

# Instantons, Euclidean supersymmetry and Wick rotations.

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## Abstract

We discuss the reality properties of the fermionic collective coordinates in Euclidean space in an instanton background and construct hermitean actions for  $\mathcal{N} = 4$  and  $\mathcal{N} = 2$  supersymmetric Euclidean Yang-Mills theories.

Keywords: Euclidean supersymmetry, reality condition, instantons, zero modes

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# 1 Introduction.

In Euclidean supersymmetric Yang-Mills field theories (SYM) [1, 2, 3], the choice between an instanton or anti-instanton background determines whether the chiral or anti-chiral fermions acquire zero mode solutions parametrized by Grassmann collective coordinates. In order that the action written in two-component formalism with Weyl spinors  $\lambda^\alpha$  and  $\bar{\lambda}_{\dot{\alpha}}$  be hermitean, we need a reality condition on the fermions in Euclidean space; however, the latter is usually not specified. If we were to continue using the same action with spinors  $\lambda^\alpha$  and  $\bar{\lambda}_{\dot{\alpha}}$  as in Minkowski space we run immediately into a problem: if  $\lambda_\alpha$  contains fermionic zero modes in the presence of an anti-instanton,  $\bar{\lambda}_{\dot{\alpha}}$  does not have any zero modes, and this cannot be reconciled with the Majorana condition  $\bar{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^\dagger$ . If one just would forget about this condition on the fermions, one would find a similar problem for the bosons at the level of field equations of motion. For example, in the  $\mathcal{N} = 4$  supersymmetric model [4], the six scalars  $\phi^{AB}$  in the **6** of  $SU(4)$  are usually taken to satisfy the condition

$$(\phi^{AB})^* \equiv \bar{\phi}_{AB} = \frac{1}{2}\epsilon_{ABCD}\phi^{CD}, \quad (1)$$

and their field equation in Minkowski space-time reads

$$\mathcal{D}^2\phi^{AB} + \sqrt{2}\{\lambda^{\alpha,A}, \lambda_\alpha^B\} + \frac{1}{\sqrt{2}}\epsilon^{ABCD}\{\bar{\lambda}_{\dot{C}}, \bar{\lambda}_{\dot{D}}\} - \frac{1}{2}[\bar{\phi}_{CD}, [\phi^{AB}, \phi^{CD}]] = 0. \quad (2)$$

But if one were to use the same field equation in Euclidean space with an instanton solution (only changing  $\frac{d}{dt}$  and  $A_0$  into  $i\frac{d}{dt}$  and  $iA_4$ , respectively) one would find that since only  $\lambda^{\alpha,A}$  carries fermionic zero modes, the reality condition in (1) is violated. Some physicists are confident that, as far as explicit calculations are concerned, no problem arises if one views all fields as complex without reality conditions. (The action is holomorphic: it depends on fields, not their complex conjugates. For fermions an independent  $\bar{\psi}$  or a dependent  $\bar{\psi} = (\psi)^*$  gives the same Grassmann integration [3].)

Our resolution of this problem is consistent with both points of view: the action contains complex fields and is the same in Euclidean and Minkowski spaces, but consistent reality conditions for the fields exist in Euclidean space and are different from those in Minkowski space. These reality conditions involve both the space-time and internal  $R$ -symmetry groups. To derive the correct reality conditions on spinors and scalars such that the resulting action is hermitean, we use dimensional reduction on a torus with one time coordinate from the  $\mathcal{N} = 1$  SYM in (9,1) or (5,1) Minkowski space to (4,0) Euclidean space with 16 and 8 supercharges, respectively [5]. We shall see that in the reality condition of fermions the “space-time metric”  $\gamma^4$  is replaced by a metric  $\eta_{AB}$  of the  $R$ -symmetry group. This metric appears since the compact  $R$  symmetry group of the  $\mathcal{N} = 4$  Minkowski model becomes non-compact when the non-compact space-time group  $SO(3,1)$  is converted to the compact Euclidean group  $SO(4)$ . We discuss in the following two

sections both the  $\mathcal{N} = 4$  and the  $\mathcal{N} = 2$  model and will comment on the  $\mathcal{N} = 1$  model in the conclusion.

## 2 $\mathcal{N} = 4$ Euclidean model.

To construct the  $\mathcal{N} = 4$  supersymmetric YM model in Euclidean  $d = (4, 0)$  space, we start with the  $\mathcal{N} = 1$  SYM model in  $d = (9, 1)$  Minkowski space-time, and reduce it on a six-torus with one time and five space coordinates. In this way the Euclidean action and reality conditions on Euclidean fields are automatically produced and can be compared with their Minkowski counterparts [4] by reducing the same  $\mathcal{N} = 1$  theory on a torus with six space coordinates. This reduction is expected to lead to an internal non-compact  $SO(5, 1)$  symmetry group in Euclidean space which is the Wick rotation of the  $SU(4) = SO(6)$   $R$ -symmetry group in  $d = (3, 1)$ . The reality conditions on bosons and fermions will both use an internal metric for this non-compact internal symmetry group, and since for the instanton solution one uses the 't Hooft symbols  $\eta_{\mu\nu}^a$  (self-dual) and  $\bar{\eta}_{\mu\nu}^a$  (anti-self-dual) it is natural to use them also for the internal metric. For spinors the internal metric follows by dimensional reduction from the properties of the Dirac matrices in 10, 6 and 4 dimensions, hence we use  $\eta$  and  $\bar{\eta}$  symbols in the construction of Dirac matrices in 6 dimensions. In 4 dimensions we take the usual off-diagonal representation in terms of  $\sigma_\mu$  and  $\bar{\sigma}_\mu$ , but note that all four  $\gamma_\mu$  are hermitean and square to +1. One of the Dirac matrices in 6 dimensions will be associated to the time coordinate and has square  $-1$ ; all other are again hermitean with square +1. We shall need properties of the charge conjugation matrices in  $d = (9, 1)$ ,  $d = (5, 1)$  and  $d = (4, 0)$ . It has been shown by means of finite group theory [6] that all these properties are representation independent. In particular, there are two charge conjugation matrices  $C^+$  and  $C^-$  in even dimensions, satisfying  $C^\pm \Gamma^\mu = \pm (\Gamma^\mu)^T C^\pm$ , and  $C^+ = C^- * \Gamma$ , where  $*\Gamma$  is the product of all Dirac matrices normalized to  $(*\Gamma)^2 = +1$ . Then  $*\Gamma$  is a hermitean matrix. These charge conjugation matrices do not depend on the signature of space-time and  $C^- * \Gamma = \pm (*\Gamma)^T C^-$  with  $-$  sign in  $d = 10, 6$  but with  $+$  sign in  $d = 4$ . In  $d = 10$   $(C^\pm)^T = \pm C^\pm$  while for  $d = 6$   $(C^\pm)^T = \mp C^\pm$ , finally for  $d = 4$   $(C^\pm)^T = -C^\pm$ .

We start with the  $d = (9, 1)$   $\mathcal{N} = 1$  Lagrangian

$$\mathcal{L}_{10} = \frac{1}{g_{10}^2} \text{tr} \left\{ \frac{1}{2} F_{MN} F^{MN} + \bar{\Psi} \Gamma^M \mathcal{D}_M \Psi \right\}, \quad (3)$$

where  $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$  and  $\Psi$  is a Majorana-Weyl spinor

$$\Gamma^{11} \Psi = \Psi, \quad \Psi^T C_{10}^- = \Psi^\dagger i \Gamma^0 \equiv \bar{\Psi}, \quad (4)$$

where  $\Gamma^{11} \equiv *\Gamma$ . Both  $A$  and  $\Psi$  are Lie algebra valued,  $A = A^a T_a$  and  $\Psi = \Psi^a T_a$  with anti-hermitean  $SU(N)$  generators  $T^a$  normalized according to  $\text{tr} T^a T^b = -\frac{1}{2} \delta^{ab}$ . The  $\Gamma$ -matrices obey

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$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^{\mu,\alpha\beta'} \\ i\bar{\sigma}^\mu_{\alpha'\beta} & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_4^- = \gamma^4\gamma^2 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\alpha'\beta'} \end{pmatrix}$$

$$\hat{\gamma}^a = \begin{pmatrix} 0 & \Sigma^{a,AB} \\ \bar{\Sigma}^a_{AB} & 0 \end{pmatrix}, \quad \hat{\gamma}^7 = \hat{\gamma}^1 \dots \hat{\gamma}^6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_6^- = i\hat{\gamma}^4\hat{\gamma}^5\hat{\gamma}^6 = \begin{pmatrix} 0 & \delta_A^B \\ \delta^A_B & 0 \end{pmatrix}$$


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Table 1: The Dirac matrices in four and six dimensions and corresponding charge conjugation matrices [7]. Here  $\sigma^\mu = (\vec{\sigma}, i)$  and  $\bar{\sigma}^\mu = (\vec{\sigma}, -i)$  for  $\mu = 1, \dots, 4$ , and  $\Sigma^{a,AB} = \{-i\eta^{1,AB}, \eta^{2,AB}, \eta^{3,AB}, i\bar{\eta}^{k,AB}\}$ ,  $\bar{\Sigma}^a_{AB} = \{i\eta^1_{AB}, -\eta^2_{AB}, -\eta^3_{AB}, i\bar{\eta}^k_{AB}\}$  are expressed in terms of 't Hooft symbols [8]. Numerically,  $\eta^{a,AB} = \eta^a_{AB}$  and the same for  $\bar{\eta}$ . Furthermore,  $\epsilon_{\alpha\beta} = -\epsilon^{\alpha'\beta'}$ ,  $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ .

the Clifford algebra  $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$  with metric  $\eta^{MN} = \text{diag}(-, +, \dots, +)$ . The Lagrangian is a density under the following transformation rules

$$\delta A_M = \bar{\zeta} \Gamma_M \Psi, \quad \delta \Psi = -\frac{1}{2} F_{MN} \Gamma^{MN} \zeta, \quad (5)$$

with  $\Gamma^{MN} = \frac{1}{2}[\Gamma^M \Gamma^N - \Gamma^N \Gamma^M]$  and  $\bar{\zeta} = \zeta^T C_{10}^-$ . (Of course,  $\zeta^T C_{10}^- = \zeta^\dagger i\Gamma^0$ .)

To proceed with the dimensional reduction we choose a particular representation of the gamma matrices in  $d = (9, 1)$ , namely<sup>1</sup>,

$$\Gamma^M = \{\hat{\gamma}^a \otimes \gamma^5, \mathbb{1}_{[8 \times 8]} \otimes \gamma^\mu\}, \quad \Gamma^0 = \begin{pmatrix} 0 & -i\eta^1 \\ i\eta^1 & 0 \end{pmatrix} \otimes \gamma^5, \quad \Gamma^{11} = \Gamma^0 \dots \Gamma^9 = \hat{\gamma}^7 \otimes \gamma^5, \quad (6)$$

where the  $8 \times 8$  Dirac matrices  $\hat{\gamma}^a$  and  $\hat{\gamma}^7$  of  $d = (5, 1)$  with  $a = 1, \dots, 6$  and the usual  $\gamma^\mu$  and  $\gamma^5$  of  $d = (4, 0)$  are given in the Table 1. The charge conjugation matrix  $C_{10}^-$  is given by  $C_6^- \otimes C_4^-$ . Upon compactification to Euclidean  $d = (4, 0)$  space the 10-dimensional Lorentz group  $SO(9, 1)$  reduces to  $SO(4) \times SO(5, 1)$  with compact space-time group  $SO(4)$  and  $R$ -symmetry group  $SO(5, 1)$ . In these conventions a 32-component chiral Weyl spinor  $\Psi$  decomposes as follows into 8 and 4 component chiral-chiral and antichiral-antichiral spinors

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \lambda^{\alpha,A} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar{\lambda}_{\alpha',A} \end{pmatrix}, \quad (7)$$

where  $\lambda^{\alpha,A}$  ( $\alpha = 1, 2$ ) transforms only under the first  $SU(2)$  in  $SO(4) = SU(2) \times SU(2)$ , while  $\bar{\lambda}_{\alpha',A}$  changes only under the second  $SU(2)$ . Furthermore,  $\bar{\lambda}_{\alpha',A}$  transforms in the complex conjugate of

<sup>1</sup>Use  $\eta^a_{AB} \eta^b_{BC} = -\delta^{ab} \delta_{AC} - \epsilon^{abc} \eta^c_{AC}$ . The same relation holds for  $\bar{\eta}^a_{AB}$ . Further,  $[\eta^a, \bar{\eta}^b] = 0$ .

the  $SO(5, 1)$  representation of  $\lambda^{\alpha, A}$ , namely,  $(\lambda^*)^{\alpha, B} \eta_{BA}^1$  transforms like  $\bar{\lambda}_{\alpha, A}$ , and the two spinor representation of  $SO(5, 1)$  are pseudoreal, i.e.  $[\hat{\gamma}^a, \hat{\gamma}^b]_L^* \eta^1 = \eta^1 [\hat{\gamma}^a, \hat{\gamma}^b]_R$  where  $L$  ( $R$ ) denotes the upper (lower) 4-component spinor. The Majorana condition (4) on  $\Psi$  leads in Euclidean space to reality conditions<sup>2</sup> on  $\lambda^\alpha$  which are independent of those on  $\bar{\lambda}_{\alpha'}$ , namely,

$$\left(\lambda^{\alpha, A}\right)^\dagger = -\lambda^{\beta, B} \epsilon_{\beta\alpha} \eta_{BA}^1, \quad \left(\bar{\lambda}_{\alpha', A}\right)^\dagger = -\bar{\lambda}_{\beta', B} \epsilon^{\beta'\alpha'} \eta^{1, BA}. \quad (8)$$

These reality conditions are consistent [6] and define a symplectic Majorana spinor in Euclidean space. The  $SU(2) \times SU(2)$  covariance of (8) is obvious from the pseudoreality of the  $\mathbf{2}$  of  $SU(2)$ , but covariance under  $SO(5, 1)$  can also be checked (use  $[\eta^a, \bar{\eta}^b] = 0$ ).

Substituting these results, the action reduces to

$$\begin{aligned} \mathcal{L}_E^{\mathcal{N}=4} = \frac{1}{g^2} \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} - 2i \bar{\lambda}_A^{\alpha'} \bar{\sigma}_{\mu, \alpha'\beta} \mathcal{D}_\mu \lambda^{\beta, A} + \frac{1}{2} \left( \mathcal{D}_\mu \bar{\phi}_{AB} \right) \left( \mathcal{D}_\mu \phi^{AB} \right) \right. \\ \left. - \sqrt{2} \bar{\phi}_{AB} \left\{ \lambda^{\alpha, A}, \lambda_\alpha^B \right\} - \sqrt{2} \phi^{AB} \left\{ \bar{\lambda}_A^{\alpha'}, \bar{\lambda}_{\alpha', B} \right\} + \frac{1}{8} \left[ \phi^{AB}, \phi^{CD} \right] \left[ \bar{\phi}_{AB}, \bar{\phi}_{CD} \right] \right\}, \quad (9) \end{aligned}$$

with  $\bar{\phi}_{AB} \equiv \frac{1}{2} \epsilon_{ABCD} \phi^{CD}$ . The anti-symmetric scalar  $\phi^{AB}$  is defined by  $\frac{1}{\sqrt{2}} \Sigma^{a, AB} A_a$  where  $A_a$  are the first six real components of the ten dimensional gauge field potential  $A_M$ . Since the first  $\Sigma$  matrix has an extra factor  $i$  in order that  $(I^0)^2 = -1$ , see (6), the reality condition on  $\phi^{AB}$  involves  $\eta_{AB}^1$

$$\left(\phi^{AB}\right)^* = \eta_{AC}^1 \phi^{CD} \eta_{DB}^1. \quad (10)$$

The Euclidean action in (9) is hermitean under the reality conditions in (8) and (10). In fact, one obtains the same action for the Minkowski case by reducing on a torus with 6 space coordinates, but then we find the reality conditions in (1) and the usual Majorana condition:  $\bar{\lambda}_{\dot{\alpha}, A} = \left(\lambda_\alpha^A\right)^\dagger$ .

The action is invariant under the dimensionally reduced supersymmetry transformation rules<sup>3</sup>

$$\begin{aligned} \delta A_\mu &= -i \bar{\zeta}_A^{\alpha'} \bar{\sigma}_{\mu, \alpha'\beta} \lambda^{\beta, A} + i \bar{\lambda}_{\beta', A} \sigma_\mu^{\alpha\beta'} \zeta_\alpha^A, \\ \delta \phi^{AB} &= \sqrt{2} \left( \zeta^{\alpha, A} \lambda_\alpha^B - \zeta^{\alpha, B} \lambda_\alpha^A + \epsilon^{ABCD} \bar{\zeta}_C^{\alpha'} \bar{\lambda}_{\alpha', D} \right), \\ \delta \lambda^{\alpha, A} &= -\frac{1}{2} (\sigma^{\mu\nu})^\alpha{}_\beta F_{\mu\nu} \zeta^{\beta, A} - i \sqrt{2} \bar{\zeta}_{\alpha', B} \sigma_\mu^{\alpha\alpha'} \mathcal{D}_\mu \phi^{AB} + \left[ \phi^{AB}, \bar{\phi}_{BC} \right] \zeta^{\alpha, C}, \\ \delta \bar{\lambda}_{\alpha', A} &= -\frac{1}{2} (\bar{\sigma}^{\mu\nu})_{\alpha'}{}^{\beta'} F_{\mu\nu} \bar{\zeta}_{\beta', A} + i \sqrt{2} \zeta^{\alpha, B} \bar{\sigma}_{\mu, \alpha'\alpha} \mathcal{D}_\mu \bar{\phi}_{AB} + \left[ \bar{\phi}_{AB}, \phi^{BC} \right] \bar{\zeta}_{\alpha', C}, \quad (11) \end{aligned}$$

where  $\sigma_{\mu\nu} \equiv \frac{1}{2} [\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu]$  is anti-self-dual,  $\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = -\sigma_{\mu\nu}$ , and  $\bar{\sigma}_{\mu\nu} \equiv \frac{1}{2} [\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu]$  is self-dual,  $\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \bar{\sigma}_{\rho\sigma} = \bar{\sigma}_{\mu\nu}$ , and  $\frac{1}{2} \epsilon_{ABCD} \Sigma^{a, CD} = -\bar{\Sigma}_{AB}^a$ . Also these rules are the same as in Minkowski case, but with modified reality conditions.

<sup>2</sup>Unless specified otherwise, equations which involve hermitean or complex conjugation of fields will be understood as not Lie algebra valued, i.e. they hold for the components  $\lambda^{\alpha, \alpha, A}$ , etc.

<sup>3</sup>In Euclidean space we use the notation  $\alpha'$  instead of Minkowskian  $\dot{\alpha}$ . Then  $(\delta \lambda^{\alpha, A})^* \epsilon_{\dot{\alpha}\beta} = -\delta \bar{\lambda}_{\dot{\beta}, A}$ , but there is no similar relation between  $(\delta \lambda^{\alpha, A})^*$  and  $\delta \bar{\lambda}_{\alpha', A}$  in Euclidean space because  $SO(4) = SU(2) \times SU(2)$  but  $SO(6)$  is simple.

One of the main motivations for studying Euclidean supersymmetric theories is that they allow one to compute non-perturbative instantons effects in correlation functions. In many correlators the one-loop perturbative corrections occur only as quantum determinants which are the products of the non-zero modes of the quantum fluctuations and cancel due to supersymmetry. In these cases the analysis can be limited to the study of zero modes. The Euclidean equations of motion

$$\begin{aligned} \mathcal{D}_\nu F_{\nu\mu} - i \left\{ \bar{\lambda}_A^{\alpha'} \bar{\sigma}_{\mu,\alpha'\beta}, \lambda^{\beta,A} \right\} - \frac{1}{2} \left[ \bar{\phi}_{AB}, \mathcal{D}_\mu \phi^{AB} \right] &= 0, \\ \mathcal{D}^2 \phi^{AB} + \sqrt{2} \left\{ \lambda^{\alpha,A}, \lambda_\alpha^B \right\} + \frac{1}{\sqrt{2}} \epsilon^{ABCD} \left\{ \bar{\lambda}_C^{\alpha'}, \bar{\lambda}_{\alpha',D} \right\} - \frac{1}{2} \left[ \bar{\phi}_{CD}, \left[ \phi^{AB}, \phi^{CD} \right] \right] &= 0, \\ \bar{\sigma}_{\mu,\alpha'\beta} \mathcal{D}_\mu \lambda^{\beta,A} + i\sqrt{2} \left[ \phi^{AB}, \bar{\lambda}_{\alpha',B} \right] = 0, \quad \sigma_{\mu,\alpha\beta'} \mathcal{D}_\mu \bar{\lambda}_A^{\beta'} + i\sqrt{2} \left[ \bar{\phi}_{AB}, \lambda_\alpha^B \right] &= 0, \end{aligned} \quad (12)$$

are consistent with the reality conditions in Eqs. (8, 10). We can obtain a complete solution to the above equations by first solving exactly the simplified field equations  $\mathcal{D}_\nu F_{\nu\mu} = 0$ ,  $\bar{\sigma}_{\mu,\alpha'\beta} \mathcal{D}_\mu \lambda^{\beta,A} = 0$ ,  $\sigma_{\mu,\alpha\beta'} \mathcal{D}_\mu \bar{\lambda}_A^{\beta'} = 0$ . Choosing the anti-instanton configuration

$$A_\mu^{\alpha,\bar{I}}(x; x_0, \rho) = 2 \frac{\rho^2 \eta_{\mu\nu}^\alpha (x - x_0)_\nu}{(x - x_0)^2 ((x - x_0)^2 + \rho^2)}, \quad (13)$$

there are only solutions for  $\lambda$ . From index theory it follows that there are  $4 \times 2 \times N$  fermionic zero modes in the background of a  $k = 1$  anti-instanton. Namely, there are  $4 \times 4$  supersymmetric and superconformal zero modes [9]

$$\lambda^{\alpha,A} = \frac{1}{2} (\sigma^{\mu\nu})^\alpha{}_\beta \left( \xi^{\beta,A} - \bar{\eta}_{\beta'}^A \sigma_\rho^{\beta\beta'} x^\rho \right) F_{\mu\nu}^{\bar{I}}, \quad (14)$$

with Grassmann collective coordinates (GCC)  $\xi$  and  $\bar{\eta}$  which to some extent are superpartners of center-of-mass coordinate  $x_0$  and scale  $\rho$  of the anti-instanton  $A_\mu^{\bar{I}}(x; x_0, \rho)$ . The remaining  $4 \times 2 \times (N - 2)$  zero modes are given by (setting  $x_0 = 0$ ) [10, 11]

$$\left( \lambda^{\alpha,A} \right)_u{}^v = \left[ \frac{\rho^2}{x^2 (x^2 + \rho^2)^3} \right]^{1/2} \left( \mu_u^A (x^\alpha)^v + (x^\alpha)_u \bar{\mu}^{A,v} \right), \quad (15)$$

where for fixed  $\alpha$  and  $A$ , the  $N$ -component vectors  $\mu_u^A$  and  $(x^\alpha)^v$  are given by

$$\mu_u^A = \left( \mu_1^A, \dots, \mu_{N-2}^A, 0, 0 \right), \quad (x^\alpha)^v = \left( 0, \dots, 0, x^\mu \sigma_\mu^{\alpha\beta'} \right) \quad \text{with} \quad N - 2 + \beta' = v. \quad (16)$$

Further,  $(x^\alpha)_u = (x^\alpha)^v \epsilon_{vu}$  and  $\bar{\mu}^{A,v}$  possesses also  $N - 2$  nonvanishing components.

Our reality conditions for spinors in (8) imply then reality conditions for the Grassmann collective coordinates. Straightforward substitution yields the following results

$$\left( \xi^{A,\alpha} \right)^\dagger = -\xi^{\beta,B} \epsilon_{\beta\alpha} \eta_{BA}^1, \quad \left( \bar{\eta}_{\alpha'}^A \right)^\dagger = -\bar{\eta}_{\beta'}^B \epsilon^{\beta'\alpha'} \eta_{BA}^1, \quad \left( \mu_u^A \right)^\dagger = \bar{\mu}^{B,u} \eta_{BA}^1, \quad \left( \bar{\mu}^{A,u} \right)^\dagger = -\mu_u^B \eta_{BA}^1, \quad (17)$$

where we have used the involution of Pauli matrices in Euclidean space  $\left( \sigma_\mu^{\alpha\beta'} \right)^* = \sigma_{\alpha\beta'}$ . One can now make an expansion of the field equations in the number of fermion fields and solve them order

by order in the number of GCC. To second order in GCC one must solve  $\mathcal{D}^2\phi^{AB} + \sqrt{2}\{\lambda^{\alpha,A}, \lambda_\alpha^B\} = 0$  [11, 12, 7]. Substitution of these solutions into the  $\mathcal{N} = 4$  action  $S = -\int d^4x \mathcal{L}$  (9) leads to an extra term, in addition to the standard one-anti-instanton action  $S^{\bar{I}} = \frac{8\pi^2}{g^2}$ , which lifts all the fermionic zero modes except the  $\xi$  and  $\bar{\eta}$  [12, 7]

$$\Delta S = \frac{\pi^2}{4g^2\rho^2}\epsilon_{ABCD}\left(\bar{\mu}^{u,A}\mu_u^B\right)\left(\bar{\mu}^{v,C}\mu_v^D\right). \quad (18)$$

This term is hermitean w.r.t. the relations (17) since  $\epsilon_{ABCD}\eta_{AA'}^1\eta_{BB'}^1\eta_{CC'}^1\eta_{DD'}^1 = (\det \eta^1)\epsilon_{A'B'C'D'}$  and  $\det \eta_{AB}^1 = 1$ .

### 3 $\mathcal{N} = 2$ Euclidean model.

Let us now address the  $\mathcal{N} = 2$  super-Yang-Mills Euclidean model [13] deduced by dimensional reduction from the  $d = (5, 1)$   $\mathcal{N} = 1$  theory. The Lagrangian reads

$$\mathcal{L}_6 = \frac{1}{g_6^2}\text{tr}\left\{\frac{1}{2}F_{MN}F^{MN} + \bar{\Psi}_i\Gamma^M\mathcal{D}_M\Psi^i\right\}, \quad (19)$$

with the symplectic Majorana-Weyl condition

$$\Gamma^7\Psi^i = \Psi^i, \quad \Psi^{i,T}C_6^-\epsilon_{ij} = \Psi^{j\dagger}i\Gamma^0. \quad (20)$$

The action is invariant under transformations

$$\delta A_M = \bar{\zeta}_i\Gamma_M\Psi^i, \quad \delta\Psi^i = -\frac{1}{2}F_{MN}\Gamma^{MN}\zeta^i, \quad i = 1, 2, \quad (21)$$

(use  $\bar{\Psi}_i\Gamma^M\zeta^i = -\bar{\zeta}_i\Gamma^M\Psi^i$ ). We choose a different representation of the Dirac matrices as compared to the previous section, namely

$$\Gamma^M = \{\tau^a \otimes \gamma^5, \mathbb{1}_{[2\times 2]} \otimes \gamma^\mu\}, \quad (22)$$

with  $\tau^0 = -i\sigma^2$  and  $\tau^1 = \sigma^1$ . Then the charge conjugation matrix is  $C_6^- = \Gamma^0\Gamma^2\Gamma^4 = i\sigma^2 \otimes C_4^-$  and  $*\Gamma = \Gamma^7 = \Gamma^5\Gamma^4 \dots \Gamma^0 = \sigma^3 \otimes \gamma^5$ . From (19) the four-dimensional  $R$ -symmetry  $U(2)$  is manifest: the fermion transforms as the  $(\mathbf{4}, \mathbf{2})$  of  $SO(5, 1) \times SU(2)$ .

The procedure of the reduction via the time direction is completely equivalent to the one discussed in the previous section with the decomposition of the  $d = (5, 1)$  Lorentz group into  $SO(5, 1) \rightarrow SO(4) \times SO(1, 1)$ . This gives

$$\begin{aligned} \mathcal{L}_E^{\mathcal{N}=2} = \frac{1}{g^2}\text{tr} \left\{ \frac{1}{2}F_{\mu\nu}F_{\mu\nu} - 2i\bar{\lambda}_i^{\alpha'}\bar{\sigma}_{\mu,\alpha'\beta}\mathcal{D}_\mu\lambda^{\beta,i} + (\mathcal{D}_\mu S)^2 - (\mathcal{D}_\mu P)^2 - [P, S]^2 \right. \\ \left. - (S - P)\{\lambda^{\alpha,i}, \lambda_{\alpha,i}\} - (S + P)\{\bar{\lambda}^{\alpha',i}, \bar{\lambda}_{\alpha',i}\} \right\}, \quad (23) \end{aligned}$$

where  $P \equiv A^0$  and  $S \equiv A^1$ . We use  $\lambda^{\alpha,i}\epsilon_{ij}\epsilon_{\alpha\beta} = \lambda_{\beta,j}$  and analogously for  $\bar{\lambda}_{\alpha'}^i$ . As expected, the  $U(1)$  of the  $U(2)$  becomes non-compact in complete agreement with the results of [1, 3]: the automorphism of the supersymmetry algebra is  $SO(1, 1)$  so that Eq. (23) is invariant under scaling transformations  $(S + P) \rightarrow \varrho(S + P)$ ,  $(S - P) \rightarrow \varrho^{-1}(S - P)$ ,  $\lambda \rightarrow \varrho^{1/2}\lambda$  and  $\bar{\lambda} \rightarrow \varrho^{-1/2}\bar{\lambda}$ . The  $SU(2)$  survives the reduction and remains compact. The  $\mathcal{N} = 2$  supersymmetry transformation is given by

$$\begin{aligned}\delta A_\mu &= i\bar{\zeta}_i^{\alpha'}\bar{\sigma}_{\mu,\alpha'\beta}\lambda^{\beta,i} - i\zeta_{\alpha,i}\sigma_\mu^{\alpha\beta'}\bar{\lambda}_{\beta'}^i, & \delta S &= \zeta_{\alpha,i}\lambda^{\alpha,i} - \bar{\zeta}_i^{\alpha'}\bar{\lambda}_{\alpha'}^i, & \delta P &= \zeta_{\alpha,i}\lambda^{\alpha,i} + \bar{\zeta}_i^{\alpha'}\bar{\lambda}_{\alpha'}^i, \\ \delta\lambda^{\alpha,i} &= -\frac{1}{2}(\sigma_{\mu\nu})^\alpha{}_\beta F_{\mu\nu}\zeta^{\beta,i} - i\bar{\zeta}_{\alpha'}^i\sigma_\mu^{\alpha\alpha'}\mathcal{D}_\mu(S + P) - [P, S]\zeta^{\alpha,i}, \\ \delta\bar{\lambda}_{\alpha'}^i &= -\frac{1}{2}(\bar{\sigma}_{\mu\nu})_{\alpha'}{}^{\beta'} F_{\mu\nu}\bar{\zeta}_{\beta'}^i - i\zeta^{\alpha,i}\bar{\sigma}_{\mu,\alpha'\alpha}\mathcal{D}_\mu(S - P) + [P, S]\bar{\zeta}_{\alpha'}^i.\end{aligned}\tag{24}$$

The Lagrangian (23) is hermitean w.r.t. the reality conditions

$$\left(\lambda^{\alpha,i}\right)^\dagger = i\lambda^{\beta,j}\epsilon_{ji}\epsilon_{\beta\alpha}, \quad \left(\bar{\lambda}_{\alpha'}^i\right)^\dagger = -i\bar{\lambda}_{\beta'}^j\epsilon_{ji}\epsilon^{\beta'\alpha'},\tag{25}$$

stemming from the  $d = 6$  constraints (20).

## 4 Conclusions.

The dimensional reduction of higher dimensional SYM theories via the time direction naturally leads to Euclidean  $\mathcal{N} > 1$  supersymmetric models with a hermitean action. The Lagrangians of the Euclidean models and the SUSY transformation rules written in covariant form w.r.t. the internal  $R$  group do not change their form as compared to the Minkowskian ones, however, the compact internal  $R$ -symmetry group becomes non-compact in Euclidean space and the reality conditions on fields involve different metrics. For fermions we end up with symplectic Majorana conditions in (4, 0). When these results are translated to the reality conditions on the Grassmann collective coordinates we obtain a real effective action for the collective coordinates induced by instantons.

We could have used other representation for the  $d = (9, 1)$  Dirac matrices, for example,

$$\Gamma^M = \{\hat{\gamma}^a \otimes \mathbf{1}_{[4\times 4]}, \hat{\gamma}^7 \otimes \gamma^\mu\}.\tag{26}$$

In this representation  $C_{10}^- = C_6^- \otimes C_4^+$  but the reality conditions in (8) are unchanged because  $C_4^+ = C_4^- \gamma^5$  while  $\Gamma^0$  differs also by a factor  $\gamma^5$ .

Obviously, our modus operandi is not applicable to the  $\mathcal{N} = 1$  model. In Euclidean space due to absence of real Dirac matrices the generators of the supersymmetry are complex four-component spinors. Since the internal  $R$ -symmetry group is Abelian for simple supersymmetry we cannot impose the symplectic Majorana condition. In this situation one accepts the idea of



complexification of all fields [14] of the theory without a reality condition (and loose hermiticity of the action). If, however, one wants to preserve a real gauge field one is forced to enhance the  $\mathcal{N} = 1$  supersymmetry to the  $\mathcal{N} = 2$  SUSY [1].

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