $\mathcal{N} = 4$ instanton supermultiplet: Secs. II.5-II.6 review the construction of the four gauginos $\lambda_\alpha^A(x)$ ($A = 1, 2, 3, 4$) [46], and Sec. II.7 reviews the construction of the six adjoint scalars $A^{AB}(x)$ [26, 45, 47].[3]

Semiclassical physics requires knowledge, not only of the relevant saddle-point configurations that dominate a given process (in this case, the multi-instanton supermultiplet of Sec. II), but also, of how properly to weight these configurations in the path integral. Specifically, one needs to construct the weighting factor

$$d\mu_{\mathrm{phys}}^k e^{-S_{\mathrm{inst}}^k} \ , \tag{1.4}$$

where S_{inst}^k is the k-instanton action, and $d\mu_{\mathrm{phys}}^k$ stands for the k-instanton bosonic and fermionic collective coordinate measure. These two quantities are studied in detail in Secs. III and IV, respectively. S_{inst}^k has a crucial role to play: it is responsible for lifting the non-exact fermion zero modes, thereby ensuring a nonzero result for the Grassmann integrations. For the gauge group $SU(N)$ with $\mathcal{N} = 4$ supersymmetry, the k-instanton ADHM configuration has $8kN$ adjoint fermion zero modes. As mentioned earlier, precisely sixteen of these modes are exact; these are saturated by the explicit insertions of the external operators in the correlators G_n. The remaining $8kN - 16$ lifted modes must be saturated instead by bringing down the appropriate power of S_{inst}^k. In previous work [48], we constructed S_{inst}^k in the $\mathcal{N} = 4$ model in a somewhat indirect way, first by implementing the supersymmetry algebra directly on the multi-instanton collective coordinates, and then by requiring that S_{inst}^k be a supersymmetric invariant quantity (among other properties). In Sec. III, we present an alternative construction of S_{inst}^k which is much more direct, albeit calculationally intensive, and leads to the same result. In brief, the k-instanton supermultiplet is inserted into the component Lagrangian, and the space-time integrations are carried out explicitly using Gauss's law. In the absence of Higgs VEVs, we find that S_{inst}^k is a pure fermion quadrilinear term (see Eqs. (3.1)-(3.2) below), with one fermion collective coordinate drawn from each of the four gaugino sectors $A = 1, 2, 3, 4$.

Section IV is a detailed multi-purpose study of the k-instanton collective coordinate measure, $d\mu_{\mathrm{phys}}^k$. Section IV.1 reviews the construction of this measure given previously in Ref. [49]; the requirements of $\mathcal{N} = 4$ supersymmetry, cluster decomposition, and renormalization group flow to the measures of lower supersymmetry (among other constraints) suffice to fix its form uniquely. The remainder of Sec. IV is completely new. In Sec. IV.2, we offer an alternative construction of $d\mu_{\mathrm{phys}}^k \exp -S_{\mathrm{inst}}^k$ which is also directly relevant to the physics of the anti-

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[3]In our convention, upper (lower) $SU(4)$ indices indicate a **4** (**4̄**).

deSitter/superconformal field theory (AdS/CFT) correspondence. Specifically, we consider $\mathcal{N} = 4$ supersymmetric $U(N)$ gauge theory as realized on a set of N parallel D3-branes. Of course, this is the starting point for the analysis leading to Maldacena's conjecture. Here, however, we will not take the large-N supergravity limit which leads to the strong-coupled $\mathcal{N} = 4$ theory, but rather stay at weak coupling and take the ordinary decoupling limit $\alpha' \to 0$ while keeping g^2 and N fixed. In fact it is well known that D-instantons located on the D3-branes are equivalent to Yang-Mills instantons in the world-volume gauge theory [50, 51]. In a similar context, Witten has shown that the ADHM construction emerges in a natural way from standard facts about D-branes [52]. This viewpoint turns out to be quite powerful and, in particular, it provides a simple rederivation of the ADHM formalism developed in our previous papers. While this work was being completed, a paper which has some overlap with Sec. IV.2 has appeared [53].

The form of the measure derived in Secs. IV.1-IV.2 can be termed the "flat measure," as the ADHM collective coordinates are integrated over as Cartesian variables. However the number of such integration variables grows with N and one must choose a more appropriate set of coordinates before a traditional saddle-point approximation can be undertaken. To this end, we transform to a smaller, gauge-invariant set of collective coordinates, which is done in Sec. IV.3. In the resulting gauge-invariant measure (Eq. (4.55) below), the number of integration variables is independent of N, and only grows with the topological number k. This form of the measure is then the appropriate starting point for the large-N saddle-point treatment that we develop beginning in Sec. V.

Section V has the largest overlap with the preliminary version of our paper, Ref. [10]. It is devoted to marrying the large-N limit, on the one hand, with supersymmetric ADHM calculus on the other. As a warm-up, in Sec. V.1 we revisit the one-instanton sector already discussed in Sec. I.3, and develop a formal saddle-point approximation to the gauge-invariant measure as $N \to \infty$. To leading order in $1/N$, the bosonic collective coordinate integration turns out to be dominated by a ten-dimensional submanifold with the geometry of $AdS_5 \times S^5$. Thus we find the intriguing result that, even at weak coupling, the large-N instanton "sees" the metric of the supergravity background. The correspondence between the instanton scale-size ρ and the radial direction on AdS_5 has been previously noted in [34, 54]. Indeed the power of ρ in the factor $d^4X \, d\rho \, \rho^{-5}$ which enters the measure (with X_n the instanton 4-position) is forced by the scale invariance of the $\mathcal{N} = 4$ model, and necessarily agrees with the volume form of the conformally invariant space AdS_5. Rather, the main novelty in our analysis is the explanation of the S^5: it arises from a scalar field χ_a transforming in the vector representation of the $SO(6)$ R-symmetry which is introduced in order to bilinearize the fermion quadrilinear action S_{inst}^k. In particular, as $N \to \infty$, the χ_a variables are increasingly peaked around an S^5 subspace of $\mathbb{R}^6$. The use of such auxiliary Gaussian variables is familiar from the standard large-N treatment of theories with fermions in the fundamental representation of $SU(N)$, such as the Gross-Neveu model [55]. It is remarkable that they are not just mathematical artifacts,

but play such a central rôle in the AdS/CFT correspondence. In fact, as explained in Section IV.2, these auxiliary fields have their origin as the gauge fields of a six-dimensional $U(k)$ gauge theory whose dimensional reduction describes k D-instantons in the presence of D3 branes.

Sections V.2-V.3 extend the saddle-point method to the k-instanton case. In Sec. V.2 we solve the coupled saddle-point equations at the strictly classical level, i.e., neglecting (unjustifiably) the prefactor contribution due to small fluctuations. In this simple approximation scheme, we find that the k instantons live in mutually commuting $SU(2)$ subgroups of the $SU(N)$ gauge group, and are described by k independent copies of $AdS_5 \times S^5$. This is the puzzle discussed earlier, and it has to be reconciled with the supergravity side of the correspondence. The puzzle is resolved in Sec. V.3, in which the small fluctuations are carefully taken into account. These generate a singular attractive effective potential which draws the k instantons to a common point, thereby reducing the moduli space to a *single* copy of $AdS_5 \times S^5$, precisely as required by the AdS/CFT correspondence. (However the k instantons remain in mutually commuting $SU(2)$'s.) And there is a further surprise: the Lagrangian which describes small fluctuations in the vicinity of this reduced moduli space turns out to be identical to ten-dimensional $\mathcal{N} = 1$ supersymmetric $SU(k)$ Yang-Mills theory, dimensionally reduced to $0 + 0$ dimensions. The collective coordinate k-instanton-measure thus factorizes into the measure on $AdS_5 \times S^5$ times the partition function $\hat{\mathcal{Z}}_k$ for this $SU(k)$ theory. Thus, the k-instanton semiclassical measure, $\int d\mu_{\text{phys}}^k \exp - S_{\text{inst}}^k$, in the $\mathcal{N} = 4$ Yang-Mills theory is reduced to the partition function of the multi-D-instanton matrix model. We then use the recent explicit evaluation of $\hat{\mathcal{Z}}_k$ [56–58] to complete the specification of the effective large-N k-instanton measure; the final expression is given in Eq. (5.45). Intriguingly, this matrix model is precisely the one posited by Ref. [59] as a definition of Type IIB string theory.

In Sec. V.4 we comment further on $\hat{\mathcal{Z}}_k$. In particular, it is remarkable that $\hat{\mathcal{Z}}_k$ is precisely the partition function that describes k-D-instantons in *flat* space [37–39]. It is also noteworthy how, having started with an $SU(N)$ gauge theory, one ends up with an $SU(k)$ gauge theory; this type of "duality" is well known from a string theory context [50–52], and is apparent from our discussion in Sec. IV.2. We also discuss the expected relation between $\hat{\mathcal{Z}}_k$ and the Euler character of the charge-k ADHM moduli space.

In Sec. VI we use this large-N effective measure to evaluate the correlators of interest. Section VI.1 is devoted to the correlators G_n, with $n = 16$, 8 or 4; as advertised above, we find exact agreement with the supergravity prediction of [34]. However the n insertions in these correlators all correspond, on the supergravity side, to fields which have no dependence on S^5, i.e. which are "s-wave". As such, they do not probe this part of the string theory background. Accordingly, in Sec. VI.2 we extend the analysis to correlators corresponding to supergravity fields which are higher partial waves on S^5, i.e. massive Kaluza-Klein modes on S^5, and verify the proposed AdS/CFT correspondence for a tower of such operators. This analysis highlights the novel rôle played by the auxiliary scalar variables χ_a in our formalism.

Section VII contains some concluding thoughts. Section VII.1 gives a slanted historical overview of the ADHM multi-instanton program: largely stymied in nonsupersymmetric applications due to the intractability of the nonlinear constraints and the lack of knowledge of the measure; recently rejuvenated in the context of supersymmetric theories; and, as the present paper demonstrates, simplified even more dramatically in the large-N limit. And Sec. VII.2 briefly discusses the source of both $1/N$ corrections and (despite the aforementioned nonrenormalization theorem) g^2 corrections to the Yang-Mills calculations.

I.2 Review of the superstring prediction

Before we describe in detail the superstring prediction, some general comments are in order. As we have alluded to in the last section, Maldacena's conjecture [2] relates Type IIB supergravity on $AdS_5 \times S^5$ to the large-N limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. The refinement of this correspondence presented in [3,4] identifies the $\mathcal{N} = 4$ theory as living on the four-dimensional boundary of AdS_5. In particular, each chiral primary operator $\mathcal{O}$ in the boundary conformal field theory is identified with a particular Kaluza-Klein mode of the supergravity fields which we denote as $\Phi_{\mathcal{O}}$. The generating function for the correlation functions of $\mathcal{O}$ is then given in terms of the supergravity action $S_{\mathrm{IIB}}[\phi_{\mathcal{O}}]$ according to

$$\left\langle \exp \int d^4x \, J_{\mathcal{O}}(x)\mathcal{O}(x) \right\rangle = \exp -S_{\mathrm{IIB}}\left[\Phi_{\mathcal{O}}; J\right] . \tag{1.5}$$

The IIB action on the right-hand side of the equation is evaluated on a configuration which solves the classical field equations subject to the condition $\Phi_{\mathcal{O}}(x) = J(x)$ on the four-dimensional boundary.

In most applications considered so far, the relation (1.5) has primarily been applied at the level of classical supergravity, which corresponds to $N \to \infty$, with $g^2 N$ fixed and large, in the boundary theory [3,4,14,15,60–69]. However, the full equivalence of IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory conjectured in [2] suggests that (1.5) should hold more generally, with quantum and stringy corrections to the classical supergravity action corresponding to g^2 and $1/g^2 N$ corrections in the $\mathcal{N} = 4$ theory, respectively. The particular comparison that concerns us here is between Yang-Mills instanton contributions to the correlators of $\mathcal{O}$ generated by the left-hand side of (1.5) and D-instanton corrections to the IIB effective action on the right-hand side. For future reference, the standard metric on $AdS_5 \times S^5$, in coordinates $x_\mu = (X_n, \rho, \hat{\Omega})$, is given as,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{\rho^2}\left(dX_n dX_n + d\rho^2\right) + d\hat{\Omega}^2 . \tag{1.6}$$

Here, $\hat{\Omega}$ is a unit $SO(6)$ vector denoting a point on the unit 5-sphere. The four-dimensional theory in question lives on the boundary of AdS_5 which is located at $\rho = 0$.

The relevant D-instanton contributions arise as $(\alpha')^3$ corrections [33] to the classical IIB theory on $AdS_5 \times S^5$. In particular, as in more conventional field theory settings, the corrections take the form of instanton-induced vertices in a local effective action. The D-instanton contribution to an n-point correlator G_n comes from a tree level Feynman diagram with one vertex, located at some point $(X_n, \rho, \hat{\Omega})$ in the bulk of $AdS_5 \times S^5$. The diagram also has n external legs corresponding to operator insertions on the boundary. The position-space Feynman rules associate an overall constant with the vertex and a bulk-to-boundary propagator, to be described below, with each external leg [3, 4, 61]. The selection rule which specifies the necessary fermionic insertions is conveniently encoded by integrating over Grassmann variables ε which correspond to the sixteen supersymmetries of the IIB theory which are broken by the D-instanton. These variables are associated to the sixteen supersymmetric and superconformal zero modes of an instanton configuration on the Yang-Mills side. The final amplitude is then obtained by integrating over the position of the vertex on $AdS_5 \times S^5$. Hence the combined bosonic and fermionic integration measure for the vertex is ,

$$\int \sqrt{g}\, d^{10}x\, d^{16}\varepsilon = \int \frac{d^4X\, d\rho}{\rho^5}\, d^5\hat{\Omega}\, d^{16}\varepsilon\,. \tag{1.7}$$

For instance, the bulk-to-boundary propagator for an $SO(6)$ singlet scalar free field of mass m on AdS_5 is,

$$K_\Delta(X, \rho; x, 0) = \frac{\rho^\Delta}{(\rho^2 + (x - X)^2)^\Delta}\,, \tag{1.8}$$

where $(mL)^2 = \Delta(\Delta - 4)$.

We now turn in detail to the IIB superstring prediction [33] (closely following the more detailed treatment in [34]). In [36], Green and Gutperle conjectured an exact form for certain non-perturbative (in g_{st}) corrections to certain terms in the Type IIB supergravity effective action. In the present application, where the string theory is compactified on $AdS_5 \times S^5$, it is important for the overall consistency of the Banks-Green prediction that the non-perturbative terms in the effective action do not alter the $AdS_5 \times S^5$ background, since the latter is conformally flat [33]. In particular, at leading order beyond the Einstein-Hilbert term in the derivative expansion, the IIB effective action is expected to contain a totally antisymmetric 16-dilatino effective vertex of the form [40, 42]

$$(\alpha')^{-1} \int d^{10}x\, \sqrt{\det g}\, e^{-\phi/2}\, f_{16}(\tau, \bar{\tau})\, \Lambda^{16}\ +\ \text{H.c.} \tag{1.9}$$

Here Λ is a complex chiral $SO(9, 1)$ spinor, and f_{16} is a certain weight $(12, -12)$ modular form under $SL(2, \mathbb{Z})$. At the same order in the derivative expansion there are other terms related to (1.9) by supersymmetry and involving other modular forms [40–42] and, in particular, $f_n(\tau, \bar{\tau})$, with $n = 8$ and 4. This modular symmetry is precisely S-duality of the Type IIB superstring, and although this does not completely determine the modular forms f_n, for the $n = 4$ term

Green and Gutperle [36] were strongly attracted to the following conjecture, later proved in Ref. [70] and generalized to $n \neq 4$ in [41, 42]:

$$f_n(\tau, \bar{\tau}) = (\text{Im}\tau)^{3/2} \sum_{(p,q)\neq(0,0)} (p + q\bar{\tau})^{n-11/2}(p + q\tau)^{-n+5/2}. \tag{1.10}$$

These rather arcane expressions turn out to have the right modular properties, i.e. weight $(n-4, -n+4)$, and also have very suggestive weak coupling expansions [36, 37, 40, 42]:

$$e^{-\phi/2}f_n = 32\pi^2\zeta(3)g^{-4} - \frac{2\pi^2}{3(9-2n)(7-2n)} + \sum_{k=1}^{\infty} \mathcal{G}_{k,n} , \tag{1.11}$$

where

$$\mathcal{G}_{k,n} = \left(\frac{8\pi^2 k}{g^2}\right)^{n-7/2} \left(\sum_{d|k} \frac{1}{d^2}\right) \left[e^{-(8\pi^2/g^2-i\theta)k} \sum_{j=0}^{\infty} c_{4-n,j-n+4} \left(\frac{g^2}{8\pi^2 k}\right)^j \right.$$
$$\left. + e^{-(8\pi^2/g^2+i\theta)k} \sum_{j=0}^{\infty} c_{n-4,j+n-4} \left(\frac{g^2}{8\pi^2 k}\right)^{j+2n-8} \right], \tag{1.12}$$

and the numerical coefficients are

$$c_{n,r} = \frac{(-1)^n\sqrt{8\pi}\,\Gamma(3/2)\Gamma(r-1/2)}{2^r\Gamma(r-n+1)\Gamma(n+3/2)\Gamma(-r-1/2)} . \tag{1.13}$$

The summation over d in (1.12) runs over the positive integral divisors of k. Notice that, having taken into account the conjectured correspondence (1.1) to the couplings of four-dimensional Yang-Mills theory, the expansion (1.11) has the structure of a semiclassical expansion: the first two terms correspond to the tree and one-loop pieces, while the sum on k is interpretable as a sum on Yang-Mills instanton number, the first and second terms in the square bracket being instantons and anti-instantons, respectively (this is dictated by the θ dependence). Each of these terms includes a perturbative expansion around the instantons, although notice that the leading order anti-instanton contributions are suppressed by a factor of g^{4n-16} (so not suppressed for $n = 4$) relative to the leading order instanton contributions.

In the present paper, our focus will be on the leading semiclassical contributions to the f_n; by this we mean, for each value of the topological number k, the leading-order contribution in g^2. For f_{16} and f_8 the leading semiclassical contributions come from instantons only and have the form

$$e^{-\phi/2}f_n\Big|_{k\text{-instanton}} = \text{const} \cdot \left(\frac{k}{g^2}\right)^{n-7/2} e^{2\pi i k\tau} \sum_{d|k} \frac{1}{d^2} , \tag{1.14}$$

neglecting g^2 corrections. For the special case of f_4 there is an identical anti-instanton contribution with $i\tau \to -i\bar{\tau}$. Later, we shall find it very significant that $e^{-\phi/2}f_n$ has a dependence on the instanton number of the form $k^{n-7/2}$. In Type IIB superstring theory, the terms in the

13

square bracket in (1.12), which are non-perturbative in the string coupling, are interpreted as being due to D-instantons.

From the effective vertex (1.9) one can construct Green's functions $G_{16}(x_1, \ldots, x_{16})$ for sixteen dilatinos $\Lambda(x_i)$, $1 \leq i \leq 16$, which live on the boundary of AdS_5:

$$G_{16} = \langle \Lambda(x_1) \cdots \Lambda(x_{16}) \rangle \sim (\alpha')^{-1} e^{-\phi/2} f_{16} t_{16} \int \frac{d^4X\, d\rho}{\rho^5} \prod_{i=1}^{16} K_{7/2}^F(X, \rho; x_i, 0) \qquad (1.15)$$

suppressing spinor indices. Here $K_{7/2}^F$ is the bulk-to-boundary propagator for a spin-$\frac{1}{2}$ Dirac fermion of mass $m = -\frac{3}{2}L^{-1}$ and scaling dimension $\Delta = \frac{7}{2}$ [3, 4, 61, 71]:

$$K_{7/2}^F(X, \rho; x, 0) = K_4(X, \rho; x, 0) \left(\rho^{1/2} \gamma_5 - \rho^{-1/2}(x - X)_n \gamma^n \right) \qquad (1.16)$$

with

$$K_4(X, \rho; x, 0) = \frac{\rho^4}{\left(\rho^2 + (x - X)^2 \right)^4} \,. \qquad (1.17)$$

In these expressions the x_i are four-dimensional space-time coordinates for the boundary of AdS_5 while ρ is the fifth, radial, coordinate. The quantity t_{16} in Eq. (1.15) is (in the notation of Ref. [34]) a 16-index antisymmetric invariant tensor which enforces Fermi statistics and ensures, *inter alia*, that precisely 8 factors of $\rho^{1/2} \gamma_5$ and 8 factors of $\rho^{-1/2} \gamma^n$ are picked out in the product over $K_{7/2}^F$. Related by supersymmetry to G_{16}, are correlation functions G_8 and G_4 which have an analogous structure to (1.15), but involving different bulk-to-boundary propagators and with overall factors that we can summarize as

$$G_n \propto (\alpha')^{-1} e^{-\phi/2} f_n(\tau, \bar{\tau}) \,. \qquad (1.18)$$

Notice that there is no explicit dependence in these expressions on the coordinates on S^5; in particular, the propagator does not depend on them (save through an overall multiplicative factor which we drop). This is because G_{16}, G_8 and G_4 are correlators of operators whose supergravity associates are constant on S^5. In Sec. VI.2 below, after we have elucidated the meaning of the S^5 in the four-dimensional picture, we will broaden the discussion to include operators that correspond to fields on the supergravity side that vary over S^5, and verify that the correct dependence on these coordinates ensues.

Notice further that in the expression (1.15), the five-dimensional space-time dependence does not mix in any way with the τ dependence, where the latter is carried solely by the modular form f_{16}. In particular this means that for arbitrary topological charge k, the D-instanton contribution to G_{16} only contains a single copy—and not k copies—of AdS_5, as evinced by the single copy of the $d^4X\, d\rho\, \rho^{-5}$ volume form in Eq. (1.15). The rational for this, as we alluded to earlier, is that when only 16 fermion zero-modes are saturated, as is apparent in the term in the effective action (1.9), it is more appropriate to think of a configuration of k D-instantons

as a single charged k bound state since the integrals over the relative positions are actually convergent [37–39]. However, it will be an interesting challenge (successfully resolved in Sec. V.3 below) to reproduce this feature from the Yang-Mills side, rather than k distinct copies which one would normally expect from a dilute instanton gas approximation (if it could be justified). Beyond this qualitative agreement, it is the principal goal of this paper to demonstrate that the identical expression for G_{16} emerges from the gauge theory side of the correspondence, to leading semiclassical order (meaning, for each topological number k, to leading order in g^2). But before developing the ADHM machinery necessary to treat the case of general k, let us review the rather simpler situation for $k = 1$.

I.3   Review of the Yang-Mills calculation at the one-instanton level

According to Maldacena's conjecture, the correlator (1.15) in the IIB theory should correspond to a certain 16-fermion correlator in four-dimensional large-N supersymmetric Yang-Mills theory. The fermion operator in the four-dimensional Yang-Mills picture with the right transformation properties to correspond to the dilatino is the gauge-invariant composite operator [3, 4, 34]

$$\Lambda^A_\alpha \; = \; g^{-2}\sigma^{mn}{}_\alpha{}^\beta \, \mathrm{tr}_N \, v_{mn} \lambda^A_\beta \;, \tag{1.19}$$

which is a spin-1/2 fermionic component of the superfield Noether current detailed in [34, 43]. Here v_{mn} is the $SU(N)$ gauge field strength while the λ^A_β are the Weyl gauginos, with the index $A = 1, 2, 3, 4$ labeling the four supersymmetries. The numerical tensor σ^{mn} projects out the self-dual component of the field strength,[4] so that, at leading order, only instantons rather than anti-instantons can contribute.

To begin to see how, in the Yang-Mills picture, a 16-point correlator of the operator (1.19) might possibly reproduce the structure of Eq. (1.15), let us recall the suggestive observation made by [34]. These authors focused on the one-instanton sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(2)$. In this particularly transparent case, the single instanton contains precisely sixteen adjoint fermion zero modes: a supersymmetric plus a superconformal zero mode, each of which is a Weyl 2-spinor, times four supersymmetries. A non-vanishing 16-fermion correlator is therefore obtained by saturating each of the fermion insertions with a distinct such zero mode. The semiclassical Yang-Mills calculation then proceeds by replacing v_{mn} in the composite operator (1.19) by the standard one-instanton field strength $v_{mn}(x - X)$, with X the center of the instanton; likewise for λ^A_β one substitutes into Eq. (1.19) the well-known expression for the gaugino:[5]

$$\lambda^A_\beta(x) \; = \; -\big(\xi^A_\alpha - \sigma^n_{\alpha\dot\alpha}\bar\eta^{\dot\alpha A} \cdot (x - X)_n\big)\sigma^{kl}{}_\beta{}^\alpha \, v_{kl}(x - X) \;. \tag{1.20}$$

---

[4]We use Wess and Bagger conventions throughout [72], continued to Euclidean space via $(x^0, \vec{x}) \to (x^0, i\vec{x})$, with $\eta^{nm}a_n a_m \to -a_n a_n$, except that our convention for integrating Weyl spinors is $\int d^2\lambda \, \lambda^2 = 2$, rather than Wess and Bagger's 1.

[5]See for instance Eqs. (4.3a) and (A.5) of [26].

Here $\xi^{\alpha A}$ and $\bar{\eta}_{\dot{\alpha}}^A$ are Grassmann parameters that multiply the supersymmetric and supercon-formal zero modes, respectively. Using the fact that

$$\text{tr}_N\, v_{mn} v_{kl}\Big|_{\text{1-inst}} = \frac{1}{3}\, \mathcal{P}_{mn,kl}^{\text{SD}}\, \text{tr}_N\, v_{pq}^2\,\Big|_{\text{1-inst}} = 32 \mathcal{P}_{mn,kl}^{\text{SD}}\, K_4(X,\rho;x,0)\ , \tag{1.21}$$

where

$$\mathcal{P}_{mn,kl}^{\text{SD}} = \tfrac{1}{4}(\delta_{mk}\delta_{nl} - \delta_{ml}\delta_{nk} + \epsilon_{mnkl}) \tag{1.22}$$

is the projector onto self-dual antisymmetric tensors, the composite fermion (1.19) becomes

$$\Lambda_\alpha^A(x)\Big|_{\text{1-inst}} = -\frac{96}{g^2}\left(\xi_\alpha^A - \sigma_{\alpha\dot{\alpha}}^n \bar{\eta}^{\dot{\alpha}A}\cdot(x-X)_n\right) K_4(X,\rho;x,0)\ . \tag{1.23}$$

We therefore find for the one-instanton contribution to the 16-fermion correlator in $\mathcal{N}=4$ $SU(2)$ gauge theory [34]:

$$\begin{aligned}
\langle\Lambda_{\alpha_1}^1(x_1)\cdots\Lambda_{\alpha_{16}}^4(x_{16})\rangle\Big|_{\text{1-inst}} &= C_2\, g^{-24}\, e^{2\pi i\tau} \int\frac{d^4X\,d\rho}{\rho^5} \prod_{A=1,2,3,4} d^2\xi^A\, d^2\bar{\eta}^A \\
&\times\left(\xi_{\alpha_1}^1 - \sigma_{\alpha_1\dot{\alpha}}^n\bar{\eta}^{\dot{\alpha}1}\cdot(x_1-X)_n\right) K_4(X,\rho;x_1,0) \\
&\times\cdots\times\left(\xi_{\alpha_{16}}^4 - \sigma_{\alpha_{16}\dot{\alpha}}^n\bar{\eta}^{\dot{\alpha}4}\cdot(x_{16}-X)_n\right) K_4(X,\rho;x_{16},0)\ .
\end{aligned} \tag{1.24}$$

Here, the power of ρ in the integration measure is fixed by the fact that the $\mathcal{N}=4$ model has vanishing β-function; the power of g comes from combining the explicit dependence in Eq. (1.23) with factors of g^{-8} and g^{16} from the bosonic and fermionic integrations, respectively [73]; and C_2 is an overall numerical constant.

As the authors of Ref. [34] point out, the space-time dependence of the one-instanton Yang-Mills expression (1.24) precisely matches the supergravity expression (1.15). Indeed, performing the sixteen Grassmann integrations over ξ_α^A and $\bar{\eta}^{\dot{\alpha}A}$ in Eq. (1.24) produces (by definition) the totally antisymmetric tensor t_{16} tied into sixteen propagators $K_{7/2}^F$. Furthermore, the remaining integration measure for the five bosonic collective coordinates $\{\rho, X_n\}$ in Eq. (1.24) is identical to the volume form of AdS_5 that appears in Eq. (1.15); there, the scale-size ρ is re-interpreted as the inverse of the radial position of the D-instanton in AdS_5 [34, 54].

Beyond the spatial dependence, one also finds agreement at the one-instanton level in the overall strength of the couplings in front of the two expressions. On the supergravity side, from Eqs. (1.11)-(1.15) one extracts the one-instanton prefactor

$$(\alpha')^{-1}\, e^{-\phi/2}\, f_{16}\Big|_{\text{1-inst}} \sim g^{-24}\, e^{-8\pi^2/g^2+i\theta}\,\sqrt{N}\ , \tag{1.25}$$

up to an overall numerical constant. This g dependence is already present in the Yang-Mills expression (1.24) [34]. Reproducing this $\sqrt{N}$ dependence, too, means generalizing the above Yang-Mills calculation from $SU(2)$ to $SU(N)$, which was done in Ref. [44]. In $\mathcal{N}=4\ SU(N)$ gauge theory a single instanton has a total of $8N$ adjoint fermion zero modes so that for $N>2$

there are additional zero modes beyond the sixteen ξ_α^A and $\bar\eta^{\dot\alpha A}$ modes which must be lifted in order to obtain a non-zero result. As in several other cases in three [30] and four [48] dimensions, this lifting is due to the presence of a specific Grassmann quadrilinear term in the instanton action (see Sec. III below). Pulling down $2N-4$ powers of this quadrilinear and performing the Grassmann integrations over the lifted modes turns out to be straightforward, and is reviewed below in Sec. V.1. The chief result of Ref. [44] is that the Yang-Mills expression (1.24) continues to hold for the gauge group $SU(N)$, up to the replacement $C_2 \to C_N$ where[6]

$$C_N = \frac{(2N-2)!}{(N-1)!(N-2)!}\, 2^{-2N+49}\, 3^{16}\, \pi^{-10} = 2^{47} 3^{16} \pi^{-21/2} \sqrt{N}\big(1 - \tfrac{5}{8}N^{-1} + \mathcal{O}(N^{-2})\big),$$
(1.26)

the final equality following from Stirling's formula. In this algebraically nontrivial way, the overall $\sqrt{N}$ dependence predicted by supergravity is recovered in the large-N limit of the gauge theory. Moreover, since

$$\frac{1}{N} = \frac{g^2}{g^2 N} = \frac{4\pi g_{st}\,(\alpha')^2}{L^4}$$
(1.27)

from Eqs. (1.1)-(1.2), the explicit form of the $1/N$ series implied by the exact Yang-Mills one-instanton result (1.26) gives an infinite number of predictions for corrections in powers of $g_{st}(\alpha')^2$ on the IIB side of the correspondence.

Upon reflection, however, one might suspect that the agreement described above between the supergravity and Yang-Mills expressions for G_{16} is an accident of the one-instanton sector. For, only in that sector is the instanton action density $\mathrm{tr}_N v_{mn}^2$ proportional to the propagator K_4, as per Eq. (1.21). As reviewed in Sec. II.1 below, for topological number k the instanton action density is a rational function of x whose denominator generically involves a polynomial in x of degree $2k$, raised to a power. In particular, this means that for $k>1$, the k-instanton density generically looks nothing like a classical propagator. One plausible way around this problem is to posit that the dilute instanton gas approximation somehow becomes valid in the Yang-Mills k-instanton calculation. If this were the case, however, the k-instanton density would resemble a *sum* of k distinct one-instanton densities (hence a sum of supergravity propagators). Moreover, the moduli space would generically contain k distinct copies (rather than a single copy) of AdS_5, meaning integrations over k distinct instanton 4-positions and sizes. In other words, even if the dilute instanton gas approximation were valid, the k-instanton Yang-Mills expression for $k>1$ can reasonably be expected to look nothing like the supergravity result (1.15)!

Actually we will find below that precisely the right propagator-like structure of the instanton density, accompanied by the collapse of k copies of AdS_5 to just one copy, is recovered in the Yang-Mills calculation as $N \to \infty$. This is one of several pleasing simplifications that occur in the multi-instanton formalism when one passes to the large-N limit.

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[6]The constant C_N differs by a factor of 2^{-8} from that quoted in [44]. The difference is due to the present conventions of footnote 4 for integrating Weyl spinors.

II   The $\mathcal{N} = 4$ Multi-Instanton Supermultiplet

We now review the formalism of ADHM instantons, suitably supersymmetrized. There are two philosophically distinct ways to introduce the multi-instanton supermultiplet. One way is to view it as the exact solution of the coupled classical Euler-Lagrange equations in Euclidean space; in the absence of all Higgs VEVs such exact solutions do of course exist. Our ultimate goal, however, is not just to find classical solutions, but rather to calculate their quantum contributions to correlators G_n, which includes the effects of a perturbative expansion in the instanton background. An efficient and elegant way to take the leading perturbations into account automatically is to modify the background configuration itself as explained below; however, the instanton supermultiplet is then no longer an exact solution to the coupled equations of motion. This is the essence of the second approach which we follow in this paper.

The key distinction between the two methods is whether or not all of the fermion zero modes are included in the multi-instanton multiplet. In $\mathcal{N} = 4$ $SU(N)$ gauge theory there are $8kN$ fermion zero modes of the covariant Weyl operator $\bar{\slashed{\mathcal{D}}} = \bar{\sigma}^n \mathcal{D}_n$ in the background of the gauge-field instanton of topological charge k. However, most of these are flat directions of the action only at the *Gaussian* level and are lifted when the action is expanded to higher order in fluctuations. The zero modes which remain exact to all orders are those protected by symmetry, namely the eight supersymmetric and eight superconformal fermion zero modes (2.53a)-(2.53b). If one insists on working with only exact solutions, then only these sixteen fermion modes can be included in the multi-instanton super-multiplet; the k-instanton action is then simply a constant, $8\pi^2 k/g^2 - ik\theta$. The remaining $8kN - 16$ fermion zero modes are lifted and are treated in this method as fluctuations around the background. If one nevertheless attaches Grassmann collective coordinates to the lifted modes, one finds that the resulting k-instanton *effective* action will depend on these collective coordinates. In particular there will be a Grassmann quadrilinear term (Eq. (3.2) below) generated by scalar exchange through the Yukawa couplings; since this is a tree-level diagram it is not forbidden by a perturbative nonrenormalization theorem.

Instead, in the alternative approach adopted here, we will include all $8kN$ Dirac operator fermion zero modes in the multi-instanton super-multiplet from the outset. The aforementioned scalar exchange tree graph is then already built in at the classical level, so that the quadrilinear action (3.2) is generated without our having to calculate separate diagrams. However, in this more efficient approach, the coupled Euler-Lagrange equations must be solved iteratively, order by order in g; this is not surprising since the multi-instanton configuration is no longer an exact solution. In this way, our definition of the multi-instanton supermultiplet is similar to the more general case of the instanton calculus in the non-conformal cases developed in [26,45,47,48]. Recall that for a non-zero VEV non-trivial exact solutions of the coupled Euler-Lagrange equations cannot exist and instead one is instructed to solve *constrained* equations [74] iteratively in g; the two formalisms are the same in the VEV$\rightarrow 0$ limit.

As stated earlier, in this paper we restrict our attention to leading semiclassical order, meaning the first non-vanishing order in g at each topological level. For convenience we will rescale all the fields so that the only g dependence in the action is through the overall coefficient g^{-2}; the explicit g dependence in the Euler-Lagrange equations can be trivially restored by undoing this rescaling.

The hallmark of ADHM calculus is that, with the correct ansatz for the classical component fields, their defining Euler-Lagrange equations are mapped into finite-dimensional matrix equations. Thus, for the gauge field $v_m(x)$, the Yang-Mills equation (a nonlinear differential equation) is mapped into the nonlinear ADHM constraint equations (2.15a)-(2.15c) below. Similarly, for the classical gauginos $\lambda_\alpha^A(x)$, the covariant Weyl equation (a linear homogeneous differential equation) is mapped into the homogeneous linear constraint equations (2.50a)-(2.50b) below. And finally, for the classical adjoint Higgs fields $A^{AB}(x)$, the covariant Klein-Gordon equation with a Yukawa source term (an inhomogeneous linear differential equation) is mapped into the inhomogeneous linear constraint equation (2.72) below. Note that at leading semiclassical order, the anti-gauginos $\bar{\lambda}_{\dot{\alpha}A}(x)$ and the auxiliary superfield components F and D are all turned off; they do, however, turn on at a higher order in g, in the process of solving by iteration the coupled approximate Euler-Lagrange equations.

II.1 Construction of the classical gauge field

In Secs. II.1-II.4 we concern ourselves with pure $SU(N)$ gauge theory, without fermions or scalars. Gauge fields v_m are traceless anti-Hermitian $N \times N$ matrices and

$$v_{mn} = \partial_m v_n - \partial_n v_m + [v_m, v_n] \tag{2.1}$$

is the field-strength. The ADHM multi-instanton is the general solution of the self-duality equation

$$v_{mn} = {}^*v_{mn} \equiv \tfrac{1}{2}\epsilon_{mnkl}v_{kl} \tag{2.2}$$

in the sector of topological number (equivalently, winding or instanton number) k, where

$$k = -\frac{1}{16\pi^2} \int d^4x \, \mathrm{tr}_N \, v_{mn} \, {}^*v^{mn} \ . \tag{2.3}$$

The ADHM construction of such multi-instantons is discussed in Refs. [9, 46, 75, 76]. Here we follow, with minor modifications, the $SU(N)$ formalism of Refs. [45, 46].

The basic object in the ADHM construction is the $(N + 2k) \times 2k$ complex-valued matrix

$\Delta_{[N+2k]\times[2k]}$ which is taken to be linear in the space-time variable x_n :[7]

$$\Delta_{[N+2k]\times[2k]}\,(x) \;\equiv\; \Delta_{[N+2k]\times[k]\times[2]}\,(x) \;=\; a_{[N+2k]\times[k]\times[2]} \;+\; b_{[N+2k]\times[k]\times\underline{[2]}}\, x_{\underline{[2]}\times[2]}\;. \tag{2.4}$$

Here we have represented the $[2k]$ index set as a product of two index sets $[k]\times[2]$ and have used a quaternionic representation of x_n,

$$x_{[2]\times[2]} \;=\; x_{\alpha\dot\alpha} \;=\; x_n\,\sigma^n_{\alpha\dot\alpha}\;. \tag{2.5}$$

It follows that $\partial_n\Delta = b\sigma_n$. By counting the number of bosonic and fermionic zero modes, we will soon verify that k in Eq. (2.4) is indeed the instanton number while N is the parameter in the gauge group $SU(N)$. As discussed below, the complex-valued constant matrices a and b in Eq. (2.4) constitute a (highly over complete) set of k-instanton collective coordinates.

For generic x, the null-space of the Hermitian conjugate matrix $\bar\Delta(x)$ is N-dimensional, as it has N fewer rows than columns. The basis vectors for this null-space can be assembled into an $(N+2k)\times N$ dimensional complex-valued matrix $U(x)$,

$$\bar\Delta_{[2k]\times\underline{[N+2k]}}\,U_{\underline{[N+2k]}\times[N]} \;=\; 0 \;=\; \bar U_{[N]\times\underline{[N+2k]}}\,\Delta_{\underline{[N+2k]}\times[2k]}\;, \tag{2.6}$$

where U is orthonormalized according to

$$\bar U_{[N]\times\underline{[N+2k]}}\,U_{\underline{[N+2k]}\times[N]} \;=\; 1_{[N]\times[N]}\;. \tag{2.7}$$

In turn, the classical ADHM gauge field v_m is constructed from U as follows. Note first that for the special case $k=0$, the antisymmetric gauge configuration v_n defined by

$$v_{n\,[N]\times[N]} \;=\; \bar U_{[N]\times\underline{[N+2k]}}\,\partial_n\,U_{\underline{[N+2k]}\times[N]} \tag{2.8}$$

is "pure gauge" (i.e., it is a gauge transformation of the vacuum), so that it automatically solves the self-duality equation (2.2) in the vacuum sector. The ADHM ansatz is that Eq. (2.8) continues to give a solution to Eq. (2.2), even for nonzero k. As we shall see, this requires the additional condition

$$\bar\Delta_{[2]\times[k]\times\underline{[N+2k]}}\,\Delta_{\underline{[N+2k]}\times[k]\times[2]} \;=\; 1_{[2]\times[2]}\,f^{-1}_{[k]\times[k]}\;, \tag{2.9}$$

where f is an arbitrary x-dependent $k\times k$ dimensional Hermitian matrix.

To check the validity of the ADHM ansatz, we first observe that Eq. (2.9) combined with the null-space condition (2.6) imply the completeness relation

$$\begin{aligned}
\mathcal{P}_{[N+2k]\times[N+2k]} &\;\equiv\; U_{[N+2k]\times\underline{[N]}}\,\bar U_{\underline{[N]}\times[N+2k]} \\
&\;=\; 1_{[N+2k]\times[N+2k]} \;-\; \Delta_{[N+2k]\times\underline{[k]}\times[2]}\,f_{\underline{[k]}\times\underline{[k]}}\,\bar\Delta_{[2]\times\underline{[k]}\times[N+2k]}\;.
\end{aligned} \tag{2.10}$$

---

[7]For clarity, in Sec. II we will occasionally show matrix sizes explicitly, e.g. the $SU(N)$ gauge field will be denoted $v^m_{[N]\times[N]}$. To represent matrix multiplication in this notation we will underline contracted indices: $(AB)_{[a]\times[c]} = A_{[a]\times\underline{[b]}}\,B_{\underline{[b]}\times[c]}$. Also we adopt the shorthand $X_{[m}Y_{n]} = X_mY_n - X_nY_m$.

Note that $\mathcal{P}$, as defined, is actually a projection operator; the fact that one can write $\mathcal{P}$ in these two equivalent ways turns out to be a useful trick in ADHM algebra, used pervasively below. With the above relations together with integrations by parts, the expression for the field strength v_{mn} may be massaged as follows:

$$
\begin{aligned}
v_{mn} &\equiv \partial_{[m}v_{n]} + v_{[m}v_{n]} = \partial_{[m}(\bar{U}\partial_{n]}U) + (\bar{U}\partial_{[m}U)(\bar{U}\partial_{n]}U) = \partial_{[m}\bar{U}(1 - U\bar{U})\partial_{n]}U \\
&= \partial_{[m}\bar{U}\Delta f\bar{\Delta}\partial_{n]}U = \bar{U}\partial_{[m}\Delta f\partial_{n]}\bar{\Delta}U = \bar{U}b\sigma_{[m}\bar{\sigma}_{n]}f\bar{b}U = 4\bar{U}b\sigma_{mn}f\bar{b}U \ .
\end{aligned}
\tag{2.11}
$$

Self-duality of the field strength then follows automatically from the well-known self-duality property of the numerical tensor σ_{mn}, defined by [72]

$$
\sigma_{mn\alpha}{}^{\beta} = \tfrac{1}{4}(\sigma_{m\alpha\dot{\alpha}}\bar{\sigma}_n^{\dot{\alpha}\beta} - \sigma_{n\alpha\dot{\alpha}}\bar{\sigma}_m^{\dot{\alpha}\beta}) \ .
\tag{2.12}
$$

A technical point: the above construction does not actually distinguish between the gauge group $U(N)$ and $SU(N)$, i.e., the classical gauge field constructed in this way is not automatically traceless (unlike the field strength). However, it can be made so by a gauge transformation $U \to Ug^\dagger$, where $g^\dagger \in U(1)$.

Already at this stage we can infer the behavior of $\mathrm{tr}_N\, v_{mn}^2$ as a rational function of x. From Eq. (2.9) we see that the $k \times k$ matrix f^{-1} is a quadratic polynomial in x. Consequently the denominator of f itself is a polynomial in x of degree $2k$ (i.e., like $\det f^{-1}$); the denominator of $\mathrm{tr}_N\, v_{mn}^2$ then generically goes like the fourth power of this polynomial, as follows from Eqs. (2.10)-(2.11). This explains our comments at the end of Sec. I.3, that only at the one-instanton level can this quantity look like a classical supergravity propagator, which only involves quadratic forms in the denominator. We will see in Sec. V below how this pessimistic observation is averted when the ADHM formalism is combined with the large-N limit.

In the next subsection we will count the independent degrees of freedom of the ADHM configuration and confirm that it has precisely the number of collective coordinates needed to describe a k-instanton solution.

II.2 Constraints, collective coordinates and canonical forms

We have seen that the ADHM construction for $SU(N)$ makes essential use of matrices of various sizes: $(N + 2k) \times N$ matrices U, $(N + 2k) \times 2k$ matrices Δ, a and b, $k \times k$ matrices f, and 2×2 matrices $\sigma_{\alpha\dot{\alpha}}^n$, $\bar{\sigma}^{n\,\dot{\alpha}\alpha}$, $x_{\alpha\dot{\alpha}}$, etc. (Notice that when $N = 2$, the dimensionalities of U and Δ differ from the "$SU(2)$ as $Sp(1)$" formalism reviewed in Ref. [26].) Accordingly, we introduce

a variety of index assignments:

$$
\begin{aligned}
\text{Instanton number indices } [k]: & \quad 1 \le i, j, l \cdots \le k \\
\text{Color indices } [N]: & \quad 1 \le u, v \cdots \le N \\
\text{ADHM indices } [N+2k]: & \quad 1 \le \lambda, \mu \cdots \le N + 2k \\
\text{Quaternionic (Weyl) indices } [2]: & \quad \alpha, \beta, \dot{\alpha}, \dot{\beta} \cdots = 1, 2 \\
\text{Lorentz indices } [4]: & \quad m, n \cdots = 0, 1, 2, 3
\end{aligned}
$$

No extra notation is required for the $2k$ dimensional column index attached to Δ, a and b, since it can be factored as $[2k] = [k] \times [2] = j\dot{\beta}$, etc., as in Eq. (2.4). With these index conventions, Eq. (2.4) reads

$$
\Delta_{\lambda i \dot{\alpha}}(x) = a_{\lambda i \dot{\alpha}} + b^{\alpha}_{\lambda i} x_{\alpha \dot{\alpha}} , \qquad \bar{\Delta}^{\dot{\alpha}\lambda}_i(x) = \bar{a}^{\dot{\alpha}\lambda}_i + \bar{x}^{\dot{\alpha}\alpha} \bar{b}^{\lambda}_{\alpha i} , \tag{2.13}
$$

while the factorization condition (2.9) becomes

$$
\bar{\Delta}^{\dot{\beta}\lambda}_i \Delta_{\lambda j \dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}} (f^{-1})_{ij} . \tag{2.14}
$$

Combining Eqs. (2.13)-(2.14), and noting that $f_{ij}(x)$ is arbitrary, one extracts the three x-independent conditions on a and b:

$$
\bar{a}^{\dot{\alpha}\lambda}_i a_{\lambda j \dot{\beta}} = (\tfrac{1}{2}\bar{a}a)_{ij} \, \delta^{\dot{\alpha}}_{\dot{\beta}} , \tag{2.15a}
$$

$$
\bar{a}^{\dot{\alpha}\lambda}_i b^{\beta}_{\lambda j} = \bar{b}^{\beta\lambda}_i a^{\dot{\alpha}}_{\lambda j} , \tag{2.15b}
$$

$$
\bar{b}^{\lambda}_{\alpha i} b^{\beta}_{\lambda j} = (\tfrac{1}{2}\bar{b}b)_{ij} \, \delta^{\beta}_{\alpha} . \tag{2.15c}
$$

These three conditions are known as the ADHM constraints [46, 75]. They define a set of coupled quadratic conditions on the matrix elements of a, $\bar{a}$, b and $\bar{b}$. As such, for $k > 3$, they cannot be solved in closed form in terms of algebraic functions. This unfortunate fact is the single biggest historical impediment that has hindered progress in multi-instanton calculus.

The elements of the matrices a and b comprise the collective coordinates for the k-instanton gauge configuration. Clearly the number of independent such elements grows as k^2, even after accounting for the constraints (2.15a)-(2.15c). In contrast, the number of physical collective coordinates should equal $4kN$ which scales linearly with k. [8] It follows that a and b constitute a highly redundant set. Much of this redundancy can be eliminated by noting that the ADHM construction is unaffected by x-independent transformations of the form

$$
\begin{aligned}
\Delta_{[N+2k]\times[k]\times[2]} & \rightarrow \Lambda_{[N+2k]\times[N+2k]} \Delta_{[N+2k]\times\underline{[k]}\times[2]} B^{-1}_{\underline{[k]}\times[k]} , \\
U_{[N+2k]\times[N]} & \rightarrow \Lambda_{[N+2k]\times[N+2k]} U_{[N+2k]\times[N]} , \\
f_{[k]\times[k]} & \rightarrow B_{[k]\times\underline{[k]}} f_{\underline{[k]}\times\underline{[k]}} B^{\dagger}_{\underline{[k]}\times[k]} ,
\end{aligned} \tag{2.16}
$$

___

[8]To see this, consider the limit of k far-separated (distinguishable) instantons; each individual instanton is then described by four positions, one scale size, and $4N - 5$ iso-orientations, totaling $4N$ collective coordinates. Usually, as in [75, 77], the number of collective coordinates of the k-instanton is quoted as $4kN - N^2 + 1$ for $k \ge N/2$ and $4k^2 + 1$ for $k < N/2$. These formulae represent only true collective coordinates, that is, excluding the global gauge rotations of the k-instanton configuration. In contrast, in our counting we include such global rotations since they appear in our k-instanton measure. Then the total number of collective coordinates is $4kN$.

provided $\Lambda \in U(N + 2k)$ and $B \in Gl(k, \mathbf{C})$. (These are in addition to the usual space-time gauge symmetries reviewed in Sec. 6 of [26].) Exploiting these symmetries, one can choose a representation in which b assumes a simple canonical form [46]:

$$b_{[N+2k]\times[2k]} = \begin{pmatrix} 0_{[N]\times[2k]} \\ 1_{[2k]\times[2k]} \end{pmatrix} , \qquad a_{[N+2k]\times[2k]} = \begin{pmatrix} w_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix} . \tag{2.17}$$

We can make this canonical form a little more explicit with a convenient representation of the index set $[N + 2k]$. We decompose each ADHM index $\lambda \in [N + 2k]$ into[9]

$$\lambda = u + l\beta , \quad 1 \le u \le N , \quad 1 \le l \le k , \quad \beta = 1, 2 . \tag{2.18}$$

In other words, the top $N \times 2k$ submatrices in Eq. (2.17) have rows indexed by $u \in [N]$, whereas the bottom $2k \times 2k$ submatrices have rows indexed by the pair $l\beta \in [k] \times [2]$. Equation (2.17) then becomes

$$a_{\lambda\,i\dot\alpha} = a_{(u+l\beta)\,i\dot\alpha} = \begin{pmatrix} w_{u\,i\dot\alpha} \\ (a'_{\beta\dot\alpha})_{li} \end{pmatrix} , \tag{2.19a}$$

$$\bar{a}_i^{\dot\alpha\lambda} = \bar{a}_i^{\dot\alpha\,(u+l\beta)} = \begin{pmatrix} \bar{w}_{iu}^{\dot\alpha} , & (\bar{a}'^{\dot\alpha\beta})_{il} \end{pmatrix} , \tag{2.19b}$$

$$b_{\lambda\,i}^\alpha = b_{(u+l\beta)\,i}^\alpha = \begin{pmatrix} 0 \\ \delta_\beta{}^\alpha \delta_{li} \end{pmatrix} , \tag{2.19c}$$

$$\bar{b}_{\alpha i}^\lambda = \bar{b}_{\alpha i}^{u+l\beta} = \begin{pmatrix} 0 , & \delta_\alpha{}^\beta \delta_{il} \end{pmatrix} . \tag{2.19d}$$

With a and b in the canonical form (2.19a), the third ADHM constraint of (2.15c) is satisfied automatically, while the remaining constraints (2.15a) and (2.15b) boil down to:

$$\mathrm{tr}_2\, \tau^c \bar{a} a = 0 \tag{2.20a}$$

$$(a'_n)^\dagger = a'_n . \tag{2.20b}$$

In Eq. (2.20a) we have contracted $\bar{a}^{\dot\beta} a_{\dot\alpha}$ with the three Pauli matrices $(\tau^c)^{\dot\alpha}{}_{\dot\beta}$, while in Eq. (2.20b) we have decomposed $(a'_{\alpha\dot\alpha})_{li}$ and $(\bar{a}'^{\dot\alpha\alpha})_{il}$ in our usual quaternionic basis of spin matrices:

$$(a'_{\alpha\dot\alpha})_{li} = (a'_n)_{li}\, \sigma^n_{\alpha\dot\alpha} , \quad (\bar{a}'^{\dot\alpha\alpha})_{il} = (a'_n)_{il}\, \bar{\sigma}^{n\,\dot\alpha\alpha} . \tag{2.21}$$

Note that the canonical form for b given in Eq. (2.19a) is preserved by a residual $U(k)$ subgroup of the $U(N + 2k) \times Gl(k, \mathbf{C})$ symmetry group (2.16), namely:

$$\Delta_{[N+2k]\times[2k]} \rightarrow \begin{pmatrix} 1_{[N]\times[N]} & 0_{[2k]\times[N]} \\ 0_{[N]\times[2k]} & \mathcal{R}^\dagger_{[2k]\times[2k]} \end{pmatrix} \Delta_{[N+2k]\times[2k]}\, \mathcal{R}_{[2k]\times[2k]} \tag{2.22}$$

---

[9]The Weyl index β in this decomposition is raised and lowered with the ϵ tensor as always [72], whereas for the $[N]$ and $[k]$ indices u and l there is no distinction between upper and lower indices.

where $\mathcal{R}_{[2k]\times[2k]} = R_{ij}\,\delta^{\dot\beta}_{\dot\alpha}$ and $R_{ij} \in U(k)$. In terms of w and a', this residual transformation acts as

$$w_{ui\dot\alpha} \rightarrow w_{uj\dot\alpha}\,R_{ji}\,, \quad (a'_{\alpha\dot\alpha})_{ij} \rightarrow R^\dagger_{il}\,(a'_{\alpha\dot\alpha})_{lp}\,R_{pj}\,. \tag{2.23}$$

It follows that the physical moduli space, $\mathrm{M}^k_{\mathrm{phys}}$, of inequivalent self-dual gauge configurations in the topological sector k is the quotient of the space M^k of all solutions of the ADHM canonical constraints (2.20a) and (2.20b), by this residual symmetry group $U(k)$:

$$\mathrm{M}^k_{\mathrm{phys}} = \frac{\mathrm{M}^k}{U(k)}\,. \tag{2.24}$$

Finally we can count the independent collective coordinate degrees of freedom of the ADHM multi-instanton. A general complex matrix $a_{[N+2k]\times[2k]}$ has $4k(N+2k)$ real degrees of freedom. The two ADHM conditions (2.20a) and (2.20b) impose $3k^2$ and $4k^2$ real constraints, respectively, while modding out by the residual $U(k)$ symmetry removes another k^2 degrees of freedom. In total we therefore have

$$4k(N+2k) - 3k^2 - 4k^2 - k^2 = 4kN \tag{2.25}$$

real degrees of freedom, precisely as required. Of these, the four real degrees of freedom $X_{\alpha\dot\alpha} = X_n\sigma^n_{\alpha\dot\alpha}$ corresponding to

$$a_{\lambda i\dot\alpha} = -b^\alpha_{\lambda i}\,X_{\alpha\dot\alpha} \tag{2.26}$$

are the translational collective coordinates, as is obvious from Eq. (2.13).

II.3   Asymptotics of the multi-instanton

Let us determine the instanton v_n more explicitly, in a particularly useful gauge. This entails solving for U, and hence v_n itself via (2.8), in terms of Δ. It is convenient to make the decomposition:

$$U_{[N+2k]\times[N]} = \begin{pmatrix} V_{[N]\times[N]} \\ U'_{[2k]\times[N]} \end{pmatrix}\,, \quad \Delta_{[N+2k]\times[2k]} = \begin{pmatrix} w_{[N]\times[2k]} \\ \Delta'_{[2k]\times[2k]} \end{pmatrix}\,. \tag{2.27}$$

Then from the completeness condition (2.10) one finds

$$V_{[N]\times[N]}\,\bar V_{[N]\times[N]} = 1_{[N]\times[N]} - w_{[N]\times\underline{[k]}\times\underline{[2]}}\,f_{\underline{[k]}\times\underline{[k]}}\,\bar w_{\underline{[2]}\times\underline{[k]}\times[N]}\,. \tag{2.28}$$

For any V that solves this equation, one can find another by right-multiplying it by an x-dependent $U(N)$ matrix. A specific choice of V corresponds to fixing the space-time gauge. The "instanton singular gauges" correspond to taking any one of the 2^N choices of matrix square roots:

$$V = (1 - wf\bar w)^{1/2}\,. \tag{2.29}$$

Next, U' in Eq. (2.27) is determined in terms of V via

$$U' = -\Delta' f \bar{w} \bar{V}^{-1} \tag{2.30}$$

which likewise follows from Eq. (2.10).

Equations (2.29) and (2.30) determine U in (2.27), and hence the gauge field v_n via Eq. (2.8). We list for later use the leading large-$|x|$ asymptotic behavior of several key ADHM quantities, assuming instanton singular gauge (2.29):

$$\Delta \rightarrow bx, \qquad f_{kl} \rightarrow \frac{1}{|x|^2} \delta_{kl}, \qquad U' \rightarrow -\frac{1}{|x|^2} x\, \bar{w}, \qquad V \rightarrow 1_{[N]\times[N]}. \tag{2.31}$$

II.4 Connection to the usual one-instanton collective coordinates, and the dilute instanton gas limit

It is illuminating to compare the ADHM collective coordinates contained in the matrix a, to the more familiar variables describing the instanton position, size, and iso-orientation. Let us first focus on the one-instanton sector. In particular, let us verify that the usual one instanton solution [73, 78] follows from the general ADHM formalism reviewed above. We adopt the canonical form (2.19a)-(2.19d) and set the instanton number $k = 1$, thus dropping the i, j indices. Contrary to the "$SU(2)$ as $Sp(1)$" treatment of Ref. [26], now the ADHM constraints (2.20a) and (2.20b) do not disappear in the one-instanton sector. Instead, Eq. (2.20b) says that a'_n is real,

$$a'_n \equiv -X_n \in \mathbb{R}^4, \tag{2.32}$$

after which Eq. (2.20a) collapses to

$$\bar{w}_u^{\dot\alpha} w_{u\dot\beta} = \rho^2 \delta^{\dot\alpha}_{\dot\beta}. \tag{2.33}$$

The quantities ρ and X_n will soon be identified with the instanton scale size and space-time position, respectively. It is convenient to put w in the form:

$$w_{u\dot\alpha} = \rho \Upsilon_{[N]\times[N]} \begin{pmatrix} 0_{[N-2]\times[2]} \\ 1_{[2]\times[2]} \end{pmatrix}, \qquad \Upsilon \in \frac{SU(N)}{SU(N-2)}. \tag{2.34}$$

Setting $\Upsilon = 1$ initially, we find for Δ and f:

$$\Delta_{[N+2]\times[2]} = \begin{pmatrix} 0_{[N-2]\times[2]} \\ \rho \cdot 1_{[2]\times[2]} \\ y_{[2]\times[2]} \end{pmatrix}, \qquad f = \frac{1}{y^2 + \rho^2}, \tag{2.35}$$

with $y = (x - X)$. Equations (2.29)-(2.30) then amount to

$$V_{[N]\times[N]} = \begin{pmatrix} 1_{[N-2]\times[N-2]} & 0 \\ 0 & \left(\frac{y^2}{y^2+\rho^2}\right)^{1/2} 1_{[2]\times[2]} \end{pmatrix} \tag{2.36}$$

and

$$U'_{[2]\times[N]} = \left(0_{[2]\times[N-2]} \quad , \quad -\left(\frac{\rho^2}{y^2(y^2+\rho^2)} \right)^{1/2} y_{[2]\times[2]} \right) . \tag{2.37}$$

The gauge field then follows from Eq. (2.8):

$$v_n = \begin{pmatrix} 0 & 0 \\ 0 & v_n^{SU(2)} \end{pmatrix} . \tag{2.38}$$

Here $v_n^{SU(2)}$ is the standard singular-gauge $SU(2)$ instanton [73] with space-time position X_n, scale-size ρ, and in a fixed "reference" iso-orientation:

$$v_n^{SU(2)}(x) = \frac{\rho^2 \, \bar{\eta}_{nm}^a \, (x-X)^m \, \tau^a}{(x-X)^2 \, ((x-X)^2+\rho^2)} , \tag{2.39}$$

where $\bar{\eta}_{mn}^a$ is an 't Hooft eta symbol (see the Appendix). For a general iso-orientation matrix Υ we obtain instead

$$v_n = \Upsilon \begin{pmatrix} 0 & 0 \\ 0 & v_n^{SU(2)} \end{pmatrix} \Upsilon^\dagger , \qquad \Upsilon \in \frac{SU(N)}{U(1) \times SU(N-2)} . \tag{2.40}$$

We see that the instanton always lives in an $SU(2)$ subgroup of the $SU(N)$ gauge group. An explicit representation of this embedding is formed by the three composite $SU(2)$ generators

$$(t^c)_{uv} = \rho^{-2} \, w_{u\dot{\alpha}} \, (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \, \bar{w}_v^{\dot{\beta}}, \qquad c = 1, 2, 3. \tag{2.41}$$

Next let us consider the general k-instanton configuration. In order to make contact with familiar non-ADHM collective coordinates, it is necessary to pass to the dilute instanton gas limit. In ADHM language this corresponds to the limit

$$[a'_n, a'_m] \rightarrow 0 \quad \forall \, m, n . \tag{2.42}$$

Note that this condition is invariant under the residual $U(k)$ symmetry (2.23). However, Eq. (2.42) means that there exists a preferred $U(k)$ transformation which simultaneously diagonalizes the four a'_n matrices:

$$a'_n = \text{diag}\big((a'_n)_{11}, \ldots, (a'_n)_{kk}\big) . \tag{2.43}$$

With this choice, the individual collective coordinates of the k far-separated instantons are the obvious analogs of the one-instanton expressions (2.32), (2.33) and (2.41):

$$X_n^i = -(a'_n)_{ii}, \tag{2.44a}$$

$$\rho_i^2 = \tfrac{1}{2}\bar{w}_{iu}^{\dot{\alpha}} \, w_{ui\dot{\alpha}}, \tag{2.44b}$$

$$(t_i^c)_{uv} = \rho_i^{-2} \, w_{ui\dot{\alpha}} \, (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \, \bar{w}_{iv}^{\dot{\beta}}, \tag{2.44c}$$

where the (unsummed) index $i = 1, \ldots, k$ labels the individual instantons.

26

II.5 Construction of the adjoint fermion zero modes

In a supersymmetric theory the gauge field v_m is accompanied by a gaugino λ^A, where the index A runs over the number of independent supersymmetries. In particular, the $\mathcal{N} = 4$ model has an $SU(4)_R \cong SO(6)_R$ R-symmetry, and the λ^A transform in the fundamental representation of the $SU(4)_R$ or equivalently in the spinor representation of the $SO(6)_R$.

In the leading semiclassical approximation, the λ^A are replaced by the non-trivial solutions to the covariant Weyl equation in the ADHM background,

$$\bar{\slashed{\mathcal{D}}}\lambda^A \;=\; 0 \;, \tag{2.45}$$

where $\bar{\slashed{\mathcal{D}}} = \bar{\sigma}^n \mathcal{D}_n$. By the index theorem, the zero modes of $\bar{\slashed{\mathcal{D}}}$ comprise $2kN$ independent Grassmann degrees of freedom for each supersymmetry. As discussed in [79] in the one-instanton context, these zero modes can be thought of as the superpartners of the instanton. Explicit expressions for the adjoint fermion zero modes in the ADHM background were first obtained in [46]. In our notation they read (cf. Eq. (7.1) of [26]):

$$(\lambda^A_\alpha)_{uv} \;=\; \bar{U}^\lambda_u \mathcal{M}^A_{\lambda i} f_{ij} \bar{b}^\rho_{\alpha j} U_{\rho v} \;-\; \bar{U}^\lambda_u b_{\lambda i \alpha}\, f_{ij} \bar{\mathcal{M}}^{\rho A}_j U_{\rho v} \;. \tag{2.46}$$

Here $\mathcal{M}^A_{\lambda i}$ and $\bar{\mathcal{M}}^{\rho A}_j$ are constant $(N + 2k) \times k$ and $k \times (N + 2k)$ matrices of Grassmann collective coordinates; they can be viewed as either two real Grassmann matrices or as two complex Grassmann matrices which are Hermitian conjugates of one another.

In order to verify that the ansatz (2.46) satisfies the Weyl equation (2.45), we use the general differentiation formulae (see Eqs. (2.8), (2.10), (2.13) and (2.14) above)

$$\partial_n f \;=\; -f \cdot \partial_n(\tfrac{1}{2}\bar{\Delta}^{\dot\alpha}\Delta_{\dot\alpha}) \cdot f \;=\; \begin{cases} -f \cdot \bar{\sigma}^{\dot\alpha\alpha}_n \bar{b}_\alpha \Delta_{\dot\alpha} \cdot f \\ -f \cdot \bar{\Delta}^{\dot\alpha} b^\alpha \sigma_{n\alpha\dot\alpha} \cdot f \end{cases} \tag{2.47}$$

together with

$$\begin{aligned}
\mathcal{D}_n(\bar{U}\mathcal{J}U) &\equiv \partial_n(\bar{U}\mathcal{J}U) + [v_n, \bar{U}\mathcal{J}U] \\
&= \bar{U}\,\partial_n \mathcal{J}\,U - \bar{U}b^\alpha \sigma_{n\alpha\dot\alpha} f \bar{\Delta}^{\dot\alpha} \mathcal{J}U - \bar{U}\mathcal{J}\Delta_{\dot\alpha} f \bar{\sigma}^{\dot\alpha\alpha}_n \bar{b}_\alpha U \;,
\end{aligned} \tag{2.48}$$

valid for any $\mathcal{J}(x)$. From Eqs. (2.46)-(2.48) we then calculate:

$$\bar{\slashed{\mathcal{D}}}^{\dot\alpha\alpha}\lambda^A_\alpha \;=\; 2\bar{U}b^\alpha f\big(\bar{\Delta}^{\dot\alpha}\mathcal{M}^A + \bar{\mathcal{M}}^A \Delta^{\dot\alpha}\big) f \bar{b}_\alpha U \;. \tag{2.49}$$

Hence the condition for a gaugino zero mode is the following two sets of linear constraints on $\mathcal{M}^A$ and $\bar{\mathcal{M}}^A$ which ensure that the right-hand side vanishes (expanding $\Delta(x)$ as $a + bx$) [46]:

$$\bar{\mathcal{M}}^{\lambda A}_i\, a_{\lambda j \dot\alpha} \;=\; -\bar{a}^\lambda_{i\dot\alpha}\,\mathcal{M}^A_{\lambda j} \;, \tag{2.50a}$$

$$\bar{\mathcal{M}}^{\lambda A}_i\, b^\alpha_{\lambda j} \;=\; \bar{b}^{\alpha\lambda}_i\,\mathcal{M}^A_{\lambda j} \;. \tag{2.50b}$$

In a formal sense discussed in Ref. [45], these fermionic constraints are the "spin-$\frac{1}{2}$" superpartners of the original "spin-1" ADHM constraints (2.15a) and (2.15b), respectively. Note further that Eq. (2.50b) is easily solved when b is in the canonical form (2.19a). With the ADHM index decomposition (2.18), we set

$$
\mathcal{M}^A_{\lambda i} \equiv \mathcal{M}^A_{(u+l\beta)\,i} = \begin{pmatrix} \mu^A_{ui} \\ (\mathcal{M}'^A_\beta)_{li} \end{pmatrix}, \quad \bar{\mathcal{M}}^{\lambda A}_i \equiv \bar{\mathcal{M}}^{u+l\beta,A}_i = \left(\bar\mu^A_{iu}\,, \ (\bar{\mathcal{M}}'^{\beta A})_{il} \right). \tag{2.51}
$$

Equation (2.50b) then collapses to

$$
\bar{\mathcal{M}}'^A_\alpha = \mathcal{M}'^A_\alpha \tag{2.52}
$$

which allows us to eliminate $\bar{\mathcal{M}}'^A$ in favor of $\mathcal{M}'^A$.

II.6 Classification and overlap formula for the fermion zero modes

Let us classify these Weyl spinor zero modes. Two can be immediately distinguished, namely those proportional to the ADHM matrices a and b. We choose the following linear combinations

$$
\mathcal{M}^A_{\lambda i} = 4b^\alpha_{\lambda i}\,\xi^A_\alpha\,, \qquad \bar{\mathcal{M}}^{\lambda A}_i = 4\bar{b}^\lambda_{\alpha i}\,\xi^{\alpha A} \tag{2.53a}
$$

and

$$
\begin{aligned}
\mathcal{M}^A_{\lambda i} &= 4\,a_{\lambda i\dot\alpha}\bar\eta^{\dot\alpha A} - 4k^{-1}b^\alpha_{\lambda i}\mathrm{tr}_k(a'_{\alpha\dot\alpha})\bar\eta^{\dot\alpha A} \\
\bar{\mathcal{M}}^{\lambda A}_i &= -4\bar{a}^{\dot\alpha\lambda}_i\,\bar\eta^A_{\dot\alpha} + 4k^{-1}\bar{b}^\lambda_{\alpha i}\mathrm{tr}_k(\bar{a}'^{\dot\alpha\alpha})\bar\eta^A_{\dot\alpha}\,,
\end{aligned} \tag{2.53b}
$$

where ξ^A_β and $\bar\eta^{\dot\alpha A}$ are arbitrary spinor parameters. These are the so-called "supersymmetric" and "superconformal" zero modes,[10] respectively [79]; they satisfy the fermionic constraints (2.50a)-(2.50b) by virtue of the ADHM constraints (2.15a)-(2.15b).

As for the remaining $8kN-16$ modes, it is simplest to focus first on the one-instanton sector, $k=1$, with the instanton oriented as in Eqs. (2.35)-(2.38), and with Υ originally set to unity. Apart from the sixteen modes (2.53a)-(2.53b), there are $8N-16$ additional fermionic zero modes which are the superpartners to gauge orientations [23]. They are constructed by setting $\mathcal{M}'^A = 0$ and also $\mu^A_u = 0$ for $u = N-1$ or N, with arbitrary choices for μ^A_u for $u \leq N-2$; by inspection, these satisfy the constraints (2.50a)-(2.50b). Turning on the orientation matrix Υ as in Eq. (2.40) simply rotates these choices of $\mathcal{M}^A$ by Υ.

Next let us construct these modes for $k > 1$. Our strategy is first to consider the bosonic gauge orientation modes, and then to act on them with supersymmetry. By definition, these orientation modes must preserve all gauge-invariant combinations of the bosonic collective

---

[10]Subtracting the trace terms in (2.53b), which amounts to an admixture of the supersymmetric mode, corresponds to performing the superconformal transformation about the center of the multi-instanton, rather than about the arbitrary origin of space-time.

coordinates contained in the matrix a. The (global) gauge dependence of a collective coordinate is carried by the index u which transforms in the fundamental representation of $SU(N)$; this index is attached to the submatrix w but not to the submatrix a' (see Eq. (2.19a)). A natural set of gauge-invariant collective coordinates (used pervasively below) is obtained by constructing bosonic bilinear variables W in which u is summed over:

$$\left(W^{\dot{\alpha}}_{\ \dot{\beta}}\right)_{ij} = \bar{w}^{\dot{\alpha}}_{iu} w_{uj\dot{\beta}}\,, \quad W^0 = \mathrm{tr}_2\, W, \quad W^c = \mathrm{tr}_2\, \tau^c W, \quad c = 1, 2, 3\,. \tag{2.54}$$

By definition the infinitesimal gauge orientation modes δw are the ones which preserve all the W's, i.e., which satisfy

$$\bar{w}^{\dot{\alpha}}_{iu} \delta w_{uj\dot{\beta}} + \delta \bar{w}^{\dot{\alpha}}_{iu} w_{uj\dot{\beta}} = 0\,. \tag{2.55}$$

Now we consider the fermionic superpartners of these modes. Under a supersymmetry transformation one simply has [45]:

$$\delta w_{ui\dot{\alpha}} = i\bar{\xi}_{\dot{\alpha}A}\mu^A_{ui}\,, \qquad \delta \bar{w}^{\dot{\alpha}}_{iu} = -i\bar{\mu}^A_{iu}\bar{\xi}^{\dot{\alpha}}_A\,. \tag{2.56}$$

Inserting Eq. (2.56) into Eq. (2.55) produces the gauge-invariant conditions

$$\bar{\xi}_{\dot{\beta}A}\bar{w}^{\dot{\alpha}}_{iu}\mu^A_{uj} + \bar{\xi}^{\dot{\alpha}}_A\bar{\mu}^A_{iu}w_{uj\dot{\beta}} = 0 \tag{2.57}$$

or equivalently,

$$\bar{w}^{\dot{\alpha}}_{iu}\mu^A_{uj} = 0 \quad \text{and} \quad \bar{\mu}^A_{iu}w_{uj\dot{\alpha}} = 0\,. \tag{2.58}$$

To satisfy these constraints, it is convenient to decompose μ^A as follows:

$$\mu^A_{iu} = w_{uj\dot{\alpha}}(\zeta^{\dot{\alpha}A})_{ji} + \nu^A_{iu}, \qquad \bar{\mu}^A_{iu} = (\bar{\zeta}^A_{\dot{\alpha}})_{ij}\bar{w}^{\dot{\alpha}}_{uj} + \bar{\nu}^A_{iu}\,, \tag{2.59}$$

where ν^A lies in the orthogonal subspace to w:

$$\bar{w}^{\dot{\alpha}}_{iu}\nu^A_{uj} = 0, \qquad \bar{\nu}^A_{iu}w_{uj\dot{\alpha}} = 0\,. \tag{2.60}$$

The superpartners of the bosonic coset coordinates are then precisely the variables $\{\nu^A, \bar{\nu}^A\}$.[11]

Now let us count the number of fermion modes of these various types. It is easily checked (see Eq. (4.53) below) that the fermionic ADHM constraint (2.50a) only involves the $\{\mathcal{M}'^A, \zeta^A, \bar{\zeta}^A\}$ modes, and not the $\{\nu^A, \bar{\nu}^A\}$ modes. Our convention will be to use this constraint to eliminate the $\bar{\zeta}^A$ in favor of the others. Counting real independent Grassmann degrees of freedom, we thus find $8k^2$ of the $\mathcal{M}'^A$ modes (which include the eight exact supersymmetric modes (2.53a)), $8k^2$ of the ζ^A modes (which, linearly combined with the $\mathcal{M}'^A$ modes, include the eight exact superconformal modes (2.53b)), and $8kN - 16k^2$ of the ν^A modes, totaling $8kN$. This is precisely as expected: one initially has $8k(N + 2k)$ real Grassmann parameters in $\mathcal{M}^A$ and $\bar{\mathcal{M}}^A$, subject

---

[11]It is worth mentioning that although the coset coordinates correspond to bosonic zero modes which are generated by Lagrangian symmetries, this is not true of their fermionic partners.

to $8k^2$ constraints from each of Eqs. (2.50a) and (2.50b), for a net of $8kN$ independent real gaugino zero modes.

Finally, we will need the expression for the overlap of two adjoint fermion zero modes. Thanks to the ADHM formalism, the space-time integration of the product of two classical fermion fields can be equated to an ordinary trace over the product of the associated collective coordinate matrices. The expression to be proved reads:

$$\int d^4x \, \mathrm{tr}_N \, \lambda_\alpha \zeta^\alpha \; = \; -\frac{\pi^2}{2} \mathrm{tr}_k \big[\bar{\mathcal{M}}(\mathcal{P}_\infty + 1)\mathcal{N} + \bar{\mathcal{N}}(\mathcal{P}_\infty + 1)\mathcal{M} \big] \; . \tag{2.61}$$

Here

$$\mathcal{P}_\infty \; = \; \lim_{|x|\to\infty} \mathcal{P} \; = \; 2 - b\bar{b} \, , \tag{2.62}$$

as per Eqs. (2.10) and (2.31), and $\mathcal{M}$ and $\mathcal{N}$ are the collective coordinate matrices corresponding to λ_α and ζ_α, respectively.

The formula (2.61), attributed in Ref. [76] to E. Corrigan [80], is proved in a similar way to the overlap formula for two bosonic vector zero modes presented in Appendices B-C of [26]. The strategy of the proof is to show that the integrand is actually a total derivative,

$$\mathrm{tr}_N \, \lambda_\alpha \zeta^\alpha \; = \; \tfrac{1}{8} \partial_n \partial^n \, \mathrm{tr}_k \big[\bar{\mathcal{M}}(\mathcal{P} + 1)\mathcal{N} f + \bar{\mathcal{N}}(\mathcal{P} + 1)\mathcal{M} f \big] \; , \tag{2.63}$$

after which Eq. (2.61) follows from Gauss's law, together with the asymptotic formulae of Sec. II.3. To verify this, let us first write out the left-hand side of Eq. (2.63):[12]

$$\mathrm{tr}_N \, \lambda_\alpha \zeta^\alpha \; = \; \mathrm{tr}\big[(\bar{\mathcal{N}}\mathcal{P}\mathcal{M} + \bar{\mathcal{M}}\mathcal{P}\mathcal{N})f\bar{b}_\alpha \mathcal{P}b^\alpha f + \bar{\mathcal{M}}\mathcal{P}b^\alpha f\bar{\mathcal{N}}\mathcal{P}b_\alpha f + \mathcal{P}\mathcal{M}f\bar{b}_\alpha \mathcal{P}\mathcal{N}f\bar{b}^\alpha \big]. \tag{2.64}$$

We have used the cyclicity of the trace, together with the definition (2.10) for the projector $\mathcal{P}$. Turning to the right-hand side of Eq. (2.63), one calculates:

$$\begin{aligned}
\tfrac{1}{8}\partial_n\partial^n \, \mathrm{tr}\big[\bar{\mathcal{M}}(\mathcal{P}+1)\mathcal{N}f\big] \; &= \; \tfrac{1}{4}\mathrm{tr}\big[2\bar{\mathcal{M}}\{\mathcal{P}, b^\alpha f\bar{b}_\alpha\}\mathcal{N}f - 2\bar{\mathcal{M}}\Delta_{\dot\alpha}f\bar{b}_\alpha \mathcal{P}b^\alpha f\bar{\Delta}^{\dot\alpha}\mathcal{N}f \\
&\qquad + 2\bar{\mathcal{M}}(\mathcal{P}+1)\mathcal{N}f\bar{b}_\alpha \mathcal{P}b^\alpha f - \bar{\mathcal{M}}\Delta_{\dot\alpha}f\bar{\sigma}^{n\dot\alpha\alpha}\bar{b}_\alpha \mathcal{P}\mathcal{N}\partial_n f \\
&\qquad - \bar{\mathcal{M}}\mathcal{P}b^\alpha \sigma^n_{\alpha\dot\alpha}f\bar{\Delta}^{\dot\alpha}\mathcal{N}\partial_n f\big] \\
&= \; \tfrac{1}{2}\mathrm{tr}\big[\mathcal{M}f\bar{b}_\alpha \mathcal{P}b^\alpha \bar{\mathcal{N}}(1-\mathcal{P}) + \bar{\mathcal{M}}(\mathcal{P}+1)\mathcal{N}f\bar{b}_\alpha \mathcal{P}b^\alpha f \\
&\qquad - \mathcal{M}f\bar{b}^\alpha \mathcal{P}\mathcal{N}f\bar{b}_\alpha \mathcal{P} - \bar{\mathcal{M}}\mathcal{P}b_\alpha f\bar{\mathcal{N}}\mathcal{P}b^\alpha f\big] \\
&= \; \tfrac{1}{2}\mathrm{tr}_N\lambda_\alpha\zeta^\alpha + \tfrac{1}{2}\mathrm{tr}\big[(\bar{\mathcal{M}}\mathcal{N} - \bar{\mathcal{N}}\mathcal{M})f\bar{b}_\alpha \mathcal{P}b^\alpha f\big] \; .
\end{aligned} \tag{2.65}$$

Here the expressions on the right-hand sides follow from the differentiation formulae

$$\partial_n\partial^n f \; = \; 4f\bar{b}_\alpha \mathcal{P}b^\alpha f \; , \tag{2.66a}$$

$$\partial^n \mathcal{P} \; = \; -\Delta_{\dot\alpha}f\bar{\sigma}^{n\dot\alpha\alpha}\bar{b}_\alpha \mathcal{P} - \mathcal{P}b^\alpha \sigma^n_{\alpha\dot\alpha}f\bar{\Delta}^{\dot\alpha} \; , \tag{2.66b}$$

$$\partial_n\partial^n \mathcal{P} \; = \; 4\{\mathcal{P}, b^\alpha f\bar{b}_\alpha\} - 4\Delta_{\dot\alpha}f\bar{b}_\alpha \mathcal{P}b^\alpha f\bar{\Delta}^{\dot\alpha} \; , \tag{2.66c}$$

_____

[12]Here, and in the following, the trace on the right-hand side is either over instanton or ADHM indices, depending on the context.

together with the two alternate expressions in Eq. (2.47); we have also invoked the relations (2.50a), (2.50b) and (2.10) and, once again, cyclicity under the trace. From the final rewrite in Eq. (2.65), the desired result (2.63) follows by inspection upon symmetrization in $\mathcal{M}$ and $\mathcal{N}$, QED.

II.7 Construction of the adjoint Higgs bosons

In addition to the gauge field v_m and the four Weyl spinors λ_α^A, the $\mathcal{N}=4$ vector supermultiplet contains three complex scalars which likewise transform in the adjoint representation of $SU(N)$. In $\mathcal{N}=2$ language, one of these three complex scalars comes from the $\mathcal{N}=2$ gauge multiplet while the other two live in a single adjoint matter hypermultiplet. It is convenient to assemble these three complex scalars into an adjoint-valued antisymmetric tensor field $A^{AB}(x)$, endowed with the canonical kinetic energy

$$\tfrac{1}{4}\epsilon_{ABCD}\int d^4x\,\mathrm{tr}_N\,\mathcal{D}_n A^{AB}\mathcal{D}_n A^{CD}\ . \tag{2.67}$$

The antisymmetric field is subject to a specific reality condition:

$$\tfrac{1}{2}\epsilon_{ABCD}A^{CD}\ =\ (A^{AB})^\dagger\,, \tag{2.68}$$

where $\dagger$ acts only on gauge indices, which implies that it transforms in the vector $\mathbf{6}$ representation of $SO(6)_R$. We can transform to explicit vector components via

$$A^{AB}=\frac{1}{\sqrt{8}}\bar{\Sigma}_a^{AB}A_a\ , \tag{2.69}$$

where the coefficients $\bar{\Sigma}_a^{AB}$ are defined in the Appendix.

In the leading semiclassical approximation, $A^{AB}(x)$ is replaced by the solution of the classical Euler-Lagrange equation

$$\mathcal{D}^2 A^{AB}\ =\ \sqrt{2}\,i\,[\,\lambda^A,\lambda^B\,]\ , \tag{2.70}$$

where $\mathcal{D}^2$ is the covariant Klein-Gordon operator in the multi-instanton background, and λ^A is the adjoint fermion zero mode (2.46). The construction of the solution to this equation for arbitrary Higgs VEVs is one of the principal technical results of Refs. [26, 45]. Here we will content ourselves to summarize the solution in the exact conformal case of vanishing VEVs. One finds that A^{AB} has the additive form

$$i\,A^{AB}(x)\ =\ \frac{1}{2\sqrt{2}}\bar{U}\left(\mathcal{M}^B f\bar{\mathcal{M}}^A-\mathcal{M}^A f\bar{\mathcal{M}}^B\right)U\ +\ \bar{U}\cdot\begin{pmatrix}0_{[N]\times[N]} & 0_{[N]\times[2k]}\\ 0_{[2k]\times[N]} & \mathcal{A}^{AB}_{[k]\times[k]}\otimes 1_{[2]\times[2]}\end{pmatrix}\cdot U\ . \tag{2.71}$$

The $k \times k$ anti-Hermitian collective coordinate matrix $\mathcal{A}^{AB}$ is defined as the solution to the inhomogeneous linear equation

$$\mathbf{L} \cdot \mathcal{A}^{AB} = \Lambda^{AB} . \tag{2.72}$$

Here

$$\Lambda^{AB} = \frac{1}{2\sqrt{2}} \left(\bar{\mathcal{M}}^A \mathcal{M}^B - \bar{\mathcal{M}}^B \mathcal{M}^A \right) \tag{2.73}$$

and $\mathbf{L}$ is a positive self-adjoint linear operator that maps the space of $k \times k$ scalar-valued (anti-)Hermitian matrices onto itself.[13] Explicitly, if Ω is such a matrix, then $\mathbf{L}$ is defined as [26]

$$\begin{aligned}
\mathbf{L} \cdot \Omega &= \tfrac{1}{2}\{\Omega, W^0\} - \tfrac{1}{2}\mathrm{tr}_2\left([\bar{a}', \Omega]a' - \bar{a}'[a', \Omega]\right) \\
&= \tfrac{1}{2}\{\Omega, W^0\} + [a'_n, [a'_n, \Omega]] ,
\end{aligned} \tag{2.74}$$

where W^0 is the Hermitian $k \times k$ matrix

$$W^0_{ij} = \bar{w}^{\dot\alpha}_{iu} w_{uj\dot\alpha} , \quad W^{0\dagger} = W^0 \tag{2.75}$$

which we introduced in Eq. (2.54). From Eqs. (2.72)-(2.75) one sees that $\mathcal{A}^{AB}$ and Λ^{AB} transform in the adjoint representation of the residual $U(k)$ (2.22) (i.e., like a' and $\mathcal{M}'^A$).

Defined in this way, the Higgs field $A^{AB}(x)$ correctly satisfies the equation (2.70); see Sec. 7 of Ref. [26] for calculational details. We also note that the constraints (2.15a), (2.15b), (2.50a), (2.50b) and (2.72) may be thought of as the "spin-1," "spin-$\tfrac{1}{2}$," and "spin-0" components of an $\mathcal{N} = 4$ supermultiplet of constraints [47, 48].

III   Construction of the Multi-Instanton Action

Having constructed the gauge, gaugino, and scalar components of the $\mathcal{N} = 4$ superinstanton in Sec. II, we now derive the k-instanton action S^k_{inst}. The expression we will derive reads [48]:

$$S^k_{\mathrm{inst}} = \frac{8\pi^2 k}{g^2} - ik\theta + S^k_{\mathrm{quad}} . \tag{3.1}$$

Here S^k_{quad} is a particular fermion quadrilinear term, with one fermion collective coordinate chosen from each of the four gaugino sectors $A = 1, 2, 3, 4$:

$$S^k_{\mathrm{quad}} = \frac{\pi^2}{g^2} \epsilon_{ABCD} \, \mathrm{tr}_k \, \Lambda^{AB} \mathcal{A}^{CD} = \frac{\pi^2}{g^2} \epsilon_{ABCD} \, \mathrm{tr}_k \, \Lambda^{AB} \mathbf{L}^{-1} \Lambda^{CD} , \tag{3.2}$$

---

[13]Positivity of $\mathbf{L}$ follows from the Hermiticity of a'_n and the fact that W^0 is the product of a (non-square) matrix with its Hermitian conjugate: $W^0 = \bar{w}^{\dot\alpha} w_{\dot\alpha}$.

with $\mathbf{L}$ as in Eq. (2.74). As an important consistency check, note that this expression does not lift any of the sixteen exact supersymmetric and superconformal zero modes (2.53a)-(2.53b); indeed, thanks to the constraints (2.50a)-(2.50b) these do not appear in Λ^{AB}. In contrast, the remaining $8kN - 16$ fermionic modes do enter into S^k_{quad}, and can therefore be lifted by bringing down powers of the action from the exponent.

We know of three distinct derivations of Eqs. (3.1)-(3.2). The first, given in Ref. [48], is somewhat indirect, as it uses the supersymmetry properties of the collective coordinates; the second, presented below, is more straightforward, albeit calculationally intensive. In Ref. [48], we considered the $\mathcal{N} = 4$ model with $SU(2)$ gauge group, on the Coulomb branch in which an adjoint Higgs VEV v breaks the gauge symmetry down to $U(1)$. Using methods from previous work [26, 47] we calculated the fermion bilinear terms in the k-instanton action; they are proportional to either v or $\bar{\mathrm{v}}$. Then we reasoned as follows. It is known that the supersymmetry algebra can be implemented directly on the (unconstrained) multi-instanton collective coordinates $\{a, \mathcal{M}^A\}$ [47, 79]. Physical quantities such as the multi-instanton collective coordinate action should be a supersymmetric invariant. The fermion bilinear terms calculated in Ref. [48] can be shown not to form a supersymmetric invariant expression by themselves. However, the unique addition of the quadrilinear term (3.2), taken together with the bilinear terms, does produce a supersymmetric invariant expression. Furthermore, in the conformal limit v $\to 0$, only the quadrilinear term survives; in this limit it becomes a supersymmetric invariant by itself. Note that, without first introducing the VEV, we would not have been able to fix the overall multiplicative constant in front of S^k_{quad} using supersymmetry alone. Note further that while the derivation detailed in Ref. [48] was only for the gauge group $SU(2)$, with the notational redefinitions of [45] it immediately extends to general $SU(N)$. An alternative derivation of the action will be presented in Sec. IV.2 below using D-brane methods.

Here, instead, we will present a more straightforward derivation of the quadrilinear action (3.2). Our strategy is simply to plug the semiclassical expressions (2.8), (2.46) and (2.71) for the gauge, gaugino and scalar fields into the $\mathcal{N} = 4$ component Lagrangian, and to carry out the spatial integrations. Such a direct approach might appear unpromising, since in general the functional form of the gauge field cannot be written down explicitly. (Remember that for $k > 3$ the nonlinear ADHM constraints (2.20a) cannot be solved in closed form; see Eqs. (2.15a)-(2.15c) *ff.*) However, as in the $\mathcal{N} = 2$ models considered in [26, 47, 81], it turns out that the integrations can nevertheless be accomplished using Gauss's law, and that only the asymptotic expressions (2.31) are actually needed. Here are the details.

At leading semiclassical order, the relevant parts of the $\mathcal{N} = 4$ component Lagrangian are

the gauge, Higgs and Yukawa terms

$$S = S_{\text{gauge}} + S_{\text{higgs}} + S_{\text{yuk}}$$

$$= -\frac{1}{2g^2} \int d^4x \, \text{tr}_N \, v_{mn}^2 + \frac{i\theta}{16\pi^2} \int d^4x \, \text{tr}_N \, v_{mn} \, {}^*v^{mn}$$

$$+ \frac{1}{4g^2} \epsilon_{ABCD} \int d^4x \, \text{tr}_N \, \mathcal{D}_n A^{AB} \mathcal{D}_n A^{CD} + \frac{i}{\sqrt{2}\,g^2} \epsilon_{ABCD} \int d^4x \, \text{tr}_N \, A^{AB} [\lambda^C, \lambda^D] \,. \tag{3.3}$$

As discussed at the opening of Sec. II, the other terms in the component Lagrangian, involving the antifermions $\bar{\lambda}$, or the auxiliary components F and D, only turn on at higher order in g^2 and can be neglected for present purposes. The first line in Eq. (3.3) yields the usual k-instanton contribution

$$S_{\text{gauge}} = -2\pi i k \tau = \frac{8\pi^2 k}{g^2} - ik\theta \,. \tag{3.4}$$

With an integration by parts together with the scalar equation of motion (2.70), the last two terms may be rewritten as

$$S_{\text{higgs}} + S_{\text{yuk}} = g^{-2} \epsilon_{1BCD} \int d^4x \left(\partial_n (\text{tr}_N \, A^{1B} \mathcal{D}_n A^{CD}) + i\sqrt{2} \, \text{tr}_N \, \lambda^1 [\lambda^B, A^{CD}] \right) \,. \tag{3.5}$$

For definiteness we have singled out the supersymmetry index '1'; alternatively this and subsequent equations may be rewritten in a manifestly $SU(4)_R$ invariant way. The first term here, being a total derivative, may be converted to a surface integral on the sphere at infinity. Since it is gauge-invariant we can evaluate it in any convenient gauge. In particular, in the instanton singular gauge defined in Sec. II.3 above, Eqs. (2.31) and (2.71) imply that in the limit $|x| \to \infty$

$$A^{AB} \sim \frac{1}{x^2} \,, \qquad \mathcal{D}_n A^{AB} \sim \frac{1}{x^3} \tag{3.6}$$

so that this surface contribution vanishes.

We will now show that the commutator term in Eq. (3.5) can likewise be rewritten as a surface term. The general construction parallels that used in Ref. [81] for deriving the fermion quadrilinear in the simpler case of the $\mathcal{N} = 2$ model coupled to fundamental hypermultiplets. The basic idea is to construct quantities $\bar{\psi}^{\dot\alpha}$ and Ξ_α such that

$$\tfrac{1}{2}\epsilon_{1BCD}[\lambda_\alpha^B, A^{CD}] = \slashed{\mathcal{D}}_{\alpha\dot\alpha}\bar{\psi}^{\dot\alpha} + \Xi_\alpha \,, \tag{3.7}$$

where $\slashed{\mathcal{D}}_{\alpha\dot\alpha} = \sigma_{\alpha\dot\alpha}^n \mathcal{D}_n$, and Ξ_α is itself an adjoint fermion zero mode of the form

$$(\Xi_\alpha)_{uv} = \bar{U}_u^\lambda \mathcal{N}_{\lambda i} f_{ij} \bar{b}_{\alpha j}^\rho U_{\rho v} - \bar{U}_u^\lambda b_{\lambda i\alpha} f_{ij} \bar{\mathcal{N}}_j^\rho U_{\rho v} \,. \tag{3.8}$$

The collective coordinate matrix $\mathcal{N}$ (which will be seen to depend in a nontrivial way on the original collective coordinates $\{a, \mathcal{M}^A\}$) is subject to the usual zero mode conditions from Sec. II.5:

$$\bar{\mathcal{N}}_i^\lambda a_{\lambda j\dot\alpha} = -\bar{a}_{i\dot\alpha}^\lambda \mathcal{N}_{\lambda j} \,, \tag{3.9a}$$

$$\bar{\mathcal{N}}_i^\lambda b_{\lambda j}^\alpha = \bar{b}_i^{\alpha\lambda} \mathcal{N}_{\lambda j} \,. \tag{3.9b}$$

With the rewrite (3.7), the commutator term in Eq. (3.5) becomes

$$
\begin{aligned}
2\sqrt{2}\,i\int d^4x\,\mathrm{tr}_N\,\lambda^{1\alpha}(\slashed{\mathcal{P}}_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}+\Xi_\alpha) \;&=\; 2\sqrt{2}\,i\int d^4x\,\partial_n(\mathrm{tr}_N\,\lambda^{1\alpha}\sigma^n_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}) \\
&+\; \sqrt{2}\,i\pi^2\mathrm{tr}_k\big[\bar{\mathcal{M}}^1(\mathcal{P}_\infty+1)\mathcal{N}+\bar{\mathcal{N}}(\mathcal{P}_\infty+1)\mathcal{M}^1\big]
\end{aligned}
\tag{3.10}
$$

In the first term we have used the fact that $\slashed{\bar{\mathcal{P}}}\lambda^1=0$ to pull the derivative outside the trace, while in the second term we have invoked the zero mode overlap formula (2.61) (which likewise follows from a surface integration). The reader can verify below *a posteriori*, using the asymptotics of Sec. II.3, that $\lambda^1\sim x^{-3}$ and $\bar\psi\sim x^{-2}$; hence the first term on the right-hand side of Eq. (3.10), like its Higgs counterpart in Eq. (3.5), gives a vanishing contribution at infinity and may be dropped. Instead, it is the second term which is entirely responsible for the final answer (3.2).

Let us return to Eq. (3.7), and solve for $\bar\psi^{\dot\alpha}$ and Ξ_α. As usual in ADHM calculus, some educated guesswork is required. To this end we expand the left-hand side, using Eqs. (2.46) and (2.71):

$$
\begin{aligned}
\tfrac{1}{2}\epsilon_{1BCD}[\lambda^B_\alpha,\mathcal{A}^{CD}] \;=\; \tfrac{1}{2}\epsilon_{1BCD}\Big(&\frac{1}{\sqrt{2}}\bar U\mathcal{M}^B f\bar b_\alpha\mathcal{P}\mathcal{M}^D f\bar{\mathcal{M}}^C U \;-\; \frac{1}{\sqrt{2}}\bar U b_\alpha f\bar{\mathcal{M}}^B\mathcal{P}\mathcal{M}^D f\bar{\mathcal{M}}^C U \\
&+\; \bar U\mathcal{M}^B f\bar b_\alpha\mathcal{P}\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U \;-\; \bar U b_\alpha f\bar{\mathcal{M}}^B\mathcal{P}\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U\Big) \;-\; \text{H.c.}
\end{aligned}
\tag{3.11}
$$

Here $\mathcal{P}$ is the projection operator defined in Eq. (2.10) above. Now, for an arbitrary quantity of the type $\bar U\mathcal{J}U$, the chain-rule identity

$$
\slashed{\mathcal{P}}_{\alpha\dot\alpha}(\bar U\mathcal{J}U) \;=\; \bar U(\slashed{\partial}_{\alpha\dot\alpha}\mathcal{J})U + 2\bar U b_\alpha f\bar\Delta_{\dot\alpha}\mathcal{J}U + 2\bar U\mathcal{J}\Delta_{\dot\alpha}f\bar b_\alpha U
\tag{3.12}
$$

follows trivially from Eq. (2.48). Since $\partial_n\Delta=b\sigma_n$, a comparison of Eqs. (3.11) and (3.12) motivates the ansatz:

$$
\bar\psi_{\dot\alpha} \;=\; \bar\psi^{(1)}_{\dot\alpha}+\bar\psi^{(2)}_{\dot\alpha}+\bar\psi^{(3)}_{\dot\alpha}
\tag{3.13}
$$

where

$$
\bar\psi^{(1)}_{\dot\alpha} \;=\; -\frac{1}{8\sqrt{2}}\epsilon_{1BCD}\bar U\mathcal{M}^B f\bar\Delta_{\dot\alpha}\mathcal{M}^D f\bar{\mathcal{M}}^C U \;-\; \text{H.c.} ,
\tag{3.14a}
$$

$$
\bar\psi^{(2)}_{\dot\alpha} \;=\; -\tfrac{1}{4}\epsilon_{1BCD}\bar U\mathcal{M}^B f\bar\Delta_{\dot\alpha}\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U \;-\; \text{H.c.} ,
\tag{3.14b}
$$

and

$$
\bar\psi^{(3)}_{\dot\alpha} \;=\; \bar U\cdot\begin{pmatrix}0&0\\0&\mathcal{G}_{\dot\alpha}\end{pmatrix}\cdot U , \quad \bar{\mathcal{G}}_{\dot\alpha}=-\mathcal{G}_{\dot\alpha} , \quad \partial_n\mathcal{G}_{\dot\alpha}=0 .
\tag{3.14c}
$$

We expect $\bar\psi^{(1)}_{\dot\alpha}$ to account (more or less) for the first line of Eq. (3.11), and $\bar\psi^{(2)}_{\dot\alpha}$ to account (more or less) for the next line. The presence of $\bar\psi^{(3)}_{\dot\alpha}$, while less obviously motivated at this

stage, will be needed to ensure that the quantity Ξ_α defined in Eq. (3.7) obeys the zero-mode constraints (3.9a)-(3.9b).

By explicit calculation using Eqs. (2.47), (2.50a), (2.50b), (3.12) and (3.14a), one finds:

$$
\not{D}_{\alpha\dot\alpha}\bar\psi^{(1)\dot\alpha} = \frac{1}{2\sqrt{2}}\,\epsilon_{1BCD}\Big(\bar U\mathcal{M}^B f\bar b_\alpha\mathcal{P}\mathcal{M}^D f\bar{\mathcal{M}}^C U
$$
$$
-\;\bar U b_\alpha f\bar{\mathcal{M}}^B(\mathcal{P}-1)\mathcal{M}^D f\bar{\mathcal{M}}^C U\Big)\;-\;\text{H.c.}
\tag{3.15}
$$

Except for the "-1" in the final term, this reproduces the first line of Eq. (3.11), as expected. Similarly one calculates:

$$
\not{D}_{\alpha\dot\alpha}\bar\psi^{(2)\dot\alpha} = \tfrac{1}{2}\epsilon_{1BCD}\Big(\bar U\mathcal{M}^B f\bar b_\alpha(\mathcal{P}+1)\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U - \bar U b_\alpha f\bar{\mathcal{M}}^B(\mathcal{P}-1)\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U
$$
$$
-\;\bar U b_\alpha f\cdot\mathrm{tr}_2\,\bar\Delta\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\Delta\cdot f\bar{\mathcal{M}}^B U\Big)\;-\;\text{H.c.}
$$
$$
= \tfrac{1}{2}\epsilon_{1BCD}\Big(\bar U\mathcal{M}^B f\bar b_\alpha\mathcal{P}\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U \;-\; \bar U b_\alpha f\bar{\mathcal{M}}^B\mathcal{P}\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\cdot U
$$
$$
+\;\bar U b_\alpha f\Big(\bar{\mathcal{M}}^B\cdot\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}-\mathcal{A}^{CD}\bar{\mathcal{M}}^B\Big)U
$$
$$
-\;\frac{1}{\sqrt{2}}\bar U b_\alpha f\bar{\mathcal{M}}^B\mathcal{M}^D f\bar{\mathcal{M}}^C U\Big)\;-\;\text{H.c.}
\tag{3.16}
$$

so that the second line in Eq. (3.11) is accounted for as well. Here the final equality follows from the commutator identity

$$
\mathrm{tr}_2\,\bar\Delta\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\Delta \;=\; -\mathbf{L}\cdot\mathcal{A}^{CD}\;+\;\{\mathcal{A}^{CD},f^{-1}\}
$$
$$
=\;-\frac{1}{2\sqrt{2}}\,(\bar{\mathcal{M}}^C\mathcal{M}^D-\bar{\mathcal{M}}^D\mathcal{M}^C)\;+\;\{\mathcal{A}^{CD},f^{-1}\}
\tag{3.17}
$$

implied by Eqs. (2.13)-(2.14) and (2.72)-(2.74). Finally,

$$
\not{D}_{\alpha\dot\alpha}\bar\psi^{(3)\dot\alpha} = 2\bar U b_\alpha f\bar\Delta_{\dot\alpha}\cdot\begin{pmatrix}0&0\\0&\mathcal{G}^{\dot\alpha}\end{pmatrix}\cdot U\;-\;\text{H.c.}
$$
$$
= 2\bar U b_\alpha f\Big(\bar a_{\dot\alpha}\cdot\begin{pmatrix}0&0\\0&\mathcal{G}^{\dot\alpha}\end{pmatrix}-\mathcal{G}^{\dot\alpha}\bar a_{\dot\alpha}\Big)U\;-\;\text{H.c.}
\tag{3.18}
$$

where to obtain the final equality we have decomposed $\bar\Delta=\bar a+\bar x\bar b$, commuted $\bar x\bar b$ through $\mathcal{G}$, and replaced it by $-\bar a$ thanks to the orthogonality relation (2.6).

Next we sum the expressions (3.15), (3.16) and (3.18), and compare to the right-hand side of Eq. (3.11). By inspection, we confirm the ansätze (3.7)-(3.8), where the Grassmann zero mode matrix has the form

$$
\mathcal{N} \;=\; \tfrac{1}{2}\epsilon_{1BCD}\Big(\begin{pmatrix}0&0\\0&\mathcal{A}^{CD}\end{pmatrix}\mathcal{M}^B\;-\;\mathcal{M}^B\mathcal{A}^{CD}\Big)+2\begin{pmatrix}0&0\\0&\mathcal{G}^{\dot\alpha}\end{pmatrix}a_{\dot\alpha}-2a_{\dot\alpha}\mathcal{G}^{\dot\alpha}\,.
\tag{3.19}
$$

Up till this point we have yet to solve for $\mathcal{G}^{\dot{\alpha}}$. This is accomplished by inserting $\mathcal{N}$ into the fermionic constraints (3.9a)-(3.9b). One finds that Eq. (3.9b) is satisfied automatically by the expression (3.19). In contrast, Eq. (3.9a) amounts to $2k^2$ independent real linear constraints, which is precisely the number required to fix the anti-Hermitian $k \times k$ matrices $\mathcal{G}^{\dot{\alpha}}$ completely (the explicit form for $\mathcal{G}^{\dot{\alpha}}$ is not required).

Finally we insert this expression for $\mathcal{N}$ into the final term in Eq. (3.10). Thanks to the constraints (2.50a)-(2.50b) on $\mathcal{M}^1$, the last two terms in Eq. (3.19) do not contribute to the inner product. It is easily checked, using the self-adjointness of $\mathbf{L}$, that the first two terms in Eq. (3.19) precisely reproduce the $SU(4)_R$ invariant quadrilinear (3.2), as claimed.

IV   The Multi-Instanton Collective Coordinate Integration Measure

In this section we present an in-depth discussion of the multi-instanton collective coordinate measure. We begin with the so-called "flat measure," in which the various bosonic and fermionic collective coordinates described in Sec. II are integrated over as Cartesian variables; in the flat measure, the ADHM supermultiplet of constraints is implemented simply as a product of δ-functions. Section IV.1 reviews the field-theory derivation of the flat measure given in Refs. [45, 49, 82] using uniqueness arguments, whereas Sec. IV.2 presents an independent rederivation of the flat measure directly from superstring theory by exploiting the world-volume description of D-instantons in the presence of D3-branes.

However, the flat measure is not optimal for our present purposes, for the following reason. Starting in Sec. V we will explore how the ADHM multi-instanton formalism combines in a natural way with the large-N limit. Unfortunately, for large N, the number of collective coordinates contained in the matrices $\{a, \mathcal{M}^A\}$ grows linearly with N. In order to implement a traditional saddle-point approach, it is desirable to change to a new set of collective coordinates whose number stays fixed as N grows. A strong clue for how to proceed comes from anticipating Sec. VI, where we finally calculate the correlators G_n ($n = 16, 8$ or 4) and related higher-partial-wave correlators, and put Maldacena's conjecture to the test. A common feature of these n-point functions is that the n insertions are all gauge-invariant composite operators. This suggests that we change integration variables to a smaller set of gauge-invariant collective coordinates. With this application in mind, Sec. IV.3 is devoted to the construction of the so-called "gauge-invariant measure" from the flat measure. Specifically, this entails calculating the Jacobians needed to switch from the original multiplet of ADHM variables, to the gauge-invariant bosonic and fermionic collective coordinates which we have already identified in Sec. II.6. Pleasingly, it turns out that after one integrates out the gauge coset, the number of remaining gauge-invariant variables no longer grows with N, but only grows with the topological number k.

IV.1 The ADHM multi-instanton measure

We begin by reviewing the "flat measure" on the space of ADHM variables. As the small-fluctuations determinants in a self-dual background cancel between the bosonic and fermionic sectors in a four-dimensional supersymmetric theory [83], the relevant measure is the one inherited from the Feynman path integral on changing variables from the fields to the collective coordinates which parametrize the k-instanton moduli space $\mathrm{M}^k_{\mathrm{phys}}$. In principle the super-Jacobian for this change of variables can be calculated by evaluating the normalization matrices of the appropriate bosonic and fermionic zero-modes. In practice, this involves resolving the ADHM constraints and can only be accomplished explicitly for $k \leq 2$.

Instead, the heuristic strategy adopted in Refs. [45,49,82] is to postulate a measure, and to verify *a posteriori* that its form is fixed uniquely by various consistency requirements enumerated below. In particular, in the $\mathcal{N} = 4$ supersymmetric theory, the measure has the following form [45,49]:

$$
\int d\mu^k_{\mathrm{phys}} = \frac{2^{-k^2/2}(C_1'')^k}{\mathrm{Vol}\, U(k)} \int d^{4k^2}a'\; d^{2kN}\bar{w}\; d^{2kN}w \prod_{A=1,2,3,4} d^{2k^2}\mathcal{M}'^A\; d^{kN}\bar{\mu}^A\; d^{kN}\mu^A
$$

$$
\times \prod_{B=2,3,4} d^{k^2}\mathcal{A}^{1B} \prod_{r=1,\ldots,k^2} \Bigg[\prod_{c=1,2,3} \delta\big(\tfrac{1}{2}\mathrm{tr}_k\, T^r(\mathrm{tr}_2\, \tau^c\bar{a}a)\big) \prod_{A=1,2,3,4} \prod_{\dot{\alpha}=1,2} \delta\big(\mathrm{tr}_k\, T^r(\bar{\mathcal{M}}^A a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}}\mathcal{M}^A)\big)
$$

$$
\times \prod_{B=2,3,4} \delta\big(\mathrm{tr}_k\, T^r(\mathbf{L}\cdot\mathcal{A}^{1B} - \Lambda^{1B})\big)\Bigg] .
$$

$$(4.1)$$

In writing the above, we have used a normalization for the integrations that is different, but more convenient for our purposes, from that in [49]. This difference accounts for the overall factor of $2^{-k^2/2}$. The integrals over the $k \times k$ matrices a'_n, $\mathcal{M}'^A$ and $\mathcal{A}^{AB}$ are defined as the integral over the components with respect to a Hermitian basis of $k \times k$ matrices T^r normalized so that $\mathrm{tr}_k\, T^r T^s = \delta^{rs}$. These matrices also provide explicit definitions of the δ-function factors in the way indicated. Encoded in this measure is information about the $SU(2)$ embeddings of the instantons inside $SU(N)$; it is pleasing that the ADHM parametrization replaces the traditional trigonometric variables of the coset matrix Υ describing this embedding, by Cartesian variables endowed with a flat measure (apart from the δ-function insertions).

We can offer the following consistency checks on this measure:

(i) In the 1-instanton sector, Eq. (4.1) reduces to

$$
\int d\mu^1_{\mathrm{phys}} = \frac{C_1''}{2^{3/2}\pi} \int d^4a'\; d^{2N}\bar{w}\; d^{2N}w \prod_{A=1,2,3,4} d^2\mathcal{M}'^A\; d^N\bar{\mu}^A\; d^N\mu^A \prod_{B=2,3,4} d\mathcal{A}^{1B}
$$

$$
\times \prod_{c=1,2,3} \delta\big(\tfrac{1}{2}\mathrm{tr}_2\, \tau^c\bar{w}w\big) \prod_{A=1,2,3,4} \prod_{\dot{\alpha}=1,2} \delta\big(\bar{\mu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\mu^A\big) \prod_{B=2,3,4} \delta\big(\mathbf{L}\cdot\mathcal{A}^{1B} - \Lambda^{1B}\big) .
$$

$$(4.2)$$

The first δ-function is then resolved as per Eqs. (2.33)-(2.34); also the third δ-function is trivial since in the one-instanton sector $\mathbf{L} \equiv 2\rho^2$ as per Eqs. (2.74), (2.75) and (2.33). In this way, the bosonic part of Eq. (4.2) precisely reproduces the standard one-instanton measure [73, 84]; in particular, the position X_n, size ρ, and group iso-orientation Υ of the instanton follows from the identifications made in Sec. II.4 above. Likewise, the fermionic collective coordinates in Eq. (4.2) can be identified with the supersymmetric and superconformal modes (2.53a)-(2.53b), and with the superpartners of the iso-orientations zero modes discussed in Sec. II.6. To summarize, in the one-instanton sector we have

$$\det{}_2 W = \rho^4 \,, \qquad \det \mathbf{L} \equiv \mathbf{L} = 2\rho^2 \qquad \zeta^{\dot\alpha A} = 4\bar\eta^{\dot\alpha A} \,, \qquad \mathcal{M}_\alpha'^A = 4\xi_\alpha^A \,. \qquad (4.3)$$

After integrating over the global iso-orientation of the instanton (as we do for the multi-instanton measure in Sec. IV.3), the one-instanton measure is

$$\int d\mu_{\mathrm{phys}}^1 = \frac{2^{2N-69/2}\pi^{2N-2}C_1''}{(N-1)!(N-2)!} \int \rho^{4N-13}\, d^4X\, d\rho \prod_{A=1,2,3,4} d^2\xi^A\, d^2\bar\eta^A\, d^{(N-2)}\nu^A\, d^{(N-2)}\bar\nu^A. \qquad (4.4)$$

Comparing this with the one-instanton Bernard measure [84] suitably generalized to an $\mathcal{N} = 4$ theory (see Eq. (13) of Ref. [44][14]), we extract the normalization factor

$$C_1'' = 2^{-2N+1/2}\pi^{-6N}g^{4N} \qquad (4.5)$$

needed below.

(ii) The k-instanton measure should be dimensionless, since the β-function vanishes in this $\mathcal{N} = 4$ model. And indeed this is satisfied by the expression (4.1), since the engineering length dimensions are $[a] = [\Lambda^{1B}] = 1$, $[\mathcal{M}] = 1/2$, $[d\mathcal{M}] = -1/2$, and $[\mathcal{A}^{1B}] = -1$.

(iii) If one inserts into the $\mathcal{N} = 4$ measure (4.1) a mass term for (say) gauginos $\mathcal{M}^3$ and $\mathcal{M}^4$, of the form

$$\exp\, im\,\mathrm{tr}_N\big(\bar{\mathcal{M}}^4(\mathcal{P}_\infty + 1)\mathcal{M}^3 + \bar{\mathcal{M}}^3(\mathcal{P}_\infty + 1)\mathcal{M}^4\big)\,, \qquad (4.6)$$

(see Eq. (2.61)) and passes to the decoupling limit $m \to \infty$ in the manner described in Sec. 5 of Ref. [49], the measure properly flows down to the $\mathcal{N} = 2$ measure given in Ref. [45].

(iv) The measure should preserve a net of $8kN$ unsaturated Grassmann integrations at the k-instanton level, in other words, $8kN$ exact fermion zero modes. It is easy to see that this counting is obeyed by the right-hand side of Eq. (4.1): $8k^2$ fermionic δ-functions saturate $8k^2$ out of the $8k^2 + 8kN$ fermionic integrations over $\{\mathcal{M}'^A, \mu^A, \bar\mu^A\}$, leaving $8kN$ exact fermion zero modes. While there is no index theorem in the $\mathcal{N} = 4$ model guaranteeing this number, due to the absence of a $U(1)$ anomaly, the fact that this theory can be deformed into an $\mathcal{N} = 2$ model, where the anomaly exists, forces this naïve counting nonetheless.

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[14]We have multiplied that result by 2^8 due to the difference in conventions for integrating Weyl fermions.

(v) The invariance of the measure (4.1) under the residual $U(k)$ symmetry (2.22) is obvious.

(vi) Cluster decomposition in the dilute-gas limit of large space-time separation between instantons fixes the overall constant in the k-instanton measure (4.1) in terms of the one instanton factor C_1''. The derivation is analogous to that given in [45, 82].

(vii) As with all physically relevant quantities, the k-instanton measure has to be a supersymmetric invariant. This important requirement can be directly checked by performing the supersymmetry transformations on the collective coordinates detailed in Ref. [45]. The key point here is that the three δ-functions in Eq. (4.1) represent the spin-1, spin-$\frac{1}{2}$ and spin-0 ADHM constraints (2.15a), (2.50a) and (2.72), respectively, which together form an $\mathcal{N} = 4$ supermultiplet of constraints. Note that in the final δ-function in Eq. (4.1) we have singled out the supersymmetry index '1', so that, naïvely, the invariance is only guaranteed for supersymmetry transformations $\xi_A Q^A + \bar{\xi}_A \bar{Q}^A$ with $A = 1$. However, since the $\mathcal{A}^{1B}$ integration is trivially accomplished using

$$\int \prod_{B=2,3,4} d^{k^2} \mathcal{A}^{1B} \prod_{r=1,\dots,k^2} \prod_{B=2,3,4} \delta\big(\mathrm{tr}_k T^r(\mathbf{L} \cdot \mathcal{A}^{1B} - \Lambda^{1B})\big) = \big(\det{}_{k^2}\mathbf{L}\big)^{-3} \qquad (4.7)$$

which is independent of the '1' direction, the expression (4.1) is actually invariant under all four supersymmetries; i.e. it is $SU(4)_R$ invariant.

(viii) Finally we can make the following uniqueness argument [49]. Since the δ-functions in Eq. (4.1) are dictated by the ADHM formalism, and since the resulting measure turns out to be a supersymmetric invariant and also preserves the correct number of fermion zero modes, we claim that the ansatz (4.1) is in fact unique. To see why, let us consider including an additional function of the collective coordinates, $F(a, \mathcal{M}^A)$, in the integrand of Eq. (4.1). To preserve supersymmetry, we require that F be a supersymmetry invariant. It is a fact that any non-constant function that is a supersymmetry invariant must contain fermion bilinear pieces (and possibly higher powers of fermions as well). By the rules of Grassmann integration, such bilinears would necessarily lift some of the adjoint fermion zero modes contained in the collective coordinate matrices $\mathcal{M}^A$. But since Eq. (4.1) contains precisely the right number of unlifted fermion zero modes as per (iv), this argument rules out the existence of a non-constant function F. Moreover, any constant F would be absorbed into the overall multiplicative factor, which is already fixed by cluster decomposition.

In the following subsection, we shall give an alternative, more direct, derivation of the $\mathcal{N} = 4$ measure (4.1), as well as the quadrilinear term (3.2), from string theory. (Note that the $\mathcal{N} = 2$, $\mathcal{N} = 1$ and (classical) $\mathcal{N} = 0$ measures presented in [45, 49, 82] may in turn be derived from the $\mathcal{N} = 4$ measure, and thus ultimately from string theory, by renormalization group decoupling; see item (iii) above and Sec. 5 of [49].) But first, we should note that the results of the $\mathcal{A}^{1B}$ integration, given by Eq. (4.7), may be elegantly combined with the quadrilinear action (3.2) derived in Sec. III, via the introduction of auxiliary Gaussian $k \times k$ matrices of variables χ_{AB}

transforming in the vector $\mathbf{6}$ of $SO(6)_R$:

$$(\det{}_{k^2}\mathbf{L})^{-3} \exp{-S_{\text{quad}}^k} \;=\; \pi^{-3k^2} \int d^{6k^2}\chi \exp\big[-\text{tr}_k\,\chi_a\mathbf{L}\chi_a + 4\pi ig^{-1}\text{tr}_k\,\chi_{AB}\Lambda^{AB}\big]. \tag{4.8}$$

As in Sec. II.7, the antisymmetric tensor χ_{AB} satisfies a pseudo-reality condition

$$\tfrac{1}{2}\epsilon^{ABCD}\chi_{CD} = \chi_{AB}^\dagger\,, \tag{4.9}$$

where $\dagger$ acts only on instanton indices and not on $SU(4)_R$ indices; χ_{AB} can be written as an explicit $SO(6)_R$ vector χ_a, $a = 1,\ldots,6$, by using the coefficients Σ_{AB}^a defined in the Appendix:

$$\chi_{AB} = \frac{1}{\sqrt{8}}\Sigma_{AB}^a\chi_a\,. \tag{4.10}$$

In the $SO(6)$ representation the six $k \times k$ matrices χ_a are Hermitian. This kind of transformation is a well-known tool for analyzing the large-N limit of field theories with four-fermion interactions, like the Gross-Neveu and Thirring models [55]. We shall find that this transformation is absolutely crucial in our analysis, since it introduces a new set of bosonic collective coordinates, namely χ_a, which will have a very important rôle to play in the relation of the Yang-Mills theory to the superstring theory.

IV.2   D-instantons and the ADHM Measure

In this section we will show that the $\mathcal{N} = 4$ ADHM measure of [49], described in Sec. IV.1, can actually be derived using D-brane techniques. We will consider $\mathcal{N} = 4$ supersymmetric Yang-Mills theory realized on the world-volume N D3-branes in Type IIB string theory. According to [50,51], D-instantons located on the D3-branes are equivalent to Yang-Mills instantons in the world-volume gauge theory. In the following we determine the collective coordinate measure for k D-instantons, $d\mu_{\text{phys}}^k \exp{-S_{\text{inst}}^k}$, in the presence of N D3-branes. In a limit where the gauge theory on the D3-branes decouples from gravity, we find that the D-instanton measure coincides with our earlier results for the k-instanton ADHM measure. In this Section, unlike the preceding one, we will not keep track of the overall normalization of the measure.

We begin by briefly reviewing some basic facts about D-branes in Type II string theory [85]. A Dp-brane is defined in the first instance as a p-dimensional hyperplane in nine spatial dimensions on which open strings can end. The massless states in the open string spectrum give rise to massless fields which propagate on the $p + 1$ dimensional world-volume of the D-brane. Specifically, the massless modes or collective coordinates of a single Dp-brane come from dimensional reduction to $d = p + 1$ of a $U(1)$ vector multiplet of $\mathcal{N} = 1$ supersymmetry in $d = 10$. In contrast, closed string modes propagate in the ten dimensional bulk. After dimensional reduction, the ten-dimensional gauge field A_μ, $\mu = 0, 1, 2, \ldots 9$, yields a $p + 1$ dimensional gauge field and $9 - p$ real scalars. The scalars specify the location of the D-brane

in the $9 - p$ dimensions transverse to its world-volume. The $d = 10$ multiplet also includes a Majorana-Weyl fermion, Ψ. The sixteen independent components of Ψ correspond to sixteen fermion zero modes of the D-brane. This number reflects the fact that the D-brane is a BPS configuration which breaks half of the 32 supersymmetries of the Type II theory.

When k parallel Dp-branes are located at different points in their common transverse dimensions their massless degrees of freedom are simply k copies of the collective coordinates described above, which corresponds to gauge group $U(1)^k$. However, when two or more D-branes coincide, additional states corresponding to open strings stretched between the two branes become massless leading to enhanced gauge symmetry [86]. In the maximal case, where all k Dp-branes coincide, the unbroken gauge group is $U(k)$. The low-energy effective action for the world-volume theory can be obtained from dimensional reduction of ten dimensional super-Yang-Mills theory (SYM$_{10}$) with gauge group $U(k)$. The ten-dimensional action is,

$$S_k^{(10)} = \frac{1}{g_{10}^2} \int d^{10}x \, \mathrm{tr}_k \left(\tfrac{1}{2} F_{\mu\nu}^2 + i \bar{\Psi} \Gamma_\mu \mathcal{D}_\mu \Psi \right), \tag{4.11}$$

where $\mathcal{D}_\mu \Psi = \partial_\mu \Psi - i[A_\mu, \Psi]$ is the covariant derivative in the adjoint representation and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the ten-dimensional field strength. The Γ-matrices obey the ten-dimensional Euclidean Clifford algebra, $\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$. The particular representation of the Γ_μ used below is given explicitly in the Appendix.

Dimensional reduction to $p + 1$ dimensions proceeds by setting all spacetime derivatives in the reduced directions to zero. As in the case of a single Dp-brane, the ten dimensional gauge field yields a $p + 1$ dimensional gauge field and $9 - p$ real scalars, all in the adjoint representation of the gauge group. Configurations with some or all of the D-branes separated in spacetime then correspond the Coulomb branch of the world-volume gauge theory. In terms of string theory parameters, the Yang-Mills coupling constant in $d = p + 1$ is identified as $g_{p+1}^2 = 2(2\pi)^{p-2} e^\phi \alpha'^{(p-3)/2}$. This concludes the discussion of features which are common to all Dp-branes. The two cases which interest us most here are the D-instanton ($p = -1$) and the D3 brane (along the way we will also meet the $p = 5$ and $p = 9$ cases). First of all we recall that in the case of N parallel D3-branes the low-energy effective theory on the brane world-volume is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $U(N)$. This configuration is the starting point for Maldacena's analysis [2]. The dimensionless four-dimensional gauge coupling is related to the string coupling as per Eq. (1.1). In general the four-dimensional fields which propagate on the brane also have couplings to the ten-dimensional graviton and the other closed string modes. To decouple the four-dimensional theory from gravity it is necessary to take the limit $\alpha' \to 0$. In particular we will take this limit with the string-coupling held fixed and small. This gives a weakly coupled gauge theory on the D3-brane. Note that this limit is quite different from the large-N decoupling limit considered by Maldacena in which the D3-branes can reliably be described as classical solutions of Type IIB supergravity. As emphasized in previous sections, Maldacena's limit leads instead to the strong coupling limit ($g^2 N \gg 1$) of the four-dimensional gauge theory.

A D-instanton (or D(-1)-brane) corresponds to the extreme case where the dimensional reduction is complete and the world-volume is a single point in Euclidean spacetime. Correspondingly, rather than finite mass or tension, a single D-instanton has finite action $S_{\text{cl}} = 2\pi(e^{-\phi} - ic^{(0)})$ where $c^{(0)}$ is the Ramond-Ramond scalar of the IIB theory. As above, the collective coordinates of a charge-k D-instanton correspond to a $U(k)$ vector multiplet of SYM_{10}. The diagonal components of the ten-dimensional gauge field A_μ specify the location of k D-instantons in $\mathbb{R}^{10}$. As we dimensionally reduce to $d = 0$, these degrees of freedom are c-numbers rather than fields. In addition to a constant part equal to kS_{cl}, the action of a charge-k D-instanton also depends on the collective coordinates via the dimensional reduction of (4.11),

$$S_k = -\frac{1}{2g_0^2}\text{tr}_k\,[A_\mu, A_\nu]^2 + \frac{1}{g_0^2}\text{tr}_k\,\bar\Psi\Gamma_\mu[A_\mu, \Psi], \tag{4.12}$$

where $g_0 \sim (\alpha')^{-2}$.

In addition to the manifest $SO(10)$ symmetry under ten-dimensional rotations, the action is trivially invariant under translations of the form $A_\mu \to A_\mu + x_\mu 1_{[k]\times[k]}$. Hence $k^{-1}\text{tr}_k\,A_\mu$, which corresponds to the abelian factor of the $U(k)$ gauge group, is identified with the center-of-mass coordinate of the charge k D-instanton. The fermionic symmetries of the collective coordinate action are,

$$\delta A_\mu = i\bar\eta\Gamma_\mu\Psi\ ,$$
$$\delta\Psi = -[A_\mu, A_\nu]\Gamma^{\mu\nu}\eta + 1_{[k]\times[k]}\,\varepsilon\ ,$$

where $\Gamma_{\mu\nu} = i[\Gamma_\mu, \Gamma_\nu]/4$. The sixteen components of the Majorana-Weyl $SO(10)$ spinor ε correspond to the sixteen zero modes of the D-instanton configuration generated by the action of the $D = 10$ supercharges. Like the bosonic translation modes, these modes live in the abelian factor of the corresponding $U(k)$ field, $k^{-1}\text{tr}_k\,\Psi$. In contrast, the Majorana-Weyl spinor η parametrizes the sixteen supersymmetries left unbroken by the D-instanton. These are analogous to the collective coordinate supersymmetries of the ADHM formalism described in [47].

In ordinary field theory, instantons yield non-perturbative corrections to correlation functions via their saddle-point contribution to the Euclidean path integral. In the semiclassical limit, the path-integral in each topological sector reduces to an ordinary integral over the instanton moduli space. The extent to which similar ideas apply to D-instantons is less clear, in part because string theory lacks a second-quantized formalism analogous to the path integral. Despite this, there is considerable evidence that D-instanton contributions to string-theory amplitudes also reduce to integrals over collective coordinates at weak string coupling [36]. In this case the relevant collective coordinates are the components the ten-dimensional $U(k)$ gauge field A_μ and their superpartners Ψ. According to Green and Gutperle [36–39], the charge-k D-instanton contributions to the low-energy correlators of the IIB theory are governed by the

partition function,

$$\mathcal{Z}_k = \frac{1}{\mathrm{Vol}\, U(k)} \int_{U(k)} d^{10}A\, d^{16}\Psi\, \exp{-S_k}\; . \tag{4.13}$$

This partition function can be thought of as the collective coordinate integration measure for k D-instantons. In particular, the leading semiclassical contribution of k D-instantons to the correlators of the low-energy supergravity fields is obtained by inserting the classical value of each field in the integrand of (4.13). Because of the symmetries described above, the collective coordinate action S_k does not depend on the $U(1)$ components of the fields A_μ and Ψ. Hence, to obtain a non-zero answer, the inserted fields must include at least sixteen fermions to saturate the corresponding Grassmann integrations. As in field theory instanton calculations, the resulting amplitudes can be interpreted in terms of an instanton-induced vertex in the low-energy effective action. The spacetime position of the D-instanton, x_μ, is interpreted as the location of the vertex. In particular, the work of Green and Gutperle [36] has focused on a term of the form R^4 in the IIB effective action (here R is the ten-dimensional curvature tensor) and its supersymmetric completion.

So far, we have only considered D-instantons in the IIB theory in a flat ten dimensional background and in the absence of other branes. In order to make contact with four-dimensional gauge theory we need to understand how these ideas apply to D-instantons in the presence of D3-branes. In particular we wish to determine how the D-instanton measure (4.13) is modified by the introduction of N parallel D3-branes. Conversely, in the absence of D-instantons, the theory on the four-dimensional world volume of the D3-branes is $\mathcal{N} = 4\ U(N)$ super Yang-Mills theory. Hence a related question is how the D-instantons appear from the point of view of the four-dimensional theory on the D3 branes. In fact, the brane configuration considered here is a special case of a system which has been studied intensively in the past. The general case involves a configuration of k Dp-branes in the presence of $N\ D(p+4)$-branes, with all branes parallel. As we will review below, in each of these cases the lower-dimensional brane corresponds to a Yang-Mills instanton in the world-volume gauge theory of the higher [51]. We begin by reviewing the maximal case $p = 5$, which was first considered (in the context of Type I string theory) by Witten [52]. The cases with $p < 5$ then follow straightforwardly by dimensional reduction.

We start by considering a theory of k parallel D5-branes in isolation. As above, the world-volume theory is obtained by dimensional reduction of ten dimensional $\mathcal{N} = 1$ super-Yang-Mills theory with gauge group $U(k)$. The resulting theory in six dimensions has two Weyl supercharges of opposite chirality and is conventionally denoted as $\mathcal{N} = (1,1)$ SYM.[15] After dimensional reduction, the $SO(10)$ Lorentz group of the Euclidean theory in ten dimensions is broken to $H = SO(6) \times SO(4)$. The $SO(6)$ factor is the Lorentz group of the six dimensional theory while the $SO(4)$ is an R-symmetry. The ten dimensional gauge field A_μ splits up into

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[15]Some convenient facts about six-dimensional supersymmetry are reviewed in [87] (See page 67 in particular).

an adjoint scalar a' in the vector representation of $SO(4)$ and a six dimensional gauge field χ_a with $a = 1, \ldots, 6$. Explicitly we set $A_\mu = (a'_n, \chi_a)$ where $n = 0, 1, 2, 3$ is an $SO(4)$ vector index.

In order to describe the fermion content of the theory we will consider the covering group of H, $\bar{H} = SU(4) \times SU(2)_L \times SU(2)_R$. We introduce indices $A = 1, 2, 3, 4$ and $\alpha, \dot{\alpha} = 1, 2$ corresponding to each factor. As mentioned above, a ten dimensional Majorana-Weyl spinor can be decomposed into two Weyl spinors of opposite chirality in six dimensions. The corresponding representation of $SO(10)$ decomposes as,

$$\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \tag{4.14}$$

under $\bar{H}$. An explicit decomposition of the ten-dimensional spinor Ψ in terms of the six dimensional spinors, $\mathcal{M}'^A_\alpha$ and $\lambda_{\dot{\alpha}A}$ is given in the Appendix. As in the previous sections of this paper, it will often be convenient to rewrite the $SO(6)$ vector χ_a as a (quasi-real) antisymmetric tensor χ_{AB} using Eq. (4.10), where χ_{AB} is subject to the reality condition (4.9).

The fields $(\chi_a, \mathcal{M}'^A_\alpha, \lambda^{\dot{\alpha}}_A, a'_n)$ form a vector multiplet of $\mathcal{N} = (1, 1)$ supersymmetry in six dimensions. In terms of an $\mathcal{N} = (0, 1)$ subalgebra, the $\mathcal{N} = (1, 1)$ vector multiplet splits up into an $\mathcal{N} = (0, 1)$ vector multiplet containing χ_a and $\lambda^{\dot{\alpha}}_A$ and an adjoint hypermultiplet containing a'_n and $\mathcal{M}'^A_\alpha$. The action of the $\mathcal{N} = (1, 1)$ theory is

$$S^{(6)} = \frac{1}{g_6^2} S_{\text{gauge}} + S^{(\text{a})}_{\text{matter}} , \tag{4.15}$$

where

$$S_{\text{gauge}} = \int d^6x \, \text{tr}_k \left(\tfrac{1}{2} F_{ab}^2 - \sqrt{2}\pi \lambda_{\dot{\alpha}A} \left(\bar{\Sigma}_a^{AB} \mathcal{D}_a \right) \lambda_B^{\dot{\alpha}} - \tfrac{1}{2} D_{mn}^2 \right) , \tag{4.16}$$

and

$$S^{(\text{a})}_{\text{matter}} = \int d^6x \, \text{tr}_k \left((\mathcal{D}_a a'_n)^2 - \sqrt{2}\pi \mathcal{M}'^{\alpha A} \left(\Sigma^a_{AB} \mathcal{D}_a \right) \mathcal{M}'^B_\alpha + i\pi[a'_{\alpha\dot{\alpha}}, \mathcal{M}'^{\alpha A}]\lambda^{\dot{\alpha}}_A + iD_{mn}[a'_m, a'_n] \right) . \tag{4.17}$$

In the above, we have rescaled the fields so that the six-dimensional coupling constant, $g_6^2 \sim \alpha'$, only appears in front of the action of the $\mathcal{N} = (1, 1)$ vector multiplet. The reason for the unconventional normalization of the fermion kinetic terms will become apparent below. For later convenience we have also introduced a real anti-self dual auxiliary field for the vector multiplet, $D_{mn} = -(^*D_{mn})$. D transforms in the adjoint representation of $SU(2)_R$, which can be made explicit by writing $D_{mn} = -D^c \bar{\eta}^c_{mn}$, where the eta symbol is defined in the Appendix.

Following [51], the next step is to introduce N D9-branes of the Type IIB theory whose world-volume fills the ten dimensional spacetime.[16] These are analogous to the N D3-branes

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[16]In fact a IIB background with non-vanishing D9-brane charge suffers from inconsistencies at the quantum level. This is not relevant here because the D9-branes in question are just a starting point for a classical dimensional reduction.

in the $p = -1$ case on which we will eventually focus. The world volume theory of the D9-branes in isolation (i.e. in the absence of the D5-branes) is simply ten dimensional $U(N)$ supersymmetric Yang-Mills. As explained by Douglas [51], the effective action for this system contains a coupling between the field strength V_{mn} of the world-volume gauge field and the the rank six antisymmetric tensor field $C_{\mu\nu\cdots\rho}$ which comes from the Ramond-Ramond sector of the Type IIB theory. The latter field is dual to the three-form field strength which appears in the Type IIB supergravity action. This coupling has the form,

$$\int C \wedge V \wedge V, \tag{4.18}$$

where C and V are written as a six-form and a two-form respectively and the integration is over the ten-dimensional world volume of the D9-branes. The same six-form field also couples minimally to the Ramond-Ramond charge carried by D5-branes. Hence a configuration of the $U(N)$ gauge fields with non-zero second Chern class, $V \wedge V$, acts as a source for D5-brane charge. More concretely, if the D9-brane gauge field is chosen to be independent of six of the world-volume dimensions and an ordinary Yang-Mills instanton is embedded in the remaining four dimensions, then the resulting configuration has the same charge-density as a single D5-brane. Both objects are also BPS saturated and therefore they also have the same tension. Further confirmation of the identification of D5-branes on a D9-brane as instantons was found in [51] where the gauge-field background due to a Type I D5-brane was shown to be self-dual via its coupling to the world-volume of a D1-brane probe.

As described above, D5-branes appear as BPS saturated instanton configurations on the D9-brane which break half of the supersymmetries of the world-volume theory. Conversely, the presence of D9-branes also break half the supersymmetries of the D5-brane world-volume theory described by the action (4.15). Specifically, the $\mathcal{N} = (1, 1)$ supersymmetry of the six-dimensional theory is broken down to the $\mathcal{N} = (0, 1)$ subalgebra described above equation (4.15). To explain how this happens we recall that open strings stretched between branes give rise to fields which propagate on the D-brane world-volume. So far we have only included the adjoint representation fields which arise from strings stretching between pairs of D5-branes. As our configuration now includes both D5- and D9-branes there is the additional possibility of states corresponding to strings with one end on each of the two different types of brane. As the D5-brane and D9-brane ends of the string carry $U(k)$ and $U(N)$ Chan-Paton indices respectively, the resulting states transform in the $(\boldsymbol{k}, \boldsymbol{N})$ representation of $U(k) \times U(N)$.

In fact the additional degrees of freedom comprise kN hypermultiplets of $\mathcal{N} = (0, 1)$ supersymmetry in six dimensions [52]. As these hypermultiplets cannot be combined to form multiplets of $\mathcal{N} = (1, 1)$ supersymmetry, the residual supersymmetry of the six dimensional theory is $\mathcal{N} = (0, 1)$ as claimed above. Each hypermultiplet contains two complex scalars $w_{ui\dot\alpha}$. Here, as previously, i and u are fundamental representation indices of $U(k)$ and $U(N)$ respectively. The fact that hypermultiplet scalars transform as doublets of the $SU(2)$ R-symmetry is familiar from $\mathcal{N} = 2$ theories in four dimensions. Each hypermultiplet also contains a pair of

complex Weyl spinors, μ_{ui}^A and $\bar{\mu}_{iu}^A$. The six-dimensional action for the hypermultiplets is,

$$S_{\text{matter}}^{(\text{f})} = \int d^6x \, \text{tr}_k \big(\mathcal{D}_a^F \bar{w}_u^{\dot\alpha} \mathcal{D}_a^F w_{u\dot\alpha} - 2\sqrt{2}\pi \bar{\mu}_u^A \left(\Sigma_{AB}^a \mathcal{D}_a^F \right) \mu_u^B$$
$$+ i\pi \left(\bar{\mu}_u^A w_{u\dot\alpha} + \bar{w}_{u\dot\alpha} \mu_u^A \right) \lambda_A^{\dot\alpha} + D^c \bar{w}_{u\dot\alpha} (\tau^c)_{\ \dot\beta}^{\dot\alpha} w_u^{\dot\beta} \big) \ . \tag{4.19}$$

The scalar and fermion kinetic terms in the above action include the fundamental representation covariant derivative, $\mathcal{D}_a^F w = \partial_a w - i w \chi_a$. The remaining two terms in (4.19) are fundamental representation versions of the Yukawa coupling and D-term which appear in (4.17). The complete action of the $d = 6$ theory is $g_6^{-2} S_{\text{gauge}} + S_{\text{matter}}^{(\text{a})} + S_{\text{matter}}^{(\text{f})}$.

The $\mathcal{N} = (0,1)$ supersymmetry transformations for this action can be deduced from the supersymmetry transformations of ten dimensional Yang-Mills theory in Eq. (4.11). The latter are

$$\delta A_\mu = i\bar{\eta}\Gamma_\mu\Psi \ ,$$
$$\delta\Psi = -iF_{\mu\nu}\Gamma^{\mu\nu}\eta \ , \tag{4.20}$$

The $\mathcal{N} = (0,1)$ supersymmetry of the six-dimensional action (4.16), (4.17) and (4.19) is then obtained as the subalgebra of the ten-dimensional $\mathcal{N} = 1$ algebra obtained by choosing, in the notation of the Appendix,

$$\eta = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar{\xi}_A^{\dot\alpha} \end{pmatrix} \ . \tag{4.21}$$

This yields the transformations,

$$\delta a_n' = \frac{1}{2}\bar{\xi}_{\dot\alpha A}\bar{\sigma}_n^{\dot\alpha\alpha}\mathcal{M}_\alpha'^A, \qquad \delta\mathcal{M}_\alpha'^A = \frac{2\sqrt{2}}{\pi}(\mathcal{D}_a a_n')\bar{\Sigma}_a^{AB}\sigma_{\alpha\dot\alpha}^n\bar{\xi}_B^{\dot\alpha},$$

$$\delta\chi_a = i\sqrt{2}\bar{\xi}_{\dot\alpha A}\bar{\Sigma}_a^{AB}\lambda_B^{\dot\alpha}, \qquad \delta\lambda_A^{\dot\alpha} = \frac{1}{\pi}F^{ab}\bar{\Sigma}_{a\ A}^{b\ B}\bar{\xi}_B^{\dot\alpha} - \frac{i}{\pi}[a_m', a_n']\bar{\sigma}_{\ \dot\beta}^{mn\dot\alpha}\bar{\xi}_A^{\dot\beta}, \tag{4.22}$$

$$\delta w_{\dot\alpha u} = \bar{\xi}_{\dot\alpha A}\mu_u^A, \qquad \delta\mu_u^A = \frac{2\sqrt{2}}{\pi}\bar{\Sigma}_a^{AB}(\mathcal{D}_a^F w_{\dot\alpha u})\bar{\xi}_B^{\dot\alpha}.$$

After dimensional reduction to $d = 0$ and elimination of χ_a and $\lambda_A^{\dot\alpha}$ by their equations of motion, these transformation rules for the remaining variables a_n', $\mathcal{M}_\alpha'^A$, $w_{u\dot\alpha}$ and μ_u^A agree with the collective coordinate supersymmetry transformations Eqs. (13a) and (13b) of [49] (suitably generalized to gauge group $SU(N)$).

The various fields of the six dimensional theory and their transformation properties under $U(k) \times U(N) \times \bar{H}$ are tabulated below:

	$U(k)$	$U(N)$	$SU(4)$	$SU(2)_L$	$SU(2)_R$
χ	**adj**	**1**	**6**	**1**	**1**
λ	**adj**	**1**	**$\bar{4}$**	**1**	**2**
D	**adj**	**1**	**1**	**1**	**3**
a'	**adj**	**1**	**1**	**2**	**2**
$\mathcal{M}'$	**adj**	**1**	**4**	**2**	**1**
w	**k**	**N**	**1**	**1**	**2**
$\bar{w}$	**$\bar{k}$**	**$\bar{N}$**	**1**	**1**	**2**
μ	**k**	**N**	**4**	**1**	**1**
$\bar{\mu}$	**$\bar{k}$**	**$\bar{N}$**	**4**	**1**	**1**

Table 1: Transformation properties of the fields

The reader will have noticed that we have chosen our notation so that each six-dimensional field has a counterpart, denoted by the same letter, in our discussion of the ADHM construction in Section III. The only exceptions are the fields λ and D whose significance will be explained below. Also the group indices on each $d = 6$ field are in one-to-one correspondence with those on the ADHM variable of the same name. The physical reason for this correspondence is simple: the six dimensional fields are the collective coordinates of the D5-branes. As the D5-branes are equivalent to the Yang-Mills instantons, the vacuum moduli space of the $U(k)$ gauge theory on the D5-branes should coincide with the k-instanton moduli space described by the ADHM construction. As the only scalar fields in the six-dimensional theory lie in $\mathcal{N} = (0,1)$ hypermultiplets, the relevant vacuum moduli space is conventionally referred to as a Higgs branch. Precisely how the proposed equivalence arises was explained in [52] where it was shown that the D-term vacuum conditions which follow from the six-dimensional action coincide with the non-linear constraint equations of the ADHM construction. This and several new aspects of this correspondence will be demonstrated explicitly below.

Starting from the configuration of D5 and D9-branes described above, the general case of parallel Dp- and D$(p + 4)$-branes with $p < 5$ can be obtained by dimensional reduction on the brane world-volumes. In particular the case of D0- and D4-branes in the Type IIA theory has been studied extensively because of its application as a lightcone matrix model of the $(2, 0)$ theory in six dimensions [88]. In this case we will perform one further dimensional reduction and study k D-instantons in the presence of N D3-branes. Following the above discussion, this corresponds to k Yang-Mills instantons in the four-dimensional $\mathcal{N} = 4$ theory. The symmetry group $U(k) \times U(N) \times \bar{H}$ of the six dimensional system now has a simple interpretation in terms of the four-dimensional theory: $U(k)$ is the internal symmetry of the ADHM construction, $U(N)$

is the gauge group, $SU(4)$ is the R-symmetry group of the $\mathcal{N} = 4$ supersymmetry algebra and $SO(4) = SU(2)_L \times SU(2)_R$ is the four-dimensional Lorentz group. In six dimensions the scalar fields were all in $\mathcal{N} = (0, 1)$ hypermultiplets, and hence the resulting vacuum moduli space was a Higgs branch. After dimensional reduction to $d = 0$, the six-dimensional gauge fields χ_a become scalars which can acquire VEVs. The resulting Coulomb branch corresponds to motion of the D-instantons in the six dimensions transverse to the D3-branes. We will see below that χ_a is just the auxiliary field introduced at the end of Sec. IV.1 to bilinearize the four-fermion term in the instanton action.

We will now write down the collective-coordinate measure which determines the leading semiclassical contribution of k D-instantons to correlation functions of the low-energy fields of the IIB theory in the presence of N D3-branes. From the above discussion, the appropriate generalization of (4.13), is obtained by dimensionally reducing the partition function of the six-dimensional theory with action $g_6^{-2} S_{\text{gauge}} + S_{\text{matter}}^{(a)} + S_{\text{matter}}^{(f)}$ defined in (4.16), (4.17) and (4.19) to zero dimensions. Thus we have,

$$\mathcal{Z}_{k,N} = \frac{1}{\text{Vol}\, U(k)} \int d^6\chi\, d^8\lambda\, d^3 D\, d^4 a'\, d^8\mathcal{M}'\, d^2 w\, d^2\bar{w}\, d^4\mu\, d^4\bar{\mu}\, e^{-S_{k,N}} , \qquad (4.23)$$

where $S_{k,N} = g_0^{-2} S_G + S_K + S_D$ with,

$$S_G = \text{tr}_k\left(- [\chi_a, \chi_b]^2 + \sqrt{2} i\pi \lambda_{\dot{\alpha}A} [\chi_{AB}^\dagger, \lambda_B^{\dot{\alpha}}] + 2D^c D^c \right), \qquad (4.24a)$$

$$S_K = -\text{tr}_k\left([\chi_a, a_n']^2 + \chi_a \bar{w}_u^{\dot{\alpha}} w_{u\dot{\alpha}} \chi_a + \sqrt{2} i\pi \mathcal{M}'^{\alpha A} [\chi_{AB}, \mathcal{M}_\alpha'^B] + 2\sqrt{2} i\pi \bar{\mu}_u^A \chi_{AB} \mu_u^B \right), \qquad (4.24b)$$

$$S_D = i\pi \text{tr}_k\left([a_{\alpha\dot{\alpha}}', \mathcal{M}'^{\alpha A}] \lambda_A^{\dot{\alpha}} + \bar{\mu}_u^A w_{u\dot{\alpha}} \lambda_A^{\dot{\alpha}} + \bar{w}_{u\dot{\alpha}} \mu_u^A \lambda_A^{\dot{\alpha}} + D^c \left(W^c - i[a_n', a_m'] \bar{\eta}_{nm}^c \right) \right). \qquad (4.24c)$$

Note that S_G arises from dimensional reduction of S_{gauge} and $S_K + S_D$ comes from dimensional reduction of $S_{\text{matter}}^{(a)} + S_{\text{matter}}^{(f)}$. Specifically S_K contains the six-dimensional gauge couplings of the hypermultiplets, while S_D contains the Yukawa couplings and D-terms.

Semiclassical correlation functions of the light fields can be calculated by replacing each field with its value in the D-instanton background and performing the collective coordinate integrations with measure (4.23). In the case of the low-energy gauge fields on the D3-brane, the relevant classical configuration is simply the charge-k Yang-Mills instanton specified by the ADHM data which appears explicitly in the action (4.24a)-(4.24c). Note that the measure (4.23) depends explicitly on the string length-scale through the zero-dimensional coupling $g_0^2 \sim (\alpha')^{-2}$ which appears in the action. As a consequence correlation functions which include fields inserted at distinct spacetime points x_i and x_j will have a non-trivial expansion in powers of $\sqrt{\alpha'}/|x_i - x_j|$. In order to decouple the world-volume gauge theory from gravity must take the limit $\alpha' \to 0$. Hence we take the strong coupling limit $g_0^2 \to \infty$, in the exponent of (4.23).[17] Our final answer for the modified D-instanton measure is therefore,

$$\mathcal{Z}_{k,N} = \frac{1}{\text{Vol}\, U(k)} \int d^6\chi d^8\lambda d^3 D\, d^4 a'\, d^8\mathcal{M}'\, d^4 w\, d^4\mu\, d^4\bar{\mu}\, \exp\left(-S_K - S_D \right). \qquad (4.25)$$

---

[17]See page 4 of [88] for the corresponding discussion in the $p = 0$ case.

As mentioned above, we have not kept track of the overall normalization of this expression which includes a numerical constant that depends on k and N as well as some power of the four-dimensional gauge coupling.

We can now make contact with our results for the ADHM measure in Sec. IV.1.[18] In particular, the equation of motion for D is precisely the non-linear ADHM constraint (2.20a). Similarly the equation of motion for λ is the fermionic constraint (2.50a). Integration over these variables yields the first two sets of δ-functions in the measure formula (4.1). Further, S_K can be compactly rewritten in the notation of Sec. IV.1 as,

$$S_K = \mathrm{tr}_k\, \chi_a \mathbf{L} \chi_a - 4\pi i \mathrm{tr}_k\, \chi_{AB} \Lambda^{AB} \ . \tag{4.26}$$

This is equal to (minus) the exponent appearing in equation (4.8). On integrating out the gauge field χ_a, the instanton action reduces to the fermion quadrilinear term (3.2). We have therefore reproduced our result for the ADHM measure, up to an overall normalization constant. In addition the action $S_K + S_D$ is invariant under eight supercharges which are inherited from the $\mathcal{N} = (0, 1)$ theory in six dimensions. The resulting supersymmetry transformations are identical to the collective coordinate supersymmetries of the $\mathcal{N} = 4$ ADHM measure given in [49] (suitably generalized for gauge group $U(N)$ as in [45]). There are also eight additional fermionic symmetries which only appear after taking the decoupling limit $\alpha' \to 0$. These correspond to the half of the superconformal transformations of the $\mathcal{N} = 4$ theory which leave the instanton invariant.

It is interesting to compare the above discussion of D-instantons in the presence of D3-branes with the corresponding results for the Dp/D$(p+4)$ system with $p > 0$. In the $p = 0$ case considered in [88], we have D0-branes which correspond to solitons in the $4 + 1$ dimensional gauge theory on the D4-brane world-volume. These solitons are just four-dimensional Yang-Mills instantons thought of as static finite-energy configurations in five dimensions. At weak coupling, we expect the dynamics of these solitons to be correctly described by supersymmetric quantum mechanics on the instanton moduli space. Hence, it is not surprising that precisely this quantum mechanical system is obtained in [88] by dimensionally reducing the six-dimensional $\mathcal{N} = (0, 1)$ theory described above down to a single time dimension. On adding another world-volume dimension, we obtain the D1/D5 system discussed by Witten in [89]. The D1-branes are now strings in a six dimensional Yang-Mills theory and, in the decoupling limit, their world-sheet dynamics is described by a two-dimensional $\mathcal{N} = (4, 4)$ non-linear σ-model with the ADHM moduli space as the target manifold. This σ-model has kinetic terms for the coordinates on the target and their superpartners and as usual the supersymmetric completion of the action involves a four-fermion term which couples to the Riemann tensor of the target. If we reduce this action back to $d = 1$ by discarding spatial derivatives we obtain the quantum mechanics of [88]. If we then reduce to $d = 0$ by discarding time derivatives also, the only term

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[18]The difference between the gauge groups $U(N)$ and $SU(N)$ is irrelevant in this context because the instantons always live in the nonabelian factor.

of the σ-model action which survives is the four-fermion term. This is the origin of the fermion quadrilinear term (3.2) in the collective coordinate action.

IV.3   The gauge-invariant measure

In Secs. IV.1-IV.2 we have presented field-theory and string-theory derivations of the flat measure, Eq. (4.1). However, as stated earlier, this form of the measure is inadequate to our present purposes, for two reasons:

(i) The number of bosonic and fermionic variables grows with N. Such a growth is inconvenient in light of our stated aim, in Sec. V below, to treat the large-N limit in a traditional saddle-point approximation.

(ii) The collective coordinates w and μ^A carry a u index which transforms in the fundamental representation of the (global) $SU(N)$ gauge group. In other words, they do not define a gauge-invariant set of variables. This is inefficient, given that in Sec. VI below, we intend to evaluate only correlators of gauge-invariant operators.

Accordingly, our present task is to change variables from w and μ^A to the gauge-invariant set of collective coordinates already introduced in Sec. II.6. After integrating out the degrees of freedom corresponding to the gauge coset[19]

$$\frac{SU(N)}{SU(N-2k)} \ , \tag{4.27}$$

we will find that the number of remaining integration variables no longer grows with N, but only with k. The main calculational aim of this subsection is to derive the bosonic and fermionic Jacobians associated with these changes of variables. The reader who is uninterested in the details of our algebra can skip directly to our final result for the gauge-invariant measure given in Eq. (4.55) below. Note that this expression incorporates the auxiliary Gaussian variables χ_a introduced in Eqs. (4.8)-(4.10).

We begin the Jacobian calculation in the bosonic sector where we need to change variables from the collective coordinates $w_{ui\dot\alpha}$, whose u index transforms in the fundamental representation of the gauge group $SU(N)$, to the gauge-invariant bilinear variables $(W^{\dot\alpha}_{\ \dot\beta})_{ij}$ defined in Eq. (2.75).

Let us consider the more general mathematical problem of an $N \times K$ dimensional complex matrix v^l_u where $u = 1, \ldots, N$ and $l = 1, \ldots, K$, and let us always assume $N \geq K$ (which is

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[19]Actually physical gauge transformations involve modding out by an additional $U(1)$ corresponding to overall phase rotations since these transformations lie in the auxiliary group $U(k)$. However, we shall loosely refer to (4.27) as the "gauge coset".

appropriate for the large-N limit to follow). In the case at hand, l stands for the composite index $(i, \dot\alpha)$ and so

$$K = 2k , \quad N \geq 2k , \tag{4.28}$$

where k is the topological number as always. Furthermore we assume that these variables (read: the w's) are endowed with a flat integration measure

$$\mathcal{D}v = \prod_{u=1}^{N}\prod_{l=1}^{K} dv_u^l \, dv_u^{l*} . \tag{4.29}$$

A suitable $SU(N)$ gauge transformation Υ puts the matrix v_u^l into upper-triangular form:

$$
\begin{pmatrix} v_1^1 & \cdots & v_1^K \\ \vdots & \ddots & \vdots \\ v_N^1 & \cdots & v_N^K \end{pmatrix}
= \Upsilon(\omega_x) \cdot
\begin{pmatrix}
\xi_1^1 & \xi_1^2 & \cdots & \xi_1^K \\
0 & \xi_2^2 & \cdots & \xi_2^K \\
 & 0 & \ddots & \vdots \\
\vdots & & \ddots & \xi_K^K \\
 & & & 0 \\
 & & & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix} . \tag{4.30}
$$

The ξ_k^l are complex except for the diagonal elements ξ_l^l which we can choose to be real. The group parameters ω_x coordinatize the coset $SU(N)/SU(N - K)$ which acts non-trivially on the ξ matrix; the index x runs over $1, \ldots, x_{\max}$ where

$$x_{\max} = 2KN - K^2 . \tag{4.31}$$

Obviously the number of independent real parameters, namely $2KN$, is the same on both sides of Eq. (4.30).

While *a priori* the ξ_l^k are only defined for $l \leq k \leq K$, it is convenient to extend them to $k < l \leq K$ as well, by defining $\xi_l^k = (\xi_k^l)^*$. In terms of these extended variables we define the gauge-invariant bilinears W_{kl} via the matrix equation

$$
W = v^\dagger v =
\begin{pmatrix}
\xi_1^1 & 0 & \cdots & 0 \\
\xi_2^1 & \xi_2^2 & & \vdots \\
\vdots & \vdots & \ddots & 0 \\
\xi_K^1 & \xi_K^2 & \cdots & \xi_K^K
\end{pmatrix}
\begin{pmatrix}
\xi_1^1 & \xi_1^2 & \cdots & \xi_1^K \\
0 & \xi_2^2 & \cdots & \xi_2^K \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 0 & \xi_K^K
\end{pmatrix} . \tag{4.32}
$$

Note that there are as many real degrees of freedom in the $\{W_{kl}\}$ as in the $\{\xi_k^l\}$. From Eq. (4.32) it follows that

$$\det{}_K W = \left(\prod_{l=1}^{K} \xi_l^l \right)^2 . \tag{4.33}$$

We wish to calculate the Jacobian induced by changing integration variables from the original collective coordinates v_u^l, to the gauge-invariant bilinears W_{kl} together with the coset parameters ω_x. It is useful to pass through an intermediate change of variables involving the ξ_k^l rather than the W_{kl}. The desired Jacobian can then be expressed as a quotient:

$$\frac{\partial\big(\{v_u^l, v_u^{l*}\}\big)}{\partial\big(\{W_{kl}, \omega_x\}\big)} \;=\; \frac{\partial\big(\{v_u^l, v_u^{l*}\}\big)}{\partial\big(\{\xi_l^k, \omega_x\}\big)} \Big/ \frac{\partial\big(\{W_{kl}\}\big)}{\partial\big(\{\xi_l^k\}\big)} \;. \tag{4.34}$$

What we aim to show is that this Jacobian actually factors cleanly into a function of the gauge-invariant parameters W_{kl}, times a function of the coset variables ω_x; integrating over the gauge coset then turns the latter function into some overall normalization constant that we will evaluate explicitly.

To this end, let us analyze, in turn, the denominator and the numerator on the right-hand side of Eq. (4.34). The denominator is readily evaluated from Eq. (4.32) by induction in K. For $K = 1$ one has simply $\partial W_{11}/\partial\xi_1^1 = 2\xi_1^1$. From Eq. (4.32) we can also easily relate the Jacobian for K to that for $K-1$; one finds

$$\begin{aligned}
\frac{\partial\big(\{W_{kl}\}\big)}{\partial\big(\{\xi_l^k\}\big)}\bigg|_K &= \frac{\partial W_{KK}}{\partial\xi_K^K} \bigg(\prod_{l=1}^{K-1} \frac{\partial W_{lK}}{\partial\xi_l^K} \frac{\partial W_{Kl}}{\partial\xi_K^l}\bigg) \frac{\partial\big(\{W_{kl}\}\big)}{\partial\big(\{\xi_l^k\}\big)}\bigg|_{K-1} \\
&= 2\xi_K^K \bigg(\prod_{l=1}^{K-1} (\xi_l^l)^2\bigg) \frac{\partial\big(\{W_{kl}\}\big)}{\partial\big(\{\xi_l^k\}\big)}\bigg|_{K-1} \;.
\end{aligned} \tag{4.35}$$

It follows by induction that

$$\frac{\partial\big(\{W_{kl}\}\big)}{\partial\big(\{\xi_l^k\}\big)} \;=\; 2^K\, \xi_K^K\, (\xi_{K-1}^{K-1})^3 \cdots (\xi_1^1)^{2K-1} \;. \tag{4.36}$$

Next, we evaluate the numerator on the right-hand side of Eq. (4.34). At this stage, it is useful to make an explicit choice of coset coordinates ω_x. We choose them to be simply the independent entries of the first K orthonormal columns u^l of Υ:

$$u_u^l = \Upsilon_{ul} \;, \quad u^{l\dagger} \cdot u^{l'} = \delta^{ll'} \;. \tag{4.37}$$

The N-vectors u^l then provide the well-known parametrization of the coset as a product of spheres [90]. To see this, u^1 is a unit vector in an N-dimensional complex space and consequently parametrizes S^{2N-1}. The second vector u^2 is a similar unit vector, but one which is orthogonal to u^1, and consequently parameterizes S^{2N-3}. Continuing this chain of argument, we see that the vectors $\{u^l\}$ parametrize the product of spheres

$$\frac{SU(N)}{SU(N-K)} \simeq S^{2N-1} \times S^{2N-3} \times \cdots \times S^{2N-2K+1} \;. \tag{4.38}$$

From (4.30) we can read off the expansion of the vectors v^l in terms of this basis:

$$v^l = \sum_{k=1}^{l} \xi_k^l u^k \ . \tag{4.39}$$

The numerator on the right-hand side of Eq. (4.34) can be determined through the following iterative process. First we start with $v^1 = \xi_1^1 u^1$, whose measure can be written in polar coordinates as

$$\prod_{u=1}^{N} dv_u^1 dv_u^{1*} = 2^N (\xi_1^1)^{2N-1} \, d\xi_1^1 \, d^{2N-1}\hat{\Omega}_1 \ . \tag{4.40}$$

Here $d^{2N-1}\hat{\Omega}_1$ is the usual measure for the solid angles on S^{2N-1} parametrized by u^1. Continuing the process on the next vector $v^2 = \xi_1^2 u^1 + \xi_2^2 u^2$, we have

$$\prod_{u=1}^{N} dv_u^2 dv_u^{2*} = d\xi_1^2 d\xi_1^{2*} \cdot 2^{N-1}(\xi_2^2)^{2N-3} \, d\xi_2^2 \, d^{2N-3}\hat{\Omega}_2 \ . \tag{4.41}$$

In general

$$\prod_{u=1}^{N} dv_u^l dv_u^{l*} = \prod_{k=1}^{l-1} d\xi_k^l d\xi_k^{l*} \cdot 2^{N-l-1}(\xi_l^l)^{2N-2l+1} \, d\xi_l^l \, d^{2N-2l+1}\hat{\Omega}_l \ . \tag{4.42}$$

where $\hat{\Omega}_l$ is parametrized by u^l. Hence

$$
\begin{aligned}
\mathcal{D}v = & 2^{NK-K(K-1)/2}(\xi_1^1)^{2N-1}(\xi_2^2)^{2N-3}\cdots(\xi_K^K)^{2N-2K+1}d\xi_1^1\cdots d\xi_K^K \prod_{k<l} d\xi_k^l \, d\xi_k^{l*} \\
& \times d^{2N-1}\hat{\Omega}_1 \, d^{2N-3}\hat{\Omega}_2 \cdots d^{2N-2K+1}\hat{\Omega}_K \ .
\end{aligned}
\tag{4.43}
$$

Using Eqs. (4.33), (4.34), (4.36) and (4.43), we therefore obtain

$$\mathcal{D}v = 2^{NK-K(K+1)/2}(\det{}_K W)^{N-K} \, d^{K^2}W \, d^{2N-1}\hat{\Omega}_1 \, d^{2N-3}\hat{\Omega}_2 \cdots d^{2N-2K+1}\hat{\Omega}_K \ . \tag{4.44}$$

Thus, when this measure is used to integrate only gauge-invariant quantities, as we shall do in Sec. VI, then only the $\det W$ piece survives; the remaining terms are integrated over the gauge coset to give an overall normalization constant reflecting the volume of the coset (4.27). Re-introducing $k = K/2$ and the vectors w, we state our final result as

$$\int_{\substack{\text{gauge}\\\text{coset}}} d^{2kN}w \, d^{2kN}\bar{w} = c_{k,N} \, (\det{}_{2k}W)^{N-2k} \, d^{k^2}W^0 \prod_{c=1,2,3} d^{k^2}W^c \, . \tag{4.45}$$

The integral over the $2k \times 2k$ matrix W has been written as four separate integrals over the $k \times k$ matrices W^0 and W^c, defined in (2.54), with respect to the basis T^r, defined below (4.1).[20] The normalization constant is precisely

$$
\begin{aligned}
c_{k,N} &= 2^{2kN-4k^2-k} \operatorname{Vol} S^{2N-1} \cdot \operatorname{Vol} S^{2N-3} \cdots \operatorname{Vol} S^{2N-4k+1} \\
&= \frac{2^{2kN-4k^2+k} \pi^{2kN-2k^2+k}}{\prod_{i=1}^{2k}(N-i)!} \ .
\end{aligned}
\tag{4.46}
$$

---

[20]This accounts for an additional factor of 2^{-2k^2}.

In changing variables from the w's to the W's, one must also, of course, specify the integration domain for the latter. Since these variables are the inner products of vectors they will be constrained by various triangle inequalities. Fortunately, these technicalities will not be important for us, the reason being that the saddle-point values of the W's obtained in Sec. 5.2 satisfy all such triangle inequalities by inspection.

We now turn to the fermionic sector. As explained in Sec. II.6, the fermionic equivalent of integrating out the gauge coset is to integrate out the Grassmann variables ν^A and $\bar{\nu}^A$ defined in Eqs. (2.59)-(2.60). Since these coordinates are orthogonal to the $\bar{w}$ and w vectors, respectively, it is easy to see from (2.50a) that they do not appear in the fermionic ADHM constraints; however, they do appear in the fermion quadrilinear term (3.2). The Jacobian for the change of variables from the original fermionic coordinates $\{\mu^A, \bar{\mu}^A\}$ to the variables of Sec. II.6, namely $\{\zeta^A, \bar{\zeta}^A, \nu^A, \bar{\nu}^A\}$, is, for each value of A,[21]

$$\frac{\partial\big(\{\mu^A, \bar{\mu}^A\}\big)}{\partial\big(\{\zeta^A, \bar{\zeta}^A, \nu^A, \bar{\nu}^A\}\big)} = (\det_{2k} W)^{-k} \ . \tag{4.47}$$

In order to integrate out the ν^A and $\bar{\nu}^A$ variables it is useful to split the fermion bilinear (2.73) that couples to χ_{AB} in (4.8) as

$$\Lambda^{AB} = \hat{\Lambda}^{AB} + \tilde{\Lambda}^{AB}, \tag{4.48}$$

where the first term has components just depending on $\{\nu^A, \bar{\nu}^A\}$:

$$\hat{\Lambda}^{AB}_{ij} = \frac{1}{2\sqrt{2}} \left(\bar{\nu}^A_{iu} \nu^B_{uj} - \bar{\nu}^B_{iu} \nu^A_{uj} \right), \tag{4.49}$$

and the second term depends on the remaining variables

$$\tilde{\Lambda}^{AB} = \frac{1}{2\sqrt{2}} \left(\bar{\zeta}^A_{\dot{\alpha}} W^{\dot{\alpha}}_{\dot{\beta}} \zeta^{\dot{\beta}B} - \bar{\zeta}^B_{\dot{\alpha}} W^{\dot{\alpha}}_{\dot{\beta}} \zeta^{\dot{\beta}A} + \{\mathcal{M}'^{\alpha A}, \mathcal{M}'^B_\alpha\} \right) . \tag{4.50}$$

We can now explicitly integrate out the ν^A's and $\bar{\nu}^A$'s:

$$\int \prod_{A=1,2,3,4} d^{k(N-2k)} \nu^A \, d^{k(N-2k)} \bar{\nu}^A \, \exp\left[\sqrt{8}\pi i g^{-1} \mathrm{tr}_k \, \chi_{AB} \bar{\nu}^A \nu^B \right] = \left(\frac{8\pi^2}{g^2} \right)^{2k(N-2k)} (\det_{4k} \chi)^{N-2k} \ , \tag{4.51}$$

where the determinant is of $(\chi_{AB})_{ij}$ viewed as a $4k \times 4k$ matrix.

Actually, going to a gauge invariant measure carries with it a very significant advantage: we can integrate out the ADHM δ-functions explicitly. In the bosonic sector, one notices that the ADHM constraints are written as

$$0 = W^c + [\, a'_n \, , \, a'_m \,] \, \mathrm{tr}_2 \, \tau^c \bar{\sigma}^{nm} = W^c - i[\, a'_n \, , \, a'_m \,] \, \bar{\eta}^c_{nm} \ . \tag{4.52}$$

---

[21]We define the integrals over the $k \times k$ matrices $\zeta^{\dot{\alpha}A}$ and $\bar{\zeta}^{\dot{\alpha}A}$ with respect to the Hermitian basis T^r, as for the bosonic variables a'_n and W^0, and the fermionic variables $\mathcal{M}'^A_\alpha$, above.

in terms of the gauge invariant coordinates. Notice that they are *linear* in W^c and consequently the W^c integrals simply remove the bosonic ADHM δ-functions in (4.1) (giving rise to the numerical factor of 2^{3k^2} from the $\frac{1}{2}$'s in the arguments of the δ-functions). There is a similar happy story in the fermionic sector: in terms of the new variables the fermionic ADHM constraints (2.50a) are

$$\bar{\zeta}^A_{\dot{\beta}} W^{\dot{\beta}}_{\dot{\alpha}} + W_{\dot{\alpha}\dot{\beta}} \zeta^{\dot{\beta}A} + \left[\mathcal{M}'^{\alpha A}, a'_{\alpha\dot{\alpha}} \right] = 0. \tag{4.53}$$

These equations can be used to eliminate the $8k^2$ variables $\bar{\zeta}^A_{\dot{\alpha}}$ The relevant integral is simply

$$\int d^{2k^2}\sigma^A \prod_{r=1,\dots,k^2} \prod_{\dot{\alpha}=1,2} \delta\left(\mathrm{tr}_k\, T^r (\bar{\zeta}^A_{\dot{\beta}} W^{\dot{\beta}}_{\dot{\alpha}} + W_{\dot{\alpha}\dot{\beta}} \zeta^{\dot{\beta}A} + \left[\mathcal{M}'^{\alpha A}, a'_{\alpha\dot{\alpha}} \right]) \right) = (\det_{2k} W)^k. \tag{4.54}$$

Notice that the factor on the right-hand side conveniently cancels the Jacobian of (4.47).

Putting everything together we now have a much simpler form for the measure for integrating gauge invariant quantities:

$$\int d\mu^k_{\mathrm{phys}}\, e^{-S^k_{\mathrm{inst}}} = \frac{g^{8k^2} 2^{4kN - 19k^2/2 + k/2} e^{2\pi i k \tau} c_{k,N}}{\pi^{2kN + 11k^2}\, \mathrm{Vol}\, U(k)} \int d^{k^2} W^0\, d^{4k^2} a'\, d^{6k^2}\chi \prod_{A=1,2,3,4} d^{2k^2}\mathcal{M}'^A\, d^{2k^2}\zeta^A$$

$$\times\; (\det_{2k} W \det_{4k}\chi)^{N-2k} \exp\left[4\pi i g^{-1}\, \mathrm{tr}_k\, \chi_{AB} \tilde{\Lambda}^{AB} - \mathrm{tr}_k\, \chi_a \mathbf{L} \chi_a \right]. \tag{4.55}$$

In particular, since the number of integration variables no longer grows with N (in contradistinction to the flat measure), this expression is immediately amenable to a large-N saddle-point analysis, to which we now turn.

V   The Large-N Limit in a Saddle-Point Approximation

V.1   The one-instanton measure revisited, and the emergence of the S^5

As a prelude to the technically demanding case of multi-instantons, it is instructive to revisit the one-instanton sector, and in particular, to derive the result (1.26) cited in Sec. I.3 above [44]. In this simple setting, we will formulate a large-N saddle-point method which naturally extends to $k > 1$, and we will already see the emergence of the S^5 factor expected from the AdS/CFT correspondence.

Consider the gauge invariant measure (4.55) for $k = 1$. In this case we can greatly simplify the expression: in addition to the relations (4.3) one has $\tilde{\Lambda}^{AB} = 0$, and also [44]

$$\det_4 \chi = \tfrac{1}{64}\left(\epsilon^{ABCD} \chi_{AB} \chi_{CD} \right)^2 \equiv \tfrac{1}{64} \sum_{a=1}^{6} (\chi_a)^2. \tag{5.1}$$

The first equality in Eq. (5.1) is a well-known factorization property of 4×4 antisymmetric matrices, while the second equality involves the $SO(6)$ rewrite (4.10). Introducing six-dimensional polar coordinates $\chi_a = \{r, \hat{\Omega}\}$, one then finds for the measure:

$$\int d\mu_{\text{phys}}^1 \, e^{-S_{\text{inst}}^1} \;=\; \frac{g^8 e^{2\pi i \tau}}{2^{31} \pi^{13} (N-1)!(N-2)!} \int d^4 X \, d\rho \, d^5 \hat{\Omega} \, \rho^{4N-7} I_N \prod_{A=1,2,3,4} d^2 \xi^A \, d^2 \bar{\eta}^A \; . \quad (5.2)$$

Here I_N denotes the r integration, which it is instructive to separate out:

$$I_N = \int_0^\infty dr \, r^{4N-3} \, e^{-2\rho^2 r^2} = \tfrac{1}{2}(2\rho^2)^{-2N+1} \int_0^\infty dx \, x^{2N-2} \, e^{-x} = \tfrac{1}{2}(2\rho^2)^{1-2N}(2N-2)! \; . \quad (5.3)$$

From Eqs. (5.2)-(5.3) one sees that the X_n and ρ integrations assemble into the scale-invariant AdS_5 volume form $d^4 X \, d\rho \, \rho^{-5}$. Moreover we can perform the integration over $\hat{\Omega}$, yielding $\text{Vol } S^5 = \pi^3$, to recapture the numerical factor C_N quoted in Eq. (1.26) (where the remaining powers of two come from the factors of 96 associated to each of the operator insertions).

Equations (5.2)-(5.3) are the principal result of [44], and, we stress, are exact for all N. However, this exactness can be traced to the factorization property (5.1), which is special to the one-instanton sector. In order to generalize the above to $k > 1$, at least in the large-N limit, it is important to reproduce these results in an alternative way, using saddle-point methods. To this end, note that the integral I_N is nothing but a Gamma function. For large N it is well approximated by Stirling's formula, which, the reader will recall, is derived as follows. First one rescales $r \to \sqrt{N} r$, or equivalently

$$\chi_a \to \sqrt{N} \chi_a \; , \quad (5.4)$$

so that N factors out of the exponent. The integral then becomes

$$I_N = N^{2N-1} \int_0^\infty dr \, r^{-3} \, e^{2N(\log r^2 - \rho^2 r^2)} \; , \quad (5.5)$$

which is in a form amenable to a standard saddle-point evaluation. The saddle-point is at $r = \rho^{-1}$ and, to leading order, a Gaussian integral around the solution gives

$$\lim_{N \to \infty} I_N = \rho^{2-4N} N^{2N-1} e^{-2N} \sqrt{\frac{\pi}{N}} \; , \quad (5.6)$$

which is valid up to $1/N$ corrections.

While this is all very elementary, we have uncovered something truly surprising. A single Yang-Mills instanton in an $\mathcal{N} = 4$ supersymmetric gauge theory is effectively parametrized at large N by its position X_n, scale size ρ, and a point $\hat{\Omega}$ on S^5 (as well as the sixteen fermionic collective coordinates). Notice that this parametrization *only* emerges in the large-N limit where the solution of the saddle-point equations identifies the radius of the sphere with the inverse of the instanton scale size; i.e. the radius of the sphere with the radial variable of AdS_5. These

parameters are precisely the coordinates required to specify the position of an object in the ten-dimensional space $AdS_5 \times S^5$, where ρ^{-1} is a radial coordinate on the AdS_5. Given Maldacena's conjectured correspondence between correlation functions in the supergravity theory in ten dimensions and the four-dimensional gauge theory on its boundary, this leads us to identify a large-N Yang-Mills instanton with a (singly-charged) D-instanton. In this identification, the sixteen supersymmetric and superconformal zero-modes of the Yang-Mills instanton $\{\xi^A_\alpha, \bar{\eta}^{\dot{\alpha}A}\}$ are identified with the sixteen supersymmetries of the Type IIB string theory broken by the D-instanton denoted ε in Eq. (1.7).

V.2 Naïvely the k-instanton moduli space contains k copies of $AdS_5 \times S^5$

We now turn to the case of general topological number k. For $k > 1$ one quickly abandons any hope of performing the integrations in Eq. (4.55) exactly. Instead, let us extend the large-N saddle-point treatment developed in Sec. V.1. As in the one-instanton sector, we first perform the rescaling (5.4). The gauge-invariant measure (4.55) then becomes:

$$
\int d\mu^k_{\text{phys}}\, e^{-S^k_{\text{inst}}} \;=\; \frac{g^{8k^2} N^{k^2} e^{2\pi i k \tau}}{2^{27k^2/2 - k/2}\, \pi^{13k^2}\, \text{Vol}\, U(k)} \int d^{k^2} W^0\, d^{4k^2} a'\, d^{6k^2} \chi \prod_{A=1,2,3,4} d^{2k^2} \mathcal{M}'^A\, d^{2k^2} \zeta^A
$$
$$
\times \;\; (\det{}_{2k} W \det{}_{4k} \chi)^{-2k} \, \exp\left[4\pi i g^{-1} \sqrt{N}\, \text{tr}_k\, \chi_{AB} \tilde{\Lambda}^{AB} \;-\; N S_{\text{eff}} \right] .
$$
$$\tag{5.7}$$

Here, in anticipation of the large-N limit, we have already applied Stirling's formula to the numerical prefactor (4.46). We have also introduced the "effective k-instanton action" S_{eff} in which all the terms which scale with N are collected:[22]

$$
S_{\text{eff}} = -2k(1 + 3\log 2) - \log \det{}_{2k} W - \log \det{}_{4k} \chi + \text{tr}_k\, \chi_a \mathbf{L} \chi_a .
$$
$$\tag{5.8}$$

This expression involves the $11k^2$ bosonic variables comprising the eleven independent $k \times k$ Hermitian matrices W^0, a'_n and χ_a. The remaining components W^c, $c = 1, 2, 3$, are eliminated in favor of the a'_n via the ADHM constraint (4.52). The action is invariant under the $U(k)$ symmetry (2.23) which acts by adjoint action on all the variables.

With N factored out of the exponent, the measure is in a form which is amenable to a saddle-point treatment as $N \to \infty$. The coupled saddle-point equations read:

$$
\epsilon^{ABCD} \left(\mathbf{L} \cdot \chi_{AB} \right) \chi_{CE} \;=\; \tfrac{1}{2} \delta^D_E\, 1_{[k] \times [k]} ,
$$
$$\tag{5.9a}$$
$$
\chi_a \chi_a \;=\; \tfrac{1}{2} (W^{-1})^0 ,
$$
$$\tag{5.9b}$$
$$
[\chi_a, [\chi_a, a'_n]] \;=\; i\bar{\eta}^c_{nm} [a'_m, (W^{-1})^c] .
$$
$$\tag{5.9c}$$

---

[22]As previously, we translate back and forth as convenient between the antisymmetric tensor representation χ_{AB} and the $SO(6)_R$ vector representation χ_a, $a = 1, \dots, 6$ (4.10).

These are obtained by varying S_{eff} with respect to the matrix elements of χ, W^0 and a'_n, respectively, and rewriting "log det" as "tr log." We have employed the $k \times k$ matrices

$$(W^{-1})^0 = \text{tr}_2 \, W^{-1}, \qquad (W^{-1})^c = \text{tr}_2 \, \tau^c W^{-1}. \tag{5.10}$$

The general solution to these coupled saddle-point equations is easily found. It has the simple property that all the quantities are diagonal in instanton indices,

$$\begin{align}
W^0 &= \text{diag}\left(2\rho_1^2, \ldots, 2\rho_k^2\right), \tag{5.11a}\\
\chi_a &= \text{diag}\left(\rho_1^{-1}\hat{\Omega}_a^1, \ldots, \rho_k^{-1}\hat{\Omega}_a^k\right), \tag{5.11b}\\
a'_n &= \text{diag}\left(-X_n^1, \ldots, -X_n^k\right), \tag{5.11c}
\end{align}$$

up to a common adjoint action by the $U(k)/U(1)^k_{\text{diag}}$ residual symmetry. For each value of the instanton index $i = 1, \ldots, k$, the six-dimensional unit vector $\hat{\Omega}_a^i$ parametrizes an independent point on S^5,

$$\hat{\Omega}_a^i \hat{\Omega}_a^i = 1 \qquad \text{(no sum on } i\text{)}, \tag{5.12}$$

where the radius of the ith S^5 factor is ρ_i^{-1}.

A simple picture of this leading-order saddle-point solution emerges: it can be thought of as k independent copies of the one-instanton saddle-point solution described in Sec. V.1, where the ith instanton is parametrized by a point $(X_n^i, \rho_i, \hat{\Omega}_a^i)$ on $AdS_5 \times S^5$. Additional insight into this solution emerges from considering the $SU(2)$ generators $(t_i^c)_{uv}$ describing the embedding of the ith instanton inside $SU(N)$. From Eqs. (2.44c), (2.54) and (5.11a) one derives the commutation relations

$$\left[t_i^a, t_j^b\right] = 2i\delta_{ij}\,\epsilon_{abc}\,t_i^c \tag{5.13}$$

so that at the saddle-point, thanks to the Kronecker-δ, the k individual instantons lie in k mutually commuting $SU(2)$ subgroups. Actually this feature follows from large-N statistics alone,[23] and has nothing to do with either the existence of supersymmetry or with the details of the ADHM construction; nevertheless the fact that Eq. (5.13) emerges from the saddle-point equations is a reassuring check on our formalism. In this one particular respect, the solution (5.11a)-(5.11c) can be said to be dilute-gas-like. Another important property is that the effective action (5.8) evaluated on these saddle-point solutions is zero; hence there is no exponential dependence on N in the final result. Finally we should make the technical point that, thanks to the diagonal structure of these solutions, they are automatically consistent with the triangle inequalities on the boson bilinear W discussed in the paragraph following Eq. (4.46); hence we never need to specify more explicitly the integration limits on the W variables.

---

[23]Consider the analogous problem of k randomly oriented vectors in $\mathbb{R}^N$ in the limit $N \to \infty$; clearly the dot products of these vectors tend to zero simply due to statistics.

We are now squarely faced with the puzzle described in Sec. I: the fact that we have obtained k copies of $AdS_5 \times S^5$ ostensibly contradicts the supergravity side of the AdS/CFT correspondence, where only one copy of this moduli space appears at each topological level k (e.g., see the supergravity expression (1.15) for the correlator G_{16}). In the following subsection we show that this k-fold degeneracy is lifted when one considers small fluctuations in the neighborhood of the generic solution (5.11a)-(5.11c), and that the Yang-Mills multi-instanton moduli space indeed collapses to a single copy, precisely as dictated by Maldacena's conjecture.

V.3 The exact k-instanton moduli space collapses to one copy of $AdS_5 \times S^5$

Let us repeat some general lore about saddle-point methods. To evaluate a multi-dimensional integral in saddle-point approximation, it never suffices merely to identify the space of saddle-point solution(s). One must also evaluate the leading non-vanishing small-fluctuations integral in the background of the solution(s). Typically this small-fluctuations integral is Gaussian, and produces a prefactor proportional to $(\det \mathrm{M})^{-1/2}$ where M is the quadratic form. If M has null-vector directions, one needs to go beyond quadratic order in small fluctuations, specifically up to the order where the degeneracy in these directions is lifted. Of course, the degeneracy may persist to all orders reflecting the existence of a moduli-space of saddle-point solutions. One must then introduce collective coordinates which parametrize the flat directions of the leading-order action[24] and restrict the small-fluctuations integration to directions orthogonal to the moduli. However, the flat directions of the leading-order action can still be lifted at higher order in $1/N$. For example, this happens when the small fluctuations determinant described above depends explicitly on the moduli. On exponentiating the determinant we find a potential term of order N^0 which corrects the leading-order action and lifts the flat directions. This effect is particularly important when the space of saddle-point solutions is non-compact as the potential term can render the otherwise divergent integral over the collective coordinates finite. Finally, there may be flat directions which remain unlifted to all orders in $1/N$ (typically because they are protected by a symmetry). In this case the corresponding collective coordinates parametrize a moduli-space of exactly degenerate stationary-points of the effective action to all orders in $1/N$.

Now let us apply these general considerations to the particular space of saddle-point solutions obtained in Sec. V.2 above. At the k-instanton level, to leading order in $1/N$, this solution space has the geometry of $(AdS_5 \times S^5)^k$. However, a small-fluctuations analysis in the background of a generic solution point confirms that most of this degeneracy is actually lifted at the next order by the effect described above. Specifically, the flat directions corresponding to the *relative* positions of the k instantons on $AdS_5 \times S^5$ are lifted at order N^0. Thus we find

---

[24]In the following, "leading-order action" denotes those terms in the exponent of the integrand which are of order N.

that the exact, unlifted, bosonic moduli space consists of only one copy of $AdS_5 \times S^5$ which corresponds to the center-of-mass position of the instantons. Furthermore, we will see that the large-N saddle-point expansion can be reformulated as an expansion in fluctuations around the exact saddle-points parametrized by these moduli. In particular, the leading order in this expansion can be elegantly reinterpreted as a partition function of a standard supersymmetric $SU(k)$ gauge theory, and in this way, reduced to previously known results. We will see that this matches precisely the description of D-instantons in string theory.

In order to verify these claims, let us first perform a small-fluctuations expansion in the neighborhood of a generic solution point on $(AdS_5 \times S^5)^k$. By "generic" we mean that for all i and j,

$$X_n^i \neq X_n^j , \quad \hat{\Omega}_a^i \neq \hat{\Omega}_a^j , \quad \rho_i \neq \rho_j . \tag{5.14}$$

We will truncate the expansion at Gaussian order, and restrict ourselves for now to the bosonic degrees of freedom. Let us introduce $k \times k$ fluctuating fields $\{ \delta W_{ij}^0 , \delta\chi_{aij} , \delta a_{nij}' \}$, in the background of the solution (5.11a)-(5.11c), by defining:

$$W^0 = \text{diag}\big(2\rho_1^2, \ldots, 2\rho_k^2\big) + \delta W^0 , \tag{5.15a}$$

$$\chi_a = \text{diag}\big(\rho_1^{-1}\hat{\Omega}_a^1, \ldots, \rho_k^{-1}\hat{\Omega}_a^k\big) + \delta\chi_a , \tag{5.15b}$$

$$a_n' = \text{diag}\big(-X_n^1, \ldots, -X_n^k\big) + \delta a_n' . \tag{5.15c}$$

Inserting these expressions into S_{eff}, one finds that the Gaussian integration is governed by the effective quadratic action

$$S^{(2)} = \sum_{1 \leq i,j \leq k} \Big(v_{ij} \cdot v_{ij} \left| \sigma_{ij}' \cdot \sigma_{ij}' \right| - \left| v_{ij} \cdot \sigma_{ij}' \right|^2 + \left| u_{ij} \cdot \sigma_{ij} \right|^2 \Big). \tag{5.16}$$

Here σ and σ' denote the matrix-valued 11-vectors of fluctuating fields[25]

$$\sigma_{ij} = \big(\rho_i\rho_j\delta\chi_{aij} , \, \delta a_{nij}' , \, (2\rho_i\rho_j)^{-1}\delta W_{ij}^0\big) , \qquad \sigma_{ij}' = \big(\rho_i\rho_j\delta\chi_{aij} , \, \delta a_{nij}' , \, 0\big) , \tag{5.17}$$

while v_{ij} and u_{ij} are matrix-valued 11-vectors formed from parameters of the saddle-point solution:

$$v_{ij} = \big(\rho_i^{-1}\hat{\Omega}_a^i - \rho_j^{-1}\hat{\Omega}_a^j , \, (\rho_i\rho_j)^{-1}(X_n^i - X_n^j) , \, 0\big) , \qquad u_{ij} = \big(\rho_i^{-1}\hat{\Omega}_a^i + \rho_j^{-1}\hat{\Omega}_a^j , \, 0 , \, 1\big). \tag{5.18}$$

By inspection, the quadratic form (5.16) is positive semi-definite. The null-vectors can be classified as follows. For the diagonal elements, the variations

$$\sigma_{ii} = \big(\rho_i^2\delta\chi_{aii} , \, \delta a_{nii}' , \, -2\rho_i\hat{\Omega}^i \cdot \delta\chi_{ii}\big) , \tag{5.19}$$

---

[25]Inner products on such 11-vectors are defined as $(v^1, v^2, v^3) \cdot (u^1, u^2, u^3) = v_a^1 u_a^1 + v_n^2 u_n^2 + v^3 u^3$.

have zero eigenvalue. These correspond to motion along the $10k$-dimensional solution manifold (5.11a)-(5.11c) parametrized by $\{X_n^i, \hat{\Omega}_a^i, \rho_i\}$. As for the off-diagonal elements, the only null-vectors are the $k(k-1)$ independent variations σ_{ij} in the directions

$$\left(\rho_j \hat{\Omega}_a^i - \rho_i \hat{\Omega}_a^j, \, -X_n^i + X_n^j, \, \rho_i/\rho_j - \rho_j/\rho_i\right), \qquad i \neq j, \tag{5.20}$$

with no sum on i and j. These are precisely the variations induced by the adjoint action of $U(k)/U(1)_{\text{diag}}^k$ on the solution. In summary, at a generic point (5.14), all the Gaussian null-vectors correspond to directions which are already explicitly taken into account by the moduli for $(AdS_5 \times S^5)^k \times U(k)/U(1)_{\text{diag}}^k$. Conversely, since there are no additional null-vectors, we conclude that at a generic solution point this moduli space is complete (at least locally).

However the generic case (5.14) is not the full story. In particular, whenever a vector v_{ij} associated to a pair of instantons is zero, additional flat directions appear at quadratic order. The condition $v_{ij} = 0$ requires that the two instantons live at the same point in $AdS_5 \times S^5$, i.e., $X_n^i = X_n^j$, $\rho_i = \rho_j$, and $\hat{\Omega}_a^i = \hat{\Omega}_a^j$. At this special point, all the variations σ_{ij} which are orthogonal to the vector u_{ij} are flat to quadratic order. Correspondingly, the determinantal prefactor $(\det M)^{-1/2}$ blows up. Additional zeroes of this determinant appear whenever the positions of additional instantons coalesce, leading to a greater divergence. As above, if one were to exponentiate this determinantal prefactor, one would naturally obtain a correction to the leading-order action of order N^0. In the present case this leads to an attractive singular potential which partially lifts the degeneracy in the moduli space, and draws the k distinct instantons to a *common center* in $AdS_5 \times S^5$! Loosely speaking, the attractive interaction results in a k-instanton "bound state". However, the divergent nature of potential indicates that the above analysis in terms of quadratic fluctuations around the generic saddle-point is not adequate. In particular, the extra zero modes appearing when two instantons approach each other mean that, in these regions of the moduli space, we must go to higher order in the fluctuation expansion.

To perform a consistent saddle-point analysis, we should now expand to next-to-leading order around the generic solution and augment the quadratic action $S^{(2)}$ with a term $S^{(4)}$ quartic in δW^0, $\delta \chi_a$ and $\delta a_n'$. After integrating out these fluctuations, we should finally obtain a convergent integral over those moduli of the generic saddle-point solution which describe the relative positions of the k instantons on $AdS_5 \times S^5$. In fact we will meet an important simplification which means that we will not need to perform these steps in full. We will find that the leading order in $1/N$ can actually be obtained by a much simpler proceedure: rather than expanding in fluctuations around the generic solution we will instead expand around the maximally degenerate solution identified above. This amounts to treating the $k-1$ relative positions on $AdS_5 \times S^5$ as fluctuations around the maximally-degenerate solution rather than moduli of the generic solution. One way of seeing that the two proceedures are equivalent at leading order in $1/N$ is to note that $S^{(2)}$, which by definition is quadratic in fluctuations around the generic solution, also happens to depend only quadratically on the moduli. This means that the exact expression (5.16) for $S^{(2)}$ can also be viewed as a term of quartic order in

an expansion around the maximally-degenerate solution where the additional moduli are now thought of as fluctuations. Thus the leading-order term in our new expansion of the action automatically contains the leading-order term in the old one. However, the leading term in the new expansion also contains additional quartic terms which lift the extra zero modes of the degenerate configurations thereby remove the divergent behaviour discovered above. In the following we shall find, up to an overall integral over one copy of $AdS_5 \times S^5$, a finite well-defined answer for the leading term in the large-N expansion of the ADHM measure.

As discussed above we will now begin our small fluctuations analysis afresh, this time in the background of the *maximally degenerate* saddle-point solution

$$W^0 = 2\rho^2 \, 1_{[k]\times[k]}, \qquad \chi_a = \rho^{-1}\hat{\Omega}_a \, 1_{[k]\times[k]}, \qquad a'_n = -X_n \, 1_{[k]\times[k]} \, , \qquad (5.21)$$

which corresponds to k instantons living at a common point $\{X_n, \hat{\Omega}_a, \rho\}$ in $AdS_5 \times S^5$. (From the ADHM constraint (4.52) it follows that the remaining components of W vanish: $W^c = 0$ for $c = 1, 2, 3$.) This degenerate solution, unlike Eqs. (5.11a)-(5.11c), is invariant under the residual $U(k)$. With the instantons sitting on top of one another, it looks like the complete opposite of the dilute instanton gas limit; however the instantons still live in k mutually commuting $SU(2)$ subgroups of $SU(N)$ as per Eq. (5.13), which is a dilute-gas-like feature. Notice that, just as in the one instanton sector, the radius of the S^5 is fixed to be ρ^{-1} at the saddle-point.

In order to expand about this special solution, one first needs to factor out the exact moduli corresponding to the common position of the k-instanton "bound state" on $AdS_5 \times S^5$. This is done in the following way. For each $k \times k$ matrix, we introduce a basis of traceless Hermitian matrices $\hat{T}^r$, $r = 1, \ldots, k^2 - 1$, normalized by $\mathrm{tr}_k \hat{T}^r \hat{T}^s = \delta^{rs}$. For each $k \times k$ matrix v we separate out the "scalar" component v_0 by taking

$$v = v_0 1_{[k]\times[k]} + \hat{v}^r \hat{T}^r \, . \qquad (5.22)$$

The change of variables from the T^r basis used in (4.1) $\{1_{[k]\times[k]}, \hat{T}^r\}$ involves a Jacobian

$$d^{k^2}v = k^{\pm 1/2}dv_0 \, d^{k^2-1}\hat{v}, \qquad (5.23)$$

where ± 1 refers to bosons and fermions, respectively. For the moment we continue to focus on the bosonic variables, which are decomposed as follows:

$$a'_n = -X_n 1_{[k]\times[k]} + \hat{a}'_n \, , \qquad (5.24\mathrm{a})$$

$$\chi_a = \rho^{-1}\hat{\Omega}_a 1_{[k]\times[k]} + \hat{\chi}_a \, . \qquad (5.24\mathrm{b})$$

By definition the traceless matrix variables $\hat{a}'_n$ and $\hat{\chi}_a$ are the fluctuating fields (there is no need to write $\delta\hat{a}'_n$ or $\delta\hat{\chi}_a$). Note that, in terms of the generic solution considered above, the diagonal entries of the matrices $\hat{a}'_n$ and $\hat{\chi}_a$ correspond to the moduli u and v while the off-diagonal elements correspond to fluctuations.

Inserting Eqs. (5.24a)-(5.24b) into Eq. (5.8) and Taylor expanding is a tedious though straightforward exercise. As anticipated above, we will now expand to fourth order in the fluctuating fields around the solution parametrized by the ten exact $AdS_5 \times S^5$ moduli. The expansion of the determinant terms in (5.8) is facilitated by first writing "log det" as "tr log" and then expanding the logarithm:

$$
\begin{aligned}
\mathrm{tr}_{2k} \log W &= 2k \log \rho^2 + \frac{1}{\rho^2} \mathrm{tr}_k(\delta W^0) - \frac{1}{4\rho^4} \mathrm{tr}_k(\delta W^0)^2 + \frac{1}{12\rho^6} \mathrm{tr}_k(\delta W^0)^3 \\
&\quad - \frac{1}{32\rho^8} \mathrm{tr}_k(\delta W^0)^4 + \frac{1}{2\rho^4} \mathrm{tr}_k \left[\hat{a}'_n, \hat{a}'_m\right]^2 + \cdots ,
\end{aligned}
\tag{5.25}
$$

and

$$
\mathrm{tr}_{4k} \log \chi = -2k \log(8\rho^2) - 2^5 \rho^2 \mathrm{tr}_{4k} \left(\hat{\Omega}^* \hat{\chi}\right)^2 + \frac{2^9 \rho^3}{3} \mathrm{tr}_{4k} \left(\hat{\Omega}^* \hat{\chi}\right)^3 - 2^{10} \rho^4 \mathrm{tr}_{4k} \left(\hat{\Omega}^* \hat{\chi}\right)^4 + \cdots .
\tag{5.26}
$$

In these expansions we have dropped fifth- and higher-order terms in the fluctuating fields. Here we are anticipating the fact that these terms are not needed to regularize the small-fluctuations integration. On obtaining a finite answer from the leading-order terms we may immediately rescale the integration variables in a standard way which shows that the fluctuations around the maximally degenerate saddle point are of order $N^{-1/4}$. The higher order terms in the exponent therefore yield subleading contributions in the large-N expansion. In particular, this is true for the diagonal components of $\hat{a}'_n$ and $\hat{\chi}_a$ which correspond to the moduli of the generic saddle-point solution discussed earlier in this Section. This shows that our large-N expansion around the maximally degenerate saddle-point is self-consistent. In Eq. (5.25) we have used Eq. (4.52), while in Eq. (5.26) and in subsequent equations we move back and forth as convenient between the 6-vector and the antisymmetric tensor representations of $\hat{\Omega}$ and χ using Eq. (4.10). In particular, the $SO(6)$ orthonormality condition $\hat{\Omega} \cdot \hat{\Omega} = 1$ becomes, in 4×4 matrix language,

$$
\hat{\Omega}\hat{\Omega}^* = -\tfrac{1}{8} 1_{[4]\times[4]} , \quad \text{or} \quad \hat{\Omega}^{-1} = -8\hat{\Omega}^* ,
\tag{5.27}
$$

which has been implemented in Eq. (5.26).

Next we need a systematic method for re-expressing the traces over $4k \times 4k$ matrices in Eq. (5.26) as traces over $k \times k$ matrices. We will exploit the following "moves":[26]

$$
\hat{\Omega}^* \hat{\chi} = -\hat{\chi}^\dagger \hat{\Omega} - \tfrac{1}{4}(\hat{\Omega} \cdot \hat{\chi}) 1_{[4]\times[4]} ,
\tag{5.28a}
$$

$$
\mathrm{tr}_4 E^\dagger F = \mathrm{tr}_4 F^\dagger E = -\tfrac{1}{2}(E \cdot F) ,
\tag{5.28b}
$$

$$
\mathrm{tr}_4 E^\dagger F G^\dagger H = \tfrac{1}{16}\left(E_a F_a G_b H_b - E_a F_b G_a H_b + E_a F_b G_b H_a\right) .
\tag{5.28c}
$$

On the left-hand sides of Eq. (5.28b)-(5.28c), the $4k \times 4k$ matrices $\{E, F, G, H\}$ are assumed to be antisymmetric in $SU(4)_R$ indices and subject to the usual conditions (4.9)-(4.10); the

---

[26] In the following, we should emphasize that $\dagger$ only acts on instanton indices, as per the reality condition (4.9), and *not* on $SU(4)$ matrix indices.

identity (5.28a) follows from a double application of Eq. (4.9). Using Eqs. (5.27)-(5.28c) in an iterative fashion, it is then easy to derive the following trace identities:

$$
\begin{aligned}
\mathrm{tr}_{4k}\left(\hat{\Omega}^*\hat{\chi}\right)^2 &= -\mathrm{tr}_{4k}\,\hat{\chi}^\dagger\hat{\Omega}\hat{\Omega}^*\hat{\chi} - \tfrac{1}{4}\mathrm{tr}_k\big(\hat{\Omega}\cdot\hat{\chi}\,\mathrm{tr}_4(\hat{\Omega}^*\hat{\chi})\big) \\
&= \frac{1}{2^3}\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^2 - \frac{1}{2^4}\mathrm{tr}_k\,\hat{\chi}\cdot\hat{\chi}\,, \qquad\qquad\qquad (5.29\text{a})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}_{4k}\left(\hat{\Omega}^*\hat{\chi}\right)^3 &= \tfrac{1}{8}\mathrm{tr}_{4k}(\hat{\chi}^\dagger\hat{\chi}\hat{\Omega}^*\hat{\chi}) + \tfrac{1}{64}\mathrm{tr}_k\,(\hat{\Omega}\cdot\hat{\chi})^2\hat{\chi}\cdot\hat{\chi} - \tfrac{1}{32}\mathrm{tr}_k\,(\hat{\Omega}\cdot\hat{\chi})^3 \\
&= -\frac{1}{2^5}\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^3 + \frac{3}{2^7}\mathrm{tr}_k\,\hat{\chi}\cdot\hat{\chi}\,\hat{\Omega}\cdot\hat{\chi}\,, \qquad\qquad (5.29\text{b})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}_{4k}\left(\hat{\Omega}^*\hat{\chi}\right)^4 &= \mathrm{tr}_{4k}\big(\tfrac{1}{64}\hat{\chi}^\dagger\hat{\chi}\hat{\chi}^\dagger\hat{\chi} - \tfrac{1}{32}\hat{\chi}^\dagger\hat{\chi}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi} \\
&\qquad\quad - \tfrac{1}{32}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi}\hat{\chi}^\dagger\hat{\chi} + \tfrac{1}{16}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi}\big) \\
&= \frac{1}{2^7}\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^4 - \frac{1}{2^7}\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^2\hat{\chi}\cdot\hat{\chi} + \frac{1}{2^9}\mathrm{tr}_k\left(\hat{\chi}\cdot\hat{\chi}\right)^2 - \frac{1}{2^{10}}\mathrm{tr}_k\,\hat{\chi}_a\hat{\chi}_b\hat{\chi}_a\hat{\chi}_b\,.
\end{aligned}
$$
$$(5.29\text{c})$$

As before, on the left-hand side of these formulae, $4k \times 4k$ matrix multiplication is implied, whereas on the right-hand side, all $SO(6)$ indices are saturated in standard 6-vector inner products, leaving the traces over $k \times k$ matrices.

From Eqs. (5.8), (5.25), (5.26), and (5.29a)-(5.29c), one obtains for the bosonic effective action:

$$
S_{\mathrm{b}} = S^{(2)} + S^{(3)} + S^{(4)} \tag{5.30}
$$

where the quadratic, cubic and quartic actions are now given entirely as k-dimensional (rather than $2k$ or $4k$-dimensional) traces:

$$
S^{(2)} = \mathrm{tr}_k\,\varphi^2\,, \qquad \varphi = 2\rho\hat{\Omega}\cdot\hat{\chi} + \frac{1}{2\rho^2}\delta W^0\,, \tag{5.31a}
$$

$$
\begin{aligned}
S^{(3)} &= -\frac{1}{12\rho^6}\mathrm{tr}_k\,(\delta W^0)^3 + 4\rho^3\mathrm{tr}_k\,\hat{\Omega}\cdot\hat{\chi}\,\hat{\chi}\cdot\hat{\chi} - \frac{16\rho^3}{3}\mathrm{tr}_k\,(\hat{\Omega}\cdot\hat{\chi})^3 + \mathrm{tr}_k\,\delta W^0\hat{\chi}\cdot\hat{\chi} \\
&= 2\rho^2\,\mathrm{tr}_k\,\varphi\big(\hat{\chi}\cdot\hat{\chi} - 4(\hat{\Omega}\cdot\hat{\chi})^2\big) + \cdots\,, \tag{5.31b}
\end{aligned}
$$

$$
\begin{aligned}
S^{(4)} &= -\frac{1}{2\rho^4}\mathrm{tr}_k\left[\hat{a}_n',\hat{a}_m'\right]^2 + \frac{1}{32\rho^8}\mathrm{tr}_k(\delta W^0)^4 - \mathrm{tr}_k\left[\hat{\chi}_a,\hat{a}_n'\right]\left[\hat{\chi}_a,\hat{a}_n'\right] + 8\rho^4\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^4 \\
&\qquad - 8\rho^4\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^2\hat{\chi}\cdot\hat{\chi} + 2\rho^4\mathrm{tr}_k\left(\hat{\chi}\cdot\hat{\chi}\right)^2 - \rho^4\mathrm{tr}_k\,\hat{\chi}_a\hat{\chi}_b\hat{\chi}_a\hat{\chi}_b \\
&= -\frac{1}{2\rho^4}\mathrm{tr}_k\left[\hat{a}_n',\hat{a}_m'\right]^2 - 8\rho^4\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^2\hat{\chi}\cdot\hat{\chi} + 2\rho^4\mathrm{tr}_k\left(\hat{\chi}\cdot\hat{\chi}\right)^2 \\
&\qquad + 16\rho^4\mathrm{tr}_k\left(\hat{\Omega}\cdot\hat{\chi}\right)^4 - \rho^4\mathrm{tr}_k\,\hat{\chi}_a\hat{\chi}_b\hat{\chi}_a\hat{\chi}_b - \mathrm{tr}_k\left[\hat{\chi}_a,\hat{a}_n'\right]\left[\hat{\chi}_a,\hat{a}_n'\right] + \cdots\,. \tag{5.31c}
\end{aligned}
$$

Notice that only k^2 fluctuations, denoted φ, are actually lifted at quadratic order. This, in turn, implies that certain terms in $S^{(3)}$ and $S^{(4)}$ are subleading, and can be omitted. Specifically, the omitted terms in the final rewrites in Eqs. (5.31b)-(5.31c) contain, respectively, two or more, and one or more, factors of the quadratically lifted φ modes, and consequently are suppressed in large N (as a simple rescaling argument again confirms).

Now let us perform the elementary Gaussian integration over the φ's. Changing integration variables in Eq. (5.7) from $d^{k^2}W^0$ to $d^{k^2}\varphi$ using Eq. (5.31a), one finds:

$$\int d^{k^2}W^0\, e^{-N(S^{(2)}+S^{(3)})} \;=\; \left(\frac{4\pi\rho^4}{N}\right)^{k^2/2} e^{-NS^{(4)\prime}} \tag{5.32}$$

where the new induced quartic coupling reads

$$S^{(4)\prime} \;=\; -\rho^4 \mathrm{tr}_k \left(\hat{\chi}\cdot\hat{\chi} - 4(\hat{\Omega}\cdot\hat{\chi})^2\right)^2 . \tag{5.33}$$

Combining $S^{(4)\prime}$ with the original quartic coupling (5.31c) gives for the effective bosonic small-fluctuations action:

$$S_{\mathrm{b}} = -\frac{1}{2}\mathrm{tr}_k \left(\rho^{-4}[\hat{a}'_n,\hat{a}'_m]^2 + 2[\hat{\chi}_a,\hat{a}'_n]^2 + \rho^4[\hat{\chi}_a,\hat{\chi}_b]^2\right) . \tag{5.34}$$

Remarkably, all dependence on the unit vector $\hat{\Omega}_a$ has canceled out.

Notice that apart from the absence of derivative terms, the expression (5.34) looks like a Yang-Mills field strength for the gauge group $SU(k)$! We can make this explicit by introducing a ten-dimensional vector field A_μ,

$$A_\mu = N^{1/4}\left(\rho^{-1}\hat{a}'_n, \rho\hat{\chi}_a\right), \quad \mu = 0,\dots,9 , \tag{5.35}$$

in terms of which

$$NS_{\mathrm{b}}(A_\mu) \;=\; -\frac{1}{2}\mathrm{tr}_k\, [A_\mu, A_\nu]^2 . \tag{5.36}$$

We recognize this as precisely the action of ten-dimensional $SU(k)$ gauge theory, reduced to $0+0$ dimensions, i.e., with all derivatives set to zero.

Now let us turn to the fermions. Since $\mathcal{N}=4$ supersymmetry in four dimensions descends from $\mathcal{N}=1$ supersymmetry in ten dimensions, and since all our saddle-point manipulations commute with supersymmetry, we expect to find the $\mathcal{N}=1$ supersymmetric completion of the ten-dimensional dimensionally-reduced action (5.36), namely

$$NS_{\mathrm{f}}(A_\mu, \Psi) = \mathrm{tr}_k\, \bar{\Psi}\Gamma_\mu\, [A_\mu, \Psi] , \tag{5.37}$$

where Ψ is a ten-dimensional Majorana-Weyl spinor, and Γ_μ is an element of the ten-dimensional Clifford algebra. To see how this comes about, we first separate out from the fermionic collective coordinates the exact zero modes, in analogy to Eqs. (5.24a)-(5.24b):

$$\mathcal{M}'^A_\alpha = 4\xi^A_\alpha 1_{[k]\times[k]} + 4\hat{a}'_{\alpha\dot\alpha}\bar{\eta}^{\dot\alpha A} + \hat{\mathcal{M}}'^A_\alpha , \tag{5.38a}$$

$$\zeta^{\dot\alpha A} = 4\bar{\eta}^{\dot\alpha A}1_{[k]\times[k]} + \hat{\zeta}^{\dot\alpha A} . \tag{5.38b}$$

Here ξ^A_α and $\bar{\eta}^{\dot\alpha A}$ are the supersymmetric and superconformal fermion modes (2.53a)-(2.53b). Expanding the fermion coupling in the exponent of (5.7) around the special solution (5.21) and

using the relations (4.52) and (5.31a), we find

$$NS_{\rm f} = i\left(\frac{8\pi^2 N}{g^2}\right)^{1/2} {\rm tr}_k\left[\left(\varphi - 2\rho(\hat{\Omega}\cdot\hat{\chi})\right)\rho\hat{\Omega}_{AB}\hat{\zeta}^{\dot\alpha A}\hat{\zeta}^B_{\dot\alpha} + \rho^{-1}\hat{\Omega}_{AB}\left[\hat{a}'_{\alpha\dot\alpha}, \hat{\mathcal{M}}'^{\alpha A}\right]\hat{\zeta}^{\dot\alpha B}\right.$$
$$\left. + \hat{\chi}_{AB}\left(\rho^2\hat{\zeta}^{\dot\alpha A}\hat{\zeta}^B_{\dot\alpha} + \hat{\mathcal{M}}'^{\alpha A}\hat{\mathcal{M}}'^B_\alpha\right)\right] . \tag{5.39}$$

If we now define the $d = 10$ Majorana-Weyl fermion field Ψ

$$\Psi = \sqrt{\frac{\pi}{2g}}\, N^{1/8}\left(\rho^{-1/2}\hat{\mathcal{M}}'^A_\alpha \,,\, \rho^{1/2}\hat{\zeta}^{\dot\alpha A}\right) , \tag{5.40}$$

and the Γ_μ matrices according to Eq. (A.12) below, we do in fact recover the simple form (5.37). In moving from Eq. (5.39) to Eq. (5.37) we have dropped the term depending on φ; since φ is a quadratically lifted bosonic mode its contribution is suppressed in large N compared to the other couplings in Eq. (5.39), as a simple rescaling argument confirms. Note that unlike the bosonic sector, the $\hat{\Omega}_a$ dependence of the fermionic action does not actually disappear; as discussed in the Appendix, it is simply subsumed into the representation of the ten-dimensional Clifford algebra.

Finally our effective measure for the k instantons has the form

$$\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} \underset{N\to\infty}{=} \frac{\sqrt{N}g^8 e^{2\pi i k\tau}}{k^3 2^{17k^2/2-k/2+25}\pi^{9k^2/2+9}}\int \rho^{-5}d\rho\, d^4X\, d^5\hat{\Omega}\prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A \cdot \hat{\mathcal{Z}}_k , \tag{5.41}$$

where $\hat{\mathcal{Z}}_k$ is the partition function of an $\mathcal{N} = 1$ supersymmetric $SU(k)$ gauge theory in ten dimensions dimensionally reduced to zero dimensions:

$$\hat{\mathcal{Z}}_k = \frac{1}{{\rm Vol}\, SU(k)}\int_{SU(k)} d^{10}A\, d^{16}\Psi\, e^{-S(A_\mu,\Psi)} ,$$
$$S(A_\mu,\Psi) = N(S_{\rm b} + S_{\rm f}) = -\frac{1}{2}{\rm tr}_k\, [A_\mu, A_\nu]^2 + {\rm tr}_k\, \bar{\Psi}\Gamma_\mu\, [A_\mu, \Psi] . \tag{5.42}$$

Notice that the rest of the measure, up to numerical factors, is independent of the instanton number k. When integrating expressions which are independent of the $SU(k)$ degrees-of-freedom, $\hat{\mathcal{Z}}_k$ is simply an overall constant factor. A calculation of Ref. [57] revealed that $\hat{\mathcal{Z}}_k$ is proportional to $\sum_{d|k} d^{-2}$, a sum over the positive integer divisors d of k. However, the constant of proportionality was fixed definitively in Ref. [56] to give[27]

$$\hat{\mathcal{Z}}_k = 2^{17k^2/2-k/2-8}\pi^{9k^2/2-9/2}k^{-1/2}\sum_{d|k}\frac{1}{d^2} . \tag{5.43}$$

In evaluating $\hat{\mathcal{Z}}_k$, we have used

$${\rm Vol}\left(SU(k)\right) = \frac{2^{k-1}\pi^{k(k+1)/2-1}}{\prod_{i=1}^{k-1} i!} . \tag{5.44}$$

---

[27]In comparing to Ref. [56], it is important to note that our convention for the normalization of the generators is ${\rm tr}_k\hat{T}^r\hat{T}^s = \delta^{rs}$, rather than $\frac{1}{2}\delta^{rs}$ in Ref. [56].

In summary, on gauge invariant and $SU(k)$ singlet operators, our effective large-N collective coordinate measure has the following simple form:

$$\int d\mu_{\text{phys}}^k \, e^{-S_{\text{inst}}^k} \underset{N\to\infty}{=} \frac{\sqrt{N}g^8}{2^{33}\pi^{27/2}} k^{-7/2} e^{2\pi i k\tau} \sum_{d|k} \frac{1}{d^2} \int \frac{d^4X \, d\rho}{\rho^5} d^5\hat{\Omega} \prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A \ . \quad (5.45)$$

As expected from the AdS/CFT correspondence (albeit counter-intuitive from the point of view of the field theory), only one copy of the $AdS_5 \times S^5$ moduli space appears at any k.

V.4 Comments on the ten-dimensional $SU(k)$ partition function

It is worthwhile making some remarks about the $SU(k)$ partition function $\hat{\mathcal{Z}}_k$ defined in Eq. (5.42). To begin with, it may surprise the reader that this integral even exists! The following naïve argument suggests that it diverges [56]: consider the basis where A_1 (say) is diagonal; then fluctuations in the directions where the other A_μ are also diagonal are unsuppressed. However, this argument does not take into account that for sufficiently large k and/or D (D being the dimension of space-time), these fluctuations are of measure zero. The question of convergence has only recently been definitively settled in Refs. [56]. For the purely bosonic version of the integral, it converges in the cases $\{D = 3, k \geq 4\}$, $\{D = 4, k \geq 3\}$ and $\{D \geq 5, k \geq 2\}$, and diverges otherwise. In the supersymmetric version the convergence is even better; in fact no divergent cases are believed to exist.

Second, we invite the reader to compare our gauge invariant measure (5.41) with the measure for D-instanton in flat space, the $U(k)$ partition function (4.13). Clearly the $U(1)$ part of this partition function, describing the center-of-mass coordinates of the D-instanton configuration has been generalized to the (supersymmetrized) volume measure on $AdS_5 \times S^5$, as in (1.7); however, the $SU(k)$ part of the partition function is identical in both cases. Hence, we find a very attractive matching of large-N Yang-Mills instantons with string theory D-instantons. In particular, on the D-instanton side, the intuition of [36–39], suggests that, as far as the contribution to the correlation functions G_n, the D-instantons contribution can be thought of as being due to a charge k D-instanton bound state, is matched by our large-N instanton analysis and our identification of the moduli space as one copy of $AdS_5 \times S^5$.

Third, it is interesting that the large-N description of k Yang-Mills instantons, in an $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory, is described by a $SU(k)$ gauge theory. This kind of "duality" between gauge group and ADHM auxiliary group has been noted previously in a string theory context [50–52], as is apparent from our discussion in Sec. IV.2. For the other classical groups, it is a well-known feature of the ADHM construction that the duality exchanges groups, i.e. $O(N) \leftrightarrow Sp(k)$ and $Sp(N) \leftrightarrow O(k)$ [76], although it turns out that in each of these cases large-N instantons are described by a unitary gauge theory.

Finally, we should comment on the fact that $\hat{\mathcal{Z}}_k$ is proportional to the non-integer expression $\sum_{d|k} d^{-2}$, rather than an integer as would normally be expected from the Gauss-Bonnet-Chern (GBC) theorem. The normal expectation goes as follows. We start by recalling that the coefficient of the four-fermion term in the instanton action can be identified with the Riemann tensor of the instanton moduli space; this is familiar from supersymmetric quantum mechanics [91] and the analysis of the three-dimensional theory with sixteen supercharges given in [30]. To formally evaluate the k-instanton contribution to a correlator such as G_{16}, one must bring down powers of the quadrilinear term to saturate the extra Grassmann integrations which are left over after the explicit fermion insertions are accounted for. The resulting bosonic integrand involves a power of the Riemann tensor with various indices contracted and is just the Gaussian curvature of the moduli space. The relevant moduli space here is actually a relative moduli space where eight center-of-mass degrees of freedom, which are the superpartners of the sixteen exact zero modes, have been modded out. This space, which we will denote $\tilde{\mathcal{M}}_{k,N}$ is obtained by taking a quotient of the full moduli space of k ADHM instantons for gauge group $U(N)$, by translations, dilatations and global $SU(2)$ gauge rotations.

If the instanton moduli space were both compact and smooth the integral of the Gaussian curvature would simply be equal to its Euler character by the GBC theorem. In this well-behaved case the Euler character is also equal to the index of the Laplacian, or in physical terms, the Witten index of supersymmetric quantum mechanics defined on the manifold in question. In reality the ADHM moduli space is non-compact and has singular points where instantons coincide or shrink to zero size. The three-dimensional case considered in [30] was somewhat better behaved as the moduli space of three-dimensional instantons (which are BPS monopoles), while still non-compact, is smooth. In these cases, one can still formally define a Witten index which now counts the number of *normalizable* zero energy states (weighted by fermion number). However, this index is in general no longer equal to the GBC integral which arises in the instanton calculation. Instead, the GBC integral is what is known as the principal or bulk contribution to the index which differs from the index itself by a surface term.

Taking various normalizations into account, the result of Sec. V.3 can be interpreted as saying that the principal contribution to the generalized index of supersymmetric quantum mechanics on $\tilde{\mathcal{M}}_{k,N}$ is equal to $\sum_{d|k} d^{-2}$ in the large-N limit. Interestingly, the principal contribution to the corresponding index in M(atrix) quantum mechanics, which is actually just the $N = 0$ case of the configuration considered in this section, is also known to be equal to $\sum_{d|k} d^{-2}$ [57]. In both these cases the surface terms must certainly be non-zero as the principal contribution is fractional. This is to be contrasted with the three-dimensional example discussed in [30], where it was argued that the corresponding surface terms vanish (this vanishing was demonstrated explicitly in the two-instanton case). Finally, we note that the occurrence of integrals which formally give the Euler character of instanton moduli space suggests an interesting connection with the partition function of the twisted $\mathcal{N} = 4$ theory evaluated in [32].

VI Large-N Instanton Correlation Functions

Finally, we can use our gauge-invariant large-N measure to calculate correlation functions $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$ of gauge-invariant composite operators. Before we consider specific n-point functions, let us make the following general comments. At leading order in N, we can replace each operator insertion with its classical k-instanton saddle-point value. Since the saddle-point solution of Sec. V.3 is relatively simple, this observation greatly streamlines the form of the operator insertions. We will restrict our attention to operators $\mathcal{O}(x)$ consisting of a single trace on the gauge group index of a product of adjoint scalars, fermions and field strengths. Each of these three adjoint quantities is of the type $\bar{U}\boldsymbol{X}U$ where $\boldsymbol{X}$ is some matrix of ADHM variables; consequently $\mathcal{O}$ has the generic form

$$\mathcal{O}(x) = \mathrm{tr}_N\big[\bar{U}\boldsymbol{X}_1 U\bar{U}\boldsymbol{X}_2 U \cdots \bar{U}\boldsymbol{X}_p U\big] = \mathrm{tr}_{N+2k}\big[\mathcal{P}\boldsymbol{X}_1\mathcal{P}\boldsymbol{X}_2\mathcal{P}\cdots\mathcal{P}\boldsymbol{X}_p\big] \,, \qquad (6.1)$$

where $\mathcal{P} = U\bar{U}$ is the projection operator (2.10). It is easily checked that at the saddle-point, the bosonic ADHM quantities f, a'_n, $\mathbf{L}$ and $\mathcal{P}$ collapse to

$$f \to \frac{1}{y^2 + \rho^2}1_{[k]\times[k]} \,, \quad a'_n \to -X_n 1_{[k]\times[k]} \,, \quad \mathbf{L} \to 2\rho^2 \,, \qquad (6.2a)$$

$$\mathcal{P} \to 1_{[N+2k]\times[N+2k]} - \frac{1}{y^2 + \rho^2}\begin{pmatrix} w_{\dot\alpha}\bar{w}^{\dot\alpha} & w_{\dot\alpha}y^{\dot\alpha\alpha} \\ y_{\alpha\dot\alpha}\bar{w}^{\dot\alpha} & y^2 1_{[2k]\times[2k]} \end{pmatrix} \,, \qquad (6.2b)$$

where $y = x - X$ as before. Deviations from these saddle-point values are suppressed by powers of $N^{-1/4}$ as follows from the rescaling (5.35).

The analogous replacement prescription for the fermionic ADHM quantities is, in general, somewhat trickier. While these, too, are amenable to a saddle-point analysis as described in Sec. VI.2 below, they are also subject to the stringent selection rules of Grassmann integration. In Sec. VI.1 we will focus on the relatively straightforward correlators G_{16}, G_8 and G_4 for which a large-N evaluation of the fermionic quantities is actually moot; this is because all the Grassmann variables in the insertions must be replaced by the sixteen exact supersymmetric and superconformal zero modes in order to obtain a nonzero result. In contrast, in Sec. VI.2 we will discuss a tower of higher partial-wave operators in which, in addition to these sixteen exact modes, there are left-over Grassmann collective coordinates which must be carefully analyzed in large N.

VI.1   Multi-instanton contributions to the correlators G_n

In this subsection we analyze the three gauge-invariant chiral correlators G_n, $n = 16$, 8 or 4, defined by [34]:

$$G_{16} = \langle \Lambda^1_{\alpha_1}(x_1) \cdots \Lambda^4_{\alpha_{16}}(x_{16}) \rangle , \quad \Lambda^A_\alpha = g^{-2} \sigma^{mn}{}_\alpha{}^\beta \operatorname{tr}_N v_{mn} \lambda^A_\beta , \tag{6.3a}$$

$$G_8 = \langle \mathcal{E}^{A_1 B_1}(x_1) \cdots \mathcal{E}^{A_8 B_8}(x_8) \rangle , \quad \mathcal{E}^{AB} = g^{-2} \operatorname{tr}_N \left(\lambda^A \lambda^B + t^{(AB)+}_{[abc]} A^a A^b A^c \right) , \tag{6.3b}$$

$$G_4 = \langle \mathcal{Q}^{a_1 b_1}(x_1) \cdots \mathcal{Q}^{a_4 b_4}(x_4) \rangle , \quad \mathcal{Q}^{ab} = g^{-2} \operatorname{tr}_N \left(A^a A^b - \tfrac{1}{6} \delta^{ab} A^c A^c \right) , \tag{6.3c}$$

where t in Eq. (6.3b) is a numerical tensor. We focus first on G_{16}, which we previously addressed in Secs. I.2-I.3. The component fields that make up the composite operator Λ^A_α have the saddle-point form

$$v_{mn}(x) \; \to \; \frac{4}{y^2 + \rho^2} \, \bar{U} \cdot \begin{pmatrix} 0_{[N] \times [N]} & 0_{[N] \times [2k]} \\ 0_{[2k] \times [N]} & \sigma_{mn\beta}{}^\gamma 1_{[k] \times [k]} \end{pmatrix} \cdot U \tag{6.4}$$

and[28]

$$\lambda^A_\alpha(x) \; \to \; \frac{1}{y^2 + \rho^2} \, \bar{U} \cdot \begin{pmatrix} 0_{[N] \times [N]} & 4 w_{\dot\alpha} \bar\eta^{\dot\alpha A} \delta_\alpha{}^\gamma \\ 4 \epsilon_{\alpha\beta} \bar\eta^A_{\dot\alpha} \bar w^{\dot\alpha} & 4 \xi^A_\beta \delta_\alpha{}^\gamma 1_{[k] \times [k]} - 4 \epsilon_{\alpha\beta} \xi^{\gamma A} 1_{[k] \times [k]} \end{pmatrix} \cdot U \; + \; \cdots \tag{6.5}$$

as follows from Eqs. (2.11) and (2.46). Here, as always, the Grassmann parameters ξ^A_α and $\bar\eta^{\dot\alpha A}$ measure the strength of the sixteen exact supersymmetric and superconformal zero modes; for a nonzero result, each of the sixteen Λ^A_α insertions must saturate a distinct such mode. The omitted terms in Eq. (6.5) stand for the remaining, lifted, fermion modes which can therefore be dropped. Putting Eqs. (6.5), (6.4) and (6.2b) together, gives us the leading order behavior of the composite operator Λ^A_α:

$$\Lambda^A_\alpha(x) = -\frac{96k}{g^2} \left(\xi^A_\alpha - \sigma^n_{\alpha\dot\alpha} \bar\eta^{\dot\alpha A} \cdot (x - X)_n \right) \frac{\rho^4}{(\rho^2 + (x - X)^2)^4} , \tag{6.6}$$

where the other fermion modes are neglected. Recalling the definition (1.8) of the supergravity propagator K_Δ, we see that this is precisely the one-instanton expression (1.23) apart from the overall factor of k, which comes trivially from tracing over a $k \times k$ unit matrix:

$$\Lambda^A_\alpha \Big|_{k\text{-inst}} = k \cdot \Lambda^A_\alpha \Big|_{1\text{-inst}} . \tag{6.7}$$

Thus the sixteen insertions account for a factor k^{16} relative to the one-instanton calculation reviewed in Sec. I.3.

Thanks to Eqs. (5.45) and (6.6), our final answer for the large-N k-instanton contribution

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[28]Here, and in the following, the Weyl indices β and γ are contracted with $\bar{U}$ and U, respectively.

to G_{16}, from the Yang-Mills side of the correspondence, therefore reads:

$$\begin{aligned}
G_{16}(x_1,\dots,x_{16})\Big|_{k\text{-inst}} &= \langle \Lambda^1_{\alpha_1}(x_1)\cdots\Lambda^4_{\alpha_{16}}(x_{16})\rangle\Big|_{k\text{-inst}} \\
&= \frac{96^{16}\sqrt{N}}{2^{33}\pi^{27/2}g^{24}}k^{25/2}e^{2\pi ik\tau}\sum_{d|k}\frac{1}{d^2}\int\frac{d^4X\,d\rho}{\rho^5}\prod_{A=1,2,3,4}d^2\xi^A\,d^2\bar\eta^A \\
&\times\ \big(\xi^1_{\alpha_1}-\sigma^n_{\alpha_1\dot\alpha}\bar\eta^{\dot\alpha 1}\cdot(x_1-X)_n\big)K_4(X,\rho;x_1,0) \\
&\times\cdots\times\ \big(\xi^4_{\alpha_{16}}-\sigma^n_{\alpha_{16}\dot\alpha}\bar\eta^{\dot\alpha 4}\cdot(x_{16}-X)_n\big)K_4(X,\rho;x_{16},0)
\end{aligned} \qquad (6.8)$$

To leading semi-classical order, this is in perfect agreement with the corresponding supergravity expression defined by Eqs. (1.14)-(1.15). In particular, note that the factor of k^{16} from Eq. (6.7), taken together with the factors of k in Eq. (5.45), precisely reproduces the leading semiclassical approximation (1.14) to the modular form $f_{16}(\tau,\bar\tau)$. Moreover, thanks to the proportionality (6.7), we have finally answered the puzzle posed in Sec. I.3; namely, we have seen how the k-instanton field strength can look like a supergravity propagator for $k>1$. For this to happen, the dominant k-instanton saddle-point configuration needed to look like k instantons not only living in mutually commuting $SU(2)$'s, but also sharing a common size and 4-position, precisely as we found in Sec. V.3.

Next we consider G_8 and G_4. In order to calculate the corresponding insertions $\mathcal{E}^{AB}$ and $\mathcal{Q}^{ab}$ one needs to utilize the ADHM expression for the scalar fields $A^{AB}(x)$ as given in Sec. II.7. For present purposes we need to saturate only the supersymmetric and the superconformal fermion zero modes by the insertions. Neglecting the other modes, one easily finds:

$$iA^{AB}\ \to\ \sqrt{2}\big(\xi^{\alpha A}-\bar\eta^A_{\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_k\cdot(x-X)^k\big)\sigma^{mn\ \beta}_{\ \ \alpha}\big(\xi^B_\beta-\bar\eta^{\dot\beta B}\sigma^l_{\beta\dot\beta}\cdot(x-X)_l\big)v_{mn}\ , \qquad (6.9)$$

where v_{mn}, in turn, is replaced by the saddle-point expression (6.4). The computation of G_4 and G_8 proceeds similarly to the computation of G_{16}. Note that only the $\lambda^A\lambda^B$ term in $\mathcal{E}^{AB}$ is needed in the leading semiclassical evaluation of G_8, as the $A^aA^bA^c$ term is suppressed by a power of the coupling g (as a careful rescaling of the component fields confirms). In particular, the k-instanton insertions are again trivially related to their one-instanton counterparts:

$$\mathcal{E}\Big|_{k\text{-inst}} = k\cdot\mathcal{E}\Big|_{1\text{-inst}}\ , \qquad \mathcal{Q}\Big|_{k\text{-inst}} = k\cdot\mathcal{Q}\Big|_{1\text{-inst}}\ . \qquad (6.10)$$

Once again, thanks to these proportionality relations, the space-time dependence of the correlators at the k-instanton level will be identical to the one-instanton dependence. Furthermore, the relations (6.10) account for a factor of k^8 and k^4 in G_8 and G_4, which conspires with the expression (5.45) to produce the leading semiclassical approximation (1.14) to the expected modular forms $f_8(\tau,\bar\tau)$ and $f_4(\tau,\bar\tau)$, respectively, as in Eq. (1.18). It follows that, just as with G_{16}, the large-N Yang-Mills results for G_8 and G_4 are in perfect agreement with the supergravity expectations [34], order by order in the instanton expansion. For G_4, there is an identical contribution from anti-instantons at the same order in g^2, as noted previously.

VI.2 Kaluza-Klein modes on S^5 and Yang-Mills correlators

Next let us consider the general problem of n-point functions in which the insertions contain more than sixteen Grassmann modes. Of course, sixteen of these must be used to saturate the exact supersymmetric and superconformal modes, as in Sec. VI.1. The question becomes how, in large N, to select the remaining, lifted, fermion zero modes.

To answer this question, we need to analyze the effective N dependence of these modes. In analogy with the bosonic quantities (6.2a)-(6.2b), it is useful to think of the unbroken ξ^A and $\bar{\eta}^A$ modes themselves as arising from a saddle-point evaluation:

$$\mathcal{M}'^A_\alpha \to 4\xi^A_\alpha 1_{[k]\times[k]}\,, \qquad \zeta^{\dot\alpha A} \to 4\bar{\eta}^{\dot\alpha A} 1_{[k]\times[k]}\ . \tag{6.11}$$

Indeed, the rescaling (5.40) implies that the remaining, fluctuating, modes in $\mathcal{M}'^A$ and ζ^A (which were denoted $\hat{\mathcal{M}}'^A$ and $\hat{\zeta}^A$ in Sec. V.3) are subleading compared to ξ^A and $\bar{\eta}^A$ by a factor of $N^{-1/8}$. There remain the modes ν^A and $\bar{\nu}^A$, which are distinct from the others in that they carry an $SU(N)$ index u. From their coupling to the χ_a field in Eq. (4.51), together with the rescaling (5.4), one sees that each $\bar{\nu}^A \nu^B$ pair in an insertion, for a fixed, unsummed value of the index u, costs a factor of $N^{1/2}$; however, summing on u (as required by gauge invariance) then turns this $N^{1/2}$ suppression into an $N^{1/2}$ enhancement. In other words, ν^A and $\bar{\nu}^A$ factors in the insertions should each be thought of as being enhanced by $N^{1/4}$. The large-N rule of thumb for choosing fermionic collective coordinates in gauge-invariant correlators is now clear: as many of the modes as possible should be ν^A and $\bar{\nu}^A$ modes, subject to sixteen of the modes being ξ^A and $\bar{\eta}^A$ modes to saturate these Grassmann integrations.

With this rule of thumb in hand, let us examine a specific set of correlators of special interest to the AdS/CFT correspondence. In Sec. VI.1, we were concerned with correlation functions of operators which on the supergravity side involved no dependence on the position on S^5. In this subsection, we will show that the dependence of operators on the position on S^5 is encoded on the Yang-Mills side in the dependence of the operators on the variables ν^A and $\bar{\nu}^A$. As just alluded to, our discussion will further elucidate the mysterious rôle played by the auxiliary variables χ_a. In general, operators depend upon the Grassmann variables ν^A and $\bar{\nu}^A$ which (we have seen) dominate in large N; however, gauge invariant operators can only depend on the gauge invariant $k \times k$ matrix combinations $\bar{\nu}^A \nu^B$. We now prove that, to leading order in N, such a combination, in the final integral with respect to the measure (5.45) is replaced by

$$\bar{\nu}^A \nu^B \to \frac{\sqrt{2}g\rho N^{1/2}}{\pi i}\epsilon^{ABCD}\hat{\Omega}_{CD}1_{[k]\times[k]}\ , \tag{6.12}$$

where the $N^{1/2}$ dependence has already been noted. To this end, consider a general insertion with a string of such combinations $\bar{\nu}^{A_1}\nu^{B_1} \otimes \cdots \otimes \bar{\nu}^{A_p}\nu^{B_p}$. We must insert this expression into the measure *before* the ν integrals have been performed, i.e. into (4.1). Performing the ν integrals as in (4.51) in the presence of the insertion leads to a modified expression involving

factors of χ^{-1} which can be derived by considering

$$\frac{\partial}{\partial x^T_{A_1 B_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^T_{A_p B_p}} (\det_{4k} x)^{N-2k} \Big|_{x=\chi}$$

$$= N^p (\det_{4k} \chi)^{N-2k} (\chi^{-1})_{B_1 A_1} \otimes \cdots \otimes (\chi^{-1})_{B_p A_p} + \cdots , \tag{6.13}$$

where T acts on instanton indices and the ellipsis represent terms of lower order in N. This shows that, after performing the ν integrals, a term of the form $\bar{\nu}^A \nu^B$ is replaced by $(gN/\sqrt{8}\pi i)(\chi^{-1})_{BA}$, to leading order. Now we rescale χ as in (5.4), and replace it with its saddle-point value (5.21) to give (6.12). The replacement shows that dependence on the position on S^5 is associated on the Yang-Mills side as dependence on combination $\bar{\nu}^A \nu^B$, where we recall from Sec. IV.3 that the $\{\nu^A, \bar{\nu}^A\}$ are superpartners of the iso-orientation of the multi-instanton.

We now consider, a class of operators which depend on the ν^A and $\bar{\nu}^B$ in an essential way. On the supergravity side, for simplicity, we will restrict our attention to a single supergravity field, the dilaton, $\phi(x, \rho, \hat{\Omega})$. Although the dilaton is a single massless scalar field in ten dimensions, it yields an infinite tower of massive fields in five dimensions via Kaluza-Klein reduction on S^5. This dimensional reduction amounts to expanding ϕ in partial waves as,

$$\phi(x, \rho, \hat{\Omega}) = \sum_{p=0}^{\infty} \phi^{(p)}_{a_1 a_2 \cdots a_p}(x, \rho) Y^{(p)}_{a_1 a_2 \cdots a_p}(\hat{\Omega}) , \tag{6.14}$$

where the scalar spherical harmonics for S^5 are defined in terms of the unit vector $\hat{\Omega}_a$ as,

$$Y^{(p)}_{a_1 a_2 \ldots a_p}(\hat{\Omega}) = \hat{\Omega}_{a_1} \hat{\Omega}_{a_2} \cdots \hat{\Omega}_{a_p} - \text{Traces} . \tag{6.15}$$

The field $\phi^{(n)}_{a_1 a_2 \ldots a_p}$ has five dimensional mass $m^2 = p(p + 4)L^{-2}$ and transforms in the pth symmetric traceless tensor representation of $SO(6)$.

The AdS/CFT correspondence suggests that each of these Kaluza-Klein modes is associated with an operator $\mathcal{O}^{(p)}_{a_1 a_2 \cdots a_p}$ in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory which transforms in the same irreducible representation of $SO(6)_R$ and has scaling dimension $\Delta = p + 4$. The required operators were identified in [3, 4] as,

$$\mathcal{O}^{(p)}_{a_1 a_2 \cdots a_p}(x) = \text{tr}_N \left[(RA)_{a_1} (RA)_{a_2} \cdots (RA)_{a_p} v^2_{mn} \right] - \text{Traces} . \tag{6.16}$$

Here, R is an $SO(6)$ rotation such that $\hat{\Omega} = R^{-1} \hat{\Omega}^{(0)}$ where $\hat{\Omega}^{(0)}$ is a reference point (e.g. the north pole). The explicit relation (1.5) between gauge theory and Type IIB supergravity correlators suggests [34] that we should compare the Yang-Mills instanton contributions to the gauge theory correlators, $\langle \mathcal{O}^{(p_1)}(x_1) \cdots \mathcal{O}^{(p_r)}(x_r) \mathcal{O}_{\mathcal{F}} \rangle$ with the D-instanton contributions to the following boundary correlation functions on the supergravity side of the correspondence,

$$G_{p_1 p_2 \ldots p_r} = \left\langle \phi^{(p_1)}(x_1, \rho = 0, \hat{\Omega}^{(1)}) \cdots \phi^{(p_r)}(x_r, \rho = 0, \hat{\Omega}^{(r)}) \mathcal{F} \right\rangle . \tag{6.17}$$

Here, $\mathcal{O}_{\mathcal{F}}$ represents an additional fermionic operator insertion which is present to saturate the exact sixteen zero modes of the multi-instanton and $\mathcal{F}$ are the corresponding insertions in

the supergravity correlator. Another requirement for these latter operator insertions is that they do not contain ν^A or $\bar{\nu}^A$ modes (otherwise the complicated issue of cross terms with the other insertions arises). For instance, $\mathcal{O}_{\mathcal{F}}$ could be a product of sixteen composite operators Λ_α^A considered in Sec. VI.1.[29] The appropriate bulk-to-boundary propagator for a field which transforms in a non-trivial representation $SO(6)$, and has scaling dimension $\Delta = p + 4$, is that which we would expect for an $SO(6)$ singlet operator, as in (1.8), augmented by the appropriate spherical harmonic for the representation in question. Hence the external leg corresponding to the field insertion $\phi_{a_1 a_2 \cdots a_p}^{(p)}(x, \rho = 0, \hat{\Omega}^{(p)})$ at the boundary, with the interior point $(X, \rho, \hat{\Omega})$, comes with the propagator

$$K_{p+4}(X, \rho; x, 0) Y_{a_1 a_2 \cdots a_p}^{(p)}(R^{(p)}\hat{\Omega}) , \qquad (6.18)$$

where we have defined the $SO(6)$ rotation $R^{(p)}$ such that $\hat{\Omega}^{(p)} = (R^{(p)})^{-1}\hat{\Omega}^{(0)}$. As before, we can now identify the degrees-of-freedom of a multi-instanton at large N, with those describing a D-instanton on $AdS_5 \times S^5$. Since, we have already demonstrated the equality between the measures, it remains only to demonstrate that replacing the operators $\mathcal{O}^{(p)}$ by their classical values in the multi-instanton background at large N is equivalent to the D-instanton "Feynman rule" (6.18) for the dual operators $\phi^{(p)}$.

Taking into account that the supersymmetric and superconformal modes are saturated by the unspecified insertion $\mathcal{O}_{\mathcal{F}}$, let us set these variables to zero in the scalar field. Consequently, at leading order,

$$iA^{AB} = \bar{U} \cdot \begin{pmatrix} \frac{1}{2\sqrt{2}(y^2+\rho^2)}(\nu^A\bar{\nu}^B - \nu^B\bar{\nu}^A) & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & \frac{1}{4\sqrt{2}\rho^2}(\bar{\nu}^A\nu^B - \bar{\nu}^B\nu^A) \end{pmatrix} \cdot U , \qquad (6.19)$$

where we have made use of the large-N behavior of $\mathbf{L}$ in (6.2a). We can now find the appropriate form for the operator (6.16) by taking the trace of a string of scalar fields and v_{mn}^2. Notice that in the replacement (6.12), the combination $\bar{\nu}^A\nu^B$ is $\mathcal{O}(N^{1/2})$, hence the leading order term in the operator comes from the product of p bottom-right-hand corners of (6.19). Note that the product of p top-left-hand corners of (6.19), which potentially gives a contribution of the same order, is canceled against the 0 in the top-left-hand corner of the ADHM matrix in (6.4). So at leading order

$$\mathcal{O}_{a_1 \cdots a_p}^{(p)} = -\frac{48\rho^4}{(2i)^p(y^2 + \rho^2)^{p+4}} \mathrm{tr}_k \left[(R^{(p)}\mathbf{L}^{-1} \cdot \hat{\Lambda})_{a_1} \cdots (R^{(p)}\mathbf{L}^{-1} \cdot \hat{\Lambda})_{a_p} - \mathrm{Traces} \right] , \qquad (6.20)$$

where $\hat{\Lambda}^{AB}$ was defined in (4.49) and, as before,

$$\hat{\Lambda}^{AB} = \frac{1}{\sqrt{8}}\bar{\Sigma}_a^{AB}\hat{\Lambda}_a . \qquad (6.21)$$

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[29] It is trivial to see that Λ_α^A does not depend on ν^A or $\bar{\nu}^A$: since it is a gauge-invariant operator that is linear in the Grassmann parameters, it could only depend on the inner products $\bar{w}\nu^A$ or $\bar{\nu}^A w$, but these vanish by definition; see Eq. (2.60).

We can now make the replacement (6.12), which can be written in the present context as

$$\hat{\Lambda}_a \to \frac{2ig\rho\sqrt{N}}{\pi}\hat{\Omega}_a 1_{[k]\times[k]} \,, \tag{6.22}$$

to give finally

$$\begin{aligned}
\mathcal{O}^{(p)}_{a_1\cdots a_p}(x) &= -48k\left(\frac{g\sqrt{N}}{\pi}\right)^p \cdot \frac{\rho^{p+4}}{(y^2+\rho^2)^{p+4}}\left[(R^{(p)}\hat{\Omega})_{a_1}\cdots(R^{(p)}\hat{\Omega})_{a_p} - \text{Traces}\right] \\
&\propto K_{p+4}(X,\rho;x,0)Y^{(p)}_{a_1\cdots a_p}(R^{(p)}\hat{\Omega}) \,.
\end{aligned} \tag{6.23}$$

Thus, at leading order in N, we have reproduced the "Feynman rule" (6.18) for the equivalent operator insertion on the the supergravity side of the correspondence.

VII   Comments and Conclusions

VII.1   Living large in a large-N world

Prior to the recent applications to supersymmetric theories, the multi-instanton program has had a checkered history (see Sec. I of Ref. [76]). On the positive side, the most noteworthy technical achievements include the ADHM construction of the full space of k-instanton solutions [9] as well as the fermion zero modes and propagators in the ADHM background [46]. On the negative side, the ADHM construction has the major drawback that the collective coordinate matrix a is highly overcomplete, and must be accompanied by the nonlinear constraints (2.20a) which are only explicitly resolvable for $k \leq 3$ [75,92]. A closely related problem was the lack of a collective coordinate measure beyond $k = 2$ [76]. Largely as a consequence of these difficulties, the ADHM construction languished for many years as a largely mathematical achievement with virtually no practical application to quantum field theory. Instead, much of the multi-instanton work in the physics community focused on the dilute instanton gas regime where single-instanton methods suffice and these obstructions do not appear. Unfortunately—by any measure—successful phenomenological applications to QCD require an instanton density much greater than this regime allows, as well as considerations of configurations of instantons and anti-instantons [93].

The addition of supersymmetry improves this situation substantially. On the one hand, as reviewed in Sec. IV.1, the fact that the bosonic and fermionic small-fluctuations determinants cancel, together with the stringent requirement of supersymmetric invariance, means that the collective coordinate integration measure can be fixed uniquely [45,49,82]. On the other hand, a host of new exact solutions to the low-energy dynamics of models with extended supersymmetry have been proposed [25,27] which have the property that physical measurables receive contributions from all orders in multi-instantons. These testable predictions have provided

fresh impetus for explicit ADHM calculations [26, 47, 81] which in certain cases have suggested emendations to the original solutions.

As the present paper demonstrates, the situation improves even more dramatically when supersymmetric ADHM calculus is accompanied by the large-N limit. Common lore holds that instanton physics is unimportant in this limit [94]; after all, a single instanton is weighted by the factor $\exp(-8\pi^2/g^2) = \exp(-8\pi^2 N/(g^2 N))$, which vanishes rapidly as $N \to \infty$ with fixed 't Hooft coupling $g^2 N$. In the present paper, we have been considering quantities which get all their contributions from instantons, except for tree and one-loop effects. They are indeed tiny contributions at large N, as the standard lore would have it, but nevertheless non-zero and with a precise, calculable, form fixed by the AdS/CFT correspondence.

As we have seen, with the additional ingredient of large N, the multi-instanton series can be evaluated explicitly, for the first time in a four-dimensional theory. We envision that the simplifications in multi-instanton calculus in this limit will have widespread applications, and are therefore worth emphasizing. Let us enumerate the various lessons we have learned at large N:

(i) A significant calculational advantage of the gauge-invariant measure introduced in Sec. IV.3 is that the ADHM constraints (2.20a), which are quadratic constraints in terms of the original collective coordinates w, become *linear* constraints (hence trivially resolvable) in terms of the gauge-invariant variables W. However, the gauge-invariant form of the measure is only available for $k \leq N/2$ (see Eqs. (4.28) and (4.30)). The large-N limit (with k fixed, or at least k growing more slowly than $N/2$) is therefore the regime in which these nonlinear constraints—which, we reiterate, have been the main technical impediment to progress in ADHM calculus—entirely disappear.

(ii) The detailed saddle-point analysis of Secs. V.2-V.3 confirms ones initial intuition regarding k-instanton configurations at large N, namely, that the dominant configurations consist of k single instantons embedded in mutually commuting $SU(2)$ subgroups of $SU(N)$. Since this conclusion follows from statistics alone, we expect it to hold equally for other gauge theories at large N. In this respect the large-N solutions are dilute-gas-like as, in the absence of other component fields, instantons that live in mutually commuting $SU(2)$'s do not interact.

(iii) More surprising—and perhaps special to the $\mathcal{N} = 4$ theory at large N—is that these k instantons live on top of one another in space-time, and share a common scale size. In this respect, and in sharp contrast to (ii), the large-N limit is the *opposite* of the dilute instanton gas regime. From the seemingly schizophrenic features (ii) and (iii) there follows a pleasing simplification unique to large N, discussed in Sec. VI.1: classical insertions into gauge-invariant correlators can simply be replaced by k times the corresponding one-instanton insertion.

(iv) Another remarkable feature of large N is that the auxiliary variables χ_a, $a = 1, \ldots, 6$,

introduced merely as a mathematical convenience in order to bilinearize the four-fermion inter-action (3.2), become confined to S^5 in the limit $N \to \infty$. As such, they provide a "window" from the Yang-Mills theory into the ten-dimensional geometry of the string theory.

(v) Finally, the $1/N$ expansion justifies the truncation at the quartic level of the small-fluctuations action about the $AdS_5 \times S^5$ saddle-point. One of the principal results of this paper is that this truncated action precisely describes $\mathcal{N} = 1$ ten-dimensional Yang-Mills theory dimensionally reduced to zero dimensions. Since the latter theory is also the D-instanton measure (4.13), this is yet another unexpected window from large-N Yang-Mills theory into ten-dimensional string theory.

VII.2   Comments on the nonrenormalization theorem, and on higher-order corrections in $1/N$ and in g^2

As discussed in Sec. I.1, the comparison between the Yang-Mills and supergravity pictures elucidated herein can be quantitative if and only if there exists a nonrenormalization theorem that allows one to relate the small $g^2 N$ to the large $g^2 N$ behavior of chiral correlators such as G_n, as has been suggested in Refs. [44]. In the absence of such a theorem the best one can hope for is that qualitative features of the agreement persist beyond leading order while the exact numerical factor in each instanton sector does not, in analogy with the mismatch in the numerical prefactor between weak and strong coupling results for black-hole entropy [3]. In our view, however, our present results provide strong evidence in favor of such a nonrenormalization theorem for the correlators G_n, for the following reason. Consider the planar diagram corrections to the leading semiclassical (i.e. $g^2 N \to 0$) result for, say, G_{16}, Eq. (6.8). In principle, these would not only modify the above result by an infinite series in $g^2 N$, but also, at each order in this expansion, and independently for each value of k, they would produce a different function of space-time. The fact that the leading semiclassical form for G_{16} that we obtain is not only k-independent, but already reproduces the space-time dependent of the D-instanton/supergravity prediction exactly, strongly suggests that such diagrammatic corrections (planar and otherwise) must vanish. Nevertheless there are necessary subleading corrections, both in $1/N$ and in g^2, to our leading semiclassical results, as follows.

In the one-instanton sector, the complete series expansion in $1/N$, at fixed leading order in g, is encapsulated in the exact expression (1.26). At the multi-instanton level, the analogous $1/N$ corrections are encoded instead in the exact formula for the measure in (4.55). It may be that the variables W^0 can be integrated out (as at large N) to leave a dimensional reduction of a ten-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with a generalized action that is some non-trivial function of N. The action could then be expanded in $1/N$ with a leading term that is the conventional $SU(k)$ Yang-Mills action of (5.42). It is conceivable that the more general action is of the Born-Infeld type [95].

Finally let us consider the source of g^2 corrections. As is clear from Eqs. (1.11)-(1.12), these corrections are absolutely necessary if (as we fully expect) the Yang-Mills expression for the correlator G_n is to sum to the complete modular form $f_n(\tau, \bar{\tau})$, and not merely reproduce the leading semiclassical approximation thereto. Yet how can such corrections be compatible with the nonrenormalization theorem postulated above? We believe that the g^2 corrections arise, not from conventional Feynman diagrams in the multi-instanton background, but rather from the fact that the instanton supermultiplet constructed herein is *not* an exact solution to the coupled Euler-Lagrange equations. This important point, which we stressed at the beginning of Sec. II, stems from the fact that we have included for convenience all the fermion zero modes in the collective coordinate matrix $\mathcal{M}^A$, and not just the exact supersymmetric and superconformal modes. As discussed in Sec. II, this approach has the calculational advantage that the quadrilinear action (3.2) is automatically generated at leading order. But it also implies that the coupled Euler-Lagrange equations must be solved iteratively, order by order in g. In particular, at the next order, the antigauginos $\bar{\lambda}^A$ turn on, as do the auxiliary superfield components D and F; these, in turn, back-react on the classical component fields already constructed in Sec. II, resulting in corrections to the multi-instanton action. We expect these corrections to sum to the appropriate modular forms, but such checks lie beyond the scope of the present paper. However, it seems likely that these corrections will be much simpler than ordinary perturbation theory in the instanton background.

Acknowledgments

Appendix A: Clifford Algebras and fermions in 4, 6 and 10 dimensions

In the appendix we establish connections between the Clifford algebras and fermions in various dimensions of interest to us. Our conventions for such representations are taken from [96].

To begin with we establish representations for the $d = 4$ and $d = 6$ Clifford algebras in Euclidean space.[30] First of all, in $d = 4$ we take the representation of the gamma matrices from

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[30]As such we shall not distinguish between upper and lower vector indices.

Wess and Bagger [72]:

$$\gamma_n = \begin{pmatrix} 0 & \sigma_n \\ \bar{\sigma}_n & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

where we implicitly assume the analytic continuation to Euclidean space $(\sigma^0, \sigma^i) \to (\sigma^0, i\sigma^i)$. In $d = 6$ we introduce an analogous representation of the form

$$\hat{\gamma}_a = \begin{pmatrix} 0 & \Sigma_a \\ \bar{\Sigma}_a & 0 \end{pmatrix}, \qquad \hat{\gamma}_7 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.2}$$

where the 4×4 dimensional matrices Σ_a and $\bar{\Sigma}_a$, $a = 1, \ldots, 6$, are most conveniently defined in terms of the 't Hooft eta symbols [73]:

$$\begin{aligned} \Sigma_{AB}^a &= \left(\eta_{AB}^c, i\bar{\eta}_{AB}^c \right), \\ \bar{\Sigma}_a^{AB} &= \left(-\eta_{AB}^c, i\bar{\eta}_{AB}^c \right), \end{aligned} \tag{A.3}$$

where [73] for $c = 1, 2, 3$:

$$\begin{aligned} \bar{\eta}_{AB}^c &= \eta_{AB}^c = \epsilon_{cAB} \qquad A, B \in \{1, 2, 3\}, \\ \bar{\eta}_{4A}^c &= \eta_{A4}^c = \delta_{cA}, \\ \eta_{AB}^c &= -\eta_{BA}^c, \qquad \bar{\eta}_{AB}^c = -\bar{\eta}_{BA}^c. \end{aligned} \tag{A.4}$$

In both $d = 4$ and $d = 6$, we can define the associated quantities

$$\begin{aligned} \gamma_{nm} &= \tfrac{i}{4}[\gamma_n, \gamma_m] = i \begin{pmatrix} \sigma_{mn} & 0 \\ 0 & \bar{\sigma}_{nm} \end{pmatrix}, \\ \hat{\gamma}_{ab} &= \tfrac{i}{4}[\hat{\gamma}_a, \hat{\gamma}_b] = i \begin{pmatrix} \Sigma_{ab} & 0 \\ 0 & \bar{\Sigma}_{ab} \end{pmatrix}. \end{aligned} \tag{A.5}$$

The charge conjugation matrices are

$$\begin{aligned} C^{(4)} &= \gamma_1 \gamma_3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \\ C^{(6)} &= \hat{\gamma}_4 \hat{\gamma}_5 \hat{\gamma}_6 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \end{aligned} \tag{A.6}$$

From these representations in 4 and 6 dimensions, we wish to build representations of the $d = 10$ Clifford. We shall need two different representations for Secs. 4.2 and 5.2. First of all we must explain some of the subtleties of defining Majorana-Weyl fermions in $d = 10$. Majorana fermions can only be properly defined in Minkowski space in $d = 10$, so we start with a Hermitian representation Γ_μ, $\mu = 0, \ldots, 9$, of the $d = 10$ Clifford algebra. The coupling of a 32 components fermion to a vector is given by $\bar{\Psi}\Gamma_\mu \Psi$. The Majorana condition on fermions is $\Psi = C\bar{\Psi}^T$, with $\bar{\Psi} = \Psi^\dagger \Gamma_0$, where C is the charge conjugation matrix defined such that for the Euclidean space gamma matrices $(\Gamma_0 \to -i\Gamma_0)$ we have $\Gamma_\mu^* = -C^{-1}\Gamma_\mu C$. At this stage we can

continue back to Euclidean space and simply treat the Euclidean Majorana fermion Ψ as real (since we never have to make use of its complex conjugate). In Euclidean space the coupling of a Majorana fermion to a vector is

$$\Psi^T (C^{-1})^T \Gamma_\mu \Psi. \tag{A.7}$$

This coupling is not actually real, and the resulting Hamiltonian is not Hermitian. However, this is not inconsistent because the relevant requirement in Euclidean quantum field theory is Osterwalder-Schrader reflection positivity rather than Hermiticity. From now on we work in Euclidean space.

The charge conjugation matrix in $d = 10$ is

$$C = C^{(6)} \otimes C^{(4)} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \otimes \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \tag{A.8}$$

The representation of the $d = 10$ Clifford algebra needed in Sec. 4.2, is built as follows:

$$\Gamma_n = 1_{[8]\times[8]} \otimes \gamma_n, \qquad \Gamma_{a+3} = \hat{\gamma}_a \otimes \gamma_5, \tag{A.9}$$

where $n = 0, \ldots, 3$ and $a = 1, \ldots, 6$. In this representation

$$\Gamma_{11} = \hat{\gamma}_7 \otimes \gamma_5. \tag{A.10}$$

A Majorana-Weyl spinor of positive chirality in $d = 10$ satisfies $\Gamma_{11}\Psi = \Psi$. In this basis, Ψ can be decomposed as,

$$\Psi = \sqrt{\frac{\pi}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \mathcal{M}'^A_\alpha \\ 0 \end{pmatrix} + \sqrt{\frac{\pi}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \lambda^{\dot{\alpha}}_A \end{pmatrix}, \tag{A.11}$$

where $\mathcal{M}'$ and λ are Weyl spinors both of $SO(4)$ and of $SO(6)$. $\mathcal{M}'$ and λ have positive and negative chirality with respect to both groups, respectively.

The representation of the $d = 10$ Clifford algebra that is needed in Sec. 5.3 is somewhat different. The fermion coupling (5.39) can be written precisely as in Eq. (5.37) with a rotated representation of the six-dimensional gamma matrices that depends on $\hat{\Omega}_a$. We define an $SO(6)$ rotation matrix R, $RR^T = 1$, such that $\hat{\Omega}'_a = R_{ab}\hat{\Omega}_b$ lies entirely along, say, the first direction, i.e. $\hat{\Omega}'_a \propto \delta_{a1}$. In the new basis, we have a representation of the $d = 6$ Clifford algebra $\hat{\gamma}'_a = R_{ab}\hat{\gamma}_b$.

In the rotated basis, we can construct a representation of the $d = 10$ Clifford algebra as follows:

$$\Gamma_n = \hat{\gamma}'_1 \otimes \gamma_n, \qquad \Gamma'_{3+a} = \hat{\gamma}'_a \otimes \left(\delta_{a1}\gamma_5 + (1 - \delta_{a1})1_{[4]\times[4]}\right), \tag{A.12}$$

where $n = 0, \ldots, 3$ and $a = 1, \ldots, 6$. The representation of the $d = 10$ Clifford algebra that appears in the text is then found by un-doing the rotation on the $d = 6$ subspace:

$$\Gamma_{3+a} = (R^{-1})_{ab}\Gamma'_{3+b}. \tag{A.13}$$

In this basis

$$\Gamma_{11} = \hat{\gamma}_7 \otimes 1_{[4]\times[4]}, \tag{A.14}$$

and a positive chirality Weyl fermion in $d = 10$ is decomposed as a positive chirality Weyl fermion in $d = 6$ which is a Dirac fermion in $d = 2$. With the correct normalization to reproduce the fermion coupling in the text, the $d = 10$ Majorana-Weyl fermion Ψ has components

$$\Psi = N^{1/8} \sqrt{\frac{\pi}{2g}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \rho^{-1/2} \hat{\mathcal{M}}_\alpha'^A \\ \rho^{1/2} \hat{\zeta}^{\dot{\alpha}A} \end{pmatrix}. \tag{A.15}$$

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