The Map Between
Conformal Hypercomplex/Hyper-Kähler
and Quaternionic(-Kähler) Geometry

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Abstract

We review the general properties of target spaces of hypermultiplets, which are quaternionic-like manifolds, and discuss the relations between these manifolds and their symmetry generators. We explicitly construct a one-to-one map between conformal hypercomplex manifolds (i.e. those that have a closed homothetic Killing vector) and quaternionic manifolds of one quaternionic dimension less. An important role is played by 'ξ-transformations', relating complex structures on conformal hypercomplex manifolds and connections on quaternionic manifolds. In this map, the subclass of conformal hyper-Kähler manifolds is mapped to quaternionic-Kähler manifolds. We relate the curvatures of the corresponding manifolds and furthermore map the symmetries of these manifolds to each other.

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1 Introduction

Ever since Einstein's theory of gravity, differential geometry has entered research areas in theoretical physics in various places and contexts. The idea of using geometry to describe and unify the forces of nature continues to play an important role in present day high energy physics. After the invention of supersymmetry, superstrings and their compactifications, connections were found with the holonomy groups and Killing spinors of Riemannian manifolds. It was shown that supersymmetric sigma models were deeply related to geometries with complex structures. Kähler manifolds appear already in $N = 1$ theories in 4 dimensions [1]. Some theories with 8 supersymmetries, i.e. $N = 4$ in 2 dimensions and $N = 2$ in 3, 4, 5 or 6 dimensions, exhibit a hypercomplex structure. It was shown that Lagrangians for rigid supersymmetry lead to hyper-Kähler [2] manifolds and Lagrangians for supergravity theories lead to quaternionic-Kähler manifolds [3]. Many aspects of such geometries are by now well-known and studied by the supersymmetry, supergravity and superstring community, and this has led to new insights and useful applications in these fields. Recently, it has been shown that a generalization of hyper-Kähler manifolds is possible for rigid $N = 2$ supersymmetric theories if one does not demand that the field equations are derivable from an action [4, 5]. Such a situation arises when in the field equations for the scalar fields, a connection is chosen different from a Levi-Civita connection derivable from a metric on the sigma model target manifold. The corresponding geometry is called hypercomplex [6], which differs from a hyper-Kähler geometry by the fact that it does not necessarily possess a covariantly constant metric preserved by the connection induced by the hypercomplex structure, as we explain in detail below. We will further refer to a Hermitian metric that is covariantly constant with respect to the latter connection as a 'good metric'. The manifolds mentioned above are schematically represented in Table 1. There it is indicated that hypercomplex

Table 1: *Quaternionic-like manifolds of real dimension* $4r$. *These are the manifolds that have a quaternionic structure satisfying (1.2) and (1.4). The holonomy group is indicated. Generalizations for different signatures are obvious. The table is essentially taken over from [7]. The dot notation means e.g.* $\mathrm{SU}(2) \cdot \mathrm{USp}(2r) = \frac{\mathrm{SU}(2) \times \mathrm{USp}(2r)}{\mathbb{Z}_2}$.

	no good metric	with a good metric	
no SU(2) curvature	***hypercomplex*** $\mathrm{G}\,\ell(r, \mathbb{H})$	***hyper-Kähler*** $\mathrm{USp}(2r)$	rigid supersymmetry
non-zero SU(2) curvature	***quaternionic*** $\mathrm{SU}(2) \cdot \mathrm{G}\,\ell(r, \mathbb{H})$	***quaternionic-Kähler*** $\mathrm{SU}(2) \cdot \mathrm{USp}(2r)$	supergravity
	field equations	action	

and hyper-Kähler manifolds occur in rigid supersymmetric theories, while quaternionic and quaternionic-Kähler manifolds occur in supergravity. Manifolds without a good metric occur as long as the field equations are not derived from a conventional action. The conventional actions in supersymmetry and supergravity involve a metric on the manifold of scalars, and the

field equations naturally involve the Levi-Civita connection. Hypercomplex and quaternionic manifolds are endowed with a connection that is not necessarily the Levi-Civita connection. For this reason we say they are not derived from an action.

The supersymmetry algebra on hypermultiplets requires that we use torsionless connections. However, hyper-Kähler torsion (HKT) manifolds, which appear as the moduli space of supersymmetric multi-black holes [8, 9], belong also to the class of hypercomplex manifolds. Indeed, they carry a triplet of complex structures with vanishing Nijenhuis tensor, implying that there exists a torsionless connection. This is similar to the way in which we used two-dimensional non-linear sigma models on group manifolds [10] for the example in [4, Appendix C]. Also there, the connection with torsion corresponding to the three-form field strength is different from the torsionless connection used in the definition of hypercomplex manifolds.

Supergravity theories can be constructed by the 'superconformal tensor calculus' [11–13], which gives insight in the geometry of the relevant sigma models. These geometries are obtained as projective spaces, related to the dilatation symmetry in the superconformal group, and with possible further projections related to the R-symmetry group, see [14] for a review of these principles. In particular, this has been used in [15–18] in the context of $N = 2$ theories with hypermultiplets, constructing the link between hyper-Kähler manifolds and quaternionic-Kähler manifolds. This construction starts from a conformal hyper-Kähler manifold. We will use the name *conformal manifold* for a manifold which has a closed homothetic Killing vector. For a manifold with coordinates q^X and affine connection $\Gamma_{XY}{}^Z$, this is a vector k^X, satisfying[1]

$$\mathfrak{D}_Y k^X \equiv \partial_Y k^X + \Gamma_{YZ}{}^X k^Z = \tfrac{3}{2}\delta_Y{}^X. \tag{1.1}$$

The presence of such a vector allows the definition of conformal symmetries as a basic step for the superconformal tensor calculus [20].

In this paper, we will formulate and prove the 1-to-1 correspondence[2] (locally) between conformal hypercomplex manifolds of quaternionic dimension $n_H + 1$ and quaternionic manifolds of dimension n_H. Furthermore, we show that this 1-to-1 correspondence is also applicable between the subset of hypercomplex manifolds that are hyper-Kähler and the subset of quaternionic manifolds that are quaternionic-Kähler. In the mathematics literature the map between quaternionic-Kähler and hyper-Kähler manifolds is constructed by Swann [21], and its generalization to quaternionic manifolds is treated in [22]. Here, we give explicit expressions for the complex structures and connections that are needed to apply these results in the context of the conformal tensor calculus in supergravity.

To explain the 1-to-1 mapping let us first repeat the basic definitions. A *hypercomplex manifold* is a manifold with a hypercomplex structure. This means that in any local patch[3]

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[1] The factor $3/2$ is a choice of normalization which is convenient for applications of 5-dimensional supergravity theories. The translation between formulations appropriate to supergravities in other dimensions has been considered in [19].

[2] As will be explained below, the correspondence is actually 1-to-1 between families (or 'equivalence classes') of manifolds.

[3] We use the integer r for the quaternionic dimension of any quaternionic-like manifold (the number of hypermultiplets). In the application to the map, this r can be n_H or $n_H + 1$ depending on whether we are considering the quaternionic space or the hypercomplex space, respectively.

there are coordinates q^X with $X = 1, \ldots, 4r$ and a triplet of complex structures $\vec{J}_X{}^Y$ that satisfy the algebra of the imaginary quaternions, which is that for any vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \cdot \vec{J} \vec{B} \cdot \vec{J} = -\mathbb{1}_{4r} \vec{A} \cdot \vec{B} + (\vec{A} \times \vec{B}) \cdot \vec{J}, \qquad \mathrm{Tr}\, \vec{J} = 0. \tag{1.2}$$

This defines an almost hypercomplex structure.[4] The closure of the (rigid) supersymmetry algebra on hypermultiplets requires that the complex structures are covariantly constant using a *torsionless* affine connection $\Gamma_{XY}{}^Z = \Gamma_{YX}{}^Z$:

$$\mathfrak{D}_X \vec{J}_Y{}^Z \equiv \partial_X \vec{J}_Y{}^Z - \Gamma_{XY}{}^W \vec{J}_W{}^Z + \Gamma_{XW}{}^Z \vec{J}_Y{}^W = 0. \tag{1.3}$$

Stated otherwise, the hypercomplex structures should be integrable.[5]

A *quaternionic manifold* is defined by a local span of these three complex structures. This means that the three complex structures can at any point q be rotated as $\vec{J}' = R(q)\vec{J}$, where R is a 3×3 matrix of SO(3). The covariant constancy condition (i.e. the integrability) should then also be covariant with respect to these rotations. This implies that one needs a connection $\vec{\omega}_X$ and the condition (1.3) is replaced by

$$\mathfrak{D}_X \vec{J}_Y{}^Z \equiv \partial_X \vec{J}_Y{}^Z - \Gamma_{XY}{}^W \vec{J}_W{}^Z + \Gamma_{XW}{}^Z \vec{J}_Y{}^W + 2\vec{\omega}_X \times \vec{J}_Y{}^Z = 0. \tag{1.4}$$

Infinitesimal SO(3) rotations are parametrized by 3 angles $\vec{\ell}(q)$,

$$\delta_{\mathrm{SU}(2)} \vec{J}_X{}^Y = \vec{\ell} \times \vec{J}_X{}^Y, \qquad \delta_{\mathrm{SU}(2)} \vec{\omega}_X = -\tfrac{1}{2}\partial_X \vec{\ell} + \vec{\ell} \times \vec{\omega}_X. \tag{1.5}$$

The connections in (1.4) are not unique. Indeed, they can be changed simultaneously depending on an arbitrary one-form $\xi = \xi_X \mathrm{d}q^X$ as [7, 23, 24]

$$\Gamma_{XY}{}^Z \to \Gamma_{XY}{}^Z + 2\delta^Z_{(X}\xi_{Y)} - 2\vec{J}_{(X}{}^Z \cdot \vec{J}_{Y)}{}^W \xi_W, \qquad \vec{\omega}_X \to \vec{\omega}_X + \vec{J}_X{}^W \xi_W. \tag{1.6}$$

These transformations relate different connections on the same manifold and therefore define equivalence classes. Furthermore, it will turn out that for the conformal hypercomplex manifolds there are related transformations between complex structures and connections. These define equivalence classes between conformal hypercomplex structures.

There is however one important difference between the transformations on the quaternionic and on the hypercomplex space. On the quaternionic manifold, the ξ-transformation yields different torsionless connections for a given (integrable) quaternionic structure (i.e. for a given span of three complex structures). Otherwise stated, an integrable quaternionic structure does not determine the connection uniquely. On the conformal hypercomplex space, the transformations relate different (integrable) hypercomplex structures in a continuous fashion. To the best of our knowledge, these transformations between hypercomplex structures on conformal manifolds have not yet been discussed in the literature. The map

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[4]Note that the tracelessness is implied by the first relation.

[5]In the context of G-structures, condition (1.3) would rather be called the one-integrability of the hypercomplex structure [7].

from conformal hypercomplex to quaternionic manifolds is subject to these transformations, and is 1-to-1 for the equivalence classes.

We give an explicit construction of this correspondence by relating the geometric quantities in the corresponding manifolds, i.e. the complex structures, affine connections, curvatures, Killing vectors and moment maps. We prove that the existence of a 'good metric' for a hypercomplex manifold is equivalent to the existence of a 'good metric' in the corresponding quaternionic space. Also the symmetries of related manifolds are mapped 1-to-1.

Our formulation here focuses purely on the geometrical aspects. Our results have been applied already to $N = 2$ matter coupled $D = 5$ supergravity [25]. The results of this paper are applicable to the geometry of hypermultiplets independent of whether they are defined in 3, 4, 5 or 6 dimensions.

In Sect. 2 we give a summary of the properties of the geometries with a triplet of complex structures ('quaternionic-like manifolds'). In particular we discuss the curvatures determining the holonomy groups. We will devote special attention to the ξ-transformations and to the properties of the Ricci tensor. At the end of that section we will look at those manifolds that have a Hermitian Ricci tensor, which include all hypercomplex, hyper-Kähler and quaternionic-Kähler manifolds. Most results in this section are due to [7].

The new work starts in Sect. 3, where we construct the map discussed above. We start by considering conformal hypercomplex manifolds and identify new transformations between connections and the hypercomplex structure that respect the conditions for hypercomplex manifolds. These spaces are then reduced to a submanifold that turns out to be quaternionic. Coordinates are chosen in view of the gauge fixing of dilatation, SU(2) symmetry and special (S) supersymmetry in the superconformal context. We construct the geometric building blocks in this suitable basis. These are the complex structures and affine connections. The freedom of ξ-transformations comes naturally out of this map. Inversely, we associate such a conformal hypercomplex manifold to any quaternionic manifold. This finishes the proof that the mapping between these manifolds is 1-to-1. In a further subsection, we focus on the manifolds with a good metric, i.e. hyper-Kähler and quaternionic-Kähler manifolds. We show that these are also 1-to-1 related and give explicit expressions for the connections. Finally, in this section we also construct the vielbeins and the spin connections (i.e. the connections that transform under local general linear quaternionic transformations).

The relation between curvatures of the conformal hypercomplex and the quaternionic manifolds is discussed in Sect. 4. As recalled in Sect. 2, the curvatures of quaternionic-like manifolds are characterized by a symmetric 'Weyl tensor' $\mathcal{W}_{ABC}{}^D$, where $A = 1, \ldots, 2r$ are indices in the tangent space. We will therefore connect the Weyl tensors of the hypercomplex and quaternionic spaces.

Symmetries of the manifolds that preserve also the hypercomplex structure are called triholomorphic. Such symmetries of conformal hypercomplex manifolds descend to quaternionic symmetries of quaternionic manifolds (which is a statement of preservation of the span of complex structures). We define the moment maps of quaternionic manifolds, and we show in Sect. 5 that these are determined in terms of components of the symmetries of the hypercomplex manifolds in directions that are projected out by the gauge fixing.

We summarize and overview all results of this paper in Sect. 6. We have written this

in such a way that this section can be read by itself by a reader familiar with quaternionic geometry (that is recapitulated in Sect. 2). We illustrate the results with a schematic picture, Fig. 1. After this summary, we discuss some remaining issues and give some remarks. In the main part of the paper, we choose the signatures in the way that is most relevant for supergravity, i.e. such that the scalars and the graviton have positive kinetic energies. This selects quaternionic-Kähler manifolds with a positive definite metric and a negative definite scalar curvature. These are obtained in the map by starting with hyper-Kähler manifolds of quaternionic signature $(- + \cdots +)$. In the discussion section, we indicate how our formulae can be used for other signatures.

An appendix gives some useful formulae for calculus with complex structures and Hermitian tensors.

2   Quaternionic-like Manifolds

In this section, we review the four different geometries that are used in this paper. They correspond to hypercomplex, hyper-Kähler, quaternionic and quaternionic-Kähler manifolds, and are distinguished by the properties of their holonomy groups and the presence of a preserved metric, as summarized in Table 1. This section is an extension of Appendix B of [4], a paper where these geometries are discussed in the context of five dimensional conformal hypermultiplets, and which itself makes heavy use of the pioneering paper [7].

We use here the name 'quaternionic-like manifolds' for all the manifolds in Table 1. In fact, the formulae for quaternionic manifolds are the most general ones, and in this respect one can argue to just use 'quaternionic manifolds' as general terminology. The subtlety is that in principle for hypercomplex manifolds one admissible basis of quaternionic structures is selected, while in quaternionic manifolds only the local span of complex structures is used. For all practical purposes, the formulae of quaternionic manifolds are the most general, and can be applied with $\vec{\omega}_X = 0$ for hypercomplex manifolds.

Subsection 2.1 gives these general formulae for an arbitrary quaternionic manifold. The properties of the curvatures are presented. The ξ-transformations mentioned in the introduction are treated in more detail in Subsect. 2.2. Special features for the case of hypercomplex manifolds are given in Subsect. 2.3, for hyper-Kähler manifolds in Subsect. 2.4 and for quaternionic-Kähler manifolds in Subsect. 2.5. In these 3 cases, the Ricci tensor is Hermitian. We give general properties of quaternionic-like manifolds with Hermitian Ricci tensor in Subsect. 2.6.

2.1   General properties of quaternionic-like manifolds

The common property of all quaternionic-like manifolds, say of dimension $4r$, is the existence of a quaternionic structure, i.e. a triplet of endomorphisms $\vec{J}$, realizing the algebra of the imaginary quaternions (1.2).

In general relativity, it is often convenient to introduce (locally) a set of one-forms that define an orthonormal frame: $e^a = e_\mu{}^a \mathrm{d}x^\mu$. Their components $e_\mu{}^a$ are the so-called 'viel-

beins'. As such, the so-called *flat* index a takes values in the orthogonal group, and μ is the one-form index, also known as the *curved* index. In the present context, the orthogonal group is substituted by the groups mentioned in Table 1. Objects with flat indices will be said to take values in the tangent bundle.

Therefore, we locally introduce the coordinates q^X with $X = 1, \ldots, 4r$, and we assume the existence of an invertible vielbein f_X^{iA}, two matrices $\rho_A{}^B$ and $E_i{}^j$ (with $i = 1, 2$ and $A = 1, \ldots, 2r$) that satisfy

$$\rho \, \rho^* = -\mathbb{1}_{2r}, \qquad E \, E^* = -\mathbb{1}_2, \tag{2.1}$$

together with the reality condition

$$(f_X^{iA})^* = f_X^{jB} E_j{}^i \rho_B{}^A. \tag{2.2}$$

One can choose a basis such that $E_i{}^j = \varepsilon_{ij}$, see e.g. [26]. We will further always use this basis.

The transformations on variables with an A index are restricted by the reality condition to $G\ell(r, \mathbb{H}) = SU^*(2r) \times U(1)$. The inverse vielbein, denoted by f_{iA}^X, satisfies

$$f_Y^{iA} f_{iA}^X = \delta_Y^X, \qquad f_X^{iA} f_{jB}^X = \delta_j^i \delta_B^A, \tag{2.3}$$

and can be used to define the quaternionic structure as

$$\vec{J}_X{}^Y \equiv -\mathrm{i} f_X^{iA} \vec{\sigma}_i{}^j f_{jA}^Y, \tag{2.4}$$

where $\vec{\sigma}$ are the three Pauli matrices. These matrices $\vec{J}$ satisfy (1.2). We use here a slight change of notation with respect to [4] in that we will indicate the triplets by a vector symbol, rather than an α index in [4] (which will be needed below to indicate a coordinate set).

All quaternionic-like manifolds admit connections with respect to which the vielbeins are covariantly constant by definition. These are a torsionless affine connection $\Gamma_{XY}{}^Z$, a $G\ell(r, \mathbb{H})$ connection $\omega_{XA}{}^B$ on the tangent bundle and possibly an $SU(2)$ connection $\omega_{Xi}{}^j$, such that

$$\mathfrak{D}_X f_Y^{iA} \equiv \partial_X f_Y^{iA} - \Gamma_{XY}{}^Z f_Z^{iA} + f_Y^{jA} \omega_{Xj}{}^i + f_Y^{iB} \omega_{XB}{}^A = 0, \tag{2.5}$$

and similarly for the inverse vielbein.

This implies that the quaternionic structure is covariantly constant with respect to the affine connection and the $SU(2)$ connection [6],

$$\mathfrak{D}_X \vec{J}_Y{}^Z \equiv \partial_X \vec{J}_Y{}^Z - \Gamma_{XY}{}^W \vec{J}_W{}^Z + \Gamma_{XW}{}^Z \vec{J}_Y{}^W + 2\vec{\omega}_X \times \vec{J}_Y{}^Z = 0. \tag{2.6}$$

The $G\ell(r, \mathbb{H})$-connection $\omega_{XA}{}^B$ on the tangent bundle can then be determined by requiring (2.5):

$$\omega_{XA}{}^B = \tfrac{1}{2} f_Y^{iB} \left(\partial_X f_{iA}^Y + \Gamma_{XZ}{}^Y f_{iA}^Z - \omega_{Xi}{}^j f_{jA}^Y \right). \tag{2.7}$$

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[6]One can make the transition from doublet to vector notation by using the sigma matrices, $\omega_{Xi}{}^j = \mathrm{i}\vec{\sigma}_i{}^j \cdot \vec{\omega}_X$, and similarly $\vec{\omega}_X = -\tfrac{1}{2}\mathrm{i}\vec{\sigma}_i{}^j \omega_{Xj}{}^i$. This transition between doublet and triplet notation is valid for any triplet object as e.g. the complex structures.

A useful concept in describing the integrability of almost quaternionic-like structures is the Nijenhuis tensor. We will use only the 'diagonal' Nijenhuis tensor (normalization for later convenience)

$$N_{XY}{}^Z \equiv \tfrac{1}{6} \vec{J}_{[X}{}^W \cdot \left(\partial_{|W|} \vec{J}_{Y]}{}^Z - \partial_Y{}] \vec{J}_W{}^Z \right). \tag{2.8}$$

A quaternionic manifold is defined by requiring that the Nijenhuis tensor satisfies

$$(1 - 2\,r)\, N_{XY}{}^Z = -\vec{J}_{[X}{}^Z N_{Y]V}{}^W \cdot \vec{J}_W{}^V, \tag{2.9}$$

which is equivalent to requiring that

$$N_{XY}{}^Z = -\vec{J}_{[X}{}^Z \cdot \vec{\omega}_{Y]}^{\mathrm{Op}}, \tag{2.10}$$

for some $\vec{\omega}_X^{\mathrm{Op}}$. Since the trace of the Nijenhuis tensor vanishes, $\vec{\omega}_X^{\mathrm{Op}}$ obeys $\vec{J}_X{}^Y \cdot \vec{\omega}_Y^{\mathrm{Op}} = 0$. Hence, (2.9) is automatically satisfied. The condition (2.10) can be solved for $\vec{\omega}_X^{\mathrm{Op}}$, and used to define an SU(2) connection

$$(1 - 2\,r)\, \vec{\omega}_X^{\mathrm{Op}} = N_{XY}{}^Z \vec{J}_Z{}^Y. \tag{2.11}$$

The condition (2.9) ensures that there exists an appropriate affine connection such that (2.6) is satisfied. This connection is called the *Oproiu connection* [27],

$$\Gamma^{\mathrm{Op}}{}_{XY}{}^Z \equiv \Gamma^{\mathrm{Ob}}{}_{XY}{}^Z - \vec{J}_{(X}{}^Z \cdot \vec{\omega}_{Y)}^{\mathrm{Op}}, \tag{2.12}$$

$$\Gamma^{\mathrm{Ob}}{}_{XY}{}^Z \equiv -\tfrac{1}{6} \left(2\partial_{(X} \vec{J}_{Y)}{}^W + \vec{J}_{(X}{}^U \times \partial_{|U|} \vec{J}_{Y)}{}^W \right) \cdot \vec{J}_W{}^Z. \tag{2.13}$$

$\Gamma^{\mathrm{Ob}}{}_{XY}{}^Z$ is the *Obata connection*, which is the solution if $\vec{\omega}_X = 0$.

Conversely, any two connections $\Gamma_{XY}{}^Z$ and $\vec{\omega}_X$ that satisfy (2.6), necessarily imply the condition (2.9) on the Nijenhuis tensor. Moreover, as shown in [4], they must be related to the connections defined by (2.11) and (2.12) by means of the ξ-transformations mentioned in (1.6).

We proceed by discussing the curvature tensor of quaternionic manifolds. We first give our conventions:

$$\begin{aligned} R_{XYZ}{}^W &\equiv 2\partial_{[X} \Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W \Gamma_{Y]Z}{}^V, \\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X} \omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A \omega_{Y]B}{}^C, \\ \vec{\mathcal{R}}_{XY} &\equiv 2\partial_{[X} \vec{\omega}_{Y]} + 2\vec{\omega}_X \times \vec{\omega}_Y. \end{aligned} \tag{2.14}$$

The integrability condition of (2.5) implies that the total curvature on the manifold is the sum of the SU(2) curvature and the $G\,\ell(r, \mathbb{H})$ curvature. This shows that the (restricted) holonomy splits in these two factors:[7]

$$\begin{aligned} R_{XYW}{}^Z &= R^{\mathrm{SU}(2)}{}_{XYW}{}^Z &+ R^{G\,\ell(r,\mathbb{H})}{}_{XYW}{}^Z \\ &= -\vec{J}_W{}^Z \cdot \vec{\mathcal{R}}_{XY} &+ L_W{}^Z{}_A{}^B \mathcal{R}_{XYB}{}^A, \end{aligned} \tag{2.15}$$

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[7]This follows also from the Ambrose-Singer theorem [28], which says that the Lie algebra of the restricted holonomy group of the frame bundle coincides with the algebra generated by the curvature. The direct product structure of the holonomy group is then reflected in these relations.

where matrices $L_A{}^B$ appear. They are defined in Appendix A, where also various useful properties are exhibited. This implies

$$4r \, \vec{\mathcal{R}}_{XY} = R_{XYZ}{}^W \vec{J}_W{}^Z, \qquad \mathcal{R}_{XYA}{}^B = \tfrac{1}{2} L_W{}^Z{}_A{}^B R_{XYZ}{}^W. \tag{2.16}$$

Furthermore, we define the Ricci tensor as

$$R_{XY} = R_{ZXY}{}^Z. \tag{2.17}$$

For an arbitrary affine connection, it has both a symmetric and antisymmetric part. In general, the antisymmetric part can be traced back to the curvature of the $\mathbb{R}$ part in $\mathrm{G}\ell(r,\mathbb{H}) = \mathrm{S}\ell(r,\mathbb{H}) \times \mathbb{R}$. Indeed, using the cyclicity condition, we find

$$R_{[XY]} = R_{Z[XY]}{}^Z = -\tfrac{1}{2} R_{XYZ}{}^Z = -\mathcal{R}_{XY}^{\mathbb{R}}, \qquad \mathcal{R}_{XY}^{\mathbb{R}} \equiv \mathcal{R}_{XYA}{}^A. \tag{2.18}$$

Therefore, the antisymmetric part of the Ricci tensor follows completely from the $\mathbb{R}$ part.

The separate curvature terms in (2.15) do not satisfy the cyclicity condition (unless the $\mathrm{SU}(2)$ curvature vanishes), and thus are not bona-fide curvatures. Another splitting of the full curvature can be made where both terms separately satisfy the cyclicity condition,

$$R_{XYW}{}^Z = R^{\mathrm{Ric}}{}_{XYW}{}^Z + R^{(\mathrm{W})}{}_{XYW}{}^Z. \tag{2.19}$$

The first part only depends on the Ricci tensor of the full curvature, and is called the '*Ricci part*'. It is defined by

$$R^{\mathrm{Ric}}{}_{XYZ}{}^W \equiv 2\delta_{[X}{}^W B_{Y]Z} - 2\delta_Z{}^W B_{[XY]} - 4\vec{J}_Z{}^{(W} \cdot \vec{J}_{[X}{}^{V)} B_{Y]V}, \tag{2.20}$$

where [29]

$$B_{XY} \equiv \frac{1}{4r} \left(\delta_X{}^Z \delta_Y{}^W - \Pi_{XY}{}^{ZW} \right) R_{(ZW)} + \frac{1}{4(r+2)} \Pi_{XY}{}^{ZW} R_{(ZW)} + \frac{1}{4(r+1)} R_{[XY]}. \tag{2.21}$$

Here, we have introduced a projection operator

$$\Pi_{XY}{}^{ZW} \equiv \tfrac{1}{4} \left(\delta_X{}^Z \delta_Y{}^W + \vec{J}_X{}^Z \cdot \vec{J}_Y{}^W \right), \tag{2.22}$$

whose properties are discussed in the Appendix. The symmetric part of B_{XY} can be considered as the candidate for a 'good metric' for quaternionic manifolds. Indeed, if there is a good metric, then it is proportional to the symmetric part of this tensor as we will see below in (2.57).

The Ricci part does satisfy the cyclicity property, and its Ricci tensor is just R_{XY}. We can further split it as

$$R^{\mathrm{Ric}}{}_{XYZ}{}^W = \left(R^{\mathrm{Ric}}_{\mathrm{symm}} + R^{\mathrm{Ric}}_{\mathrm{antis}} \right)_{XYZ}{}^W, \tag{2.23}$$

where the first term is the construction (2.20) using only the symmetric part of B (the symmetric part of the Ricci tensor), and the second term uses only the antisymmetric part of B (i.e. of the Ricci tensor).

The second term in (2.19) is defined as the remainder, and its Ricci tensor is zero. For this reason, it is called the 'Weyl part' [7]. Following again the discussion in [4], one can rewrite the Weyl part in terms of a symmetric and traceless tensor $\mathcal{W}_{ABC}{}^D$, such that [8]

$$R_{XYZ}{}^W = R^{\text{Ric}}{}_{XYZ}{}^W - \tfrac{1}{2} f_X^{iA} \varepsilon_{ij} f_Y^{jB} f_Z^{kC} f_{kD}^W \mathcal{W}_{ABC}{}^D, \tag{2.24}$$

with

$$\mathcal{W}_{CDB}{}^A \equiv \tfrac{1}{2} \varepsilon^{ij} f_{jC}^X f_{iD}^Y f_{kB}^Z f_W^{kA} R^{(\text{W})}{}_{XYZ}{}^W. \tag{2.25}$$

The $\mathrm{G}\ell(r,\mathbb{H})$ curvature can also be decomposed in its Ricci and Weyl part:

$$\begin{aligned}
\mathcal{R}_{XYA}{}^B &= \mathcal{R}^{\text{Ric}}{}_{XYA}{}^B + \mathcal{R}^{(\text{W})}{}_{XYA}{}^B, \\
\mathcal{R}^{\text{Ric}}{}_{XYA}{}^B &\equiv \tfrac{1}{2} L_W{}^Z{}_A{}^B R^{\text{Ric}}{}_{XYZ}{}^W = 2\delta_A{}^B B_{[YX]} + 4 L_{[X}{}^V{}_A{}^B B_{Y]V}, \\
\mathcal{R}^{(\text{W})}{}_{XYA}{}^B &\equiv \tfrac{1}{2} L_W{}^Z{}_A{}^B R^{(\text{W})}{}_{XYZ}{}^W = -f_X^{iC} \varepsilon_{ij} f_Y^{jD} \mathcal{W}_{CDA}{}^B.
\end{aligned} \tag{2.26}$$

On the other hand, the $\mathrm{SU}(2)$ curvature is determined only by the Ricci tensor, or, equivalently, by the tensor B_{XY}:

$$\vec{\mathcal{R}}_{XY} = 2\vec{J}_{[X}{}^Z B_{Y]Z}. \tag{2.27}$$

We can summarize the different curvature decompositions in the following scheme:

$$\begin{aligned}
R_{XYZ}{}^W = \quad (R^{\text{Ric}}_{\text{symm}} \quad + \qquad\qquad R^{\text{Ric}}_{\text{antis}} \qquad\qquad + \quad R^{(\text{W})})_{XYZ}{}^W \\
\\
= \quad (R^{\text{SU}(2)} \quad + \qquad\qquad R^{\mathbb{R}} \qquad\qquad\qquad + \quad R^{\mathrm{S}\ell(r,\mathbb{H})})_{XYZ}{}^W.
\end{aligned} \tag{2.28}$$

The terms in the second line depend only on specific terms of the first line as indicated by the arrows. This is the general scheme and thus applicable for quaternionic manifolds, which is the general case, but for specific other quaternionic-like manifolds some parts are absent as can be seen in Table 2.

2.2 The ξ-transformations

The requirement (1.4) for a fixed complex structure does not determine the connections uniquely. Indeed, the affine and $\mathrm{SU}(2)$ connections can be changed simultaneously depending on an arbitrary one-form $\xi = \xi_X \mathrm{d}q^X$ as

$$\tilde{\Gamma}_{XY}{}^Z = \Gamma_{XY}{}^Z + S_{XY}{}^{WZ} \xi_W, \qquad \vec{\tilde{\omega}}_X = \vec{\omega}_X + \vec{J}_X{}^W \xi_W. \tag{2.29}$$

Here we have introduced the S-symbols, which can be read of from (1.6). In terms of the projection operator in (2.22), they are

$$S_{XY}{}^{ZW} \equiv 2\delta_{(X}^Z \delta_{Y)}^W - 2\vec{J}_X{}^{(Z} \cdot \vec{J}_Y{}^{W)} = 4\delta_{(X}^Z \delta_{Y)}^W - 8\Pi_{(XY)}{}^{ZW}. \tag{2.30}$$

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[8]The case of four-dimensional quaternionic manifolds must be treated separately, but they are defined such that (2.24) is still satisfied. See [4] for more details.

Table 2: *The curvatures in quaternionic-like manifolds. The first line gives the decomposition according to Ricci and Weyl curvatures, while the second line gives the decomposition in accordance with the holonomy groups.*

hypercomplex	*hyper-Kähler*
$R_{\mathrm{antis}}^{\mathrm{Ric}} + R^{(\mathrm{W})}$	$R^{(\mathrm{W})}$
$R^{\mathbb{R}} + R^{\mathrm{S}\,\ell(r,\mathbb{H})}$	$R^{\mathrm{S}\,\ell(r,\mathbb{H})}$
quaternionic	*quaternionic-Kähler*
$R_{\mathrm{symm}}^{\mathrm{Ric}} + R_{\mathrm{antis}}^{\mathrm{Ric}} + R^{(\mathrm{W})}$	$R_{\mathrm{symm}}^{\mathrm{Ric}} + R^{(\mathrm{W})}$
$R^{\mathrm{SU}(2)} + R^{\mathbb{R}} + R^{\mathrm{S}\,\ell(r,\mathbb{H})}$	$R^{\mathrm{SU}(2)} + R^{\mathrm{S}\,\ell(r,\mathbb{H})}$

Further properties are given in (A.13)–(A.17).

Obviously, since the complex structures do not transform, the Nijenhuis tensor is invariant, and so the geometry remains quaternionic. When transforming the affine and the SU(2) connection as in (2.29), also the connection $\omega_{XA}{}^{B}$ transforms, according to its value in (2.7):

$$\tilde{\omega}_{XA}{}^{B} = \omega_{XA}{}^{B} + \tfrac{1}{2} L_Y{}^{Z}{}_A{}^{B} S_{XZ}{}^{YW} \xi_W. \tag{2.31}$$

Note in particular that the $\mathbb{R}$ connection transforms as

$$\tilde{\omega}_{XA}{}^{A} = \omega_{XA}{}^{A} + 2(r+1)\xi_X. \tag{2.32}$$

Clearly, all curvatures will transform under these deformations of the connections, with terms at most quadratic in ξ_X. A direct computation shows that the full Riemann curvature transforms as

$$R_{XYZ}{}^{W}(\tilde{\Gamma}) = R_{XYZ}{}^{W}(\Gamma) + 2S_{Z[Y}{}^{WU} D_{X]}\xi_U + 2S_{T[X}{}^{WU} S_{Y]Z}{}^{TV} \xi_U \xi_V. \tag{2.33}$$

Defining furthermore[9]

$$\eta_{XY} \equiv -D_X \xi_Y + \tfrac{1}{2} S_{(XY)}{}^{UV} \xi_U \xi_V = -\tilde{D}_X \xi_Y - \tfrac{1}{2} S_{XY}{}^{UV} \xi_U \xi_V, \tag{2.34}$$

we find

$$\begin{aligned}
R_{XY}(\tilde{\Gamma}) &= R_{XY}(\Gamma) + 4r\eta_{(XY)} + 8\Pi_{(XY)}{}^{UV}\eta_{UV} - 4(r+1)\partial_{[X}\xi_{Y]}, \\
\vec{\mathcal{R}}_{XY}(\tilde{\vec{\omega}}) &= \vec{\mathcal{R}}_{XY}(\vec{\omega}) - 2\vec{J}_{[Y}{}^{Z}\eta_{X]Z}, \\
B_{XY}(\tilde{\Gamma}) &= B_{XY}(\Gamma) + \eta_{XY}.
\end{aligned} \tag{2.35}$$

Using the last expression, one finds that the Ricci part, (2.20), transforms as the full curvature (2.33). Therefore, the Weyl part, $R^{(\mathrm{W})}$, does not transform, and the $\mathcal{W}$-tensor is invariant.

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[9]The quadratic terms in these equations can be understood from the consistency of applying these formulae with Γ and $\tilde{\Gamma}$ interchanged.

The ξ-transformations can be used to fix the form of the $\mathbb{R}$ connection. Indeed, looking at (2.32), we see we can transform away the $\mathbb{R}$ connection completely. The $\mathbb{R}$ curvature then vanishes. Furthermore, there are residual ξ-transformations depending on a scalar function $f(q)$, i.e. $\xi_X = \partial_X f$ that leave the $\mathbb{R}$ curvature invariant.

An alternative ξ-choice yields the Oproiu connection that satisfies

$$\vec{J}_X{}^Y \cdot \vec{\omega}_Y^{\text{Op}} = 0, \tag{2.36}$$

which leads to the connection (2.12) [4, 7].

2.3 Hypercomplex manifolds

Hypercomplex manifolds were introduced in [6]. A very thorough paper on the subject is [7]. Examples of homogeneous hypercomplex manifolds that are not hyper-Kähler, can be found in [10, 30], and are further discussed in [4]. Non-compact homogeneous manifolds are dealt with in [31]. Various aspects have been treated in two workshops with mathematicians and physicists [32, 33].

A hypercomplex manifold has no SU(2) connection:

$$\vec{\omega}_X = 0. \tag{2.37}$$

This implies that we do not allow for local SU(2) redefinitions of the hypercomplex structure.

The quaternionic structure should thus be covariantly constant with respect to the affine connection only. The unique solution of (2.6) with (2.37) is

$$\Gamma_{XY}{}^Z = \Gamma^{\text{Ob}}{}_{XY}{}^Z + N_{XY}{}^Z, \tag{2.38}$$

where the first term is symmetric and the second one (the Nijenhuis tensor) is antisymmetric in XY. The first term is called the *Obata connection* [34], and is given in terms of the complex structures and their derivatives in (2.13). As mentioned before, and motivated by the supersymmetry algebra, we consider torsionless (symmetric) connections, which requires

$$N_{XY}{}^Z = 0. \tag{2.39}$$

By definition, *a hypercomplex manifold is a 4r-dimensional manifold $\mathcal{M}$, equipped with a hypercomplex structure with vanishing Nijenhuis tensor.*

Clearly, hypercomplex manifolds are those quaternionic manifolds with vanishing SU(2) connection. If the quaternionic manifold has a non-vanishing SU(2) connection, it might still be possible to define from it a hypercomplex manifold. For instance, consider the class of quaternionic manifolds with vanishing SU(2) curvature, i.e.,

$$\vec{\mathcal{R}}_{XY} = 0. \tag{2.40}$$

This only requires the SU(2) connection to be pure gauge. The complex structures are then covariantly constant with respect to this SU(2) connection, and one can still act with

ξ-transformations. However, a small calculation shows that one can redefine the complex structures with a local SO(3) matrix R according to $\vec{J}' = RJ$, such that no SU(2) connection is needed anymore. In such a basis, also the freedom of doing ξ-transformations is fixed. The resulting manifold is then hypercomplex.

One can in fact further relax the vanishing condition on the SU(2) curvature by requiring only

$$\vec{\mathcal{R}}_{XY} = 2\vec{J}_{[X}{}^Z\eta_{Y]Z}, \tag{2.41}$$

for some η_{XY} that can be written as (2.34). In particular, the SU(2) connection is non-vanishing. But now, looking at (2.35), this implies that we can do a ξ-transformation such that the SU(2) curvature vanishes, and so we are back in the situation discussed above.

In summary, for those quaternionic manifolds that satisfy (2.41) we can associate and define a hypercomplex manifold by making use of the quaternionic, local span of the complex structures, together with the ξ-transformations.

Further properties of hypercomplex manifolds can be derived. Since they have vanishing SU(2) curvature, the Riemann tensor can be decomposed as

$$R_{XYW}{}^Z = -\tfrac{1}{2}f_X^{iA}\varepsilon_{ij}f_Y^{jB}f_W^{kC}f_{kD}^Z W_{ABC}{}^D. \tag{2.42}$$

The tensor W is defined as

$$W_{CDB}{}^A \equiv \varepsilon^{ij}f_{jC}^X f_{iD}^Y \mathcal{R}_{XYB}{}^A = \tfrac{1}{2}\varepsilon^{ij}f_{jC}^X f_{iD}^Y f_{kB}^Z f_W^{kA}R_{XYZ}{}^W, \tag{2.43}$$

and is symmetric in its lower indices. It is however not traceless, and its trace determines the Ricci tensor,

$$R_{XY} = R_{[XY]} = \tfrac{1}{2}\varepsilon_{ij}f_X^{iB}f_Y^{jC}W_{ABC}{}^A = -\mathcal{R}_{XYA}{}^A. \tag{2.44}$$

The tensor B is then antisymmetric, and is just the last term of (2.21). This form of the Ricci tensor implies that it is Hermitian

$$\Pi_{XY}{}^{UV}R_{UV} = R_{XY}. \tag{2.45}$$

This follows from

$$\vec{J}_X{}^Z f_Z{}^{iA} = -\mathrm{i}\vec{\sigma}_j{}^i f_X^{jA}, \qquad \vec{\sigma}_k{}^i \cdot \vec{\sigma}_\ell{}^j \varepsilon_{ij} = -3\varepsilon_{k\ell}. \tag{2.46}$$

The tensor W in (2.43) should not be confused with the traceless tensor $\mathcal{W}$ defined in (2.25). The precise relation is given by

$$\mathcal{W}_{ABC}{}^D = W_{ABC}{}^D - \frac{3}{2(r+1)}\delta_{(A}^D W_{BC)E}{}^E. \tag{2.47}$$

Thus, for hypercomplex manifolds, the Ricci tensor is antisymmetric and Hermitian. Conversely, a quaternionic manifold with antisymmetric and Hermitian Ricci tensor is necessarily hypercomplex. Indeed, using the general result (A.11) for any Hermitian bilinear form, (2.27) then implies that the SU(2) curvature vanishes, and so a basis can be chosen such that it is hypercomplex. We come back to the hermiticity properties of quaternionic-like manifolds at the end of this section.

Nowhere in this section have we assumed the existence of a (covariantly constant) metric. When such a tensor exists, hypercomplex manifolds are promoted to hyper-Kähler manifolds.

2.4 Hyper-Kähler manifolds

The crucial difference between hyper-Kähler and hypercomplex geometries is that hyper-Kähler manifolds admit a Hermitian metric g. This involves 3 conditions:

1. g should be Hermitian. This can be expressed as

$$\vec{J}_X{}^Z g_{ZY} = -\vec{J}_Y{}^Z g_{ZX}. \tag{2.48}$$

2. g should be invertible.

3. g should be covariantly constant using the Obata connection.

If this metric is preserved using the Obata connection, then the hypercomplex manifold is promoted to a hyper-Kähler manifold. Equivalently, when the Levi-Civita connection preserves the quaternionic structure, then the manifold is hyper-Kähler. It is clear from the discussion of the previous section, that the Levi-Civita and Obata connection on a hyper-Kähler manifold must coincide.

The Ricci tensor of the Levi-Civita connection is always symmetric. Combined with the fact that the Obata connection has an antisymmetric Ricci tensor, it follows that hyper-Kähler manifolds are Ricci flat,

$$R_{XY} = 0. \tag{2.49}$$

Using (2.18), this is equivalent to saying that the trace of $W_{ABC}{}^D$ vanishes, and so,

$$\mathcal{W}_{ABC}{}^D = W_{ABC}{}^D. \tag{2.50}$$

As a consequence, the curvature takes values in $\mathrm{USp}(2r)$.

The existence of a metric allows us to define the covariantly constant antisymmetric tensors

$$C_{AB} = \tfrac{1}{2} f_{iA}^X g_{XY} \varepsilon^{ij} f_{jB}^Y, \qquad C^{AB} = \tfrac{1}{2} f_X^{iA} g^{XY} \varepsilon_{ij} f_Y^{jB}, \tag{2.51}$$

which satisfy

$$C_{AC} C^{BC} = \delta_A{}^B, \tag{2.52}$$

and which can be used to raise and lower indices according to the NW–SE convention similar to ε_{ij}:

$$A_A = A^B C_{BA}, \qquad A^A = C^{AB} A_B. \tag{2.53}$$

The integrability condition on C_{AB} implies

$$\mathcal{R}_{XY[A}{}^C C_{B]C} = 0, \tag{2.54}$$

and it follows that

$$W_{ABCD} \equiv W_{ABC}{}^E C_{ED}, \tag{2.55}$$

is fully symmetric in its four lower indices.

2.5 Quaternionic-Kähler manifolds

For some basic references on quaternionic-Kähler manifolds, we refer to [35, 36], or the earlier references [34, 37–40].

Similar to hyper-Kähler spaces, quaternionic-Kähler manifolds admit a Hermitian and invertible metric satisfying (2.48). The connection that preserves this metric, i.e. the Levi-Civita connection, must be related to the Oproiu connection (2.12) by a ξ-transformation. It is a well known fact that *quaternionic-Kähler spaces are Einstein*:

$$R_{XY} = \frac{1}{4r} g_{XY} R. \tag{2.56}$$

From the Bianchi identity, one easily shows that the Ricci scalar is constant. The Ricci tensor is obviously symmetric, such that the $\mathbb{R}$ part of the $G\ell(r, \mathbb{H})$ curvature is zero. The B-tensor in (2.21) can easily be computed to be

$$B_{XY} = \tfrac{1}{4}\nu g_{XY}, \qquad \nu \equiv \frac{1}{4r(r+2)} R. \tag{2.57}$$

Using (2.27), one then finds that the SU(2) curvature is proportional to the quaternionic 2-form:

$$\vec{\mathcal{R}}_{XY} = \tfrac{1}{2}\nu \vec{J}_{XY}. \tag{2.58}$$

The Ricci part of the curvature is determined by the curvature of a quaternionic projective space of the same dimension:

$$
\begin{aligned}
\left(R^{\mathbb{H}P^n}\right)_{XYWZ} &\equiv \tfrac{1}{2} g_{Z[X} g_{Y]W} + \tfrac{1}{2}\vec{J}_{XY} \cdot \vec{J}_{ZW} - \tfrac{1}{2}\vec{J}_{Z[X} \cdot \vec{J}_{Y]W} \\
&= \tfrac{1}{2}\vec{J}_{XY} \cdot \vec{J}_{ZW} + L_{[ZW]}{}^{AB} L_{[XY]AB}.
\end{aligned}
\tag{2.59}
$$

The full curvature decomposition is then

$$R_{XYWZ} = \nu(R^{\mathbb{H}P^n})_{XYWZ} + \tfrac{1}{2} L_{ZW}{}^{AB} \mathcal{W}_{ABCD} L_{XY}{}^{CD}, \tag{2.60}$$

with $\mathcal{W}_{ABCD}$ completely symmetric. In supergravity, the supersymmetry connects the value of ν to the normalization of the Einstein term in the action. This fixes the value of ν to $-\kappa^2$, where κ is the gravitational coupling constant. The quaternionic-Kähler manifolds appearing in supergravity thus have negative scalar curvature, and this implies that all such manifolds that have at least one isometry are non-compact.

Properties of the connections of all the quaternionic-like manifolds are summarized in Table 3.

2.6 Hermitian Ricci tensor

In this subsection, we study the properties of quaternionic-like manifolds with a Hermitian Ricci tensor. First of all, it is immediate from the relation between B and the Ricci tensor, see (2.21), that B is Hermitian if and only if R is Hermitian.

Table 3: *The affine connections in quaternionic-like manifolds*

hypercomplex	hyper-Kähler
Obata connection	Obata connection
	= Levi-Civita connection
quaternionic	quaternionic-Kähler
Oproiu connection or	Levi-Civita connection =
other related by ξ_X transformation	connection related to Oproiu
	by a particular choice of ξ_X

The important relation we now look at is (2.27). Using (A.8), one can see that, for Hermitian B, the antisymmetric part of B does not contribute. We can therefore conclude that an antisymmetric and Hermitian Ricci tensor is equivalent to the requirement of hypercomplex. Indeed, the SU(2) curvature is then zero, and we use the argument in Sect. 2.3. The other direction was also shown in that section.

Furthermore, for the symmetric part of B, the antisymmetrization in (2.27) is automatic in the right-hand side due to (A.11). For a Hermitian Ricci tensor, we thus have,

$$\vec{\mathcal{R}}_{XY} = 2\vec{J}_X{}^Z B_{(YZ)}, \qquad R_{(XY)} = 4(r+2)B_{(XY)}. \tag{2.61}$$

It is appropriate to define a 'candidate metric',

$$g_{XY} = \frac{4}{\nu}B_{(XY)} = -\frac{1}{\nu}h_{XY}, \tag{2.62}$$

where ν is an undetermined number. We define h_{XY} such that this number can be avoided in most of our formulae. With the usual normalization in supergravity where $\kappa = 1$, this is the metric anyway. We now have

$$\vec{\mathcal{R}}_{XY} = -\tfrac{1}{2}\vec{J}_X{}^Z h_{YZ}. \tag{2.63}$$

The identification of the symmetric part of the Ricci tensor as a metric becomes even more appropriate due to the property that it is covariantly constant. To prove this, we start with acting with a D_U derivative on (2.63) and antisymmetrizing in $[XYU]$ using the Bianchi identity for the SU(2) curvature. This leads to

$$3\vec{J}_{[X}{}^Z D_U h_{Y]Z} = \vec{J}_X{}^Z D_U h_{YZ} + \vec{J}_U{}^Z D_Y h_{XZ} + \vec{J}_Y{}^Z D_X h_{UZ} = 0. \tag{2.64}$$

Multiplying this with $(\vec{J}_V{}^U \times \vec{J}_W{}^Y)$ and taking the symmetric part in (XW), after using several times the antisymmetry of $\vec{J}_X{}^Z h_{YZ}$, it leads to

$$D_V h_{XW} = 0. \tag{2.65}$$

We can summarize this section as follows:

1. If the Ricci tensor is Hermitian and the symmetric part is invertible, then it defines a good metric. Therefore the antisymmetric part of the Ricci tensor is zero in this case (with respect to the Levi-Civita connection of h_{XY}). On the other hand, if the symmetric part is zero, then a Hermitian Ricci tensor implies zero SU(2) curvature.

2. A quaternionic manifold is quaternionic-Kähler if and only if the Ricci tensor is Hermitian and its symmetric part is non-degenerate.

Thus, there are 3 cases of Hermitian Ricci tensors on quaternionic-like manifolds:

1. symmetric part is invertible (quaternionic-Kähler manifold): there is no antisymmetric part,

2. symmetric part is zero, i.e. antisymmetric Ricci (hypercomplex manifold),

3. symmetric part is non-zero but non-invertible. Then an antisymmetric part is still possible.

3 The Map

We now start to discuss the map between the hypercomplex/hyper-Kähler and quaternionic(-Kähler) manifolds. The first will be called the *large space* and will be taken to be $4(n_H + 1)$ real dimensional, while the latter will be of real dimension $4n_H$ and be called the *small space*. Objects on the large space will be denoted by hats, either on their indices or on the objects themselves or on both.

The content of this section is as follows. In Subsect. 3.1 we discuss some special properties of conformal hypercomplex manifolds that are important for our discussion. We show that the holonomy of such manifolds is further restricted and we discuss a continuous deformation of the hypercomplex structure. After choosing coordinates that are adapted to our setting and rewriting the hypercomplex structure, we prove in Subsects. 3.2 and 3.3 that the large space contains a $4n_H$ dimensional quaternionic subspace. This is the central part of our discussion. Moreover, in Subsect. 3.3.2 we clarify the origin of the quaternionic local SU(2) symmetry. In the following Subsect. 3.4 we show how a quaternionic manifold can be used to construct a hypercomplex one. Subsect. 3.5 considers the map for hypercomplex manifolds that possess a 'good metric', and are therefore hyper-Kähler manifolds. The conditions for this metric to be a 'good metric' are equivalent to the condition that the quaternionic manifold has a 'good metric' and is thus promoted to a quaternionic-Kähler manifold. Therefore, we prove that the image of the map is a quaternionic-Kähler manifold if and only if the original manifold is a conformal hyper-Kähler manifold. We show that in order for the affine connection to agree with the Levi-Civita connection, as it should be for quaternionic-Kähler manifolds, one has to choose a particular ξ-transformation between the allowed connections for quaternionic manifolds. This choice is different from the one that leads to the Oproiu connection. We also prove the inverse map: for all quaternionic manifolds we will define a candidate metric, and if this is a good metric, which is the condition

that the Ricci tensor is Hermitian and invertible, then we can construct a good metric for the conformal hypercomplex manifold. After having completed our discussion of the map, we conclude in Subsect. 3.6 by listing the vielbeins and all connection coefficients on the large and small space in the adapted coordinates.

3.1 Hypercomplex manifolds with conformal symmetry

The starting point of our map is given by hypercomplex manifolds that admit a conformal symmetry. By definition, this means there exists a closed homothetic Killing[10] vector $k^{\widehat{X}}$, satisfying (1.1).

Given this homothetic Killing vector $k^{\widehat{X}}$, three more vectors can be constructed naturally:

$$\vec{k}^{\widehat{X}} \equiv \tfrac{1}{3}\widehat{\vec{J}}_{\widehat{Y}}{}^{\widehat{X}} k^{\widehat{Y}}, \tag{3.1}$$

which generate an SU(2) algebra and satisfy

$$\widehat{\mathfrak{D}}_{\widehat{Y}}\vec{k}^{\widehat{X}} = \tfrac{1}{2}\widehat{\vec{J}}_{\widehat{Y}}{}^{\widehat{X}}. \tag{3.2}$$

It then follows that, under dilatations and SU(2) transformations, the hypercomplex structure is scale invariant and rotated into itself,

$$\left(\mathcal{L}_{\Lambda_D k}\widehat{\vec{J}}\right)_{\widehat{X}}{}^{\widehat{Y}} \equiv \Lambda_D k^{\widehat{Z}}\partial_{\widehat{Z}}\,\widehat{\vec{J}}_{\widehat{X}}{}^{\widehat{Y}} - \Lambda_D \partial_{\widehat{Z}}k^{\widehat{Y}}\,\widehat{\vec{J}}_{\widehat{X}}{}^{\widehat{Z}} + \Lambda_D \partial_{\widehat{X}}k^{\widehat{Z}}\,\widehat{\vec{J}}_{\widehat{Z}}{}^{\widehat{Y}} \;=\; 0, \tag{3.3}$$

$$\left(\mathcal{L}_{\vec{\Lambda}\cdot\vec{k}}\widehat{\vec{J}}\right)_{\widehat{X}}{}^{\widehat{Y}} \equiv (\vec{\Lambda}\cdot\vec{k}^{\widehat{Z}})\partial_{\widehat{Z}}\,\widehat{\vec{J}}_{\widehat{X}}{}^{\widehat{Y}} - (\vec{\Lambda}\cdot\partial_{\widehat{Z}}\vec{k}^{\widehat{Y}})\,\widehat{\vec{J}}_{\widehat{X}}{}^{\widehat{Z}} + (\vec{\Lambda}\cdot\partial_{\widehat{X}}\vec{k}^{\widehat{Z}})\,\widehat{\vec{J}}_{\widehat{Z}}{}^{\widehat{Y}} \;=\; -\vec{\Lambda}\times\widehat{\vec{J}}_{\widehat{X}}{}^{\widehat{Y}}.$$

Here, we introduced parameters Λ_D and $\vec{\Lambda}$ (local in spacetime but not dependent on the coordinates of the quaternionic space) to generate the infinitesimal dilatations and SU(2) transformations on the coordinates,

$$\delta_D q^{\widehat{X}} = \Lambda_D k^{\widehat{X}}, \qquad \delta_{\mathrm{SU(2)}} q^{\widehat{X}} = \tfrac{1}{3}\vec{\Lambda}\cdot(k^{\widehat{Y}}\widehat{\vec{J}}_{\widehat{Y}}{}^{\widehat{X}}) = \vec{\Lambda}\cdot\vec{k}^{\widehat{X}}. \tag{3.4}$$

Notice that all this follows from (1.1) and the covariant constancy of the quaternionic structure.

We demand (here and everywhere below) that the vectors k and $\vec{k}$ are 'symmetries' (see (5.1) below, which is mathematically the statement that they define affine transformations). This leads to

$$k^{\widehat{X}}\widehat{R}_{\widehat{X}\widehat{Y}\widehat{Z}}{}^{\widehat{W}} = 0, \qquad \vec{k}^{\widehat{X}}\,\widehat{R}_{\widehat{X}\widehat{Y}\widehat{Z}}{}^{\widehat{W}} = 0. \tag{3.5}$$

When the connection is metric, then these equations are integrability conditions for (1.1) and (3.2) using the symmetries of the Riemann tensor.

Hence, the four vector fields now introduced are zero eigenvectors of the curvature. This implies that the holonomy of a $4(n_H + 1)$ dimensional conformal hypercomplex manifold is

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[10]Although there is not necessarily a metric defined, we use the same terminology.

contained in $\mathrm{SU}(2) \cdot \mathrm{G}\ell(n_H, \mathbb{H})$, which can easily be understood as follows. On a hypercomplex manifold, we can group all vectors of a given fibre into quaternions. Let us call the fibre F at a certain point p. Given then a vector $X \in F$, we may construct a quaternion as $\{X, \widehat{J}^1(p)X, \widehat{J}^2(p)X, \widehat{J}^3(p)X\} \equiv \{X, \vec{\widehat{J}}(p)X\}$. In general, the holonomy group yields a $\mathrm{G}\ell(n_H + 1, \mathbb{H})$ action on F that is generated by the curvature. Since $\{k(p), 3\vec{k}(p)\}$ form such a quaternion, the relation (3.5) implies that the holonomy group should leave that quaternion fixed and thus should be included in $\mathrm{SU}(2) \cdot \mathrm{G}\ell(n_H, \mathbb{H})$.

Moreover, if one looks at the components of the hypercomplex structures that lie along the other quaternions, it is easy to see that the holonomy group induces an $\mathrm{SU}(2) \cdot \mathrm{G}\ell(n_H, \mathbb{H})$ action on these components. This observation is the heart of our construction, since it strongly suggests that the submanifold along the other $4n_H$ directions is quaternionic.[11]

As will be motivated in the following sections, the existence of the vector fields k and $\vec{k}$ implies that we can define a continuous family of hypercomplex structures. Suppose that we start with a conformal hypercomplex manifold with closed homothetic Killing vector k and complex structure $\vec{\widehat{J}}$. We can define

$$(\widehat{\vec{J}}_{\widehat{\xi}})_{\widehat{X}}{}^{\widehat{Y}} = \vec{\widehat{J}}_{\widehat{X}}{}^{\widehat{Y}} + \tfrac{2}{3}\left[\vec{\widehat{J}}_{\widehat{X}}{}^{\widehat{Z}}(\widehat{\xi}_{\widehat{Z}}k^{\widehat{Y}}) - (\widehat{\xi}_{\widehat{X}}k^{\widehat{Z}})\vec{\widehat{J}}_{\widehat{Z}}{}^{\widehat{Y}}\right], \qquad (3.6)$$

for a one-form with components $\widehat{\xi}_{\widehat{X}}$ that satisfies $k^{\widehat{X}}\widehat{\xi}_{\widehat{X}} = \vec{k}^{\widehat{X}}\widehat{\xi}_{\widehat{X}} = 0$.

The new complex structures still satisfy the quaternionic algebra and have vanishing Nijenhuis tensor if

$$\partial_{[\widehat{X}}\widehat{\xi}_{\widehat{Y}]} \text{ is Hermitian, and} \qquad \mathcal{L}_k\widehat{\xi}_{\widehat{X}} = \mathcal{L}_{\vec{k}}\widehat{\xi}_{\widehat{X}} = 0. \qquad (3.7)$$

The last requirement is automatic if the first two are satisfied. In conclusion, we can construct a new hypercomplex structure using (3.6) with a one-form that is constant along the flows of k and $\vec{k}$ and whose external derivative is Hermitian. One can moreover show that the new hypercomplex manifold is again conformal with the same vector field k. As far as we know this $\widehat{\xi}$-transformation has not been given before in the mathematical literature.

3.2   The map from hypercomplex to quaternionic

We now start to construct a map between $4(n_H+1)$-dimensional hypercomplex/hyper-Kähler manifolds, admitting a conformal symmetry, and $4n_H$-dimensional quaternionic(-Kähler) manifolds.

3.2.1   Suitable coordinates and almost complex structures

We first construct a set of coordinates adapted to our setting. These coordinates should allow us to solve explicitly the constraints imposed by conformal symmetry. The primary object is

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[11]More exactly, the four vector fields k and $\vec{k}$ generate a four dimensional foliation of the hypercomplex space, and we will show that the space of leaves carries a quaternionic structure.

the homothetic Killing vector (1.1). Therefore, first of all, we will choose one coordinate such that the vector k has a convenient form. One can always find local coordinates $q^{\widehat{X}} = \{z^0, y^p\}$, where $p = 1, ..., 4n_H + 3$, such that the components of the homothetic Killing vector are

$$k^{\widehat{X}} = 3z^0\delta_0^{\widehat{X}}. \tag{3.8}$$

This is obtained by choosing at any point the first coordinate in the direction of the vector $k^{\widehat{X}}$. The factor 3 in (3.8) is a convenient choice for later purposes. Notice that arbitrary coordinate transformations $y'^p(y^q)$ trivially preserve (3.8). We will make use of this freedom below.

Having singled out the 'dilatation direction', we now proceed similarly for the SU(2) vector fields (3.1). Frobenius' theorem tells that the three-dimensional hypersurface spanned by the direction of the three SU(2) vector fields can be parametrized by coordinates $(z^\alpha)_{\alpha=1,2,3}$, such that $\vec{k}^{\widehat{X}}$ is only non-zero for $\widehat{X}$ being one of the indices α. Note that these vectors point in different directions than $k^{\widehat{X}}$, due to (3.1), and the fact that the complex structures square to $-\mathbb{1}$. Therefore, they do not coincide with the direction '0' chosen above. The other $4n_H$ coordinates are indicated by q^X. Thus we have at this point

$$\begin{aligned}
q^{\widehat{X}} &= \{z^0, y^p\} = \{z^0, z^\alpha, q^X\}, && \alpha = 1, 2, 3, && X = 1, \ldots, 4n_H, \\
k^{\widehat{X}} &= 3z^0\delta_0^{\widehat{X}}, && \vec{k}^0 = \vec{k}^X = 0.
\end{aligned} \tag{3.9}$$

Now, as $\vec{k}^{\widehat{X}} = \frac{1}{3}k^{\widehat{Y}}\widehat{\vec{J}}_{\widehat{Y}}{}^{\widehat{X}}$ and due to our particular choice of coordinates, we find that

$$\widehat{\vec{J}}_0{}^\alpha = \frac{1}{z^0}\vec{k}^\alpha, \qquad \widehat{\vec{J}}_0{}^0 = 0, \qquad \widehat{\vec{J}}_0{}^X = 0. \tag{3.10}$$

Generically, we allow $\vec{k}^\alpha$ to depend on q^X (as it is the case also in group manifold reductions [41]). We assume it to be invertible as a three by three matrix, and define $\vec{m}_\alpha$ as the inverse, in the sense that we have for any vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \cdot \vec{m}_\alpha \, \vec{k}^\alpha \cdot \vec{B} = \vec{A} \cdot \vec{B} \qquad \text{or} \qquad \vec{k}^\alpha \cdot \vec{m}_\beta = \delta_\beta^\alpha. \tag{3.11}$$

It is convenient to introduce

$$\vec{A}_X \equiv \frac{1}{z^0}\widehat{\vec{J}}_X{}^0. \tag{3.12}$$

Using (3.10), we can complete the table of complex structures by requiring the quaternionic algebra (1.2). In terms of $\vec{A}_X$, we find

$$\begin{aligned}
\widehat{\vec{J}}_0{}^0 &= 0, & \widehat{\vec{J}}_\alpha{}^0 &= -z^0\vec{m}_\alpha, & \widehat{\vec{J}}_X{}^0 &= z^0\vec{A}_X, \\
\widehat{\vec{J}}_0{}^\beta &= \frac{1}{z^0}\vec{k}^\beta, & \widehat{\vec{J}}_\alpha{}^\beta &= \vec{k}^\beta \times \vec{m}_\alpha, & \widehat{\vec{J}}_X{}^\beta &= \vec{A}_X \times \vec{k}^\beta + \vec{J}_X{}^Z(\vec{A}_Z \cdot \vec{k}^\beta), \\
\widehat{\vec{J}}_0{}^Y &= 0, & \widehat{\vec{J}}_\alpha{}^Y &= 0, & \widehat{\vec{J}}_X{}^Y &= \vec{J}_X{}^Y.
\end{aligned} \tag{3.13}$$

All dependence on z^0 of these complex structures is explicitly shown in this formula. The last equation says that the components of the hypercomplex structures on the large space that lie along the small space satisfy the algebra of the imaginary quaternions.

Thus we have decomposed any almost hypercomplex structure on the large space and we find that it is expressed in 3 different quantities: the vectors $\vec{k}^\alpha$ [and their inverses, see (3.11)], vectors $\vec{A}_X$, which are arbitrary so far, and an almost quaternionic structure on the small space, $\vec{J}_X{}^Y$.

3.2.2 The complex structures and Obata connection

As was explained in Sect. 2.1, the almost hypercomplex structure $\widehat{\vec{J}}$ is hypercomplex if the Nijenhuis tensor vanishes, or, equivalently, if there exists a torsionless connection $\widehat{\Gamma}$ such that the complex structures are covariantly constant. The latter is then the Obata connection, see (2.13).

In practice it is easier to first compute that connection. First of all, the relation (1.1) in our coordinate ansatz (3.9) gives rise to

$$\widehat{\Gamma}_{\widehat{X}0}{}^{\widehat{Y}} = \frac{1}{z^0}\left(\frac{1}{2}\delta_{\widehat{X}}^{\widehat{Y}} - \delta_{\widehat{X}}^0\delta_0^{\widehat{Y}}\right). \tag{3.14}$$

Further, we can immediately find some more information on the coordinate dependence of the basic quantities. Multiplying the first relation of (3.3) by $k^{\widehat{X}}$ yields that the SU(2) vectors commute with the homothetic Killing vector field. Using (3.9), this yields

$$\partial_0\vec{k}^\alpha = 0, \qquad \partial_0\vec{m}_\alpha = 0. \tag{3.15}$$

One may further use the second line of (3.3) with $\vec{\Lambda}$ replaced by $\vec{m}_\alpha$ and obtain the z^α dependence of the SU(2) vector fields that reflect the SU(2) algebra. One can write the corresponding equation in various forms:

$$\vec{k}^\gamma \times \partial_\gamma\vec{k}^\alpha = \vec{k}^\alpha, \qquad \partial_{[\alpha}\vec{m}_{\beta]} = -\tfrac{1}{2}\vec{m}_\alpha \times \vec{m}_\beta, \qquad \vec{m}_{[\alpha} \cdot \partial_{\beta]}\vec{k}^\gamma = -\tfrac{1}{2}(\vec{m}_\alpha \times \vec{m}_\beta) \cdot \vec{k}^\gamma. \tag{3.16}$$

The connection coefficients can then be written as

$$
\begin{aligned}
&\widehat{\Gamma}_{00}{}^0 = -\tfrac{1}{2z^0}, && \widehat{\Gamma}_{00}{}^p = 0, \\
&\widehat{\Gamma}_{0p}{}^0 = 0, && \widehat{\Gamma}_{0q}{}^p = \tfrac{1}{2z^0}\delta_q^p, \\
&\widehat{\Gamma}_{\alpha\beta}{}^0 = \tfrac{1}{2}\widehat{g}_{\alpha\beta}, && \widehat{\Gamma}_{\alpha\beta}{}^\gamma = -\vec{m}_{(\alpha} \cdot \partial_{\beta)}\vec{k}^\gamma, \\
&\widehat{\Gamma}_{\alpha\beta}{}^X = 0, && \\
&\widehat{\Gamma}_{X\alpha}{}^0 = \tfrac{1}{2}z^0\vec{A}_X \cdot \vec{m}_\alpha = \tfrac{1}{2}\widehat{\vec{J}}_X{}^0 \cdot \vec{m}_\alpha, && \widehat{\Gamma}_{X\alpha}{}^\beta = \tfrac{1}{2}\widehat{\vec{J}}_X{}^\beta \cdot \vec{m}_\alpha - \vec{m}_\alpha \cdot \partial_X\vec{k}^\beta, \\
&\widehat{\Gamma}_{X\alpha}{}^Y = \tfrac{1}{2}\widehat{\vec{J}}_X{}^Y \cdot \vec{m}_\alpha, && \\
&\widehat{\Gamma}_{XY}{}^0 = \tfrac{1}{2}\widehat{g}_{XY}, \qquad \widehat{\Gamma}_{XY}{}^\alpha = -(\partial_{(X}\vec{A}_{Y)}) \cdot \vec{k}^\alpha + \widehat{\Gamma}_{XY}{}^W\vec{A}_W \cdot \vec{k}^\alpha - \tfrac{1}{2}h_{Z(X}\vec{J}_{Y)}{}^Z \cdot \vec{k}^\alpha, \\
&\widehat{\Gamma}_{XY}{}^Z = \Gamma^{\mathrm{Ob}}{}_{XY}{}^Z + \tfrac{1}{3}\widehat{\vec{J}}_{(X}{}^\delta \cdot \left(\vec{m}_\delta\delta_{Y)}^Z + \tfrac{1}{2}\vec{m}_\delta \times \vec{J}_{Y)}{}^Z\right).
\end{aligned}
\tag{3.17}
$$

Here Γ^{Ob} is the Obata connection defined by (2.13) using the $\vec{J}$ complex structures, while $\widehat{\Gamma}$ is the Obata connection using $\widehat{\vec{J}}$.

We have also introduced the following convenient notation:

$$\widehat{g}_{\alpha\beta} \equiv 2\widehat{\Gamma}_{\alpha\beta}{}^0, \qquad \widehat{g}_{\widehat{X}\widehat{Y}} \equiv 2\widehat{\Gamma}_{\widehat{X}\widehat{Y}}{}^0, \qquad h_{XY} \equiv \frac{1}{z^0}\widehat{g}_{XY} + \vec{A}_X \cdot \vec{A}_Y. \tag{3.18}$$

Note that although we have not introduced a metric, we use here suggestive notation since a 'good metric' coincides with these definitions, as we will show in Sect. 3.5.1. Hence, we have used $\widehat{g}$ as a shorthand for a complicated function of z^0, $\vec{k}^\alpha$, $\vec{A}_X$ and $\vec{J}_X{}^Y$. Considering the $\widehat{X} = 0$ and $\widehat{Y} = \alpha$ components of (3.2) in the basis (3.9) and using (3.13) leads to

$$\vec{k}_\alpha \equiv \widehat{g}_{\alpha\beta}\vec{k}^\beta = -z^0\vec{m}_\alpha. \tag{3.19}$$

This implies

$$\widehat{g}_{\alpha\beta} = -\tfrac{1}{z^0}\vec{k}_\alpha \cdot \vec{k}_\beta = -z^0\vec{m}_\alpha \cdot \vec{m}_\beta. \tag{3.20}$$

The requirements of covariantly constant $\widehat{J}$ lead to requirements on the coordinate dependence of the quantities $\vec{k}^\alpha$, $\vec{A}_X$ and $\vec{J}_X{}^Y$. From the requirements that $\widehat{\mathfrak{D}}_0\widehat{J} = 0$ and $\widehat{\mathfrak{D}}_\alpha\widehat{J} = 0$, we find that

$$\partial_0 \vec{A}_X = 0, \qquad (\partial_\alpha + \vec{m}_\alpha \times)\,\vec{A}_X + \partial_X\vec{m}_\alpha = 0, \tag{3.21}$$

$$\partial_0 \vec{J}_X{}^Y = 0, \qquad (\partial_\alpha + \vec{m}_\alpha \times)\,\vec{J}_X{}^Y = 0. \tag{3.22}$$

Note that the integrability condition for the second relation in (3.21) yields

$$(\partial_\alpha + \vec{m}_\alpha \times)\,[\vec{R}(-\tfrac{1}{2}\vec{A})]_{XY} = 0, \qquad \text{with} \qquad [\vec{R}(\vec{V})]_{XY} \equiv 2\partial_{[X}\vec{V}_{Y]} + 2\vec{V}_X \times \vec{V}_Y. \tag{3.23}$$

A main non-trivial result comes from $\widehat{\mathfrak{D}}_X\widehat{J}_Y{}^0 = 0$. We find

$$[\vec{R}(-\tfrac{1}{2}\vec{A})]_{XY} = \tfrac{1}{2}h_{Z[X}\vec{J}_{Y]}{}^Z, \tag{3.24}$$

with h as in (3.18). We can calculate this expression using the definition of the Obata connection $\widehat{\Gamma}_{\alpha\beta}{}^0$ from (2.13) and our particular decomposition (3.13). This leads to

$$h_{XY} = -\tfrac{1}{3}\left(4\vec{J}_{(X}{}^Z \cdot [\vec{R}(-\tfrac{1}{2}\vec{A})]_{Y)Z} + (\vec{J}_X{}^U \times \vec{J}_Y{}^Z) \cdot [\vec{R}(-\tfrac{1}{2}\vec{A})]_{UZ}\right). \tag{3.25}$$

However, this equation can also be obtained from solving h from (3.24). Thus, the equation implies by itself that the matrix h that appears in (3.25) is necessarily the quantity defined in (3.18). Therefore, the integrability condition is equivalent to the requirement that there should be a symmetric matrix h such that (3.24) is satisfied.

Finally, the vanishing of the components along the small space of the Nijenhuis tensor of the hypercomplex structure $\widehat{J}$ implies that the Nijenhuis tensor in the small space should be

$$6N_{XY}{}^Z = -\widehat{J}_{[X}{}^\alpha \cdot \partial_\alpha\widehat{J}_{Y]}{}^Z = \widehat{J}_{[X}{}^\alpha \cdot \vec{m}_\alpha \times \vec{J}_{Y]}{}^Z = -\left(2\vec{A}_{[X} + \vec{A}_W \times \vec{J}_{[X}{}^W\right) \cdot \vec{J}_{Y]}{}^Z. \tag{3.26}$$

This equation is the basic equation that determines that the small space is quaternionic. We will further elaborate on this in Sect. 3.3.1.

The conclusions of Sect. 3.2.1 can now be completed. There are integrable complex structures on the conformal hypercomplex space for functions

$$\vec{k}^\alpha(q^X, z^\alpha), \qquad \vec{A}_X(q^X, z^\alpha), \qquad \vec{J}_X{}^Y(q^X, z^\alpha). \tag{3.27}$$

The $\vec{k}^\alpha$ should satisfy the SU(2) algebra, i.e. (3.16), while the z^α dependence of the other quantities is determined by these vector fields using (3.21) and (3.22). Moreover there are the conditions that there should be a symmetric tensor h_{XY}, which is given by (3.25), such that (3.24) is satisfied. The Nijenhuis tensor of the complex structures in the small space should satisfy (3.26), which, as we will show in the next section, has the meaning that it is a quaternionic structure with SU(2) connection determined by $\vec{A}_X$.

We remark that, combining the previous results, the z^α-dependence of all quantities can be calculated. E.g the following are useful results:

$$\partial_\alpha h_{XY} = 0, \qquad (\partial_\alpha + \vec{m}_\alpha \times) \vec{J}_X{}^\beta - \vec{J}_X{}^\gamma \left(\vec{m}_\alpha \cdot \partial_\gamma \vec{k}^\beta \right) = 0. \tag{3.28}$$

3.3   The embedded quaternionic space

3.3.1   Proof that the small space is quaternionic

As already explained in Sect. 2, contrary to the hypercomplex case where the SU(2) connection is trivial, in the quaternionic case there is a non-trivial SU(2) connection. This means that parallel transport with respect to the affine connection rotates the three complex structures into each other. As a consequence, the integrability condition for the complex structures differs from the hypercomplex case, as the (diagonal) Nijenhuis tensor should now be proportional to the SU(2) connection (2.10). This is exactly what we obtained with (3.26), leading to the SU(2) Oproiu connection satisfying (2.36),

$$\vec{\omega}_X^{\mathrm{Op}} = -\tfrac{1}{6} \left(2\vec{A}_X + \vec{A}_Y \times \vec{J}_X{}^Y \right). \tag{3.29}$$

Hence, this shows that the small space is quaternionic.

The corresponding affine connection is, according to (2.12),

$$\begin{aligned} \Gamma^{\mathrm{Op}}{}_{XY}{}^Z &= \Gamma^{\mathrm{Ob}}{}_{XY}{}^Z - \vec{J}_{(X}{}^Z \cdot \vec{\omega}_{Y)}^{\mathrm{Op}} \\ &= \widehat{\Gamma}_{XY}{}^Z - \tfrac{1}{3}\vec{A}_V \cdot \vec{J}_{(X}{}^V \delta_{Y)}^Z + \tfrac{2}{3}\vec{A}_{(X} \cdot \vec{J}_{Y)}{}^Z + \tfrac{1}{3}\vec{A}_V \cdot \vec{J}_{(X}{}^V \times \vec{J}_{Y)}{}^Z, \end{aligned} \tag{3.30}$$

where we used the last equation of (3.17).

As we have discussed in Sect. 2.2, there is a family of torsionless connections that are compatible with that given structure. Related to that freedom, all SU(2) connections can be written as

$$\vec{\omega}_X = -\tfrac{1}{6} \left(2\vec{A}_X + \vec{A}_Y \times \vec{J}_X{}^Y \right) + \vec{J}_X{}^Y \xi_Y. \tag{3.31}$$

A particular choice that will be useful below is

$$\xi_X = \tfrac{1}{6}\vec{J}_X{}^Y \cdot \vec{A}_Y, \tag{3.32}$$

such that we find the connections

$$\vec{\omega}_X = -\tfrac{1}{2}\vec{A}_X, \qquad \Gamma_{XY}{}^Z = \widehat{\Gamma}_{XY}{}^Z + \vec{A}_{(X} \cdot \vec{J}_{Y)}{}^Z.\qquad(3.33)$$

This choice of a quaternionic connection will turn out to be special for two different reasons. First of all, if there is a good metric on the small space (i.e. if the space is quaternionic-Kähler) this connection will correspond to the Levi-Civita connection. Secondly, we will show that for this choice the $\mathbb{R}$ curvatures for both the large and the small space will be equal, $\widehat{\mathcal{R}}(\mathbb{R}) = \mathcal{R}(\mathbb{R})$. In the hypercomplex space this curvature is proportional to the Ricci tensor[12], which is Hermitian. This implies that the $\mathbb{R}$ curvature on the small space is Hermitian.

3.3.2 The local $\mathrm{SU}(2)$

One may wonder what the origin is of the local $\mathrm{SU}(2)$ invariance on the embedded quaternionic manifold. We will show that this local invariance is already present in *conformal* hypercomplex manifolds, but the transformations on the complex structures are more complicated than simple vector rotations. Then, we will show how this induces the expected local $\mathrm{SU}(2)$ on the quaternionic manifold, with $\vec{\omega}_X$ as gauge field.

Considering the equations of Sect. 3.2, they are nearly all invariant under a usual vector rotation $\delta \vec{V} = \vec{\ell} \times \vec{V}$ for all 3-vectors $\vec{V}$, especially if $\vec{\ell}$ only depends on the coordinates of the small space q^X. One troublesome equation is (3.21). $\vec{A}_X$ is an arbitrary quantity in the construction, a 'black box'. It turns out that the local invariance can be obtained by adding a gauge-type transformation for $\vec{A}_X$. Thus, we consider for the elementary quantities the following $\mathrm{SU}(2)$ transformations:

$$
\begin{aligned}
\delta_{\mathrm{SU}(2)}\vec{A}_X &= \partial_X \vec{\ell} + \vec{\ell} \times \vec{A}_X, & \delta_{\mathrm{SU}(2)}\vec{J}_X{}^Y &= \vec{\ell} \times \vec{J}_X{}^Y, \\
\delta_{\mathrm{SU}(2)}\vec{k}^\alpha &= \vec{\ell} \times \vec{k}^\alpha, & \delta_{\mathrm{SU}(2)}\vec{m}_\alpha &= \vec{\ell} \times \vec{m}_\alpha,
\end{aligned}\qquad(3.34)
$$

where the parameter $\vec{\ell}(q^X)$ cannot depend on z^0 or z^α. The main requirement for $\vec{A}_X$ was (3.24), in which the curvature of $\vec{A}$ appears. This equation is thus consistent if h_{XY} is invariant under the $\mathrm{SU}(2)$ transformations. A long calculation using the definition of the Obata connection shows that $\widehat{\Gamma}_{\widehat{X}\widehat{Y}}{}^0$ is not invariant, but precisely transforms such that h_{XY} defined as in (3.18) is invariant. This proves the local $\mathrm{SU}(2)$ symmetry of the conformal hypercomplex manifold.

Note that, due to the transformations of $\vec{A}_X$ some components of $\widehat{J}$ do not transform as an ordinary vector. These are

$$
\begin{aligned}
\delta_{\mathrm{SU}(2)}\widehat{\vec{J}}_X{}^0 &= z^0 \partial_X \vec{\ell} + \vec{\ell} \times \widehat{\vec{J}}_X{}^0, \\
\delta_{\mathrm{SU}(2)}\widehat{\vec{J}}_X{}^\beta &= -\vec{k}^\beta \times \partial_X \vec{\ell} + \vec{J}_X{}^Z \left(\vec{k}^\beta \cdot \partial_Z \vec{\ell} \right) + \vec{\ell} \times \widehat{\vec{J}}_X{}^\beta.
\end{aligned}\qquad(3.35)
$$

It turns out that the full Nijenhuis tensor is invariant under this $\mathrm{SU}(2)$.

---

[12]Observe that (3.5) implies that the Ricci tensor has only components in the directions of the small space.

The complex structures in the small space thus transform as ordinary vectors. We have seen that $\vec{A}_X$ is the gauge field of these SU(2) transformations. In the ξ-gauge where $\vec{A}_X$ is proportional to $\vec{\omega}_X$, see (3.33), this is thus the expected SU(2) gauge field, and we find

$$\delta_{\mathrm{SU(2)}}\vec{\omega}_X = -\tfrac{1}{2}\partial_X\vec{\ell} + \vec{\ell}\times\vec{\omega}_X. \tag{3.36}$$

Hence, $\vec{\omega}_X$ transforms as an true connection since (1.4) now transforms covariantly.

Another ξ-choice, as e.g. the Oproiu choice (2.36), is not invariant under SU(2). Hence, if we take this connection, it implies that the remaining SU(2) contains a compensating ξ-transformation, which is

$$\xi_X = \tfrac{1}{6}\vec{J}_X{}^Y \cdot \partial_Y\vec{\ell}. \tag{3.37}$$

This compensating transformation also contributes to the SU(2) transformation of the affine Oproiu connection Γ^{Op}, such that it cancels other terms that follow from its definition (2.12). Thus the final affine connection is SU(2) invariant, as one should expect for covariance of the covariant derivative of the complex structures (1.4).

Hence, the quaternionic local SU(2) is naturally included into the map.

3.4 The map from quaternionic to hypercomplex

3.4.1 Uplifting a quaternionic manifold

By now, we are in a position to discuss the inverse procedure, namely the construction of a conformal hypercomplex manifold starting from a quaternionic space. We may at this point choose a value for ξ, and we will see below that we have to choose one such that the $\mathbb{R}$ curvature (i.e. the antisymmetric part of the Ricci tensor) is Hermitian. This is always possible, as it is clear from (2.32) that we may even choose a ξ such that the $\mathbb{R}$ connection vanishes.

We now consider a space of 4 real dimensions bigger than the one of the quaternionic space. The extra coordinates are labeled z^0 and z^α. Let $\vec{k}^\alpha(z^\alpha, q^X)$ denote left-invariant vector fields on the SU(2) group manifold, i.e. satisfying (3.16). For now, the dependence on q^X is not fixed. One may take the SU(2) vectors independent of q, but an arbitrary dependence is allowed. It will be fixed later. Then construct their inverse $\vec{m}_\alpha(z, q)$ using (3.11). Furthermore, we define $\vec{A}_X(q) = -2\vec{\omega}_X(q)$ for $\vec{\omega}_X(q)$ the SU(2) connection for the chosen value of ξ.

We take the $\vec{k}^\alpha$, $\vec{J}_X{}^Y$ and $\vec{A}_X$ independent of z^0. The z^α dependence of $\vec{k}^\alpha$ is determined by its SU(2) property (3.16). The complex structures are taken to be covariant constant in their z^α dependence in the sense of (3.22). This means in fact that when we change the value of z^α we go to a different choice of complex structures. These different complex structures are related by an SU(2) rotation. This implies that also the SU(2) connection should change, and indeed this is in agreement with the comparison of (3.34) and (3.21). The latter equation determines the q^X-dependence of the SU(2) vector fields $\vec{k}^\alpha$. One also notices that the curvature of A_X is taken to be a covariant vector as shown in (3.23).

With these ingredients, we can construct an almost hypercomplex structure as in (3.13). In order for this structure to be integrable, the only remaining condition is (3.24). The

condition states that there should exist a symmetric object h such that the SU(2) curvature is related to the hypercomplex structure as indicated. In a quaternionic-Kähler manifold this is satisfied with h being the metric. In an arbitrary quaternionic manifold, we have (2.27). We thus just need that the antisymmetric part of B does not contribute to this equation. That is a condition of the form (A.11) for the antisymmetric part of B, which is the antisymmetric part of the Ricci tensor or $\mathbb{R}$ curvature. Hence it says that this $\mathcal{R}^{\mathbb{R}}$ should be Hermitian, which we can obtain by a ξ-transformation as mentioned in the beginning of this section.

With these choices, (3.13) defines a hypercomplex structure on a manifold parametrized by the coordinates $\{z^0, z^\alpha, q^X\}$. Moreover, it is easy to see that the vector field $3z^0\partial_0$ satisfies (1.1), hence the manifold is actually conformal hypercomplex.

3.4.2 A $\widehat{\xi}$-transformation for conformal hypercomplex manifolds

Suppose we have constructed a hypercomplex manifold from a quaternionic manifold with a Hermitian $\mathbb{R}$ curvature $\mathcal{R}^{\mathbb{R}}$. We perform a ξ-transformation with Hermitian $\partial_{[X}\xi_{Y]}$. Then the new $\mathbb{R}$ curvature $\mathcal{R}'^{\mathbb{R}}$ is also Hermitian. Hence, we can use again the procedure of Sect. 3.4.1 to obtain a hypercomplex structure on the large space. As the ξ-gauge modified the SU(2) connection, the vector $\vec{A}'_X$ is different from $\vec{A}_X$, and hence the new hypercomplex structure differs from the original one. This defines a new mapping, dependent on a one-form $\hat{\xi}$, between hypercomplex structures on conformal hypercomplex manifolds, see (3.38):

$$
\begin{array}{ccc}
\widehat{\vec{J}} & \xrightarrow{\;\hat{\xi}\;} & \widehat{\vec{J}}' \\[2pt]
{\scriptstyle\vec{A}_X}\big\downarrow & \quad {\scriptstyle\vec{A}'_X}\big\uparrow \\[2pt]
\vec{J},\,\vec{\omega} & \xrightarrow{\;\xi\;} & \vec{J},\,\vec{\omega}'
\end{array}
\tag{3.38}
$$

This is the transformation that was announced in (3.6). Remember that the ξ-transformations are defined by one forms $\xi_X \mathrm{d}q^X$ that depend only on the quaternionic coordinates q^X. This says that $\hat{\xi}_X = \xi_X$ must be constant along the flows of k and $\vec{k}$, and that $\hat{\xi}$ must transform a Hermitian $\hat{\mathcal{R}}^{\mathbb{R}}$ in another Hermitian $\hat{\mathcal{R}}^{\mathbb{R}}$, yielding the conditions in (3.7). Hence, we have re-derived the results of Sect. 3.1, and shown how its origin is necessary for the consistency of this picture. The transformation (3.6) is in these coordinates, with complex structures as in (3.13), simply given by

$$
\delta(\widehat{\vec{\xi}})\vec{A}_X = 2\vec{J}_X{}^Z\widehat{\vec{\xi}}_Z.
\tag{3.39}
$$

3.5 The map from hyper-Kähler to quaternionic-Kähler spaces

We will now restrict to the case in which there is a compatible metric. We will show that starting from a hyper-Kähler space, the small space will be automatically quaternionic-Kähler and vice-versa. The existence of a metric will moreover remove the freedom that was

implied by the ξ-transformation, since it unambiguously specifies the torsionless connection as being the Levi-Civita connection. Hence, the connection on the small space will be determined uniquely, given the structures that are defined on that manifold.

We will show how to construct the quaternionic metric from the one in the hyper-Kähler space, and show how the affine connection reduces to the Levi-Civita connection.

3.5.1 Decomposition of a hyper-Kähler metric

As we already mentioned in Sect. 2, a hypercomplex manifold is promoted to a hyper-Kähler space if there is a Hermitian metric $\widehat{g}_{\widehat{X}\widehat{Y}}$ that preserves the Obata connection, i.e.

$$\partial_{\widehat{Z}}\widehat{g}_{\widehat{X}\widehat{Y}} - 2\widehat{\Gamma}_{\widehat{Z}(\widehat{X}}{}^{\widehat{W}}\widehat{g}_{\widehat{Y})\widehat{W}} = 0, \tag{3.40}$$

$$\widehat{J}_{\widehat{X}}{}^{\widehat{Z}}\widehat{g}_{\widehat{Z}\widehat{Y}} + \widehat{J}_{\widehat{Y}}{}^{\widehat{Z}}\widehat{g}_{\widehat{Z}\widehat{X}} = 0. \tag{3.41}$$

We can now split these equations in various parts in the basis (3.9). The first equation for $\widehat{Z} = 0$, using (3.14), determines the z^0 dependence of the various parts of the metric. Considering furthermore the $\widehat{Z} = p$, $\widehat{X} = \widehat{Y} = 0$ part leads to

$$\mathrm{d}\widehat{s}^2 \equiv \widehat{g}_{\widehat{X}\widehat{Y}}\mathrm{d}q^{\widehat{X}}\mathrm{d}q^{\widehat{Y}} = -\frac{1}{z^0}\widehat{h}_{00}(\mathrm{d}z^0)^2 - 2\partial_p\widehat{h}_{00}\mathrm{d}z^0\mathrm{d}y^p + z^0\widehat{h}_{pq}\mathrm{d}y^p\mathrm{d}y^q, \tag{3.42}$$

where the $\widehat{h}$ are independent of z^0. Furthermore we want to invoke the $\widehat{X} = \widehat{Y} = 0$ part of (3.41), which using (3.13) and the invertibility of $\vec{m}_\alpha$ implies that $\partial_\alpha \widehat{h}_{00} = 0$.

One can further simplify the metric by a redefinition

$$z^{0\prime} = z^0\widehat{h}_{00}(q), \qquad z^{\alpha\prime} = z^\alpha, \qquad q^{X\prime} = q^X. \tag{3.43}$$

This redefinition preserves our previous coordinate choices (3.9). In particular, $\vec{k}$ has still only components along z^α since $\partial_\alpha \widehat{h}_{00} = 0$. This redefinition accomplishes that (using the new, primed, z^0 coordinate, but omitting the primes from now on)

$$\widehat{g}_{00} = -\frac{1}{z^0}, \qquad \widehat{g}_{0p} = 0. \tag{3.44}$$

The $\widehat{Z} = p$, $\widehat{X} = q$, $\widehat{Y} = 0$ part of (3.40) leads to

$$\widehat{g}_{\widehat{X}\widehat{Y}} = 2\widehat{\Gamma}_{\widehat{X}\widehat{Y}}{}^0. \tag{3.45}$$

This coincides with (3.18). At that time we have made a definition for arbitrary hypercomplex manifolds. Here we prove that any good metric on these manifolds is of the form (3.45) after choosing suitable coordinates.

Using the table of affine connections, (3.17) and (3.18) we thus obtain

$$\mathrm{d}\widehat{s}^2 = -\frac{(\mathrm{d}z^0)^2}{z^0} + \left\{ z^0 h_{XY}(q)\mathrm{d}q^X\mathrm{d}q^Y \right.$$
$$\left. + \widehat{g}_{\alpha\beta}[\mathrm{d}z^\alpha - \vec{A}_X(z,q)\cdot\vec{k}^\alpha\mathrm{d}q^X][\mathrm{d}z^\beta - \vec{A}_Y(z,q)\cdot\vec{k}^\beta\mathrm{d}q^Y] \right\}. \tag{3.46}$$

The metric therefore is a cone [13] and since this cone is hyper-Kähler, this implies that the base is a tri-Sasakian manifold, see e.g. [42].

We now check the remaining conditions in (3.40) and (3.41). For both it turns out that the only non-trivial components are the ones where the indices are restricted to those in the small space. We then find

$$(3.41) \quad \leftrightarrow \quad \vec{J}_{(X}{}^Z h_{Y)Z} = 0, \tag{3.47}$$
$$(3.40) \quad \leftrightarrow \quad \partial_Z h_{XY} - 2\Gamma_{Z(X}{}^W h_{Y)W} = 0, \tag{3.48}$$

where $\Gamma_{XY}{}^Z$ is given in (3.33). We can thus state that the metric h on the quaternionic space is Hermitian if and only if the metric $\widehat{g}$ on the hypercomplex manifold is Hermitian. The second result, (3.48), states that h is covariantly constant using as connection $\Gamma_{XY}{}^Z$. Therefore, it is the Levi-Civita of h. This connection also preserves the quaternionic structure, because we have shown in (3.33) that it is equivalent to the Oproiu connection up to ξ-transformations.

We have thus shown that the only ξ-choice that can be taken for quaternionic-Kähler manifolds is the one mentioned in Sect. 3.3.1, and in particular that

$$\vec{\omega}_X = -\tfrac{1}{2}\vec{A}_X. \tag{3.49}$$

It is seen from (3.46) that the induced metric in the small space is

$$g_{XY} = z^0 h_{XY} = -\frac{1}{\nu} h_{XY} = \frac{1}{\kappa^2} h_{XY}. \tag{3.50}$$

The overall factors do not play a role in the equations in this section. This metric does not depend on z^α, see (3.28).

3.5.2 The inverse map

The inverse map is of course a special case of the discussion in Sect. 3.4. Therefore, we will not repeat the complete lifting process but merely point out the facts specific to this case.

As the small space carries both a quaternionic structure $\vec{J}$ together with a good metric h, there is a unique torsionless connection compatible with that structure. Hence, we choose as a triplet of vectors $\vec{A}_X = -2\vec{\omega}_X$, where $\vec{\omega}_X$ is the SU(2) connection that corresponds to the Levi-Civita connection on the small space, and use it in (3.13). If we introduce z^0 and z^α dependence as in Sect. 3.4, the resulting almost hypercomplex structure $\widehat{\vec{J}}$ is automatically integrable. The reason for this is that Eq. (3.24) is now always satisfied since the Ricci tensor, and hence the tensor B (2.21) cannot have an antisymmetric part. Therefore, the large space is already hypercomplex.

---

[13] This follows from first extracting the z^0 dependence from $\hat{g}_{\alpha\beta}$, and then defining the radial variable $r^2 = z^0$.

We can moreover construct a good metric $\widehat{g}$ on the large space, starting from the good metric h on the small space, using the following table:

$$
\begin{aligned}
\widehat{g}_{00} &= -\frac{1}{z^0}, & \widehat{g}_{0\alpha} = \widehat{g}_{0X} &= 0, & \widehat{g}_{\alpha\beta} &= -z^0 \vec{m}_\alpha \cdot \vec{m}_\beta, \\
\widehat{g}_{\alpha X} &= z^0 \vec{A}_X \cdot \vec{m}_\alpha, & \widehat{g}_{XY} &= z^0 \left(h_{XY} - \vec{A}_X \cdot \vec{A}_Y \right).
\end{aligned} \tag{3.51}
$$

Combining this with the remarks of Sect. 3.5.1, we conclude that a conformal hyper-complex manifold is hyper-Kähler if and only if the corresponding quaternionic space is quaternionic-Kähler.

3.6 The map for the vielbeins and related connections

In the previous part, we did not yet consider the objects that are related to the $G\,\ell(r,\mathbb{H})$ structure of the manifolds. To discuss these, one needs the vielbeins. These vielbeins are also the starting point for the supersymmetry transformations of hypermultiplets. We have first to choose a suitable basis to express these objects.

3.6.1 Coordinates on the tangent space

Having singled out the z^0 and z^α coordinates in which the quaternionic structure is given by (3.13), the structure group of the frame bundle reduces from $G\,\ell(n_H + 1, \mathbb{H})$ to $\mathrm{SU}(2) \cdot G\,\ell(n_H, \mathbb{H})$, which consists of the set of frame reparametrizations on the small space. Similarly, the vielbein $\widehat{f}_{\widehat{X}}^{i\widehat{A}}$ and its inverse $\widehat{f}_{i\widehat{A}}^{\widehat{X}}$ can be decomposed. To do so, we split the index $\widehat{A}$ into (i, A) with $i = 1, 2$ and $A = 1, \ldots, 2n_H$.

To be more precise, we split any $G\,\ell(n_H + 1, \mathbb{H})$-vector $\zeta^{\widehat{A}}(q^{\widehat{X}})$ in (ζ^i, ζ^A), which is a vector of $\mathrm{SU}(2) \cdot G\,\ell(n_H, \mathbb{H})$, and where ζ^i is defined by

$$
\sqrt{\tfrac{1}{2}z^0}\, \zeta^i \equiv -\mathrm{i}\varepsilon^{ij} \widehat{f}_{j\widehat{A}}^0 \zeta^{\widehat{A}}. \tag{3.52}
$$

This choice of basis is guided by applications in supergravity, in which $\zeta^{\widehat{A}}$ are the superpartners of the coordinate scalar fields $q^{\widehat{X}}$. We will see that the choice of frame in the tangent space is useful for the identification of the quaternionic tangent space within the full hypercomplex tangent space. The formula (3.52) moreover implies

$$
\widehat{f}_{ij}^0 = -\mathrm{i}\varepsilon_{ij}\sqrt{\tfrac{1}{2}z^0}, \qquad \widehat{f}_{iA}^0 = 0. \tag{3.53}
$$

The i factors are needed for the reality conditions (2.2) where the $\rho_i{}^j$ components of $\rho_{\widehat{A}}{}^{\widehat{B}}$ are $\rho_i{}^j = -E_i{}^j = -\varepsilon_{ij}$, and ρ has no off-diagonal elements, i.e. $\rho_i^A = \rho_i^A = 0$. This is convenient for the formulation of quaternionic manifolds that appear in supergravity, see the discussion in Sect. 6.

Writing the first entry of (3.13) in terms of vielbeins, (2.4) yields

$$
\widehat{f}_0^{ij} = \mathrm{i}\varepsilon^{ij}\sqrt{\frac{1}{2z^0}}. \tag{3.54}
$$

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Furthermore, we redefine ζ^A according to

$$\zeta'^A = \zeta^A - i\sqrt{2z^0}\widehat{f}_0^{iA}\zeta^j\varepsilon_{ji}. \tag{3.55}$$

Using $\widehat{\vec{J}}_0{}^Y = 0$ and (3.54), this implies

$$f_{i\widehat{A}}^X\zeta^{\widehat{A}} = f_{iA}^X\zeta'^A. \tag{3.56}$$

Therefore, we have in the new basis $\zeta^{\widehat{A}} = (\zeta^i, \zeta'^A)$,

$$\widehat{f}_{ij}^X = 0. \tag{3.57}$$

After having established this basis, we can drop the primes. Using (2.3) and the form of components of the quaternionic structure in (3.13) combined with (2.4), we find the following table determining the vielbeins of the large space in terms of those of the small space, z^0, $\vec{k}^\alpha$ and $\vec{A}_X$:

$$
\begin{array}{lll}
\widehat{f}_{ij}^0 = -i\varepsilon_{ij}\sqrt{\tfrac{1}{2}z^0}, & \widehat{f}_{ij}^\alpha = \sqrt{\tfrac{1}{2z^0}}\vec{k}^\alpha \cdot \vec{\sigma}_{ij}, & \widehat{f}_{ij}^X = 0, \\
\widehat{f}_{iA}^0 = 0, & \widehat{f}_{iA}^\alpha = f_{iA}^X\vec{A}_X \cdot \vec{k}^\alpha, & \widehat{f}_{iA}^X = f_{iA}^X, \\
\widehat{f}_0^{ij} = i\varepsilon^{ij}\sqrt{\tfrac{1}{2z^0}}, & \widehat{f}_\alpha^{ij} = -\sqrt{\tfrac{z^0}{2}}\vec{m}_\alpha \cdot \vec{\sigma}^{ij}, & \widehat{f}_X^{ij} = \sqrt{\tfrac{z^0}{2}}\vec{A}_X \cdot \vec{\sigma}^{ij}, \\
\widehat{f}_0^{iA} = 0, & \widehat{f}_\alpha^{iA} = 0, & \widehat{f}_X^{iA} = f_X^{iA},
\end{array}
\tag{3.58}
$$

where $\vec{\sigma}_i{}^j$ denote the Pauli matrices, $\vec{\sigma}^{ij} = \varepsilon^{ik}\vec{\sigma}_k{}^j = \vec{\sigma}^{ji}$ and $\vec{\sigma}_{ij} = \vec{\sigma}_i{}^k\varepsilon_{kj} = \vec{\sigma}_{ji}$.

If there is a good metric on the hyper-Kähler space, this implies that the scalar product defined by this metric carries over to the tangent space, as was shown in (2.51). In the coordinates (3.9), this symplectic metric decomposes as

$$\widehat{C}_{AB} = C_{AB}, \qquad \widehat{C}_{ij} = \varepsilon_{ij}, \qquad \widehat{C}_{iA} = 0. \tag{3.59}$$

Note that the tangent space metric C_{AB}, which can be used to raise and lower flat indices on the small space, corresponds to the metric $g_{XY} = z^0 h_{XY}$.

3.6.2   Connections on the hypercomplex space

Since we already know the reduction rules for the $\widehat{\vec{J}}_{\widehat{X}}{}^{\widehat{Y}}$ in terms of z^0, $\vec{k}^\alpha$, $\vec{A}_X$ and $\vec{J}_X{}^Y$, see (3.13), and the form of $\widehat{\Gamma}$, we can calculate the induced connection $\widehat{\omega}$ on the tangent space from (2.7) for the hatted quantities (remember that in the large space there is no $SU(2)$ connection) using the reduction rules for the vielbeins (3.58). The induced connection has the following component expressions:

$$
\begin{array}{ll}
\widehat{\omega}_{0i}{}^j = \widehat{\omega}_{0i}{}^A = \widehat{\omega}_{0A}{}^j = 0, & \widehat{\omega}_{0A}{}^B = \tfrac{1}{2}f_Y^{iB}\partial_0 f_{iA}^Y + \tfrac{1}{2z^0}\delta_A^B, \\
\widehat{\omega}_{\alpha i}{}^A = \widehat{\omega}_{\alpha A}{}^j = 0, & \widehat{\omega}_{Xi}{}^j = -i\tfrac{z^0}{2}\vec{A}_X \cdot \vec{\sigma}_i{}^j, \\
\widehat{\omega}_{\alpha i}{}^j = i\tfrac{z^0}{2}\vec{m}_\alpha \cdot \vec{\sigma}_i{}^j, & \widehat{\omega}_{\alpha A}{}^B = \tfrac{1}{2}f_Y^{iB}\partial_\alpha f_{iA}^Y, \\
\widehat{\omega}_{Xi}{}^A = i\sqrt{\tfrac{1}{2z^0}}\varepsilon_{ik}f_X^{kA}, & \widehat{\omega}_{XA}{}^i = -i\sqrt{\tfrac{z^0}{2}}\varepsilon^{ij}f_{jA}^Y h_{YX}, \\
\widehat{\omega}_{XA}{}^B = \tfrac{1}{2}f_Y^{iB}\partial_X f_{iA}^Y + \tfrac{1}{2}f_Y^{iB}f_{iA}^Z\left(\widehat{\Gamma}_{XZ}{}^Y + \tfrac{1}{2}\vec{A}_Z \cdot \vec{J}_X{}^Y\right).
\end{array}
\tag{3.60}
$$

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3.6.3 Connections on the quaternionic space

By now, we have completely specified our reduction ansatz. This enabled us to give the component expressions for the Obata connection on the hypercomplex manifold in terms of the objects appearing in the ansatz. We now determine an expression for the $G\ell(n_H, \mathbb{H})$ connection components on the lower dimensional quaternionic space.

Using (3.60), we can dimensionally reduce the $G\ell(n_H + 1, \mathbb{H})$ connection as

$$\widehat{\omega}_{XA}{}^B = \omega^{\mathrm{Op}}{}_{XA}{}^B + \tfrac{1}{6}f^Z_{iA}\left[f^{iB}_{(X}\vec{J}_{Z)}{}^V \cdot \vec{A}_V + f^{iB}_Y\vec{A}_{(Z} \cdot \vec{J}_{X)}{}^Y - f^{iB}_Y\vec{A}_V \cdot J_{(Z}{}^V \times \vec{J}_{X)}{}^Y\right], \quad (3.61)$$

where $\omega^{\mathrm{Op}}{}_{XA}{}^B$ is the $G\ell(n_H, \mathbb{H})$ connection corresponding to the Oproiu connection (3.30) and the SU(2) connection (3.29).

As we have already explained quite extensively, there is a family of possible connections on a quaternionic manifold, related to each other by ξ-transformations, which for the $G\ell(n_H, \mathbb{H})$ connection is given by (2.31). The transformation (3.32) gives the drastic simplification

$$\omega_{XA}{}^B = \widehat{\omega}_{XA}{}^B. \tag{3.62}$$

While in quaternionic manifolds this is just one possible choice, we have seen in Sect. 3.5.1 that in quaternionic-Kähler manifolds this ξ-choice is imposed.

4 Curvatures

In the 'large space' many components of the curvatures are zero due to the existence of the homothetic Killing vector field and the SU(2) isometries, (1.1) and (3.2), yielding (3.5):

$$k^{\widehat{X}}\widehat{R}_{\widehat{X}\widehat{Y}\widehat{Z}}{}^{\widehat{W}} = \vec{k}^{\widehat{X}}\widehat{R}_{\widehat{X}\widehat{Y}\widehat{Z}}{}^{\widehat{W}} = 0. \tag{4.1}$$

This (together with the cyclicity properties of the curvatures) shows that the only possible non-zero components of the curvature of the conformal hypercomplex manifolds are $\widehat{R}_{XYZ}{}^{\widehat{W}}$. For the Ricci tensor, this implies that the only nonvanishing components are along the quaternionic directions:

$$\widehat{R}_{XY} = \widehat{R}_{ZXY}{}^Z, \tag{4.2}$$

and this is antisymmetric as in general for hypercomplex manifolds. The other part of the curvature, as shown in Table 2, is the Weyl part. The latter is generally determined by a traceless tensor, see (2.25). But for hypercomplex manifolds there is also a generically non-traceless tensor $\widehat{W}_{\widehat{A}\widehat{B}\widehat{C}}{}^{\widehat{D}}$, whose trace part determines the antisymmetric part of the Ricci tensor, and whose traceless part is $\widehat{\mathcal{W}}_{\widehat{A}\widehat{B}\widehat{C}}{}^{\widehat{D}}$. The vanishing of all the curvature components mentioned above implies that the non-vanishing parts of $\widehat{W}_{\widehat{A}\widehat{B}\widehat{C}}{}^{\widehat{D}}$ are only $\widehat{W}_{ABC}{}^{\widehat{D}}$, i.e. $\widehat{W}_{ABC}{}^D$ and $\widehat{W}_{ABC}{}^i$. The latter will not be important for the reduction to the small space. Note, from (2.47), that the traceless part, $\widehat{\mathcal{W}}_{\widehat{A}\widehat{B}\widehat{C}}{}^{\widehat{D}}$, has as non-zero components

$$\widehat{\mathcal{W}}_{ABC}{}^D = \widehat{W}_{ABC}{}^D - \frac{3}{2(n_H+2)}\delta^D_{(A}\widehat{W}_{BC)E}{}^E,$$

$$\widehat{\mathcal{W}}_{iAB}{}^j = \widehat{\mathcal{W}}_{AiB}{}^j = \widehat{\mathcal{W}}_{ABi}{}^j = -\frac{1}{2(n_H+2)}\delta^j_i\widehat{W}_{ABE}{}^E, \qquad \widehat{\mathcal{W}}_{ABC}{}^i = \widehat{W}_{ABC}{}^i. \tag{4.3}$$

This implies that the Weyl tensor of the hypercomplex space, $\widehat{\mathcal{W}}$, depends also on the trace $\widehat{W}_{ABE}{}^E$.

We now start the reduction to the small space. We use the ξ-choice (3.33), which gives the easiest formulae for the map, as explained above. The formulae for other ξ-choices follow from (2.33)–(2.35). An expression for the SU(2) curvature can be found by considering (3.24). With (3.33), this implies

$$\vec{\mathcal{R}}_{XY} = -\tfrac{1}{2}\vec{J}_{[X}{}^{Z}h_{Y]Z},\tag{4.4}$$

which is the equivalence between the SU(2) curvature and the quaternionic two-forms in the quaternionic-Kähler case. One can derive the curvature components $\widehat{R}_{XYZ}{}^{W}$ of the hypercomplex space as a function of the curvature $R_{XYZ}{}^{W}$ of $\Gamma_{XY}{}^{Z}$ by a long but straightforward calculation. We obtain

$$\widehat{R}_{XYZ}{}^{W} = R_{XYZ}{}^{W} + \tfrac{1}{2}\delta^{W}_{[X}h_{Y]Z} - \tfrac{1}{2}\vec{J}_{Z}{}^{W} \cdot \vec{J}_{[X}{}^{V}h_{Y]V} - \tfrac{1}{2}\vec{J}_{Z}{}^{V} \cdot \vec{J}_{[X}{}^{W}h_{Y]V}.\tag{4.5}$$

Taking the trace of this expression gives the Ricci tensor

$$\widehat{R}_{XY} = R_{XY} + \frac{2n_H + 1}{2}h_{XY} + \frac{1}{2}\vec{J}_X{}^{Z} \cdot \vec{J}_Y{}^{W}h_{ZW}.\tag{4.6}$$

The left hand side is antisymmetric, which determines the antisymmetric part of the quaternionic Ricci tensor R_{XY},

$$\widehat{R}_{XY} = R_{[XY]}.\tag{4.7}$$

As the $\mathbb{R}$ curvature is, up to a sign, equal to this antisymmetric part, see (2.18), we find that the $\mathbb{R}$ curvatures are the same for the large and for the small space. As explained in Sect. 3.4.2, they can both be transformed to zero, using a $\widehat{\xi}$-transformation in the large space and a ξ-transformation with $\xi_X = \widehat{\xi}_X$ in the small space.

The symmetric part of (4.6) determines the symmetric part of the Ricci tensor for the quaternionic manifold. Using this in (2.21) gives

$$B_{XY} = \frac{1}{4(n_H + 1)}\widehat{R}_{XY} - \frac{1}{4}h_{XY}.\tag{4.8}$$

Then we see that the extra terms in (4.5) constitute the Ricci part of the curvature using the symmetric part of B:

$$R^{\mathrm{Ric}}_{\mathrm{symm}\,XYZ}{}^{W} = -\tfrac{1}{2}\delta^{W}_{[X}h_{Y]Z} + \tfrac{1}{2}\vec{J}_{Z}{}^{W} \cdot \vec{J}_{[X}{}^{V}h_{Y]V} + \tfrac{1}{2}\vec{J}_{Z}{}^{V} \cdot \vec{J}_{[X}{}^{W}h_{Y]V}.\tag{4.9}$$

We can then identify

$$\widehat{R}_{XYZ}{}^{W} = \left(R - R^{\mathrm{Ric}}_{\mathrm{symm}}\right)_{XYZ}{}^{W} = \left(R^{(\mathrm{W})} + R^{\mathrm{Ric}}_{\mathrm{antis}}\right)_{XYZ}{}^{W}.\tag{4.10}$$

For the full hypercomplex manifold, we have to use $r = n_H + 1$ in (2.21) (with only the antisymmetric part), and thus

$$\widehat{B}_{XY} = \frac{1}{4(n_H + 2)}R_{[XY]} \;\Rightarrow\; \widehat{R}^{\mathrm{Ric}}_{\mathrm{antis}\,XYZ}{}^{W} = \frac{n_H + 1}{n_H + 2}R^{\mathrm{Ric}}_{\mathrm{antis}\,XYZ}{}^{W}.\tag{4.11}$$

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The Weyl parts of the curvatures are determined by the traceless $\widehat{\mathcal{W}}$ and $\mathcal{W}$ tensors, but, as mentioned above in (4.3), in the hypercomplex space this is dependent on the trace $\widehat{W}_{ABC}{}^C$. We can extract the Weyl part of the curvature of the quaternionic space from (4.10) and find

$$\widehat{W}_{ABC}{}^D = \mathcal{W}_{ABC}{}^D + \frac{3}{2(n_H + 1)} \delta^D_{(A} \widehat{W}_{BC)E}{}^E. \tag{4.12}$$

Hence, $\mathcal{W}_{ABC}{}^D$ is the traceless part of $\widehat{W}_{ABC}{}^D$.

Therefore, the main results are:

1. The antisymmetric Ricci tensor of the hypercomplex manifold is the same as the antisymmetric Ricci tensor of the quaternionic manifold.

2. The symmetric Ricci part of the quaternionic manifold is a universal expression in terms of the candidate metric h (the same as for $\mathbb{H}P^{n_H}$).

3. The traceless $\mathcal{W}$ tensor of the quaternionic space is the traceless part of the $\widehat{W}$ tensor of the hypercomplex space.

As mentioned, we derived everything for a special choice of ξ. However, we have seen in Sect. 2.2 that the Weyl tensor is not changed by a ξ-transformation. Hence, the last conclusion is valid for any ξ. Symbolically, we can represent the dependence of parts of the curvature tensors on basic tensors as follows:

$$
\begin{array}{ccccc}
\widehat{R} & = & & \widehat{R}^{\text{Ric}}_{\text{antis}} & + & \widehat{R}^{(\text{W})} \\
& & & \uparrow & \nearrow & \uparrow \\
& h_{XY} & & \widehat{W}_{ABC}{}^C & & \mathcal{W}_{ABC}{}^D \\
& \downarrow & & \downarrow & & \downarrow \\
R & = & R^{\text{Ric}}_{\text{symm}} & + & R^{\text{Ric}}_{\text{antis}} & + & R^{(\text{W})}
\end{array}
\tag{4.13}
$$

For the mapping between a hyper-Kähler manifold and a quaternionic-Kähler manifold, there is no antisymmetric part of the Ricci tensor, and hence also $\widehat{W}_{ABC}{}^D$ is traceless. Hence, the relation (4.12) reduces to

$$\widehat{W}_{ABC}{}^D = \mathcal{W}_{ABC}{}^D. \tag{4.14}$$

It implies that the hyper-Kähler curvature components along the quaternionic directions are the Weyl part of the quaternionic-Kähler curvature

$$\widehat{R}_{XYZ}{}^W = R^{(\text{W})}{}_{XYZ}{}^W. \tag{4.15}$$

The Ricci part of this quaternionic-Kähler curvature is the same expression as for $\mathbb{H}P^{n_H}$. In fact, we find

$$B_{XY} = -\tfrac{1}{4} h_{XY}. \tag{4.16}$$

The metric of the small space inherited from the large space is $g_{XY} = z^0 h_{XY}$, see (3.46). Comparing (4.16) with the relation (2.57), using this metric implies

$$\nu = -\frac{1}{z^0}. \tag{4.17}$$

In the context of supergravity, the value of z^0 determines the normalization of the Einstein term and is fixed to $z^0 = \kappa^{-2}$, where κ is the gravitational coupling constant.

Finally, note that for the 1-dimensional case, $n_H = 1$, we had restricted the definition of quaternionic manifolds with special requirements in Appendix B.4 of [4], as was also done in the mathematical literature [43]. Here we find that these relations are automatically fulfilled in the embedded quaternionic manifolds. Hence, they are unavoidable in a supergravity context.

5 Reduction of the Symmetries

5.1 Symmetries and moment maps

The main part of this subsection is a summary of the results on symmetries given in [4]. In the case of manifolds where there is no (good) metric, the question of defining symmetries needs some careful consideration. In general, we consider transformations $\delta q^X = k_I^X(q)\Lambda^I$, where the index I runs over the set of possible symmetries. We will define when these are called 'symmetries', and then we define 'quaternionic symmetries'.

A set of vector fields k_I are symmetry generators if the following condition on the connection and curvature are met:

$$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W. \tag{5.1}$$

For Riemannian manifolds with a metric g_{XY}, this is just the integrability condition that follows from the Killing equation $\mathfrak{D}_{(X} k_{Y)I} = 0$, where $k_{XI} = g_{XY} k_I^Y$. On the other hand, (5.1) does not imply a Killing equation as it is independent of a choice of metric. E.g. the conformal Killing vectors, which do not satisfy the Killing equation, satisfy (5.1). However, it is a sufficient condition to define 'symmetries' if there is no metric available.

The 'physical' origin for this condition is the following. For simplicity, consider the equations of motion for a rigid non-linear sigma model:

$$\Box q^X = \partial_\mu \partial^\mu q^X + \partial_\mu q^Y \partial^\mu q^Z \Gamma_{YZ}{}^X = 0. \tag{5.2}$$

Consider the transformation $\delta q^X = k_I^X \Lambda^I$ on this field equation:

$$\delta \Box q^X = \partial_Y k_I^X \Lambda^I \Box q^Y + \partial_\mu q^Y \partial^\mu q^Z \mathfrak{D}_Z \mathfrak{D}_Y k_I^X \Lambda^I - \partial_\mu q^Y \partial^\mu q^Z k_I^V \Lambda^I R_{ZVY}{}^X. \tag{5.3}$$

Hence, the set of equations of motion is left invariant iff the defining condition (5.1) for a 'symmetry' is satisfied .

Symmetry generators are quaternionic if the vector fields normalize the complex structures, which means

$$\mathcal{L}_{k_I} \vec{J}_X{}^Y = \vec{r}_I \times \vec{J}_X{}^Y, \qquad (5.4)$$

for some 3-vectors $\vec{r}_I$. Using the SU(2) connections, this defines moment maps $\nu \vec{P}_I(q)$ by[14]

$$\nu \vec{P}_I \equiv -\tfrac{1}{2}\vec{r}_I - k_I^X \vec{\omega}_X. \qquad (5.5)$$

See [4] for more information, where it was also shown that this leads to a decomposition of the derivatives of the k_I^X as

$$\mathfrak{D}_X k_I^Y = \nu \vec{J}_X{}^Y \cdot \vec{P}_I + L_X{}^Y{}_A{}^B t_{IB}{}^A. \qquad (5.6)$$

The so-called *moment maps* $\vec{P}_I(q)$ describe the SU(2) content of the symmetry and the $t_{IB}{}^A(q)$ describe the G$\ell(r,\mathbb{H})$ content. The $L_X{}^Y{}_A{}^B$ symbols are defined as in (A.2). We can extract $\nu \vec{P}_I$ from (5.6) as

$$4r\nu \vec{P}_I = -\vec{J}_X{}^Y \mathfrak{D}_Y k_I^X. \qquad (5.7)$$

If we project the curvature tensors along the symmetry vectors, we get the following relations:[15]

$$\vec{\mathcal{R}}_{XY} k_I^Y = -\nu \mathfrak{D}_X \vec{P}_I, \qquad \mathcal{R}_{XYB}{}^A k_I^Y = \mathfrak{D}_X t_{IB}{}^A. \qquad (5.8)$$

In supersymmetric models, the condition of quaternionic symmetries is necessary for the invariance of the full field equations including the fermions.

The vector fields generate a Lie-algebra:

$$2k_{[I}^Y \partial_Y k_{J]}^X = -f_{IJ}{}^K k_K^X. \qquad (5.9)$$

The vector at the left-hand side of this equation for any $[IJ]$ satisfies the two conditions mentioned above, (5.1) and (5.4), and this equation is thus a statement of completeness of the set of quaternionic symmetries. It leads to an important property of the moment maps, which is the 'equivariance relation'

$$-2\nu^2 \vec{P}_I \times \vec{P}_J + \vec{\mathcal{R}}_{YW} k_I^Y k_J^W - \nu f_{IJ}{}^K \vec{P}_K = 0. \qquad (5.10)$$

The absence of the SU(2) curvature parts for hypercomplex (and hyper-Kähler) manifolds implies that we can take $\nu = 0$ for these manifolds. As also $\vec{\omega}_X = 0$ in that case, we see from (5.4) and (5.5) that the Lie derivative along the symmetry generators of the complex structures vanishes. The symmetries are then called triholomorphic.

Though $\nu = 0$ implies that the moment maps do not appear for hyper-Kähler manifolds in the above relations, moment maps still appear in the Lagrangian for sigma models on these

---

[14]We define here $\nu \vec{P}_I$, where ν is a number. For quaternionic-Kähler manifolds, this number is defined by (2.57), while it is not specified in quaternionic manifolds. This normalization is convenient for comparison with other papers, and, as will be shown below, because the formulae are then applicable to hyper-Kähler manifolds upon setting $\nu = 0$.

[15]The first of these equations holds for any choice of SU(2)-connection $\vec{\omega}_X$, if a corresponding $\vec{P}_I$ is defined by (5.5).

manifolds. They are defined by a relation consistent with the equations for quaternionic manifold (5.8) and (2.58) for any ν:

$$\vec{J}_{XY}k_I^Y = -2\partial_X \vec{P}_I. \tag{5.11}$$

Notice that we have used the existence of a metric here, so this relation only defines moment maps for hyper-Kähler manifolds. They should still satisfy the $\nu = 0$ limit of the equivariance relation (5.10):

$$k_I^X \vec{J}_{XY}k_J^Y = 2f_{IJ}{}^K \vec{P}_K. \tag{5.12}$$

We can summarize the various cases as follows:

- Hypercomplex : Triholomorphic symmetries must satisfy (5.1) and

$$\mathcal{L}_{k_I}\vec{J}_X{}^Y = 0. \tag{5.13}$$

  There exist no moment maps.

- Hyper-Kähler : conditions for the triholomorphic symmetries as for hypercomplex. The condition for a triholomorphic isometry can be translated to the existence of a triplet of moment maps (5.11).

- Quaternionic-Kähler : in this case, *any* isometry normalizes the quaternionic structure [44, 45]. Indeed, if we define the moment maps for any Killing vector k_I as in (5.7), and take a covariant derivative, the first equation of (5.8) follows from (5.1), the covariant constancy of $\vec{J}$ and the decomposition (2.15). The proportionality of the SU(2) curvature and complex structure, (2.58), implies that this can be written as a covariant version of (5.11). Then the Killing equation implies

$$\vec{J}_X{}^Z\mathfrak{D}_Z k_I^Y = -\vec{J}_X{}^Z\mathfrak{D}^Y k_{IZ} = 2\mathfrak{D}^Y\mathfrak{D}_X\vec{P}_I, \qquad -\mathfrak{D}_X k_I^Z \vec{J}_Z{}^Y = -2\mathfrak{D}_X\mathfrak{D}_Y\vec{P}_I. \tag{5.14}$$

  It is straightforward to show that (5.4) is satisfied, as this reduces to the sum of these two expressions, see [4, (B.80)]. Thus, all isometries on a quaternionic-Kähler manifold are quaternionic.

- Quaternionic manifolds: Not all symmetries are quaternionic, i.e. satisfying (5.4). However, when they are, moment maps exist and are given by (5.7). Not all ξ-transformations preserve the symmetries. Indeed, the condition (5.1) is not invariant under general ξ-transformations. Writing (5.1) with ξ-transformed connections modifies it proportional to

$$S_{XY}^{ZU}\left[\xi_W \mathfrak{D}_U k_I^W + k_I^W \mathfrak{D}_W \xi_U\right]. \tag{5.15}$$

We thus conclude that a symmetry is a symmetry after a ξ-transformation iff

$$\mathcal{L}_{k_I}\xi_X = k_I^Y \partial_Y \xi_X + \xi_Y \partial_X k_I^Y = 0. \tag{5.16}$$

The condition for a symmetry to be quaternionic, (5.4), is invariant if $\vec{r}_I$ is invariant. We can obtain the transformation of the moment map from (5.7). This leads, using (A.16), to

$$\nu \widetilde{\vec{P}}_I = \nu \vec{P}_I - \vec{J}_X{}^Y k_I^X \xi_Y. \tag{5.17}$$

This implies indeed that $\vec{r}_I$ is an invariant for ξ-transformations.

5.2 Conformal hypercomplex

As explained in Sect. 3.1, hypercomplex and hyper-Kähler manifolds with a homothetic Killing vector (1.1) have additional properties. First of all, as follows from (3.5), both dilatations and SU(2) transformations define symmetries in the sense of (5.1). Notice that, when there is a metric, these symmetries are not always isometries of the metric. Precisely, the dilatations are of this type, as they satisfy the conformal Killing equation instead of the Killing equation. The dilatation symmetry is triholomorphic. The SU(2) transformations are 'quaternionic symmetries', with $\vec{r}_I = -\mathbb{1}_3$ as matrix in the 3 components of vectors and three values of I.

For the remainder of this section, we concentrate on possible additional symmetries, other than the dilatations and SU(2) transformations. We require such symmetries to commute with the dilatational symmetry. This condition is expressed as

$$k^{\hat{Y}} \widehat{\mathfrak{D}}_{\hat{Y}} \widehat{k}_I^{\hat{X}} = \tfrac{3}{2} \widehat{k}_I^{\hat{X}}. \tag{5.18}$$

For triholomorphic symmetries, this implies [4]

$$\vec{k}^{\hat{Y}} \widehat{\mathfrak{D}}_{\hat{Y}} \widehat{k}_I^{\hat{X}} = \tfrac{1}{2} \widehat{k}_I^{\hat{Y}} \widehat{\vec{J}}_{\hat{Y}}{}^{\hat{X}}, \tag{5.19}$$

which expresses the fact that they also commute with the SU(2) transformations.

For conformal hyper-Kähler manifolds, one can deduce more identities. In fact, contracting (5.18) with $k_{\hat{X}}$, one finds

$$k^{\hat{X}} \widehat{g}_{\hat{X}\hat{Y}} \widehat{k}_I^{\hat{Y}} = 0. \tag{5.20}$$

Moreover, the consistency with the conformal symmetry allows us to integrate (5.11) in terms of the moment maps [17], such that

$$-6\widehat{\vec{P}}_I = k^{\hat{X}} \widehat{\vec{J}}_{\hat{X}\hat{Y}} \widehat{k}_I^{\hat{Y}} = -\tfrac{2}{3} k^{\hat{X}} k^{\hat{Z}} \widehat{\vec{J}}_{\hat{Z}}{}^{\hat{Y}} \widehat{\mathfrak{D}}_{\hat{Y}} \widehat{k}_{I\hat{X}}. \tag{5.21}$$

A possible integration constant in $\widehat{\vec{P}}_I$ (a Fayet-Iliopoulos term) is not allowed in conformally invariant actions, see [44].

5.3 The map of the symmetries

In this section, we will discuss the reduction of triholomorphic symmetries of the large space to quaternionic symmetries of the small space. We also illustrate how triholomorphic isometries and moment maps on hyper-Kähler spaces descend to quaternionic isometries on quaternionic-Kähler manifolds.

5.3.1 Conformal hypercomplex and quaternionic symmetries

We now show that the components of the higher-dimensional symmetry generators lying along the quaternionic directions are bona-fide symmetry generators of the quaternionic space.

First of all, for any triholomorphic symmetry on a conformal hypercomplex manifold, the reduction of the Eqs. (5.18) and (5.19) leads to the z^0 and z^α dependence of the higher-dimensional symmetry generators. The form of the symmetry vectors is as follows:

$$\widehat{k}_I^0 = z^0 V_I(q), \qquad \widehat{k}_I^\alpha = \vec{k}^\alpha \cdot \vec{Q}_I, \qquad \widehat{k}_I^X = k_I^X(q), \tag{5.22}$$

where all z^0-dependence is explicitly indicated, and

$$(\partial_\alpha + \vec{m}_\alpha \times) \, \vec{Q}_I = 0, \tag{5.23}$$

while V_I and k_I^X are independent of z^α.

This can now be used to study the normalization of the complex structures (5.4) on the quaternionic space. Using the triholomorphicity of $\widehat{k}_I^{\hat{X}}$, together with (3.13), (3.22) and (5.22), it is easy to show that the vector field with components $k_I^X = \widehat{k}_I^X$ normalizes the quaternionic complex structures,

$$\mathcal{L}_{k_I} \vec{J}_X{}^Y = \vec{Q}_I \times \vec{J}_X{}^Y. \tag{5.24}$$

This identifies $\vec{Q}_I$ with the vector $\vec{r}_I$ in (5.4). The ${}_X{}^0$ component of the normalization condition gives

$$-\vec{J}_X{}^Y \partial_Y V_I + \left(\partial_X - \vec{A}_X \times \right) \left(\vec{A}_Y k_I^Y - \vec{Q}_I \right) - 2 k_I^Y [\vec{R}(-\tfrac{1}{2}\vec{A})]_{YX} = 0. \tag{5.25}$$

The relations (5.8), using footnote 15 with $\vec{\omega}_X = -\tfrac{1}{2}\vec{A}_X$, implies that the last two terms cancel, and we find

$$\partial_X V_I = 0. \tag{5.26}$$

The constant contribution in V_I can be set to zero. Indeed, this just reflects that the dilatation vector (3.8) is a triholomorphic symmetry vector. We can indicate this as the symmetry with label $I = 0$:

$$\widehat{k}_0^{\hat{X}} = 3 z^0 \delta_0^{\hat{X}}. \tag{5.27}$$

We can thus subtract this from all the other symmetries, redefining them as

$$\widehat{k}_I'^{\hat{X}} = \widehat{k}_I^{\hat{X}} - 3 z^0 \delta_0^{\hat{X}} V_I. \tag{5.28}$$

This redefined symmetry vector is of the form (5.22) with $V_I = 0$.

One may now verify explicitly that the condition for a symmetry in the large space reduces to the condition that k_I^X is a symmetry in the small space. Excluding the dilatation symmetry, the result is thus that any triholomorphic symmetry that commutes with the dilatations is of the form

$$\widehat{k}_I^0 = 0, \qquad \widehat{k}_I^\alpha = \vec{k}^\alpha \cdot \vec{r}_I, \qquad \widehat{k}_I^X = k_I^X(q), \tag{5.29}$$

40

where k_I^X is a quaternionic symmetry of the small space, satisfying (5.4). With this formula, we can thus also uplift any quaternionic symmetry of the small space to a triholomorphic symmetry of the large space preserving dilatations.

Further, we consider the algebra in the large space:

$$2\hat{k}_{[I}^{\widehat{Y}}\partial_{\widehat{Y}}\hat{k}_{J]}^{\widehat{X}} = -f_{IJ}{}^K\hat{k}_K^{\widehat{X}}. \tag{5.30}$$

Reducing this equation for the different values of $\hat{X}$ leads to

$$\begin{aligned}
X &: \quad 2k_{[I}^Y\partial_Y k_{J]}^X = -f_{IJ}{}^K k_K^X, \\
\alpha &: \quad -2\nu^2\vec{P}_I \times \vec{P}_J + \vec{\mathcal{R}}_{YW}k_I^Y k_J^W - \nu f_{IJ}{}^K\vec{P}_K = 0, \\
0 &: \quad f_{IJ}{}^K\widehat{k}_K^0 = 0.
\end{aligned} \tag{5.31}$$

The first equation says that the algebra in the small space is the same as the algebra in the large space, excluding dilatations. The second equation is the equivariance condition (5.10). The third one says that the dilatations do not appear in the right-hand side of commutator relations.

Remark that all equations obtained in this section are invariant under ξ-transformations satisfying (5.16).

5.3.2 Isometries on conformal hyper-Kähler and quaternionic-Kähler spaces

When there is a metric, we find that the only non-trivial part of the Killing equation in the large space is

$$\widehat{\mathfrak{D}}_{(X}\hat{k}_{Y)I} = z^0\mathfrak{D}_{(X}k_{Y)I}. \tag{5.32}$$

Hence, a triholomorphic symmetry preserving the dilatations is an isometry if and only if it is an isometry of the quaternionic-Kähler space.

For conformal hyper-Kähler manifolds, the moment map is defined in (5.21). Using the decomposition of the symmetry vector (5.29) and (4.17), we find

$$\widehat{\vec{P}}_I = -\frac{1}{6}k^{\hat{X}}\widehat{\vec{J}}_{\hat{X}}{}^{\hat{Y}}\widehat{k}_{I\hat{Y}} = -\frac{1}{2}\vec{k}^{\hat{X}}\widehat{g}_{\hat{X}\hat{Y}}\widehat{k}_I^{\hat{Y}} = \frac{1}{2}z^0\left(\vec{r}_I - \vec{A}_X k_I^X\right) = \vec{P}_I - z^0\left(2\vec{\omega}_X + \vec{A}_X\right)k_I^X. \tag{5.33}$$

As in the quaternionic-Kähler spaces we have (3.49), the last term vanishes and the moment map in the hyper-Kähler space is equal to the moment map in the quaternionic-Kähler space.

5.3.3 The conformal hyper-Kähler manifold of quaternionic dimension 1.

In quaternionic manifolds, the moment map is completely determined, see (5.7). Fayet-Iliopoulos (FI) terms are in general undetermined constants in the moment maps (see the review [46]). Hence, they are not present in quaternionic-Kähler manifolds, except for the 'trivial situation' $n_H = 0$. The latter corresponds to a large space of quaternionic dimension 1, which is hyper-Kähler as the metric is given in the first line of (3.51):

$$\widehat{g}_{00} = -\frac{1}{z^0}, \qquad \widehat{g}_{0\alpha} = 0, \qquad \widehat{g}_{\alpha\beta} = -z^0\vec{m}_\alpha \cdot \vec{m}_\beta. \tag{5.34}$$

This metric has $SU(2) \times SU(2)$ isometries. The first factor is generated by $\vec{k}^\alpha$, but these are not triholomorphic. However, there is a commuting set of $SU(2)$ Killing vectors, k^α_{-I}, with $I = 1, 2, 3$, and the minus sign indicating the second $SU(2)$ factor in the holonomy group, different from $\vec{k}_\alpha$. The generators are thus

$$\widehat{k}^0_I = 0, \qquad \widehat{k}^\alpha_I = k^\alpha_{-I}. \tag{5.35}$$

These are triholomorphic with respect to the complex structures defined by the first Killing vectors:

$$
\begin{aligned}
\widehat{\vec{J}}_0{}^0 &= 0, & \widehat{\vec{J}}_\alpha{}^0 &= -z^0 \vec{m}_\alpha, \\
\widehat{\vec{J}}_0{}^\beta &= \tfrac{1}{z^0} \vec{k}^\beta, & \widehat{\vec{J}}_\alpha{}^\beta &= \vec{k}^\beta \times \vec{m}_\alpha.
\end{aligned}
\tag{5.36}
$$

This leads to the moment maps

$$- 2\nu \vec{P}_I = \vec{r}_I = \vec{m}_\alpha k^\alpha_{-I}. \tag{5.37}$$

When a gauge fixing is taken, i.e. the z^α are fixed to a value, then this leads to some constants. These are the $SU(2)$ FI terms.

6 Summary and Discussion

For the convenience of the reader we summarize the main results of this paper. The map is schematically represented in Fig. 1, which will be further explained in this section.

6.1 The map from hypercomplex manifolds

The setup starts from a $4(n_H + 1)$-dimensional space that is hypercomplex and has a conformal symmetry. The latter is mathematically expressed as the presence of a closed homothetic Killing vector

$$\widehat{\mathfrak{D}}_{\hat{Y}} k^{\hat{X}} \equiv \partial_{\hat{Y}} k^{\hat{X}} + \widehat{\Gamma}_{\hat{Y}\hat{Z}}{}^{\hat{X}} k^{\hat{Z}} = \tfrac{3}{2} \delta_{\hat{Y}}{}^{\hat{X}}. \tag{6.1}$$

Observe that the coordinates and covariant derivatives in this large space are indicated by hatted quantities. The vector k, which is the generator of dilatations, yields also generators of $SU(2)$ transformations:

$$\vec{k}^{\hat{X}} \equiv \tfrac{1}{3} \widehat{\vec{J}}_{\hat{Y}}{}^{\hat{X}} k^{\hat{Y}}. \tag{6.2}$$
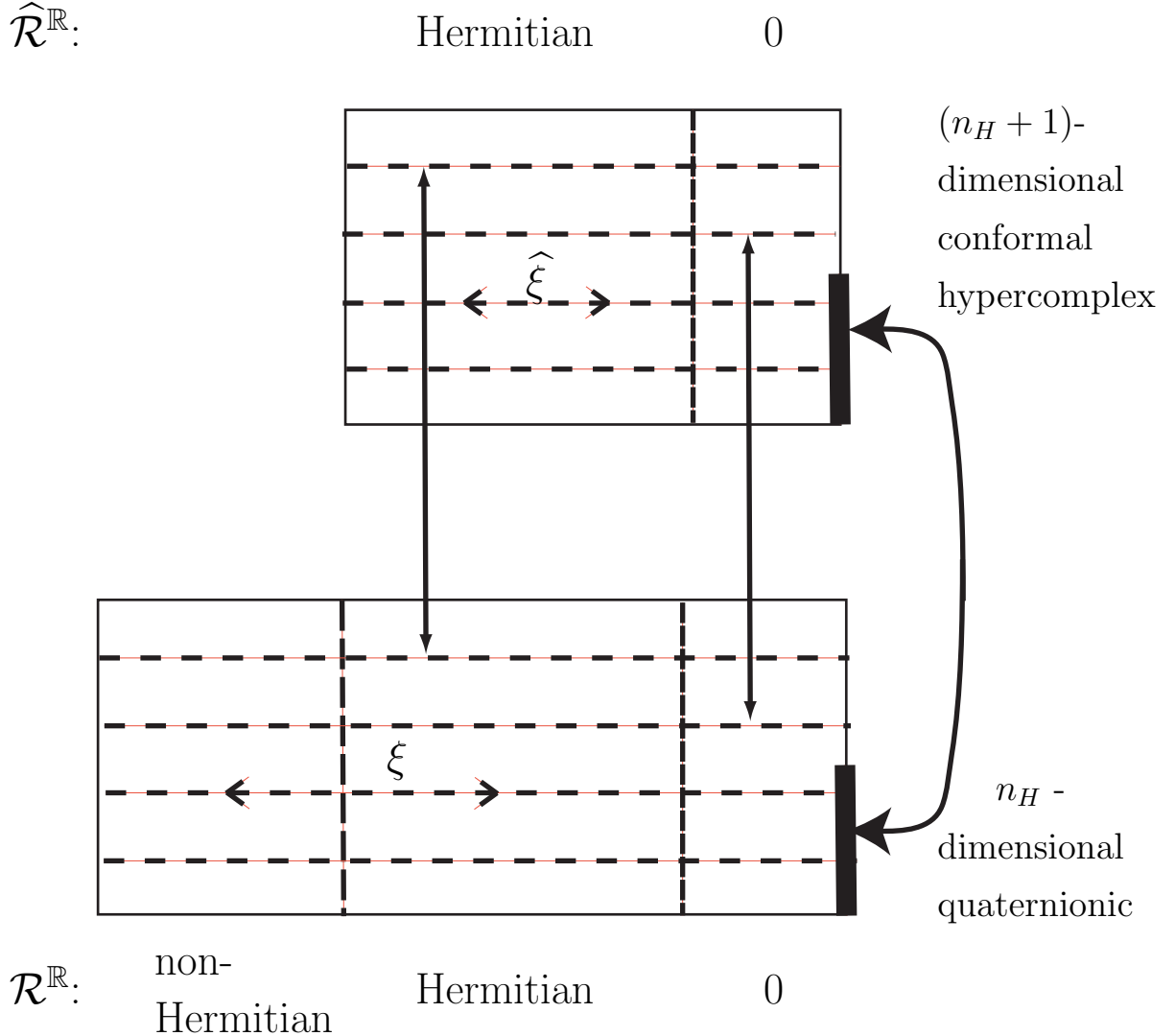
We use coordinates in this large space that are adapted to these quantities:

$$
\begin{aligned}
q^{\hat{X}} &= \{z^0,\, y^p\} = \{z^0,\, z^\alpha,\, q^X\}, & \alpha = 1, 2, 3, \quad X = 1, \ldots, 4n_H, \\
k^{\hat{X}} &= 3z^0 \delta_0^{\hat{X}}, \qquad \vec{k}^0 = \vec{k}^X = 0.
\end{aligned}
\tag{6.3}
$$

This leads to the following form of the complex structures:

$$
\begin{aligned}
\widehat{\vec{J}}_0{}^0 &= 0, & \widehat{\vec{J}}_\alpha{}^0 &= -z^0 \vec{m}_\alpha, & \widehat{\vec{J}}_X{}^0 &= z^0 \vec{A}_X, \\
\widehat{\vec{J}}_0{}^\beta &= \tfrac{1}{z^0} \vec{k}^\beta, & \widehat{\vec{J}}_\alpha{}^\beta &= \vec{k}^\beta \times \vec{m}_\alpha, & \widehat{\vec{J}}_X{}^\beta &= \vec{A}_X \times \vec{k}^\beta + \vec{J}_X{}^Z (\vec{A}_Z \cdot \vec{k}^\beta), \\
\widehat{\vec{J}}_0{}^Y &= 0, & \widehat{\vec{J}}_\alpha{}^Y &= 0, & \widehat{\vec{J}}_X{}^Y &= \vec{J}_X{}^Y.
\end{aligned}
\tag{6.4}
$$

Figure 1: *The map schematically. The two blocks represent the families of large (upper block) and small spaces (lower block), where the horizontal lines indicate how they are related by $\widehat{\xi}$, resp. ξ, transformations. They connect parametrizations of the same manifold with different complex structures for hypercomplex manifolds, and different affine and $\mathrm{SU}(2)$ connections for the quaternionic manifolds. At the right, the spaces have no $\mathbb{R}$ curvature, and part of these are hyper-Kähler, resp. quaternionic-Kähler. The latter two classes are indicated by the thick lines. The vertical arrows represent the map described in this paper, connecting the manifolds with similar parametrizations. They are a representation of the map between the horizontal lines, which is the map between hypercomplex and quaternionic manifolds. The thick arrow indicates the map between hyper-Kähler and quaternionic-Kähler spaces.*

Here, $\vec{k}^\alpha$ are SU(2) Killing vectors, and $\vec{m}_\alpha$ is the inverse as a 3×3 matrix. The result thus depends on z^0, these SU(2) Killing vectors, complex structures on the small space $\vec{J}_X{}^Y$ and a vector $\vec{A}_X$. The unique torsionless affine connection is the Obata connection

$$\widehat{\Gamma}_{\hat{X}\hat{Y}}{}^{\hat{Z}} = -\tfrac{1}{6}\left(2\partial_{(\hat{X}}\widehat{\vec{J}}_{\hat{Y})}{}^{\hat{W}} + \widehat{\vec{J}}_{(\hat{X}}{}^{\hat{U}} \times \partial_{|\hat{U}|}\widehat{\vec{J}}_{\hat{Y})}{}^{\hat{W}}\right)\cdot\widehat{\vec{J}}_{\hat{W}}{}^{\hat{Z}}. \tag{6.5}$$

The condition that this connection preserves the complex structures implies that

$$\begin{aligned}
\partial_0\vec{A}_X &= 0, & (\partial_\alpha + \vec{m}_\alpha\times)\,\vec{A}_X + \partial_X\vec{m}_\alpha &= 0, \\
\partial_0\vec{J}_X{}^Y &= 0, & (\partial_\alpha + \vec{m}_\alpha\times)\,\vec{J}_X{}^Y &= 0, \\
[\vec{R}(-\tfrac{1}{2}\vec{A})]_{XY} &\equiv -\partial_{[X}\vec{A}_{Y]} + \tfrac{1}{2}\vec{A}_X \times \vec{A}_Y = \tfrac{1}{2}h_{V[X}\vec{J}_{Y]}{}^V,
\end{aligned} \tag{6.6}$$

for some symmetric tensor h_{XY}. Furthermore it implies that the $\vec{J}_X{}^Y$ are a quaternionic structure on the small space.

Hypercomplex manifolds have no SU(2) curvature, but can have a non-vanishing $\mathbb{R}$ curvature, which is Hermitian, i.e. it commutes with the hypercomplex structure, see Appendix A. This $\mathbb{R}$ curvature is equal to the Ricci tensor of these manifolds, which is antisymmetric,

$$\mathcal{R}^{\mathbb{R}}_{\hat{X}\hat{Y}} = -R_{\hat{X}\hat{Y}}. \tag{6.7}$$

We found that in these conformal hypercomplex manifolds there is a transformation $\widehat{\xi}_{\hat{X}}$ that preserves the hypercomplex structure,

$$(\widehat{\vec{J}}_{\hat{\xi}})_{\hat{X}}{}^{\hat{Y}} = \widehat{\vec{J}}_{\hat{X}}{}^{\hat{Y}} + \tfrac{2}{3}\left[\widehat{\vec{J}}_{\hat{X}}{}^{\hat{Z}}(\widehat{\xi}_{\hat{Z}}k^{\hat{Y}}) - (\widehat{\xi}_{\hat{X}}k^{\hat{Z}})\widehat{\vec{J}}_{\hat{Z}}{}^{\hat{Y}}\right]. \tag{6.8}$$

The one-form with components $\widehat{\xi}_{\hat{X}}$ satisfies conditions implying, in the parametrization of (6.4), that the vector has only coordinates $\widehat{\xi}_X$. Furthermore this vector can only depend on q^X and is such that $\partial_{[X}\widehat{\xi}_{Y]}$ is Hermitian. The transformation is in this basis generated by

$$\delta(\widehat{\xi})\vec{A}_X = 2\vec{J}_X{}^Z\widehat{\xi}_Z, \tag{6.9}$$

while $\vec{k}^\alpha$ and $\vec{J}_X{}^Y$ are invariant. These transformations can always be used to obtain $\widehat{\mathcal{R}}^{\mathbb{R}} = 0$. They are represented by the horizontal lines in the upper part of Fig. 1.

Another invariance that is present in the conformal hypercomplex manifolds is a *local* SU(2), which acts as

$$\delta_{\mathrm{SU}(2)}\vec{A}_X = \partial_X\vec{\ell} + \vec{\ell}\times\vec{A}_X, \qquad \delta_{\mathrm{SU}(2)}\vec{J}_X{}^Y = \vec{\ell}\times\vec{J}_X{}^Y. \tag{6.10}$$

If the manifold admits a metric, and hence would be hyper-Kähler, it must be equal to

$$\widehat{g}_{\hat{X}\hat{Y}} = 2\widehat{\Gamma}_{\hat{X}\hat{Y}}{}^0. \tag{6.11}$$

In the parametrization that we use, the full form is

$$
\mathrm{d}\widehat{s}^2 \;=\; -\frac{(\mathrm{d}z^0)^2}{z^0} + \Big\{ z^0 h_{XY}(q)\mathrm{d}q^X \mathrm{d}q^Y
$$
$$
+ \widehat{g}_{\alpha\beta}[\mathrm{d}z^\alpha - \vec{A}_X(z,q)\cdot\vec{k}^\alpha \mathrm{d}q^X][\mathrm{d}z^\beta - \vec{A}_Y(z,q)\cdot\vec{k}^\beta \mathrm{d}q^Y] \Big\}, \quad (6.12)
$$

where

$$
\widehat{g}_{\alpha\beta} = -\frac{1}{z^0}\vec{k}_\alpha\cdot\vec{k}_\beta = -z^0 \vec{m}_\alpha\cdot\vec{m}_\beta, \qquad h_{XY} \equiv \frac{1}{z^0}\widehat{g}_{XY} + \vec{A}_X\cdot\vec{A}_Y. \quad (6.13)
$$

h_{XY} is the quantity that appears in (6.6). These formulae are very reminiscent of a Kaluza-Klein reduction on an SU(2) group manifold.

6.2 The quaternionic manifold

We have proven that the $4n_H$-dimensional subspace described by the q^X is a quaternionic space. This means that we restrict the $4(n_H+1)$ dimensional space by 4 gauge choices for the dilatations and SU(2) transformations spanned by $\vec{k}^\alpha$. In the context of superconformal tensor calculus, these are the transformations that are present in the superconformal group and should be gauge-fixed by the compensating hypermultiplet [26,47]. The gauge fixing of dilatations is done by fixing a value of z^0, which will set the scale of the manifold, see below. On the other hand, SU(2) is gauge-fixed by choosing a value for the z^α coordinates. For any value of z^α we thus find a quaternionic manifold. These are related by SU(2) transformations. This means that objects that depend on z^α are gauge-dependent. Some intrinsic quantities, like the affine connection and the metric on the small space, turn out to be z^α-independent.

The geometric quantities in the small space are not uniquely defined. In particular, we can take different choices for the affine connection and for the SU(2) connection $\vec{\omega}_X$, which is the gauge field of the transformations (6.10), restricted to the small space

$$
\delta_{\mathrm{SU}(2)}\vec{\omega}_X = -\tfrac{1}{2}\partial_X\vec{\ell} + \vec{\ell}\times\vec{\omega}_X. \quad (6.14)
$$

One particular choice is the *Oproiu connection* [27],

$$
\vec{\omega}_X^{\mathrm{Op}} \;=\; -\tfrac{1}{6}\Big(2\vec{A}_X + \vec{A}_Y\times\vec{J}_X{}^Y\Big),
$$
$$
\Gamma^{\mathrm{Op}}{}_{XY}{}^Z \;\equiv\; \Gamma^{\mathrm{Ob}}{}_{XY}{}^Z - \vec{J}_{(X}{}^Z\cdot\vec{\omega}_{Y)}^{\mathrm{Op}}, \quad (6.15)
$$

where $\Gamma^{\mathrm{Ob}}{}_{XY}{}^Z$ is the *Obata connection* on the small space. These connections can be changed by ξ-transformations [7,23,24],

$$
\Gamma_{XY}{}^Z \to \Gamma_{XY}{}^Z + 2\delta^Z_{(X}\xi_{Y)} - 2\vec{J}_{(X}{}^Z\cdot\vec{J}_{Y)}{}^W\xi_W, \qquad \vec{\omega}_X \to \vec{\omega}_X + \vec{J}_X{}^W\xi_W. \quad (6.16)
$$

These ξ-transformations are represented by the horizontal lines in the lower part of Fig. 1. As shown in that picture, they can e.g. be used to remove the $\mathbb{R}$ curvature in quaternionic manifolds. They can also be used to obtain a simple form for the connections and curvatures:

$$
\vec{\omega}_X = -\tfrac{1}{2}\vec{A}_X, \qquad \Gamma_{XY}{}^Z = \widehat{\Gamma}_{XY}{}^Z + \vec{A}_{(X}\cdot\vec{J}_{Y)}{}^Z. \quad (6.17)
$$

This choice of a quaternionic connection will turn out to be special for two different reasons. With this choice

$$\widehat{R}_{XY} = -\widehat{\mathcal{R}}_{XY}^{\mathbb{R}} = -\mathcal{R}_{XY}^{\mathbb{R}} = R_{[XY]}.\tag{6.18}$$

This equality implies that the map can be represented by vertical arrows in Fig. 1. Since in the hypercomplex space this curvature is proportional to the Ricci tensor, which is Hermitian, this implies that the $\mathbb{R}$ curvature on the small space is Hermitian. For uplifting a quaternionic manifold to a hypercomplex manifold using (6.4) one thus first has to apply ξ-transformations such that this condition is satisfied, as one can see in Fig. 1. The $SU(2)$ curvature is

$$\vec{\mathcal{R}}_{XY} = 2\vec{J}_{[X}{}^Z B_{Y]Z}, \qquad \text{where} \qquad B_{XY} = \frac{1}{4(n_H+1)}\widehat{R}_{XY} - \frac{1}{4}h_{XY}.\tag{6.19}$$

A quaternionic space is quaternionic-Kähler if and only if the Ricci tensor is Hermitian, and its symmetric part is invertible. This implies that the antisymmetric part vanishes, and the symmetric part is proportional to the metric. In the basis that we presented, the induced metric in the small space is

$$g_{XY} = z^0 h_{XY}.\tag{6.20}$$

This metric does not depend on z^α. In supergravity, z^0 is fixed to κ^{-2}, where κ is the gravitational coupling constant. The value of z^0 fixes the scale of the manifold in the sense that

$$\vec{\mathcal{R}}_{XY} = \tfrac{1}{2}\nu\vec{J}_{XY}, \qquad R = 4n_H(n_H+2)\nu, \qquad z^0 = -\frac{1}{\nu},\tag{6.21}$$

where R is the Ricci scalar. The Levi-Civita connection of the metric is the one in (6.17), and this fixes the ξ-transformations. Therefore, these manifolds can be represented by the vertical thick lines in Fig. 1.

6.3 Curvatures

The curvatures of conformal hypercomplex manifolds have as only non-vanishing components

$$\widehat{R}_{XYZ}{}^W, \qquad \widehat{R}_{XYZ}{}^0, \qquad \widehat{R}_{XYZ}{}^\alpha.\tag{6.22}$$

The latter two do not contribute in the map to the quaternionic space. The first one is expressed using vielbeins in a W-tensor, which plays an important role in supersymmetry:

$$\widehat{W}_{CDB}{}^A \equiv \tfrac{1}{2}\varepsilon^{ij}\widehat{f}_{jC}^{\hat{X}}\widehat{f}_{iD}^{\hat{Y}}\widehat{f}_{kB}^{\hat{Z}}\widehat{f}_{\hat{W}}^{kA}R_{\hat{X}\hat{Y}\hat{Z}}{}^{\hat{W}}.\tag{6.23}$$

It is symmetric in its lower indices. It is however not traceless in general. Its trace determines the Ricci tensor, and is thus zero for hyper-Kähler.

The curvatures can be split into a Ricci part and a Weyl part. The former further splits into a 'symmetric' and an 'antisymmetric' contribution,

$$\begin{aligned}
\widehat{R}_{XYZ}{}^W &= \left(\widehat{R}_{\text{antis}}^{\text{Ric}}\right)_{XYZ}{}^W + \widehat{R}^{(\text{W})}{}_{XYZ}{}^W,\\
R_{XYZ}{}^W &= \left(R_{\text{symm}}^{\text{Ric}} + R_{\text{antis}}^{\text{Ric}}\right)_{XYZ}{}^W + R^{(\text{W})}{}_{XYZ}{}^W.
\end{aligned}\tag{6.24}$$

There is no Ricci-symmetric part for the hypercomplex space.

With the special ξ-choice as in (6.17), we can relate the different parts as follows:

$$
\begin{aligned}
R^{\mathrm{Ric}}_{\mathrm{symm}\,XYZ}{}^{W} &= -\tfrac{1}{2}\delta^{W}_{[X}h_{Y]Z} + \tfrac{1}{2}\vec{J}_{Z}{}^{W} \cdot \vec{J}_{[X}{}^{V}h_{Y]V} + \tfrac{1}{2}\vec{J}_{Z}{}^{V} \cdot \vec{J}_{[X}{}^{W}h_{Y]V}, \\
R^{\mathrm{Ric}}_{\mathrm{antis}\,XYZ}{}^{W} &= \frac{n_{H}+2}{n_{H}+1}\widehat{R}^{\mathrm{Ric}}_{\mathrm{antis}\,XYZ}{}^{W}, \\
R^{(\mathrm{W})}{}_{XYZ}{}^{W} &= -\tfrac{1}{2}f^{iA}_{X}\varepsilon_{ij}f^{jB}_{Y}f^{kC}_{Z}f^{W}_{kD}\mathcal{W}_{ABC}{}^{D},
\end{aligned}
\tag{6.25}
$$

where

$$
\mathcal{W}_{ABC}{}^{D} = \widehat{W}_{ABC}{}^{D} - \frac{3}{2(n_{H}+1)}\delta^{D}_{(A}\widehat{W}_{BC)E}{}^{E}. \tag{6.26}
$$

Observe that the symmetric Ricci part is a universal expression in terms of h_{XY}. The Weyl part of quaternionic manifolds is determined by a tensor $\mathcal{W}_{ABC}{}^{D}$, which is the traceless part of the one mentioned in (6.23). The Weyl tensor of quaternionic manifolds is invariant under the ξ-transformations. For the case of hyper-Kähler and quaternionic-Kähler manifolds, the antisymmetric parts are absent, and the Weyl parts are identical.

6.4 Symmetries

Triholomorphic symmetries in the hypercomplex space are either the dilatations or they can be decomposed as

$$
\widehat{k}^{0}_{I} = 0, \qquad \widehat{k}^{\alpha}_{I} = \vec{k}^{\alpha} \cdot \vec{r}_{I}, \qquad \widehat{k}^{X}_{I} = k^{X}_{I}(q), \tag{6.27}
$$

where k^{X}_{I} is a quaternionic symmetry of the small space, such that

$$
\mathcal{L}_{k_{I}}\vec{J}_{X}{}^{Y} = \vec{r}_{I} \times \vec{J}_{X}{}^{Y}. \tag{6.28}
$$

Using the SU(2) connections, this defines moment maps $\nu\vec{P}_{I}$ by

$$
\nu\vec{P}_{I} \equiv -\tfrac{1}{2}\vec{r}_{I} - k^{X}_{I}\vec{\omega}_{X}. \tag{6.29}
$$

For hyper-Kähler manifolds there is also a moment map, which is equal to the one in the corresponding quaternionic-Kähler space.

Only ξ-transformations such that

$$
\mathcal{L}_{k_{I}}\xi_{X} = k^{Y}_{I}\partial_{Y}\xi_{X} + \xi_{Y}\partial_{X}k^{Y}_{I} = 0 \tag{6.30}
$$

preserve symmetries. The vectors in (6.27) do not change under these ξ-transformations. The moment map transforms as

$$
\nu\widetilde{\vec{P}}_{I} = \nu\vec{P}_{I} - \vec{J}_{X}{}^{Y}k^{X}_{I}\xi_{Y}. \tag{6.31}
$$

A conformal 1-dimensional hypercomplex manifold is always hyper-Kähler. In this case there is no quaternionic manifold after the map, but there are possible constant moment maps after a point z^{α} is fixed by the SU(2) gauge. This is the origin of the Fayet-Iliopoulos terms in supergravity when there are no physical hypermultiplets.

6.5 Remarks

In this paper, we have treated the case of negative scalar curvature for the quaternionic-Kähler space. This corresponds to $\nu < 0$, see (6.21). If there are continuous isometries, this implies that the space is non-compact. Supergravity restricts us to those manifolds, as is clear from the relation with κ^2. This is obtained by an indefinite signature in the hyper-Kähler space, as can be seen in (6.12). The terms for z^0 and z^α have negative signature, and we choose h_{XY} to be positive definite. However, we can as well generalize the equations to $z^0 < 0$, which then implies a positive definite metric for the hyper-Kähler manifold. The scalar curvature of the quaternionic-Kähler manifold is then positive, and this allows compact isometry groups. The only place where this is non-trivial is for vielbeins in Sect. 3.6, where $\sqrt{z^0}$ appears. However, this can be cured by inserting appropriate i factors. Furthermore, we never used properties of positive definiteness of h_{XY}, so that this choice can easily be relaxed, and the analysis applies to any signature. The signature of h and the sign of ν determine the signature of the large space.

We have developed this paper in relation to matter couplings to supergravity in 5 space-time dimensions. The procedure is, however, independent whether it is applied to 3, 4, 5 or 6 dimensional supersymmetry with 8 supercharges [19]. Therefore the analysis of this paper can be applied to superconformal tensor calculus in general.

The generalization of supersymmetric theories with hypercomplex or quaternionic target spaces is related to the idea that physical systems can be defined by field equations without necessity of an action. Indeed, the metric is only necessary for the construction of an action, while supersymmetry transformations and field equations are independent of a metric.

Quaternionic-Kähler manifolds appear as moduli spaces for type II superstring compactifications on Calabi-Yau 3-folds. It would be very interesting to understand the role of general quaternionic manifolds in the context of such compactifications.

In mathematics, the map that is described in this paper was investigated in [21, 22]. We have pointed out that the corresponding manifolds in the hypercomplex/hyper-Kähler picture are conformal, and discovered some new properties. E.g. the $\widehat{\xi}$ and SU(2) transformations in these manifolds played an important role. We hope that this stimulates further investigations in such conformal hypercomplex manifolds. Furthermore, we gave new contributions to the understanding of symmetries, generalizing isometries, in quaternionic manifolds.

Acknowledgments.

A Projections Depending on the Complex Structure

The complex structures are defined from the vielbeins as

$$\vec{J}_X{}^Y \equiv -\mathrm{i} f_X^{iA} \vec{\sigma}_i{}^j f_{jA}^Y, \tag{A.1}$$

where $\vec{\sigma}$ are the three Pauli matrices. Similar matrices are

$$L_W{}^Z{}_A{}^B \equiv f_{iA}^Z f_W^{iB}. \tag{A.2}$$

They project e.g. the curvature to the $\mathrm{G}\,\ell(r,\mathbb{H})$ factor as in (2.15). The matrices $L_A{}^B$ and $\vec{J}$ commute and their mutual trace vanishes

$$\vec{J}_X{}^Y L_Y{}^Z{}_A{}^B = L_X{}^Y{}_A{}^B \vec{J}_Y{}^Z, \qquad \vec{J}_Z{}^Y L_Y{}^Z{}_A{}^B = 0. \tag{A.3}$$

Other useful properties are

$$
\begin{aligned}
& L_X{}^Y{}_A{}^B L_Y{}^Z{}_C{}^D = L_X{}^Z{}_C{}^B \delta_A{}^D, \\
& L_X{}^X{}_A{}^B = 2\delta_A{}^B, \qquad L_X{}^Y{}_A{}^B L_Y{}^X{}_C{}^D = 2\delta_C{}^B \delta_A{}^D, \\
& L_Z{}^W{}_A{}^B L_X{}^Y{}_B{}^A = \tfrac{1}{2}\left(\delta_X{}^W \delta_Z{}^Y - \vec{J}_X{}^W \vec{J}_Z{}^Y\right).
\end{aligned} \tag{A.4}
$$

Bilinear forms are projected to Hermitian ones using

$$\Pi_{XY}{}^{ZW} \equiv \tfrac{1}{4}\left(\delta_X{}^Z \delta_Y{}^W + \vec{J}_X{}^Z \cdot \vec{J}_Y{}^W\right). \tag{A.5}$$

As a projection operator, it squares to itself:

$$\Pi_{XY}{}^{ZW} \Pi_{ZW}{}^{UV} = \Pi_{XY}{}^{UV}. \tag{A.6}$$

Useful relations are

$$
\begin{aligned}
4\vec{J}_Z{}^X \Pi_{XY}{}^{UV} &= \vec{J}_Z{}^U \delta_Y{}^V - \delta_Z{}^U \vec{J}_Y{}^V - \vec{J}_Z{}^U \times \vec{J}_Y{}^V \\
&= \vec{J}_{[Z}{}^X \Pi_{|X|Y]}{}^{(UV)} + \vec{J}_{(Z}{}^X \Pi_{|X|Y)}{}^{[UV]},
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
4\Pi_{XY}{}^{ZV} \vec{J}_V{}^U &= \delta_X{}^Z \vec{J}_Y{}^U - \delta_Y{}^U \vec{J}_X{}^Z - \vec{J}_X{}^Z \times \vec{J}_Y{}^U \\
&= \Pi_{[XY]}{}^{(Z|V|} \vec{J}_V{}^{U)} + \Pi_{(XY)}{}^{[Z|V|} \vec{J}_V{}^{U]}.
\end{aligned} \tag{A.8}
$$

A *Hermitian bilinear form* is a tensor F_{XY} such that $J^\alpha{}_X{}^Z J^\alpha{}_Y{}^W F_{ZW} = F_{XY}$ (no sum over α). Any bilinear form can be projected to the space of Hermitian bilinear forms by the projection Π. It is easy to prove that if

$$\Pi_{XY}{}^{ZW} F_{ZW} = F_{XY}, \tag{A.9}$$

this implies the hermiticity of F. Thus, Hermitian bilinear forms are those that satisfy (A.9). Moreover, one can prove that they satisfy

$$F_{XU} \vec{J}_Y{}^U = \tfrac{1}{2}\vec{J}_X{}^U \times \vec{J}_Y{}^V F_{UV}, \tag{A.10}$$

and thus also
$$F_{XU}\vec{J}_Y{}^U + \vec{J}_X{}^U F_{UY} = 0. \tag{A.11}$$

Inversely, this equation [or (A.10)] is also sufficient for a form to be Hermitian.

Finally, another useful matrix between bilinear forms is
$$S_{XY}{}^{ZW} \equiv 2\delta^Z_{(X}\delta^W_{Y)} - 2\vec{J}_X{}^{(Z} \cdot \vec{J}_Y{}^{W)} = 4\delta_{(XY)}{}^{ZW} - 8\Pi_{(XY)}{}^{ZW}. \tag{A.12}$$

It satisfies the following identities:
$$S_{WX}{}^{WV} = 4(r+1)\delta^V_X, \tag{A.13}$$
$$S_{XY}{}^{ZV}\vec{J}_V{}^U = 4\delta_{(X}{}^{(Z}\vec{J}_{Y)}{}^{U)} + 2\vec{J}_{(X}{}^Z \times \vec{J}_{Y)}{}^U, \tag{A.14}$$
$$S_{ZW}{}^{XV}\vec{J}_V{}^Y - \vec{J}_W{}^V S_{ZV}{}^{XY} = 2\vec{J}_Z{}^X \times \vec{J}_W{}^Y, \tag{A.15}$$
$$S_{XZ}{}^{YW}\vec{J}_W{}^Z = 4n_H\vec{J}_X{}^Y, \tag{A.16}$$
$$S_{T[X}{}^{(WU}S_{Y]Z}{}^{V)T} = 0, \tag{A.17}$$

which are used at various places in the main text.

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