

# Supergravity description of spacetime instantons

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## Abstract

We present and discuss BPS instanton solutions that appear in type II string theory compactifications on Calabi-Yau threefolds. From an effective action point of view these arise as finite action solutions of the Euclidean equations of motion in four-dimensional  $N = 2$  supergravity coupled to tensor multiplets. As a solution generating technique we make use of the c-map, which produces instanton solutions from either Euclidean black holes or from Taub-NUT like geometries.

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## 1 Introduction

Black holes in superstring theory have both a macroscopic and microscopic description. On the macroscopic side, they can be described as solitonic solutions of the effective supergravity Lagrangian. Microscopically they can typically be constructed by wrapping  $p$ -branes over  $p$ -dimensional cycles in the manifold that the string theory is compactified on. The microscopic interpretation is best understood for BPS black holes.

Apart from this solitonic sector, string theory also contains instantons. Microscopically they arise as wrapped Euclidean  $p$ -branes over  $p + 1$ -dimensional cycles of the internal manifold. The aim of this paper is to present a macroscopic picture of these instantons as solutions of the Euclidean equations of motion in the effective supergravity Lagrangian. We focus hereby on spacetime instantons, whose effects are inversely proportional to the string coupling constant  $g_s$ . The models that we will study are type II string theories compactified on a Calabi-Yau (CY) threefold. The resulting effective action is  $N = 2, D = 4$

supergravity coupled to vector multiplets and tensor multiplets. The latter can be dualized to hypermultiplets, and the geometry of the hypermultiplet moduli space - containing the dilaton - is known to receive quantum corrections, both from string loops [1] and from instantons [2]. The instanton corrections are exponentially suppressed and are difficult to compute directly in string theory. Our results yields some progress in this direction, since within the supergravity description one finds explicit formulae for the instanton action.<sup>1</sup> Related work can also be found in [4, 5], but our results are somewhat different and contain several new extensions.

Interestingly, there is a relation between black hole solutions in type IIA/B and instanton solutions in type IIB/A. Microscopically, this can be understood from T-duality between IIA and IIB. Macroscopically, this follows from the c-map [6, 7], as we will show explicitly. This defines a map between vector and tensor multiplets and as a consequence, (BPS) solutions of the vector multiplet Lagrangian are mapped to (BPS) solutions of the tensor- or hypermultiplet Lagrangian. We will use this mapping in Euclidean spacetimes. Roughly speaking, there are two classes of solutions on the vector multiplet sector: (Euclidean) black holes and Taub-NUT like solutions. These map to D-brane instantons and NS-fivebrane instantons respectively. The distinguishing feature is that the corresponding instanton actions are inversely proportional to  $g_s$  or  $g_s^2$  respectively. For both type of instantons, we give the explicit solution and the precise value of the instanton action.

The D-brane instantons are found to be the solutions to the equations obtained from c-mapping the BPS equations of [8]. Their analysis contains also  $R^2$  interactions, but they can be easily switched off. The BPS equations then obtained are similar, but not identical to the equations derived in [9]. In the derivation and description of D-brane instantons we find it convenient to make the symplectic structure of the theory and its equations manifest. The NS-fivebrane instantons are derived in a different way, not by using the c-map. This is because the BPS solutions in Euclidean supergravity coupled to vector multiplets are not fully classified. We therefore construct the NS-fivebrane instantons by extending the Bogomol'nyi-bound-formulation of [10].

Ultimately, we hope to get a better understanding of non-perturbative string theory. In particular, it is expected that instanton effects resolve conifold-like singularities in the hypermultiplet moduli space of Calabi-Yau compactifications, see e.g. [11]. These singularities are closely related - by the c-map - to the conifold singularities in the vector multiplet moduli space due to the appearance of massless black holes [12]. Moreover, in combination with the more recent relation between black holes and topological strings [13], it would

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<sup>1</sup>Instanton actions can also be studied from worldvolume theories of D-branes. For a discussion on this in the context of our paper, we refer to [3]. It would be interesting to find the precise relation to our analysis.

be interesting to study if topological string theory captures some of the non-perturbative structure of the hypermultiplet moduli space. For some hints in this direction, see [14]. Finally, we remark that instantons play an important role in the stabilization of moduli. For an example related to our discussion, we refer to [15].

This paper is organized as follows: In section 2 we treat NS-fivebrane instantons in the context of  $N = 1$  supergravity. We use this simple setup to introduce various concepts, e.g. the c-map, which we use in later sections. Section 3 is devoted to a review of instanton solutions in the universal hypermultiplet of  $N = 2$  supergravity and their relation to gravitational solutions of pure  $N = 2$  supergravity. Then in section 4 we consider instanton solutions to the theory obtained from arbitrary CY compactification of type II superstrings. Some technical details are provided in appendices at the end of this paper, including a treatment of electric-magnetic duality in tensor multiplet Lagrangians.

## 2 NS-fivebrane instantons

In this section, we give the  $N = 1$  supergravity description of the NS-fivebrane instanton. The main characteristic of this instanton is that the instanton action is inversely proportional to the square of the string coupling constant. In string theory, such instantons appear when Euclidean NS-fivebranes wrap six-cycles in the internal space, and therefore are completely localized in both space and (Euclidean) time.

It is well known that Euclidean NS-fivebranes in string theory are T-dual to Taub-NUT or more generally, ALF geometries [11] (see also [16]). We here re-derive these results from the perspective of four-dimensional (super-) gravity in a way that allows us to introduce the c-map conveniently.

### 2.1 A Bogomol'nyi bound

We start with a simple system of gravity coupled to a scalar and tensor in four spacetime dimensions,

$$\mathcal{L}^m = -\frac{1}{2\kappa^2}eR(e) + \frac{1}{2}|d\phi|^2 + \frac{1}{2}e^{2\phi}|H|^2, \quad (2.1)$$

with

$$H = dB. \quad (2.2)$$

We use form notation for the matter fields; see Appendix (A.1) for our conventions.

This model appears as a sub-sector of  $N = 1$  low-energy effective actions in which gravity is coupled to  $N = 1$  tensor multiplets. In our case we have one tensor multiplet that consists of the dilaton  $\phi$  and the NS two-form  $B$ . In four dimensions, a tensor can be dualized

into a scalar, such that only chiral multiplets appear. We will not do this dualization for reasons that become clear below.

The instanton solution can be found by deriving a Bogomol'nyi bound on the Euclidean Lagrangian [17],

$$\mathcal{L}^e = \frac{1}{2} |e^\phi * H \mp e^\phi de^{-\phi}|^2 \mp d(e^\phi H) . \quad (2.3)$$

Here, we have left out the Einstein-Hilbert term. It is well known that this term is not positive definite, preventing us to derive a Bogomol'nyi bound including gravity. In most cases, our instanton solutions are purely in the matter sector, and spacetime will be taken flat. The Bogomol'nyi equation then is

$$*H = \pm de^{-\phi} . \quad (2.4)$$

This implies that  $e^{-\phi}$  should be a harmonic function. The  $\pm$  solutions refer to instantons or anti-instantons. Notice that the surface term in (2.3) is topological in the sense that it is independent on the spacetime metric. It is easy to check that the BPS configurations (2.4) have vanishing energy momentum tensor, so that the Einstein equations are satisfied for any Ricci-flat metric.

One can now easily evaluate the instanton action on this solution. The only contribution comes from the surface term in (2.3). Defining the instanton charge as

$$\int_{S^3} H = Q , \quad (2.5)$$

with  $H$  the three-form field strength, we find<sup>2</sup>

$$S_{\text{inst}} = \frac{|Q|}{g_s^2} . \quad (2.6)$$

Here we have assumed that there is only a contribution from infinity, and not from a possible other boundary around the location of the instanton. It is easy to see this when spacetime is taken to be flat. In that case the single-centered solution for the dilaton is

$$e^{-\phi} = e^{-\phi_\infty} + \frac{|Q|}{4\pi^2 r^2} , \quad (2.7)$$

which is the standard harmonic function in flat space with the origin removed. We have furthermore related the string coupling constant to the asymptotic value of the dilaton by

$$g_s \equiv e^{-\phi_\infty/2} . \quad (2.8)$$

In our notation, this is the standard convention.

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<sup>2</sup>In the tensor multiplet formulation, the instanton action has no imaginary theta-angle-like terms. They are produced after dualizing the tensor into an axionic scalar, by properly taking into account the constant mode of the axion. In the context of NS-fivebrane instantons, this was explained e.g. in [18].

## 2.2 T-duality and the c-map

We will now re-derive the results of the previous subsection using the c-map. Though no new results, it will enable us to set the notation and to prepare for more complicated situations discussed in the next sections.

To perform the c-map, we dimensionally reduce the action (2.1) and assume that all the fields are independent of one coordinate. This can most conveniently be done by first choosing an upper triangular form of the vierbein, in coordinates  $(x^m, x^3 \equiv \tau)$ ,  $m = 0, 1, 2$ ,

$$e_\mu^a = \begin{pmatrix} e^{-\tilde{\phi}/2} \hat{e}_m^i & e^{\tilde{\phi}/2} \tilde{B}_m \\ 0 & e^{\tilde{\phi}/2} \end{pmatrix}. \quad (2.9)$$

The metric then takes the form

$$ds^2 = e^{\tilde{\phi}}(d\tau + \tilde{B})^2 + e^{-\tilde{\phi}} \hat{g}_{mn} dx^m dx^n, \quad (2.10)$$

and we take  $\tilde{\phi}$ ,  $\tilde{B}_m$  and  $\hat{g}_{mn}$  to be independent of  $\tau$ . For the moment, we take  $\tau$  to be one of the spatial coordinates, but at the end of this section, we will apply our results to the case when  $\tau$  is the Euclidean time. In our example, the Wick rotation is straightforward on the scalar-tensor sector.

We have that  $e = e^{-\tilde{\phi}} \hat{e}$ , and the scalar curvature decomposes as (ignoring terms that lead to total derivatives in the Lagrangian)

$$-eR(e) = -\hat{e}R(\hat{e}) + \frac{1}{2}|d\tilde{\phi}|^2 + \frac{1}{2}e^{2\tilde{\phi}}|\tilde{H}|^2, \quad (2.11)$$

Similarly, we require the dilaton and 2-form to be independent of  $\tau$ . The three-dimensional Lagrangian then is <sup>3</sup>

$$\mathcal{L}_3^m = -\hat{e}R(\hat{e}) + \frac{1}{2}|d\tilde{\phi}|^2 + \frac{1}{2}e^{2\tilde{\phi}}|\tilde{H}|^2 + \frac{1}{2}|d\phi|^2 + \frac{1}{2}e^{2\phi}|H|^2, \quad (2.12)$$

where  $H$  is the two-form field strength descending from the three-form  $H$  in four dimensions (see also Appendix (A.2)), so we have again that  $H = dB$  in three dimensions, where  $B$  is a one-form.

In addition, there is an extra term in the Lagrangian,

$$\mathcal{L}_3^{aux} = -\frac{1}{2}e^{2(\phi+\tilde{\phi})}|\mathbf{H} - \tilde{B} \wedge H|^2, \quad (2.13)$$

which plays no role in the three-dimensional theory. Here  $\mathbf{H} \equiv d\mathbf{B}$  is the three-form arising from the spatial component of the four-dimensional  $H$ . Being a three-form in three dimensions it is an auxiliary field. This term can therefore trivially be eliminated by its own field equation.

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<sup>3</sup>For convenience of normalization, we set  $\kappa^{-2} = 2$ .

Note that the Lagrangian  $\mathcal{L}_3$  has the symmetry

$$\phi \longleftrightarrow \tilde{\phi}, \quad B \longleftrightarrow \tilde{B}. \quad (2.14)$$

In fact, careful analysis shows that also  $\mathcal{L}_3^{aux}$  is invariant, provided we transform

$$\mathbf{B} \rightarrow \tilde{\mathbf{B}} \equiv \mathbf{B} - \frac{1}{2}\tilde{B} \wedge B. \quad (2.15)$$

The transformations in (2.14) and (2.15) define the c-map. The resulting theory can now be reinterpreted as a dimensional reduction of a four-dimensional theory of gravity coupled to a scalar  $\tilde{\phi}$  and tensor  $\tilde{B}$  obtained from the c-map, and vierbein

$$\tilde{e}_\mu^a = \begin{pmatrix} e^{-\phi/2}\tilde{e}_m^i & e^{\phi/2}B_m \\ 0 & e^{\phi/2} \end{pmatrix}, \quad (2.16)$$

where  $\phi$  and  $B$  are the original fields in (2.1) before the c-map. Our symmetry is related to the Buscher rules for T-duality [19]. We here derived these rules from an effective action approach in Einstein frame, similar to [20].

One can apply the c-map to solutions of the equations of motion. Given a ( $\tau$ -independent) solution  $\{e_\mu^a, \phi, B_{\mu\nu}\}$ , one can construct another solution after the c-map, given by  $\{\tilde{e}_\mu^a, \tilde{\phi}, \tilde{B}_{\mu\nu}\}$  as described above. This procedure can be done both in Minkowski and in Euclidean space. In the latter case, we can take the coordinate  $\tau$  to be the Euclidean time, as time-independent solutions can easily be Wick rotated. This is precisely the situation we are interested in. To be more precise, we first formulate the Euclidean four-dimensional theory based on Euclidean metrics coupled to a scalar and tensor. The dimensional reduction is still based on the decomposition of the vierbein (2.9) with  $\tau$  the Euclidean time. After dimensional reduction over  $\tau$ , the Einstein-Hilbert term now gives

$$eR(e) = \hat{e}R(\hat{e}) + \frac{1}{2}|d\tilde{\phi}|^2 + \frac{1}{2}e^{2\tilde{\phi}}|\tilde{H}|^2, \quad (2.17)$$

such that the symmetry (2.14) still holds.

### 2.3 Taub-NUT geometries and NS-fivebrane instantons

To generate instanton solutions, we will start from a time independent solution of pure Einstein gravity, and perform the c-map. This uplifts to a new solution in four dimensions with generically nontrivial scalar and tensor. In other words, we do a T-duality over Euclidean time. This of course only makes sense as a solution-generating-technique. However, such a solution is not an instanton, since it is not localized in  $\tau$ . We therefore have to uplift the solution to a  $\tau$ -dependent solution in four dimensions. This is easy if the original

solution is in terms of harmonic functions. In that case there is a natural uplifting scheme, which involves going from three- to four-dimensional harmonic functions.

We discuss now examples in the class of gravitational instantons [21]. These are vacuum solutions of the Euclidean Einstein equation, based on a three-dimensional harmonic function  $V(\vec{x})$ ,

$$ds^2 = V^{-1}(d\tau + A)^2 + Vd\vec{x} \cdot d\vec{x} . \quad (2.18)$$

Here  $A$  is a one-form in three dimensions satisfying  $*dA = \pm dV$ . The  $\pm$  solutions yields selfdual or anti-selfdual Riemann curvatures. In the notation of (2.10), we have that  $A = \tilde{B}$ ,  $\hat{e}_m^i = \delta_m^i$  and  $e^{-\tilde{\phi}} = V$ .

Multi-centered gravitational instantons correspond to harmonic functions of the form

$$V = V_0 + \sum_i \frac{m_i}{|\vec{x} - \vec{x}_i|} , \quad (2.19)$$

for some parameters  $V_0$  and  $m_i$ . For non-zero  $V_0$ , one can further rescale  $\tau = V_0\tilde{\tau}$  and  $m_i = 4V_0\tilde{m}_i$  such that one can effectively set  $V_0 = 1$ . The single-centered case corresponds to Taub-NUT geometries, or orbifolds thereof. For  $V_0 = 0$  one obtains smooth resolutions of ALE spaces (like e.g. the Eguchi-Hanson metric for the two-centered solution). For more details, we refer to [22].

Before the c-map, the dilaton and two-form are taken to be zero. The three-dimensional  $\tau$ -independent solution after the c-map is

$$e^\phi = V^{-1} , \quad *H = \pm dV , \quad \mathbf{H} = 0 , \quad (2.20)$$

and the metric is flat,  $g_{mn} = \delta_{mn}$ .

We now construct a four-dimensional  $\tau$ -dependent solution by taking  $V$  a harmonic function in four dimensions. We take the four-dimensional metric to be flat and  $H$  is still determined by  $*H = \pm dV$ , but now as a three-form field strength.

That this is still a solution for (2.1) can directly be seen from the fact that the Bogomol'nyi equations (2.4) are satisfied. The instanton action is again given by (2.6). Notice further that the difference between instantons and anti-instantons for the fivebrane corresponds to selfdual and anti-selfdual gravitational instantons.

Due to our procedure, we are making certain aspects of T-duality not explicit. We have for instance suppressed any dependence on the radius of the compactified circle parameterized by  $\tau$ . These aspects become important in order to dynamically realize the uplifting solution in terms of a decompactification limit after T-duality. It turns out that a proper T-duality of the Taub-NUT geometry, including world-sheet instanton corrections, produces a completely localized NS-fivebrane instanton based on the four-dimensional harmonic function given above. For more details, we refer to [16].

### 3 Membrane and fivebrane instantons

In the previous section, we have discussed aspects of NS-fivebrane instantons. Here, we will elaborate further on this, and also introduce membrane instantons. These appear in M-theory or type IIA string theory compactifications, and we will be interested in four-dimensional effective theories with eight supercharges such as IIA strings compactified on Calabi-Yau manifolds. The main distinction with the previous section is the presence of RR fields, and these will play an important role in this section.

General CY compactifications of type IIA strings yield  $N = 2$  supergravity theories coupled to  $h_{1,1}$  vector multiplets and  $h_{1,2} + 1$  hypermultiplets (or tensor multiplets), but in this section we will restrict ourself to the case of the universal hypermultiplet only, leaving the general case for the next section. This situation occurs when the CY space is rigid, i.e. when  $h_{1,2} = 0$ . Then there are only two three-cycles in the CY, around which the Euclidean membranes can wrap. These are the membrane instantons, and in this section we give their supergravity description. The  $h_{1,1}$  vector multiplet fields can be truncated in our setup; it suffices to have pure supergravity coupled to the universal hypermultiplet.

#### 3.1 Instantons in the double-tensor multiplet

We will describe the universal hypermultiplet in the double-tensor formulation, as this is what we get from the c-map. In this formulation it contains two tensors and two scalars, which can be thought of as two  $N = 1$  tensor multiplets coming from the NS-NS and RR sectors. Instantons in the double-tensor multiplet were already discussed in [10, 23] (see also [24]), and in the context of the c-map in [5]. In this section, we reproduce these results and extend the c-map to include also NS-fivebrane instantons.

The (Minkowskian) Lagrangian for the double-tensor multiplet can be written as [10, 25]

$$\mathcal{L}^m = -d^4x e R + \frac{1}{2}|F|^2 + \frac{1}{2}|d\phi|^2 + \frac{1}{2}e^{-\phi}|d\chi|^2 + \frac{1}{2}M_{ab} *H^a \wedge H^b . \quad (3.1)$$

The  $N = 2$  pure supergravity sector contains the metric and the graviphoton field strength  $F$ , whereas the matter sector contains two scalars and a doublet of 3-forms  $H^a = dB^a$ . The self-interactions in the double-tensor multiplet are encoded in the matrix

$$M(\phi, \chi) = e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & e^\phi + \chi^2 \end{pmatrix} . \quad (3.2)$$

From a string theory point of view, the metric,  $\phi$  and  $H^2$  come from the NS sector, while the graviphoton,  $\chi$  and  $H^1$  descend from the RR sector in type IIA strings. Notice that when we truncate to the NS sector, we obtain the Lagrangian (2.1), so the results obtained there are still valid here.



As we are interested in instanton solutions, we consider the Euclidean version of (3.1), which can be obtained by doing a standard Wick rotation  $t \rightarrow \tau = it$  and using Euclidean metrics. The form of the Lagrangian is still given by (3.1), but now the matter Lagrangian is positive definite. In [10] and [23], Bogomol'nyi equations were derived and solved for the double-tensor multiplet coupled to pure  $N = 2$  supergravity with vanishing graviphoton field strength and Ricci tensor. The solutions of these equations preserving half of the supersymmetry can be recasted into the following compact form:

$$\begin{aligned}
g_{\mu\nu} &= \delta_{\mu\nu} , \\
e^{-\phi} &= \frac{1}{4}(h^2 - p^2) , \\
*H^2 &= \frac{1}{2}(hdp - pdh) , \\
\chi &= -e^{\phi}p + \chi_c , \\
*(H^1 - \chi_c H^2) &= -dh ,
\end{aligned} \tag{3.3}$$

with  $h$  and  $p$  four-dimensional harmonic functions (satisfying  $|h| \geq |p|$ ) and  $\chi_c$  an arbitrary constant. The cases where  $h$  is negative or positive correspond to instantons or anti-instantons respectively. We have written here a flat metric  $g_{\mu\nu}$ , but it is easy to generalize this to any Ricci flat metric, as long as it admits harmonic functions. Non-trivial  $h$  and  $p$  can be obtained when one or more points are taken out of four-dimensional flat space. The solution is then of the form

$$h = h_{\infty} + \frac{Q_h}{4\pi^2|\vec{x} - \vec{x}_0|^2} , \quad p = p_{\infty} + \frac{Q_p}{4\pi^2|\vec{x} - \vec{x}_0|^2} , \tag{3.4}$$

or multi-centered versions thereof. It can be easily seen that a pole in  $p$  corresponds to a source with (electric) charge in the field equation of  $\chi$ . Similarly a pole in  $h$  corresponds to a source with (magnetic) charge in the Bianchi identity of  $H^1 - \chi_c H^2$ . For single-centered solutions, there are five independent parameters, two for each harmonic function, together with  $\chi_c$ .

### NS-fivebrane instantons with RR background fields

The general solution in (3.3) falls into two classes, depending on the asymptotic behavior of the dilaton at the origin. The first class fits into the category of NS-fivebrane instantons. The solution is characterized by

$$p = \pm(h - \alpha) , \tag{3.5}$$

with  $\alpha$  an arbitrary constant. In terms of (3.4), this condition is equivalent to

$$Q_h = \pm Q_p , \tag{3.6}$$

such that the solution only has four independent parameters. This implies that the dilaton behaves at the origin like

$$e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{r^2}\right). \quad (3.7)$$

The condition (3.5) implies that

$$*H^2 = \pm de^{-\phi}, \quad *H^1 = \pm d(e^{-\phi}\chi), \quad (3.8)$$

with  $e^{-\phi}$  the harmonic function

$$e^{-\phi} = \frac{1}{2}\alpha h - \frac{1}{4}\alpha^2, \quad (3.9)$$

and  $\chi$  is fixed in terms of  $h$  via (3.3). In this form, we get back the results of [10] and [23]. The prototype example for  $e^{-\phi}$  is of the form

$$e^{-\phi} = g_s^2 + \sum_i \frac{|Q_i|}{4\pi^2 |\vec{x} - \vec{x}_i|^2}. \quad (3.10)$$

It is easy to check that, whereas the dilaton diverges, the RR field  $\chi$  remains finite at the excised points. In fact, the BPS equations (3.3) require the values of  $\chi$  at the excised points  $\vec{x}_i$  all to be equal [18], and we denote this value by  $\chi_0$ .

The solution is characterized by the parameters  $\alpha, \chi_c, g_s$  and the charges

$$Q \equiv \int_{S_\infty^3} H^2 = \mp \sum_i |Q_i|. \quad (3.11)$$

The parameters  $\alpha$  and  $\chi_c$  can be traded for the boundary values of the RR field  $\chi_\infty$  and  $\chi_0$ . The action of the multi-centered instanton was calculated in [10, 23], and the result is

$$S_{\text{inst}} = |Q| \left( \frac{1}{g_s^2} + \frac{1}{2} (\Delta\chi)^2 \right), \quad (3.12)$$

with  $\Delta\chi \equiv \chi_\infty - \chi_0$ .

The solution above describes a generalization of the NS-fivebrane instanton discussed in section 2. Notice that the first term in the instanton action is inversely proportional to the square of the string coupling constant, as is common for NS-fivebrane instantons. The second term is the contribution from the RR background field. Only for constant  $\chi$  does one obtain a local minimum of the action <sup>4</sup>.

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<sup>4</sup>Solutions with constant  $\chi$  can be obtained from (3.3) and (3.5) by taking the limit  $\alpha \rightarrow 0$  while both  $h_\infty$  and  $Q_h \rightarrow \infty$  in such a way that  $\alpha h$  is kept fixed. Such solutions follow more directly from the Bogomol'nyi equations considered in [23].

## Membrane instantons

The remaining solutions, other than (3.6), are given by (3.3) with  $Q_h \neq Q_p$ . One can see that the asymptotic behavior of the dilaton around the origin is now

$$e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{r^4}\right). \quad (3.13)$$

Compared to the fivebrane instanton case, this behavior is more singular. However, the instanton action is still finite. As was shown in [10], the action reduces to a surface term, and the only contribution comes from infinity. One way of writing the instanton action is [23]

$$S_{\text{inst}} = \sqrt{\frac{4}{g_s^2} + (\Delta\chi)^2} \left( |Q_h| \pm \frac{1}{2} \Delta\chi Q \right), \quad (3.14)$$

with the same convention as for fivebranes, i.e.

$$\Delta\chi \equiv \chi_\infty - \chi_0 = -\frac{p_\infty}{g_s^2}, \quad (3.15)$$

and  $Q$  still defined by

$$Q \equiv \int_{S_\infty^3} H^2 = -\frac{1}{2} (h_\infty Q_p - p_\infty Q_h). \quad (3.16)$$

The plus and minus sign in (3.14) refer to instanton and anti-instanton respectively. Using the relations given above and  $g_s^2 = \frac{1}{4}(h_\infty^2 - p_\infty^2)$  one can show that (3.14) is always positive, as it should be.

Notice that the instanton action contains both the fivebrane charge  $Q$  and  $Q_h$ , which we identify with a membrane charge. For pure membrane instantons, which have vanishing NS-NS field, the second term in (3.14) vanishes. When we put  $H^2$  (and its BPS equation) to zero from the start, we can dualize  $\chi$  to a tensor and obtain a "tensor-tensor" theory. To perform this dualization we have to replace  $d\chi$  by the one-form  $D$  in the Euclideanized ( $H^2$ -less) version of (3.1) and add a Lagrange multiplier term

$$\mathcal{L}^e(\chi) \longrightarrow \mathcal{L}^e(D) + 2iB_\chi \wedge dD, \quad (3.17)$$

where  $B_\chi$  is a two-form. Integrating out  $B_\chi$  enforces  $dD = 0$  and locally  $D = d\chi$  again. Subtracting the total derivative  $2id(B_\chi \wedge D)$  and integrating out  $D$  yields the tensor-tensor theory. Using this action to evaluate the pure membrane instantons on gives

$$\begin{aligned} S'_{\text{inst}} &= S_{\text{inst}} + \Delta\chi Q_p \\ &= \frac{2}{g_s} \sqrt{Q_h^2 - Q_p^2}. \end{aligned} \quad (3.18)$$

The appearance of the second term in the first line is a result of the subtraction of the boundary term in the dualization procedure. In going from the first to the second line we used the fact that  $Q = 0$ , which allowed us to express  $\Delta\chi$  in terms of the charges  $Q_h$  and  $Q_p$  and  $g_s$ ,

$$\Delta\chi = -\frac{2}{g_s} \frac{Q_p}{\sqrt{Q_h^2 - Q_p^2}} . \quad (3.19)$$

The  $\frac{1}{g_s}$  dependence in the instanton action is typical for D-brane instantons that arise after wrapping Euclidean D-branes over supersymmetric cycles in the Calabi-Yau [2].

The microscopic interpretation of the general solution is not so clear. In the next subsection, we will see how these solutions are generated from the c-map. In this way, one can give a natural interpretation in terms of black holes and gravitational instantons.

The form of the instanton action for both fivebrane (3.12) and membrane instantons (3.14) and (3.18) was recently re-derived by solving the constraints from supersymmetry of the effective action [26]. This provides an alternative derivation of the formulas in this section and confirms that the supergravity method for computing the instanton action is correct.

### 3.2 Einstein-Maxwell theory and the c-map

In this subsection, we show that our membrane and fivebrane instanton solutions naturally follow from the c-map. To show this, we start again with the Lagrangian (3.1) for Einstein-Maxwell theory coupled to the double-tensor multiplet. After Wick rotating to Euclidean space, we perform a dimensional reduction over  $\tau = it$ . For the Minkowski theory, this was done in detail in [7] and can be easily repeated for Euclidean signatures using (2.17). We decompose the metric as in (2.9); this yields a three-dimensional metric, a one-form  $\tilde{B}$  and a scalar  $\tilde{\phi}$ . The one-form gauge potential decomposes in the standard way

$$A = (-\tilde{\chi}, \tilde{A} - \tilde{\chi}\tilde{B}) , \quad (3.20)$$

where  $\tilde{A}$  and  $\tilde{B}$  are one-forms in three dimensions, and  $\tilde{\chi}$  is the  $\tau$  component of the four-dimensional gauge field. More precisely, we have  $A = -\tilde{\chi}d\tau + (\tilde{A} - \tilde{\chi}\tilde{B})$ .

The result after dimensional reduction is <sup>5</sup>

$$\begin{aligned} \mathcal{L}_3^e = & d^3x \hat{e}R(\hat{e}) + \frac{1}{2}|d\phi|^2 + \frac{1}{2}e^{-\phi}|d\chi|^2 + \frac{1}{2}M_{ab}(\phi, \chi) *H^a \wedge H^b \\ & + \frac{1}{2}|d\tilde{\phi}|^2 + \frac{1}{2}e^{-\tilde{\phi}}|d\tilde{\chi}|^2 + \frac{1}{2}M_{ab}(\tilde{\phi}, \tilde{\chi}) *\tilde{H}^a \wedge \tilde{H}^b . \end{aligned} \quad (3.21)$$

Here we have combined the two one-forms in a doublet  $\tilde{B}^a = (\tilde{A}, \tilde{B})$  that define the (dual) two-form field strengths  $\tilde{H}^a$  in three dimensions. The matrix multiplying their

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<sup>5</sup>We are suppressing here terms like (2.13), which are irrelevant for our purpose.

kinetic energy is exactly the same as in (3.2), but now with the tilde-fields. Therefore, the Lagrangian has the symmetry

$$\phi \longleftrightarrow \tilde{\phi}, \quad \chi \longleftrightarrow \tilde{\chi}, \quad H^a \longleftrightarrow \tilde{H}^a. \quad (3.22)$$

Notice that, as observed before, setting the RR fields  $\chi$  and  $H^1$  to zero reduces to the results of the previous section.

It is now clear that one can use the same technique as in the previous section, namely to generate the instanton solutions from the solutions of the Einstein-Maxwell Lagrangian, or vice versa. The symmetry transformations (3.22) basically interchange the gravitational and the double-tensor multiplet degrees of freedom sectors of (3.21).

Since the general BPS instanton solution was given in (3.3), it is easy to translate this back to stationary BPS solutions of pure  $N = 2$  supergravity, after replacing four-dimensional by three-dimensional harmonic functions. Starting from (3.3), one obtains solutions to the equations of motion in Euclidean space. Stationary solutions of the Einstein-Maxwell Lagrangian can however easily be continued from Minkowski to Euclidean space, and vice versa. If we make the following decomposition for the metric and graviphoton vector field in Minkowski space

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -e^{\tilde{\phi}} (dt + \omega)^2 + e^{-\tilde{\phi}} \hat{g}_{mn} dx^m dx^n, \\ A &= (-\tilde{\chi}', \tilde{A} - \tilde{\chi}' \omega), \end{aligned} \quad (3.23)$$

then we can analytically continue to Euclidean space by identifying:

$$\omega = -i\tilde{B}, \quad \tilde{\chi}' = i\tilde{\chi}. \quad (3.24)$$

BPS solutions of pure  $N = 2$  supergravity were studied in [27], [28] and [9]. We are interested in stationary solutions only, as non-stationary solutions cannot be used in the c-map. Here and in the next section we will use the notation and results of the analysis of [8]. In the case of pure  $N = 2$  supergravity the BPS solutions, in terms of the variables of (3.23), read:

$$\begin{aligned} \hat{g}_{mn} &= \delta_{mn}, \\ e^{-\tilde{\phi}} &= \frac{1}{4}(h^2 + q^2), \\ *d\omega &= -\frac{1}{2}(hdq - qdh), \\ \tilde{\chi}' &= -e^{\tilde{\phi}} q + \chi'_c, \\ *(\tilde{H}^1 - \chi'_c d\omega) &= -dh, \end{aligned} \quad (3.25)$$

with  $h$  and  $q$  three-dimensional flat space harmonic functions,  $\tilde{H}^1 = d\tilde{A}$  and  $\chi'_c$  an arbitrary constant. The line element falls into the general class of Israel-Wilson-Perjés (IWP) metrics [29, 30],

$$ds^2 = -|U|^{-2}(dt + \omega)^2 + |U|^2 d\vec{x} \cdot d\vec{x} , \quad (3.26)$$

where  $U$  is any complex solution to the three dimensional Laplace equation. Comparing to (3.25) and (3.23), we have that

$$U = \frac{1}{2}(h + iq) . \quad (3.27)$$

Let  $F$  be the two-form field strength of the four-dimensional gauge field and  $G$  is its dual,

$$G_{\mu\nu} \equiv \frac{1}{2} \frac{\delta \mathcal{L}_4^m}{\delta F^{\mu\nu}} = \frac{1}{2} (*F)_{\mu\nu} . \quad (3.28)$$

Then we denote by  $F_3$  and  $G_3$  the corresponding three-dimensional one-forms

$$F_3 = d\tilde{\chi}' , \quad G_3 = \frac{1}{2} e^{\tilde{\phi}} * (\omega \wedge F_3 - \mathbf{F}) , \quad (3.29)$$

where  $\mathbf{F}$  is the two-form arising from the spatial components of the four-dimensional  $F$ . To derive the second equation in (3.29) one needs to decompose the component of (3.28) with a time-index,  $G_{tm} = \frac{1}{4} \varepsilon_{tmnl} g^{n\mu} g^{l\nu} F_{\mu\nu}$ , using the metric parameterization (3.23) (see Appendix (A.2) for our notations and conventions). The last two equations in (3.25) can now elegantly be rewritten as:

$$F_3 = -d(e^{\tilde{\phi}} q) , \quad G_3 = -\frac{1}{2} d(e^{\tilde{\phi}} h) . \quad (3.30)$$

In fact in [8] solutions were given in terms of these objects. Note that as  $F$  and  $G$  are electric-magnetic pairs, so are  $F_3$  and  $G_3$ . This will become important in the next section. It is now clear that this set of BPS Einstein-Maxwell solutions yields the same solutions as (3.3), after analytic continuation to the Euclidean fields given in (3.24), which amounts to setting  $p \equiv -iq$ . The class of IWP metrics contains many interesting examples, some of which we discuss now.

## Pure membrane instantons and black holes

We consider here solutions to (3.3) with vanishing NS-NS two-form:

$$H^2 = 0 . \quad (3.31)$$

These were the solutions that lead to the pure membrane instantons. The vanishing of  $H^2$  implies that the two harmonic functions  $h$  and  $p$  are proportional to each other,

$$p = ch , \quad (3.32)$$

for some real constant  $c$ . We take  $h$  of the form:

$$h = h_\infty + \sum_i \frac{Q_{h,i}}{4\pi^2 |\vec{x} - \vec{x}_i|^2}, \quad Q_{p,i} = c Q_{h,i}. \quad (3.33)$$

This membrane instanton is in the image of the c-map. The dual (Minkowskian) gravitational solution is static:

$$d\omega = 0, \quad (3.34)$$

and has  $q = c' h$ . The IWP metric now becomes of the Majumdar-Papapetrou type. These are multi-centered versions of the extreme Reissner-Nordström black hole. Our solutions describe the outer horizon part of spacetime in isotropic coordinates,

$$ds^2 = - \left( \gamma + \sum_i \frac{M_i}{4\pi |\vec{x} - \vec{x}_i|} \right)^{-2} dt^2 + \left( \gamma + \sum_i \frac{M_i}{4\pi |\vec{x} - \vec{x}_i|} \right)^2 (dr^2 + r^2 d\Omega^2), \quad (3.35)$$

with

$$M_i = \frac{1}{2} \sqrt{(Q_{h,i})^2 + (Q_{q,i})^2}, \quad \gamma = \frac{1}{2} \sqrt{1 + c'^2} h_\infty, \quad (3.36)$$

and  $Q_{q,i} = c' Q_{h,i}$  for each charge labeled by  $i$ . Note that in the parameterization (3.35) the event horizons are located at  $\vec{x} = \vec{x}_i$ . The metric can be made asymptotically Minkowski by a rescaling of the coordinates

$$t = \gamma t', \quad r = \frac{r'}{\gamma}. \quad (3.37)$$

### NS-fivebrane instantons and Taub-NUT with selfdual graviphoton

Here we consider the NS-fivebrane instantons with RR background fields. This solution was specified by equations (3.5), (3.8) and (3.9). Using the inverse c-map, we can relate it to a BPS solution of pure N=2 supergravity, based on the three-dimensional harmonic function:

$$e^{-\tilde{\phi}} = V \equiv v + \sum_i \frac{Q_i}{4\pi^2 |\vec{x} - \vec{x}_i|^2}. \quad (3.38)$$

The metric solution of the Taub-NUT geometry (2.18) then reappears:

$$ds^2 = V^{-1} (d\tau + \tilde{B})^2 + V d\vec{x} \cdot d\vec{x}, \quad (3.39)$$

with

$$d\tilde{B} = \pm * dV. \quad (3.40)$$

Analogously to the NS-fivebrane instanton supporting a non-trivial  $\chi$ , the Taub-NUT metric (3.39) supports a non-trivial graviphoton:

$$F_3 = \pm \frac{1}{2} \alpha V^{-2} dV, \quad (*F)_3 = -\frac{1}{2} \alpha V^{-2} dV,$$

$$\mathbf{F} = \mp \frac{1}{2} \alpha d(V^{-1} \tilde{B}), \quad \frac{1}{2} (*F)_{mn} dx^m \wedge dx^n = \frac{1}{2} \alpha d(V^{-1} \tilde{B}) . \quad (3.41)$$

We remind that, in Euclidean space, the four-dimensional field strength is given by  $F = F_3 \wedge d\tau + \frac{1}{2} \mathbf{F}_{mn} dx^m \wedge dx^n$ . For the solution (3.41), it is (anti-)selfdual. In fact, it is precisely the one found in [31] (see equation (4.15) in that reference).

The fact that the graviphoton is (anti-)selfdual implies that it has vanishing energy-momentum, which is consistent with the fact that the Taub-NUT solution is Ricci-flat. Taub-NUT solutions with (anti-)selfdual graviphoton and their T-duality relation with NS-fivebranes played an important role in a study of the partition sum of the NS-fivebrane [32].

## 4 Instantons in matter coupled $N = 2$ supergravity

In the last section we considered instantons in the double-tensor multiplet coupled to  $N = 2$  supergravity. Now we are interested in instanton solutions of the general four-dimensional low energy effective action which type II superstrings compactified on a Calabi-Yau give rise to. In the absence of fluxes, this yields (ungauged)  $N = 2$  supergravity coupled to vector and tensor multiplets (or their dual hypermultiplets). Following the spirit of the previous section, we will generate the tensor multiplets from the c-map on the gravitational and vector multiplet sector. This yields a double-tensor multiplet and  $h_{1,2}/h_{1,1}$  tensor multiplets for type IIA/B string theories. These tensor multiplets can be dualized further to hypermultiplets, but similarly to the previous section, we will not carry out this dualization. This turns out to be the most convenient way to describe instanton solutions, i.e. they are naturally described in the tensor multiplet formulation.

In this section we use the c-map, properly continued to Euclidean space, to map the BPS equations for the vector multiplets as found in [8] (with the  $R^2$ -interactions which are present in there switched off) to instantonic BPS equations for the tensor multiplet theory<sup>6</sup>.

The picture that emerges is that all BPS black hole solutions have their corresponding instantonic description after the (Euclidean) c-map. For a generic tensor multiplet theory these solutions all carry some RR-charge, and the instanton action is inversely proportional to the string coupling. There should also be NS-fivebrane instantons whose action is proportional to  $\frac{1}{g_s^2}$ . However it is not clear for a generic tensor multiplet theory how to get these from the Euclidean c-map. Therefore we derive them in a way independent of the c-map.

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<sup>6</sup>For some earlier work on vector multiplet BPS equations see [9], [33] and references therein.



## 4.1 The tensor multiplet theory

We start by discussing the tensor multiplet Lagrangian obtained after the c-map [7]. Details of the derivation can be found in the next subsection, or e.g. in [5]. The result is  $N = 2$  supergravity coupled to a double-tensor multiplet and  $n$  tensor multiplets, with  $n = h_{1,1}$  or  $h_{1,2}$  in IIA or IIB respectively. The bosonic Lagrangian, in Minkowski space, reads

$$\begin{aligned} \mathcal{L}_T^m = & -d^4x eR + \frac{1}{2}|d\phi|^2 + \frac{1}{2}e^{2\phi}|H|^2 + 2\mathcal{M}_{IJ} * dX^I \wedge d\bar{X}^J \\ & -e^{-\phi} \text{Im}\mathcal{N}_{IJ} * d\chi^I \wedge d\chi^J - e^\phi \text{Im}\mathcal{N}_{IJ} * (H^I - \chi^I H) \wedge (H^J - \chi^J H) \\ & -2 \text{Re}\mathcal{N}_{IJ} d\chi^I \wedge (H^J - \chi^J H) . \end{aligned} \quad (4.1)$$

We have left out the vector multiplet sector including the graviphoton as it is not relevant for our purposes. This sector can be easily reinstalled. The NS-NS part of the bosonic sector of the (double-)tensor multiplets consists of the dilaton,  $\phi$ , the 2-form  $B$  ( $H \equiv dB$ ) and the complex scalars  $X^I$  ( $I = 0, 1, \dots, n$ ), which are subject to the condition

$$N_{IJ} \bar{X}^I X^J = -1 . \quad (4.2)$$

This condition originates from the special Kähler geometry in the vector multiplet sector before doing the c-map. The vector multiplet sector is determined by a holomorphic prepotential  $F(X^I)$ , homogeneous of second degree, with  $N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ})$  and  $F_{IJ} = \partial_I \partial_J F(X)$ . The constraint (4.2) arises naturally as a gauge choice for dilatations in the superconformal calculus [34, 35], and together with the  $U(1)$  gauge symmetry

$$X^I \longrightarrow e^{i\alpha} X^I , \quad (4.3)$$

one can effectively eliminate one complex degree of freedom.

The RR part of the bosonic sector of the (double)-tensor multiplets is formed by the (real) scalars  $\chi^I$  and the (real) 2-forms  $B^I$  ( $H^I \equiv dB^I$ ). Altogether we have (after fixing the dilatation and  $U(1)$  gauge symmetries)  $4n + 4$  on-shell bosonic degrees of freedom, which is indeed the appropriate number for  $n$  tensor multiplets and a double-tensor multiplet.

The complex scalars  $X^I$  parameterize a manifold with metric

$$\mathcal{M}_{IJ} \equiv N_{IJ} - \frac{N_{IK} N_{JL} \bar{X}^K X^L}{N_{MN} X^M \bar{X}^N} . \quad (4.4)$$

The matrix  $\text{Im}\mathcal{N}_{IJ}$  appearing in the quadratic terms of (4.1) is determined by

$$\bar{\mathcal{N}}_{IJ} \equiv F_{IJ} - i \frac{N_{KI} \bar{X}^K N_{LJ} \bar{X}^L}{N_{MN} \bar{X}^M \bar{X}^N} . \quad (4.5)$$

One could impose the gauge choices on the matrices  $\mathcal{M}$  and  $\mathcal{N}$  to end up with the kinetic terms of the physical fields only. In the region of the special Kähler manifold where the NS-NS scalars have positive kinetic energy, one can show that also the RR scalars have positive kinetic energy [36].

The case of the double-tensor multiplet discussed in the previous section can be obtained by setting  $n = 0$  and taking the prepotential to be  $F(X) = -\frac{i}{4}(X^0)^2$ . This leads to  $N_{00} = -1$ ,  $\mathcal{M}_{00} = 0$  and  $\mathcal{N}_{00} = -\frac{i}{2}$ , and it is easy to check that the Lagrangian (4.1) reduces to (3.1).

To find instanton solutions, we first have to analytically continue to Euclidean space. For a generic  $N = 2$  system with scalars and tensors, this was discussed in [18]. The standard rules for the Wick rotation give (see Appendix (A.3) for our conventions),

$$\begin{aligned} \mathcal{L}_T^e &= +d^4x eR + \frac{1}{2}|d\phi|^2 + \frac{1}{2}e^{2\phi}|H|^2 + 2\mathcal{M}_{IJ} * dX^I \wedge d\bar{X}^J \\ &\quad - e^{-\phi} \text{Im}\mathcal{N}_{IJ} * d\chi^I \wedge d\chi^J - e^{\phi} \text{Im}\mathcal{N}_{IJ} * (H^I - \chi^I H) \wedge (H^J - \chi^J H) \\ &\quad - 2i \text{Re}\mathcal{N}_{IJ} d\chi^I \wedge (H^J - \chi^J H) . \end{aligned} \quad (4.6)$$

Notice that the last term becomes imaginary, similar to a theta-angle-like term. It will therefore be difficult to find a Bogomol'nyi bound on the action. We will return to the issue of a BPS bound in the last subsection. In fact, as we will see, we need to drop the reality conditions on the fields, as not all solutions we discuss below respect these reality conditions. For the moment, we will simply complexify all the fields<sup>7</sup>, and discuss below which instanton solutions respect which reality conditions.

It is convenient to rewrite (4.6) as

$$\begin{aligned} \mathcal{L}_T^e &= +d^4x eR \\ &\quad + \frac{1}{2}e^{2\phi}(|H|^2 + |F_I d\bar{Y}^I - Y^I d\bar{F}_I + c.c.|^2) \\ &\quad + d_I \wedge c^I - d^I \wedge c_I + D_I \wedge C^I + D^I \wedge C_I . \end{aligned} \quad (4.7)$$

Here and below, by *c.c.* we mean taking the complex conjugate before dropping the reality conditions, and then treating  $X$  and  $\bar{X}$  as independent complex fields.

The  $Y^I$  are rescaled versions of the complex scalars  $X^I$ ,

$$Y^I \equiv e^{-\frac{1}{2}\phi} \bar{h} X^I , \quad \bar{Y}^I \equiv e^{-\frac{1}{2}\phi} h \bar{X}^I , \quad (4.8)$$

with  $h$  an arbitrary (space-dependent) phase factor. Using (4.8) the condition (4.2) becomes an equation for  $e^{-\phi}$  in terms of  $Y^I$  and  $\bar{Y}^I$ ,

$$e^{-\phi} = -i \left( Y^I \bar{F}_I(\bar{Y}) - \bar{Y}^I F_I(Y) \right) . \quad (4.9)$$

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<sup>7</sup>For instance, this means that we treat  $X^I$  and  $\bar{X}^I$  as independent complex fields. The action then only depends on  $X$  and  $\bar{X}$  in a holomorphic way.

The three-forms  $(c^I, c_I)$  and the one-forms  $(d^I, d_I)$  belong to the NS-NS sector. They are defined as

$$\begin{pmatrix} c^I \\ c_I \end{pmatrix} \equiv \begin{pmatrix} -i * d(Y^I - \bar{Y}^I) \\ -i * d(F_I - \bar{F}_I) \end{pmatrix}, \quad \begin{pmatrix} d^I \\ d_I \end{pmatrix} \equiv \begin{pmatrix} d(e^\phi(Y^I + \bar{Y}^I)) \\ d(e^\phi(F_I + \bar{F}_I)) \end{pmatrix}. \quad (4.10)$$

In the RR sector we have the three-forms  $(C^I, C_I)$  and the one-forms  $(D^I, D_I)$

$$\begin{pmatrix} C^I \\ C_I \end{pmatrix} \equiv \begin{pmatrix} H^I - \chi^I H \\ ie^{-\phi} \text{Im} \mathcal{N}_{IJ} * d\chi^J + \text{Re} \mathcal{N}_{IJ} (H^J - \chi^J H) \end{pmatrix}, \\ \begin{pmatrix} D^I \\ D_I \end{pmatrix} \equiv \begin{pmatrix} -id\chi^I \\ -e^\phi \text{Im} \mathcal{N}_{IJ} * (H^J - \chi^J H) - i \text{Re} \mathcal{N}_{IJ} d\chi^J \end{pmatrix}. \quad (4.11)$$

One can show that  $C_I$  and  $D_I$  are the functional derivatives of the Lagrangian (4.7) with respect to  $D^I$  and  $C^I$  respectively

$$(*C_I)_\mu = \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_T^e}{\delta D^{\mu I}}, \quad (*D_I)_{\alpha\beta\gamma} = -\frac{1}{2} \frac{6}{\sqrt{|g|}} \frac{\delta \mathcal{L}_T^e}{C^{\alpha\beta\gamma I}}. \quad (4.12)$$

This means that the set of equations of motion and Bianchi identities of the (tensor multiplet) RR fields can be formulated as

$$d \begin{pmatrix} C^I \\ C_I \end{pmatrix} = -i \begin{pmatrix} D^I \\ D_I \end{pmatrix} \wedge H, \quad d \begin{pmatrix} D^I \\ D_I \end{pmatrix} = 0, \quad (4.13)$$

where the upper equations are Bianchi identities and the lower ones are field equations. The set of equations (4.13) is invariant under the electric-magnetic duality transformations  $Sp(2n+2, \mathbb{R})$  (see Appendix B). So  $(C^I, C_I)$  and  $(D^I, D_I)$  are symplectic vectors, as are  $(Y^I, F_I)$  (its transformation induces the appropriate transformation of  $\mathcal{N}_{IJ}$ ) and so  $(c^I, c_I)$  and  $(d^I, d_I)$ . To write our theory in terms of these symplectic vectors is useful when we consider the c-map and when deriving BPS equations through the c-map. This is because the vector multiplet theory on the other side of the c-map can be formulated in terms of symplectic vectors as well. The symplectic vectors on both sides turn out to be related in a rather simple way.

## 4.2 The c-map

Next we do a dimensional reduction of (4.7). We parameterize the metric as

$$g_{\mu\nu} dx^\mu dx^\nu = e^{\tilde{\phi}} (d\tau + \tilde{B})^2 + e^{-\tilde{\phi}} \hat{g}_{mn} dx^m dx^n. \quad (4.14)$$

This way we obtain

$$\mathcal{L}_{T3}^e = +d^3x \hat{e}R(\hat{e})$$

$$\begin{aligned}
& + \frac{1}{2} e^{2\phi} (|H|^2 + |F_I d\bar{Y}^I - Y^I d\bar{F}_I + c.c.|^2) \\
& + d_I \wedge c^I - d^I \wedge c_I + D_I \wedge C^I + D^I \wedge C_I .
\end{aligned} \tag{4.15}$$

Here we have suppressed the term  $\frac{1}{2}|d\tilde{\phi}|^2 + \frac{1}{2}e^{2\tilde{\phi}}|\tilde{H}|^2$  and terms of the form (2.13) as they play no role in our discussion.  $(d^I, d_I)$  and  $(D^I, D_I)$  are still one-forms, while  $(c^I, c_I)$  and  $(C^I, C_I)$  are now two-forms. They are again given by (4.10) and (4.11), but now with  $B$  and  $B^I$  being one-forms.

As said before we can also obtain (4.15) from a dimensional reduction of a theory of  $n$  vector multiplets and coupled to  $N = 2$  supergravity. The bosonic sector of this four-dimensional theory is given by

$$\mathcal{L}_V^m = -d^4x \tilde{e}R(\tilde{e}) + 2\mathcal{M}_{IJ} * dX^I \wedge d\bar{X}^J + F^I \wedge G_I , \tag{4.16}$$

with

$$(*G_I)_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_V^m}{\delta F^{\mu\nu I}} = (-\text{Im}\mathcal{N}_{IJ} * F^J + \text{Re}\mathcal{N}_{IJ} F^J)_{\mu\nu} . \tag{4.17}$$

We have left out the - irrelevant - tensor multiplet sector, just as we did with the vector multiplet sector in (4.1). The bosonic NS-NS sector of (4.16) consists of the metric  $\tilde{g}_{\mu\nu}$  and the complex scalars  $X^I$  (again subject to (4.2) and a  $U(1)$  gauge fixing condition). The bosonic RR sector is formed by the one-forms  $A^I$  ( $F^I \equiv dA^I$ ). As is well known, the set of field equations and Bianchi identities of  $F^I$  is invariant under  $Sp(2n + 2, \mathbb{R})$ , i.e.  $(F^I, G_I)$  is a symplectic vector.

Now we do a dimensional reduction over time, using

$$\begin{aligned}
\tilde{g}_{\mu\nu} dx^\mu dx^\nu &= -e^\phi (dt + \omega)^2 + e^{-\phi} \hat{g}_{mn} dx^m dx^n , \\
A^I &= (-\chi^{I'}, B^I - \chi^{I'} \omega) .
\end{aligned} \tag{4.18}$$

Then after identifying

$$\omega = -iB , \quad \chi^{I'} = i\chi^I , \tag{4.19}$$

complexifying all fields and multiplying the resulting three-dimensional Lagrangian by  $-1$ , we re-obtain (4.15). As we already hinted at at the end of the last subsection, we have the following simple relations between the (RR) symplectic vectors coming from both sides

$$\begin{pmatrix} F_3^I \\ G_{3I} \end{pmatrix} = - \begin{pmatrix} D^I \\ D_I \end{pmatrix} , \quad \begin{pmatrix} \mathbf{F}^I \\ \mathbf{G}_I \end{pmatrix} = \begin{pmatrix} C^I \\ C_I \end{pmatrix} - i \begin{pmatrix} D^I \\ D_I \end{pmatrix} \wedge B . \tag{4.20}$$

### 4.3 BPS equations from the c-map

In this section we use the c-map to obtain BPS instanton equations of a  $n + 1$  tensor multiplet theory from the BPS equations of a  $n$  vector multiplet theory. As said before, the latter equations are known and we use the results of [8]. In here equations were constructed for stationary solutions preserving half of the supersymmetry with parameters satisfying

$$h\epsilon_{ij} = \varepsilon_{ij}\gamma_0 e^j . \quad (4.21)$$

We remind that  $h$  is the phase factor appearing in (4.8). The metric components were found to be related to the complex scalars  $Y^I$  in the following way

$$\begin{aligned} e^{-\phi} &= -i(Y^J \bar{F}_J - \bar{Y}^J F_J) , \\ \hat{g}_{mn} &= \delta_{mn} , \\ *d\omega &= -\bar{F}_J dY^J + \bar{Y}^J dF_J + c.c. , \end{aligned} \quad (4.22)$$

while  $-i(Y^I - \bar{Y}^I)$  and  $-i(F_I - \bar{F}_I)$  should be three-dimensional harmonic functions. This fixes the NS-NS sector completely. Recall that the equation for  $e^{-\phi}$  is identically true, as follows from the definition of  $Y^I$  and the condition (4.2). For the RR fields the BPS equations of [8] are

$$\begin{pmatrix} F_3^I \\ G_{3I} \end{pmatrix} = - \begin{pmatrix} d^I \\ d_I \end{pmatrix} . \quad (4.23)$$

These equations can be equivalently formulated as

$$\begin{pmatrix} \mathbf{F}^I \\ \mathbf{G}_I \end{pmatrix} = - \begin{pmatrix} c^I \\ c_I \end{pmatrix} - i d \begin{pmatrix} e^\phi (Y^I + \bar{Y}^I) B|_{\text{BPS}} \\ e^\phi (F_I + \bar{F}_I) B|_{\text{BPS}} \end{pmatrix} , \quad (4.24)$$

where  $B|_{\text{BPS}}$  is the BPS solution of  $B$ . We remind that  $(c^I, c_I)$  and  $(d^I, d_I)$  are given by (4.10). (4.24) is the form in which the equations for the RR fields in [9] are written, however they do not have the second term on the r.h.s. Both sets of equations (4.23) and (4.24) fix the RR fields completely in terms of the complex scalars  $Y^I$ .

By construction the equations above only have stationary solutions. When  $\omega = 0$  one gets static extremal black holes. This works similar as in the pure supergravity case discussed in the last section. However there is a difference between the generic case and pure supergravity, which will become important later on in the context of NS-fivebrane instantons. We saw in last section that the pure  $N = 2$  supergravity BPS equations, after an analytic continuation to Euclidean space, gave rise to Taub-NUT solutions as well. In contrast to this, for generic functions  $F(X)$  it is far from clear if, and if yes how, this kind of solutions is contained in the general solution.

The equations above can be mapped quite easily to instanton equations of our Euclidean tensor sector. The first step is the dimensional reduction over time. As these equations only have stationary solutions, they are still valid after this dimensional reduction. Of course then they are also valid equations of the dimensionally reduced version of the tensor multiplet theory (4.15). The last step is to uplift them to equations of the full tensor theory (4.7). Basically the equations remain the same except that two-forms become three-forms and three-dimensional harmonic functions become four-dimensional harmonic functions. We find as instanton equations for the NS-NS fields

$$\begin{aligned}
e^{-\phi} &= -i(Y^J \bar{F}_J - \bar{Y}^J F_J) , \\
g_{\mu\nu} &= \delta_{\mu\nu} , \\
*H &= -i(\bar{F}_J dY^J - \bar{Y}^J dF_J + c.c.) ,
\end{aligned} \tag{4.25}$$

while  $-i(Y^I - \bar{Y}^I)$  and  $-i(F_I - \bar{F}_I)$  are now four-dimensional harmonic functions. Using (4.20) we directly read off what the instanton equations for the RR fields are

$$\begin{pmatrix} D^I \\ D_I \end{pmatrix} = \begin{pmatrix} d^I \\ d_I \end{pmatrix} , \tag{4.26}$$

or

$$\begin{pmatrix} C^I \\ C_I \end{pmatrix} = \begin{pmatrix} c^I \\ c_I \end{pmatrix} - i \begin{pmatrix} e^\phi(Y^I + \bar{Y}^I) \\ e^\phi(F_I + \bar{F}_I) \end{pmatrix} H|_{inst} . \tag{4.27}$$

Just as on the vector multiplet side both (4.26) and (4.27) fix the RR fields completely in terms of the complex scalars  $Y^I$ . For the fields appearing in (4.6) the equations take the form

$$\begin{aligned}
\chi^I &= ie^\phi(Y^I + \bar{Y}^I) + \chi_c^I , \\
*(H^I - \chi_c^I H) &= id(Y^I - \bar{Y}^I) ,
\end{aligned} \tag{4.28}$$

where  $\chi_c^I$  are arbitrary constants.

Recall from subsection (4.1) that all fields are complex. However, when we take  $-i(Y^I - \bar{Y}^I)$  and  $-i(F_I - \bar{F}_I)$  to be real, then the solutions for the dilaton and  $H^I$  are real whereas  $\chi^I$  and  $H$  become imaginary.

Let us make contact with the results of section 3. When we take the function  $F(Y)$  to be  $F(Y) = -\frac{1}{4}i(Y^0)^2$ , we see that (4.1) reduces to the action (3.1). We then get

$$Y^0 + \bar{Y}^0 = 2i(F_0 - \bar{F}_0) , \quad F_0 + \bar{F}_0 = -\frac{1}{2}i(Y^0 - \bar{Y}^0) . \tag{4.29}$$

Now we make the following identification of the harmonic functions  $-i(Y^0 - \bar{Y}^0)$  and  $-i(F_0 - \bar{F}_0)$  and the harmonic functions  $h$  and  $p$  which appeared in the BPS equations of section 3

$$-i(Y^0 - \bar{Y}^0) = h, \quad -i(F_0 - \bar{F}_0) = -\frac{1}{2}ip. \quad (4.30)$$

Equations (3.3) then follow directly. We can in this case obtain a real solution for  $\chi$  and  $H$  if we impose that  $-i(F_0 - \bar{F}_0)$  is imaginary, such that the harmonic function  $p$  is real.

#### 4.4 D-brane instantons

We now discuss the different types of solutions to the equations we obtained above. Clearly the general solution is a function of  $2n + 2$  harmonic functions. In the following we take them single-centered

$$\begin{aligned} -i(Y^I - \bar{Y}^I) &= -i(Y^I - \bar{Y}^I)_\infty + \frac{\hat{Q}^I}{4\pi^2|\vec{x} - \vec{x}_0|^2}, \\ -i(F_I - \bar{F}_I) &= -i(F_I - \bar{F}_I)_\infty + \frac{Q_I}{4\pi^2|\vec{x} - \vec{x}_0|^2}. \end{aligned} \quad (4.31)$$

However our results are easily generalized to multi-centered versions of (4.31).

In section 3 we saw that the two different types of solutions to the BPS equations for the double-tensor multiplet, membrane and the NS-fivebrane instantons, have different behavior of  $e^{-\phi}$ . For membrane instantons (having non-zero RR-charge) the dilaton behaves towards the excised point(s) as  $e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{|\vec{x} - \vec{x}_0|^4}\right)$ . For NS-fivebrane instantons (having non-zero NS-NS-, but vanishing RR-charge)  $e^{-\phi}$  is a harmonic function, which implies that towards the excised point(s) the behavior of the dilaton is  $e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{|\vec{x} - \vec{x}_0|^2}\right)$ . The different behavior of the dilaton in both types of solutions is reflected in a different dependence of the instanton action on the string coupling.

Let us now consider solutions to the general equations of last subsection (i.e. for general functions  $F(Y)$ ). The above seems to indicate that for a study of the characteristics of the instanton solutions it is good to start by analyzing the behavior of the dilaton towards the excised point(s). Doing this analysis we find that to leading order in  $\frac{1}{|\vec{x} - \vec{x}_0|}$

$$e^{-\phi}|_{\vec{x} \rightarrow \vec{x}_0} = \frac{|Z_0|^2}{16\pi^4|\vec{x} - \vec{x}_0|^4}, \quad (4.32)$$

which is as singular as the membrane instanton of section 3. Here  $Z_0$  is defined as

$$Z_0 \equiv (\hat{Q}^I F_I(X) - Q_I X^I)|_{\vec{x} \rightarrow \vec{x}_0}. \quad (4.33)$$

As seen from the c-map, the function  $Z = Q^I F_I(X) - Q_I X^I$  is the dual of the central charge function of the vector multiplet theory.

$\mathbf{Z}_0 \neq \mathbf{0}$

We first consider the case  $Z_0 \neq 0$ . Generic single-centered solutions consist of  $5n + 5$  parameters, 2 for each harmonic function and the  $n + 1$  constants  $\chi_c^I$ . The RR scalars  $\chi^I$  take the values  $\chi_c^I$  at  $\vec{x} = \vec{x}_0$ . The constants  $\hat{Q}^I$  and  $Q_I$  appearing in (4.31) can be identified with magnetic and electric charges of sources appearing in Bianchi identities and field equations respectively.  $\hat{Q}^I$  is equal to the charge of the source in the Bianchi identity of  $\hat{H}^I$

$$\hat{Q}^I = \int_{S_\infty^3} \hat{H}^I, \quad (4.34)$$

where we have defined  $\hat{H}^I \equiv H^I - \chi_c^I H$ .  $Q_I$  is up to a factor  $2i$  the charge of the source in the field equation of  $\chi^I$

$$2iQ_I = \int_{\mathbb{R}^4} d^4x \left( \frac{\delta \mathcal{L}}{\delta \chi^I} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \chi^I} \right). \quad (4.35)$$

Notice that this is consistent with the fact that the solutions for  $\chi^I$  are imaginary. As there are non-vanishing RR charges we can identify these solutions as D-brane instantons, generalizing the membrane instantons found in section 3. Also the Bianchi identity of  $H$  is sourced. The corresponding charge can be expressed in terms of the parameters appearing in the  $2n + 2$  harmonic functions

$$Q \equiv \int_{S_\infty^3} H = (F_I - \bar{F}_I)_\infty \hat{Q}^I - (Y^I - \bar{Y}^I)_\infty Q_I. \quad (4.36)$$

Evaluating (4.7) on these instantons gives

$$\begin{aligned} S_{\text{inst}}^T &= \int_{\mathbb{R}^4} (2d_I \wedge c^I - i(e^\phi(Y^I + \bar{Y}^I)d_I + e^\phi(F_I(Y) + \bar{F}_I(\bar{Y})))d^I) \wedge H|_{\text{BPS}} \\ &= \frac{2}{g_s^2} (F_I(Y) + \bar{F}_I(\bar{Y}))_\infty \hat{Q}^I - \frac{i}{g_s^4} (F_I(Y) + \bar{F}_I(\bar{Y}))_\infty (Y^I + \bar{Y}^I)_\infty Q. \end{aligned} \quad (4.37)$$

Applying (4.37) to the double-tensor multiplet theory of section 3, we have to take again  $F(Y) = -\frac{1}{4}i(Y^0)^2$ . Then using (4.29), (4.30), (3.15) and the double-tensor multiplet relation  $g_s^2 = \frac{1}{4}(h_\infty^2 - p_\infty^2)$  we re-obtain (3.14).

For  $Q = 0$  the second term in (4.37) vanishes and we find

$$S_{\text{inst}}^T = \frac{2}{g_s} (\bar{h}F_I(X) + h\bar{F}_I(\bar{X}))_\infty \hat{Q}^I. \quad (4.38)$$

This is the action for pure D-brane instantons of which the pure membrane instanton of section 3 is a specific example. We have reintroduced the variables  $X^I$  to make explicit the typical  $\frac{1}{g_s}$  dependence of D-brane instanton actions. From the c-map point of view pure



D-brane instantons are the duals of static BPS black holes living in the vector multiplet sector. Microscopically D-brane instantons come from wrapping even/odd branes over odd/even cycles in the Calabi-Yau in type IIA/B string theory.

Defining  $\Delta\varphi_I \equiv \frac{i}{g_s}(\bar{h}F_I(X) + h\bar{F}_I(\bar{X}))|_\infty$ , we can rewrite (4.38) as

$$S_{\text{inst}}^T = -2i\Delta\varphi_I Q^I . \quad (4.39)$$

In fact, one can show that  $\Delta\varphi_I = \varphi_{I\infty} - \varphi_{I0}$ , where  $\varphi_I$  is the dual (RR) scalar of  $\hat{H}^I$  and  $\varphi_{I\infty}$  and  $\varphi_{I0}$  are the asymptotic values of  $\varphi_I$  evaluated on the BPS solution

$$\varphi_I = ie^\phi(F_I(Y) + \bar{F}_I(\bar{Y})) + \varphi_{Ic} . \quad (4.40)$$

Here  $\varphi_{Ic}$  are integration constants, which coincide with  $\varphi_{I0}$ , the value of  $\varphi_I$  at the point  $\vec{x} = \vec{x}_0$ . This BPS equation is in fact implicitly stated already in the bottom equation in (4.26). Observe furthermore that the BPS solutions for  $\chi^I$ , as in (4.28), and  $\varphi_I$  are consistent with symplectic transformations, so we can write

$$\begin{pmatrix} \chi^I \\ \varphi_I \end{pmatrix} = ie^\phi \begin{pmatrix} Y^I + \bar{Y}^I \\ F_I + \bar{F}_I \end{pmatrix} + \begin{pmatrix} \chi_c^I \\ \varphi_{Ic} \end{pmatrix} . \quad (4.41)$$

Like in section 3, when we put  $H$  (and its BPS equation) to zero from the start, we can dualize all RR-scalars to tensors. This way we obtain a formulation of the theory consisting of  $2n + 2$  tensors, the "tensor-tensor" theory. The dualization procedure works similar as the one described in section (3). First we write  $D^I$  (being a one-form) instead of  $d\chi^I$  in the Euclideanized version of (4.6) (without  $H$ ) and add a Lagrange multiplier term

$$\mathcal{L}_T^e(\chi^I) \longrightarrow \mathcal{L}_T^e(D^I) + 2iB_I \wedge dD^I , \quad (4.42)$$

where  $B_I$  are  $n + 1$  two-forms. Integrating out  $B_I$  enforces  $dD^I = 0$ , giving back (locally)  $D^I = d\chi^I$ . Subtracting the total derivative  $2id(B_I \wedge D^I)$  and integrating out  $D^I$  yields the tensor-tensor theory. When we evaluate this action on the pure D-brane instantons we get

$$\begin{aligned} S_{\text{inst}}^{TT} &= -2i\Delta\varphi_I Q^I + 2i\Delta\chi^I Q_I \\ &= \frac{2}{g_s}(\bar{h}F_I(X) + h\bar{F}_I(\bar{X}))_\infty \hat{Q}^I - \frac{2}{g_s}(\bar{h}X^I + h\bar{X}^I)_\infty Q_I \\ &= \frac{4}{g_s}|Z|_\infty . \end{aligned} \quad (4.43)$$

The second term in the first line is due to the subtraction of the boundary term in the dualization procedure. To arrive at the last line we have used that  $Q = 0$ , which is consequence of the fact that we have put  $H$  to zero. The expression in the last line is (up to a factor of 4) the value of the real part of the pure D-brane instanton action as suggested in (a five-dimensional context) in [4].

$Z_0 = 0$

The case  $Z_0 = 0$  is special and needs to be analyzed separately, as the behavior of the dilaton is different. For the double-tensor multiplet of section 3 it yields NS-fivebrane instantons, which have a harmonic  $e^{-\phi}$ . This can most easily be seen from the fact that in the double-tensor multiplet case we have  $|Z_0| = \frac{1}{2}\sqrt{Q_h^2 - Q_p^2}$  (for single-centered instantons, as can be derived using (4.8), (4.29) and (4.30)). Requiring  $|Z_0|$  to vanish then gives the NS-fivebrane relation (3.6). However, for generic functions  $F(X)$  things work differently and we do not get NS-fivebrane instantons from taking  $Z_0 = 0$ . In fact in these cases the  $Z_0 = 0$  solution only differs qualitatively from the  $Z_0 \neq 0$  solution close to the excised points, which is directly related to the fact that only the asymptotic behavior of the dilaton is different. Now recall that  $Z$  is the dual of the central charge function of the vector multiplet theory. So  $Z_0 = 0$  solutions are the duals of vector multiplet solutions with vanishing central charge function at  $\vec{x} = \vec{x}_0$ . In case  $Q = 0$  these are zero-horizon black holes. Just as higher derivative corrections lift zero-horizon black holes at the two-derivative-level to finite horizon black holes [37], we expect that for  $Z_0 = 0$ -instantons higher derivative corrections have a qualitative effect on the behavior of  $e^{-\phi}$  in the limit  $\vec{x} \rightarrow \vec{x}_0$ . If this is the case the (two-derivative) differences between these solutions and the  $Z \neq 0$  instanton have no real physical significance.

## 4.5 NS-fivebrane instantons

In the previous section, we have discussed D-brane instantons. These were obtained from the c-map of the BPS solutions of [45], analytically continued to Euclidean space. We also saw that for generic functions  $F(X)$  NS-fivebrane instantons did not appear as a limiting case in a similar way as in the double-tensor multiplet theory of section 3. In fact it is not clear if they are contained at all in the general solution to the equations in subsection 4.3, just as was the case for their supposedly dual Taub-NUT solutions on the vector multiplet side.

However, we expect there to be (BPS) NS-fivebrane instantons in the general theory as well. That we have missed them so far could be understood from the fact that not all solutions in the Euclidean theory can be obtained from Wick rotating real solutions in the Lorentzian theory. Therefore, we will follow a different strategy and work directly in the Euclidean tensor multiplet Lagrangian, using a similar method as in [10]. This way we indeed find a class of NS-fivebrane instanton solutions.

We first write (4.6) as

$$\mathcal{L}_T^e = d^4x eR + 2\mathcal{M}_{IJ} * dX^I \wedge d\bar{X}^J$$

$$\begin{aligned}
& + * (N * \mathcal{H} + OE)^t \wedge A (N * H + OE) + 2\mathcal{H}^t N^t \wedge AOE \\
& - 2i \text{Re} \mathcal{N}_{IJ} d\chi^I \wedge (H^J - \chi^J H) .
\end{aligned} \tag{4.44}$$

Here we have defined the vectors

$$\mathcal{H} = \begin{pmatrix} H \\ H^I \end{pmatrix}, \quad E = \begin{pmatrix} d\phi \\ e^{-\frac{\phi}{2}} d\chi^I \end{pmatrix}, \tag{4.45}$$

and the matrices

$$N = e^{\frac{\phi}{2}} \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ -\chi^I & \delta^I_J \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\text{Im} \mathcal{N}_{IJ} \end{pmatrix}, \tag{4.46}$$

$O$  is a matrix as well, satisfying  $O^t A O = A$ . When all fields are taken real, clearly the real part of (4.44) is bounded from below by

$$\begin{aligned}
\text{Re} \mathcal{L}_{T^4}^e & \geq d^4 x e R + 2\mathcal{M}_{IJ} * dX^I \wedge d\bar{X}^J \\
& + 2\mathcal{H}^t N^t \wedge AOE .
\end{aligned} \tag{4.47}$$

Next we take the matrix  $O$  to be

$$O_{1,2} = \pm \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{4.48}$$

with  $\epsilon = \delta^I_J$  in the  $O_1$ -case and  $\epsilon = -\delta^I_J$  in the  $O_2$ -case. The plus and minus signs refer to the instanton and the anti-instanton respectively.

We now consider configurations for which the square in (4.44) is zero (i.e that saturate the bound (4.47) in case all fields are taken real). It is easy to show that for constant  $X^I$  these configurations satisfy the field equations of  $\phi$ ,  $\chi^I$  and the tensors (to the field equations of  $X^I$  we come back at a later stage). Furthermore these configurations can be shown to have vanishing energy-momentum. Therefore the gravitational background should be flat and (4.44) reduces to a total derivative. In the following we need the explicit form of this total derivative in the  $O_2$ -case

$$\begin{aligned}
\mathcal{L}_2^i & = -d(e^\phi H) - 2id(\bar{\mathcal{N}}_{IJ} \chi^I (H^J - \frac{1}{2} \chi^J H)), \\
\mathcal{L}_2^{a.i} & = +d(e^\phi H) - 2id(\mathcal{N}_{IJ} \chi^I (H^J - \frac{1}{2} \chi^J H)),
\end{aligned} \tag{4.49}$$

where the upper equation corresponds to the instanton and the lower one to the anti-instanton. In the  $O_1$ -case we get similar expressions.

Again we can make contact with the double-tensor multiplet theory by taking the function  $F$  to be  $F(X) = -\frac{1}{4}i(X^0)^2$ . The analysis above then reduces to the analysis of [10], with

the matrices  $O_{1,2}$  corresponding to their matrices  $O_{1,2}$ . The instantons related to these matrices are the NS-fivebrane instantons discussed in section 3.

Let us now consider the conditions which follow from requiring the square in (4.44) to vanish. Firstly, the  $O_1$  matrix gives

$$*\mathcal{H} = \pm \begin{pmatrix} \mathrm{d}e^{-\phi} \\ \chi^I \mathrm{d}e^{-\phi} - e^{-\phi} \mathrm{d}\chi^I \end{pmatrix}. \quad (4.50)$$

These equations are very similar to the  $O_1$  equations of [10]. Note in particular the relation  $*H = \pm \mathrm{d}e^{-\phi}$ , which is contained in both. Similarly to [10] we find that the finite-action-solution to (4.50) has a harmonic  $e^{-\phi}$  and constant  $\chi^I = \chi_0^I = \frac{Q^I}{Q}$ . Here

$$Q^I = \int_{S_\infty^3} H^I, \quad Q = \int_{S_\infty^3} H, \quad (4.51)$$

consistent with the notation we used in our treatment of D-brane instantons.

The conditions following from taking the matrix  $O_2$  in (4.44) are

$$*\mathcal{H} = \pm \mathrm{d} \begin{pmatrix} e^{-\phi} \\ e^{-\phi} \chi^I \end{pmatrix}. \quad (4.52)$$

These equations are very similar to the  $O_2$  equations of [10], with once more  $*H = \pm \mathrm{d}e^{-\phi}$  contained in both sets. The latter equation implies that  $e^{-\phi}$  is again harmonic. The remaining equations in (4.52) tell us that the same is true for  $e^{-\phi} \chi^I$ . For single-centered solutions this allows us to write  $\chi^I$  as

$$\chi^I = \chi_1^I e^\phi + \chi_0^I, \quad (4.53)$$

where the  $\chi_1^I$  are arbitrary constants. Note that  $\chi_0^I$  is the value  $\chi^I$  takes at the excised point(s). Putting (4.53) back into (4.52) we find again  $\chi_0^I = \frac{Q^I}{Q}$ . Observe that the finite-action  $O_1$ -solution is contained in this  $O_2$ -solution; we re-obtain it when we put  $\chi_1^I$  to zero. The action becomes for the single-centered instanton

$$S_{\text{inst}} = \frac{|Q|}{g_s^2} + i \bar{\mathcal{N}}_{IJ} \Delta \chi^I \Delta \chi^J Q, \quad (4.54)$$

where we have (again) defined  $\Delta \chi^I \equiv \chi^I|_\infty - \chi^I|_{\vec{x} \rightarrow \vec{x}_o}$ . For the anti-instanton  $\bar{\mathcal{N}}$  should be replaced by  $\mathcal{N}$ . The equations of motion of  $X^I$  are not automatically satisfied. Requiring this gives the extra condition that the last term in (4.54) should be extremized with respect to (the constants)  $X^I$ . Consequently the  $\chi_1^I$  and the  $X^I$  become related, unless  $\mathcal{N}_{IJ} = cst$ . The latter is for example the case in the double-tensor multiplet theory, in which we have

$\mathcal{N}_{00} = -\frac{i}{2}$  (in that case (4.54) can be seen to reduce to (3.12)). The precise relations between  $\chi_1^I$  and the  $X^I$  depend on the function  $F(X)$ . This implies that there is no general prescription for obtaining real solutions.

From  $\chi_0^I = \frac{Q^I}{Q}$  it directly follows that the charges  $\hat{Q}^I \equiv Q^I - \chi_0^I Q$  are zero. Furthermore one can show that there are no sources in the field equations of  $\chi^I$ . So there are no RR charges at all in the solutions. This means that they can be identified as (generalized) NS-fivebrane instantons. On the basis of what we know about NS-fivebrane instantons in the double-tensor multiplet [18] we expect these solutions (or at least all single-centered ones) to preserve half of the supersymmetry.

Let us finish our treatment of (generalized) NS-fivebrane instantons by considering its image under the (inverse) c-map. We find that this is a Taub-NUT geometry with  $n$  (anti-)selfdual vector fields, all of the form (3.41). It would be interesting to study if there are more general Euclidean BPS solutions of this type. We leave this for further study.

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## A Conventions

### A.1 Form notation

Greek indices are  $D$ -dimensional curved indices.  $a, b, \dots$  are  $D$ -dimensional flat indices,  $m, n, \dots$  are  $D-1$ -dimensional curved indices, and  $i, j, \dots$  are  $D-1$ -dimensional flat indices. The case relevant for us is of course  $D = 4$ , but formulae below hold for arbitrary  $D$ :

$$\begin{aligned}
a_p &\equiv \frac{1}{p!} a_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} , \\
dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} &\equiv e \epsilon^{\alpha_1 \dots \alpha_p} d^p x , \\
a_p \wedge b_q &\equiv \frac{1}{p!q!} a_{\alpha_1 \dots \alpha_p} b_{\alpha_{p+1} \dots \alpha_{p+q}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{p+q}} , \\
\epsilon^{\alpha_1 \dots \alpha_p} &\equiv e_{a_1}^{\alpha_1} \dots e_{a_p}^{\alpha_p} \epsilon^{a_1 \dots a_p} , \\
\epsilon^{0 \dots p-1} &\equiv 1 , \\
\epsilon^{1 \dots p} &\equiv 1 , \\
*a_p &\equiv \frac{1}{(D-p)!} (*a)_{\alpha_1 \dots \alpha_{D-p}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{D-p}} , \\
(*a)_{\alpha_1 \dots \alpha_{D-p}} &\equiv \frac{1}{p!} \epsilon_{\alpha_1 \dots \alpha_{D-p}}^{\beta_1 \dots \beta_p} a_{\beta_1 \dots \beta_p} ,
\end{aligned}$$

$$\begin{aligned}
|a_p|^2 &\equiv *a_p \wedge a_p \\
&= (-)^s e \frac{1}{p!} a^{\alpha_1 \dots \alpha_p} a_{\alpha_1 \dots \alpha_p} d^D x , \\
||\alpha_p||^2 &\equiv *a_p \wedge \bar{a}_p ,
\end{aligned} \tag{A.1}$$

with  $s = 1$  for Minkowskian and  $s = 0$  for Euclidean signature.

## A.2 Dimensional reduction

When  $a^m / a^e$  is a  $p$ -form in  $D$  Minkowskian/ Euclidean dimensions we denote the corresponding  $p - 1$ -form in  $D - 1$  dimensions by the same symbol

$$a^m \equiv \frac{1}{(p-1)!} a_{t m_1 \dots m_{p-1}} dx^{m_1} \wedge \dots \wedge dx^{m_{p-1}} , \tag{A.2}$$

or

$$a^e \equiv \frac{1}{(p-1)!} a_{m_1 \dots m_{p-1} \tau} dx^{m_1} \wedge \dots \wedge dx^{m_{p-1}} , \tag{A.3}$$

Only in case confusion might arise we write it as  $a_{D-1}^m / a_{D-1}^e$ .

The corresponding  $p$ -form in  $D - 1$  dimensions is denoted by

$$\mathbf{a} \equiv \frac{1}{p!} a_{m_1 \dots m_p} dx^{m_1} \wedge \dots \wedge dx^{m_p} . \tag{A.4}$$

## A.3 Wick rotation

The standard Wick rotation

$$t = -i\tau , \tag{A.5}$$

defines Euclidean Lagrangians

$$\mathcal{L}_m = i\mathcal{L}_e . \tag{A.6}$$

From their definitions it follows that

$$\begin{aligned}
e_m &= e_e , \\
\epsilon^{t\alpha_1 \dots \alpha_{D-1}} &= (-)^{D-1} \epsilon^{\tau\alpha_1 \dots \alpha_{D-1}} .
\end{aligned} \tag{A.7}$$

The Wick rotation on two-forms is

$$B_{tm} \rightarrow iB_{\tau m} , \quad B_{mn} \rightarrow B_{mn} , \tag{A.8}$$

and similarly for one-forms.

## A.4 Integration of spherically symmetric harmonic functions

$$\int_{S_\infty^{D-1}} *d \frac{Q}{(D-2)(Vol S^{D-1})r^{D-2}} = (-)^D Q . \quad (\text{A.9})$$

## B Electric-magnetic duality

Suppose we have a theory with a set of  $n + 1$  two-forms  $B^I$  and  $n + 1$  scalars  $\chi^I$  (with  $I$  running from 0 to  $n$ ), possibly accompanied by other fields (collectively denoted by  $\phi$ ). Furthermore assume we can describe the set of two-forms and scalars by (generalized) field strengths  $C^I$  and  $D^I$ . The  $C^I$  are three-forms composed of the field strengths of the two-forms with possible extra terms. The  $D^I$  are one-forms composed of the "field strengths" of the scalars with also extra terms allowed. We then introduce the objects

$$\begin{aligned} (*C_I)_\mu &= \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}}{\delta D^{\mu I}} , \\ (*D_I)_{\alpha\beta\gamma} &= -\frac{1}{2} \frac{6}{\sqrt{|g|}} \frac{\delta \mathcal{L}}{\delta C^{\alpha\beta\gamma I}} . \end{aligned} \quad (\text{B.1})$$

In case the theory under consideration has only terms quadratic in  $C^I$  and/or  $D^I$  it can be written as

$$\mathcal{L} = D_I \wedge C^I + D^I \wedge C_I . \quad (\text{B.2})$$

We now restrict ourselves to cases where the set of equations formed by the Bianchi identities of the generalized field strengths and the equations of motion of  $B^I$  and  $\chi^I$  can be formulated as

$$\begin{aligned} d \begin{pmatrix} C^I \\ C_I \end{pmatrix} &= \alpha(\phi) \wedge * \begin{pmatrix} C^I \\ C_I \end{pmatrix} + \beta(\phi) \wedge \begin{pmatrix} D^I \\ D_I \end{pmatrix} , \\ d \begin{pmatrix} D^I \\ D_I \end{pmatrix} &= \gamma(\phi) \wedge * \begin{pmatrix} C^I \\ C_I \end{pmatrix} + \eta(\phi) \wedge \begin{pmatrix} D^I \\ D_I \end{pmatrix} . \end{aligned} \quad (\text{B.3})$$

$\alpha(\phi)$  and  $\beta(\phi)$  are three-forms, while  $\gamma(\phi)$  and  $\eta(\phi)$  are one-forms<sup>8</sup>.

The set of equations (B.3) is invariant under the electric-magnetic duality transformations

$$\begin{aligned} \begin{pmatrix} C^I \\ C_I \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{C}^I \\ \tilde{C}_I \end{pmatrix} = \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} C^J \\ C_J \end{pmatrix} , \\ \begin{pmatrix} D^I \\ D_I \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{D}^I \\ \tilde{D}_I \end{pmatrix} = \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} D^J \\ D_J \end{pmatrix} , \end{aligned} \quad (\text{B.4})$$

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<sup>8</sup>In section 4 of the main text we have  $\alpha = \gamma = \eta = 0$  and  $\beta = iH$ .

and  $\alpha, \beta, \gamma$ , and  $\eta$  transforming trivially. We call (B.4) electric-magnetic duality transformations as Bianchi identities and equations of motion are rotated into each other, which is similar to the effect of conventional electric-magnetic duality transformations working on the field strengths and dual field strengths of one-form gauge fields.

A dual Lagrangian is defined by

$$\begin{aligned} (*\tilde{C}_I)_\mu &= \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\delta \tilde{\mathcal{L}}}{\delta \tilde{D}^{\mu I}}, \\ (*\tilde{D}_I)_{\alpha\beta\gamma} &= -\frac{1}{2} \frac{6}{\sqrt{|g|}} \frac{\delta \tilde{\mathcal{L}}}{\delta \tilde{C}^{\alpha\beta\gamma I}}. \end{aligned} \quad (\text{B.5})$$

In terms of the old  $(C^I, C_I)$  and  $(D^I, D_I)$  these equations are

$$\begin{aligned} \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\delta \tilde{\mathcal{L}}}{\delta D^{\mu J}} &= (U^T W)_{JK} * C_\mu^K + (U^T V)_J^K * C_{\mu K} \\ &+ \frac{\delta D_K^\eta}{\delta D^{\mu J}} (Z^T V)^{KL} * C_{\eta L} - \frac{1}{6} \frac{\delta C_K^{\alpha\beta\gamma}}{\delta D^{\mu J}} (Z^T V)^{KL} * D_{\alpha\beta\gamma L} \\ &+ \frac{\delta D_K^\eta}{\delta D^{\mu J}} (Z^T W)^K{}_L * C_\eta^L - \frac{1}{6} \frac{\delta C_K^{\alpha\beta\gamma}}{\delta D^{\mu J}} (Z^T W)^K{}_L * D_{\alpha\beta\gamma}^L, \\ \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\delta \tilde{\mathcal{L}}}{\delta C^{\mu\rho\sigma J}} &= -\frac{1}{6} (U^T W)_{JK} * D_{\mu\rho\sigma}^K - \frac{1}{6} (U^T V)_J^K * D_{\mu\rho\sigma K} \\ &- \frac{1}{6} \frac{\delta C_K^{\alpha\beta\gamma}}{\delta C^{\mu\rho\sigma J}} (Z^T V)^{KL} * D_{\alpha\beta\gamma L} + \frac{\delta D_K^\eta}{\delta C^{\mu\rho\sigma J}} (Z^T V)^{KL} * C_{\eta L} \\ &- \frac{1}{6} \frac{\delta C_K^{\alpha\beta\gamma}}{\delta C^{\mu\rho\sigma J}} (Z^T W)^K{}_L * D_{\alpha\beta\gamma}^L + \frac{\delta D_K^\eta}{\delta C^{\mu\rho\sigma J}} (Z^T W)^K{}_L * C_\eta^L. \end{aligned} \quad (\text{B.6})$$

As it turns out this set of equations can only be solved consistently in case the transformation matrix in (B.4) belongs to  $Sp(2n+2, \mathbb{R})$  when all fields are real, or otherwise a complexified version thereof. Furthermore it is important to note that generically we get  $\mathcal{L}(\tilde{C}^I, \tilde{D}^I) \neq \mathcal{L}(C^I, D^I)$ , i.e. the Lagrangian does not transform as a function. For purely quadratic theories the dual Lagrangian becomes

$$\tilde{\mathcal{L}} = \tilde{D}_I \wedge \tilde{C}^I + \tilde{D}^I \wedge \tilde{C}_I. \quad (\text{B.7})$$

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