A unified approach to sequences, bags, and trees

Introduction

This paper contains a formal definition of constructs, a new concept of which sequences, bags, and trees may be regarded as special cases. This unification of data structures commonly regarded as unrelated leads to a simpler treatment of several operations that are usually defined by structural induction for each case separately.

Constructs have been used in [1] to provide an alternative foundation for the transformational programming method proposed by Bird [2] and Meertens [4]. References in [1] are to [0], a preliminary version of the present paper.

Constructs

DEFINITION. An ordered set A is a pair $(V_A, \leq_A)$, where V_A is a set and $\leq_A$ is a relation on V_A satisfying the following laws: for arbitrary a, b, c in V_A,

(i) $a \leq_A a$ (reflexivity) ,

(ii) $a \leq_A b \wedge b \leq_A a \Rightarrow a = b$ (antisymmetry) ,

(iii) $a \leq_A b \wedge b \leq_A c \Rightarrow a \leq_A c$ (transitivity) .

For simplicity's sake, we usually write $a \in A$ instead of $a \in V_A$. We shall interpret $a \geq_A b$ as $b \leq_A a$, and $a <_A b$ as $a \leq_A b \wedge a \neq b$.

For an ordered set A and an arbitrary set X, the set $A \rightarrow X$ consists of the pairs $(\leq_A, f)$ with $f \in V_A \rightarrow X$. Note that this definition ensures that $A \neq B$ implies $(A \rightarrow X) \cap (B \rightarrow X) = \emptyset$. Once again for simplicity, we will write $f \in A \rightarrow X$ instead of $(\leq_A, f) \in A \rightarrow X$.

DEFINITION. Let A and B be ordered sets. An isomorphism of A onto B is a bijection σ of V_A onto V_B that satisfies

$$a0 \leq_A a1 \equiv \sigma(a0) \leq_B \sigma(a1)$$

for all $a0, a1$ in A .

DEFINITION. Let A and B be ordered sets, X and Y arbitrary sets. Mappings $f \in A \to X$ and $g \in B \to Y$ are called <u>equivalent</u> (notation: $f \approx g$) if there exists an isomorphism σ of A onto B such that $f = g \circ \sigma$. (Note that this implies $X = Y$.)

THEOREM 0. Let F denote the class of all mappings that belong to a set of the form $A \to X$, where A is an ordered set and X any set. Then $\approx$ is reflexive, symmetric and transitive on F .

PROOF. It is reflexive, since for any f in F ,

$$f = f \circ id \quad ,$$

where id denotes the identity mapping on the domain of f . It is symmetric, since

$$f = g \circ \sigma \Rightarrow g = f \circ \sigma^{-1} \quad .$$

Finally, it is transitive, since

$$f = g \circ \sigma \wedge g = h \circ \tau \Rightarrow f = h \circ (\tau \circ \sigma) \quad .$$

THEOREM 1. It is possible to associate with every f in F a set $C(f)$ in such a way that

$$C(f) = C(g) \equiv f \approx g \quad .$$

PROOF. If F had been a set, one could have taken $C(f)$ to be the equivalence class of f with respect to the equivalence relation $\approx$. However, F is not a set. We shall circumvent this objection by constructing nonempty sets $C(f)$ that consist of some, but not all, mappings equivalent to f .

By transfinite induction, we define for every ordinal α a set R_α in the following way: $R_0 = \emptyset$, $R_{\alpha+1} = \mathcal{P}(R_\alpha)$, and for every limit ordinal α , R_α is the union of all sets R_β with $\beta < \alpha$. For every set X , the rank of X is defined as the least α with $X \in R_\alpha$; in Zermelo-Fraenkel set theory [5], every set has a rank.

The elements of F are themselves sets: if $f \in A \to X$, then

$$f \in \mathcal{P}(V_A \times V_A) \times \mathcal{P}(V_A \times X) \quad .$$

Now define C(f) as the class of all g in F with $g \approx f$ such that g is of minimal rank. If α is the rank of the elements of C(f) , we find that every element of C(f) belongs to $P(P(R_\alpha))$, so C(f) is indeed a set.

We shall call the sets C(f) constructs; as we shall see below, they are sufficiently general for such a vague term to be appropriate.

In order to escape the obligation to give every mapping a name, we introduce an alternative notation for constructs. If A is an ordered set and E an expression in the single unbound identifier i , we define

$$(\underline{C}\ i\colon i \in A\colon E) = C(f) \quad ,$$

where f is the mapping that satisfies $f(a) = E(i{:=}\ a)$ for every a in A and is surjective (in other words, has codomain $\{E(i{:=}a) \mid a \in A\}$).

This new notation also provides us with an easy way to restrict the domain of a construct, which is sometimes called filtering. Let p be a boolean function defined on A . We define

$$(\underline{C}\ i\colon i \in A \wedge p(i)\colon E) = (\underline{C}\ i\colon i \in B\colon E) \quad ,$$

where V_B is the subset of those a in V_A that satisfy p(a) , and $\le_B$ is the restriction of $\le_A$ to $V_B \times V_B$. In case A consists of the natural numbers with their usual ordering, the conjunct $i \in A$ is usually omitted.

Sequences

EXAMPLE 0. Let A denote the set {0, 1, 2} with the usual ordering. Define $f \in A \to \{2, 3, 6\}$ by f(0) = 2, f(1) = 3, f(2) = 6 . Now a mapping $g \in B \to Y$ satisfies $g \approx f$ if and only if Y = {2, 3, 6} and V_B consists of three elements, say $V_B = \{b0, b1, b2\}$ with $b0 \le_B b1 \le_B b2$, such that g(b0) = 2, g(b1) = 3, g(b2) = 6. What these different g have in common is that they take the values 2, 3 and 6 in that order. Therefore we wish to identify C(f) with the finite sequence <2, 3, 6> . Below we shall give a definition of finite sequences that achieves precisely this effect.

DEFINITION. A finite sequence is a construct, say C(f) with $f \in A \rightarrow X$, such that f is surjective, V_A is finite and $\le_A$ is a linear ordering (i.e., for every a0 and a1 in V_A , it is true that $a0 \le_A a1 \mid a1 \le_A a0$).

REMARK. The condition that f be surjective has the effect that sequences are determined by the terms and their order alone, not by the type of these terms. For instance, there is only one sequence consisting of the numbers 0 and 1 in that order, regardless of whether these are considered as elements of the set of integers or of the set {0, 1, 25, 1988} . There is also only one empty sequence. If some application should require the introduction of typed sequences, this is achieved by the removal of the surjectivity condition. Note, however, that Theorem 2 below will then lose its validity.

In particular, if p is a boolean function on the natural numbers satisfying

$$(\underline{N}\ i:: p(i)) < \infty \quad ,$$

the construct

$$(\underline{C}\ i: p(i): x_i)$$

is a finite sequence. In the literature, many other notations can be found. Note, however, that not all authors make a clear distinction between f and C(f) ; this makes it uncertain whether dummy transformations are permitted. In case $p(i) \equiv 1 \le i \le n$, it is common practice to denote the above sequence by $\langle x_1, x_2, ..., x_n \rangle$.

THEOREM 2. Every finite sequence can be written as $\langle x_1, x_2, ..., x_n \rangle$ in precisely one way.

PROOF. Let C(f) be a finite sequence, where $f \in A \rightarrow X$. Let n be the number of elements of V_A . Define $\sigma \in V_A \rightarrow \{1, 2, ..., n\}$ by

$$\sigma(a) = (\underline{N}\ i: i \in A: i \le_A a) \quad .$$

We shall now prove that σ is an isomorphism from A onto $\{1, 2, ..., n\}$ with the usual ordering.

Take $a0$ and $a1$ in V_A with $a0 \leq_A a1$ and $a0 \neq a1$ (remember that this is abbreviated as $a0 <_A a1$). By the antisymmetry of $\leq_A$, this implies $\neg\, a1 \leq_A a0$. Now

$$\sigma(a0) < \sigma(a1)$$

$$= \{\text{definition of } \sigma\}$$

$$(\underline{N}\ i{:}\ i \in A{:}\ i \leq_A a0) < (\underline{N}\ i{:}\ i \in A{:}\ i \leq_A a1)$$

$$= \{\text{domain split}\}$$

$$(\underline{N}\ i{:}\ i \in A{:}\ i \leq_A a0) < (\underline{N}\ i{:}\ i \in A \wedge i \leq_A a0{:}\ i \leq_A a1)$$

$$+ (\underline{N}\ i{:}\ i \in A \wedge \neg\, i \leq_A a0{:}\ i \leq_A a1)$$

$$= \{ a0 \leq_A a1 \text{ , transitivity of } \leq_A \}$$

$$0 < (\underline{N}\ i{:}\ i \quad A \ \square\ \neg\, i \leq_A a0{:}\ i \leq_A a1)$$

$$= \{ \neg\, a1 \leq_A a0 \text{ , reflexivity of } \leq_A \}$$

$$\text{true} \quad .$$

This proves

$$a0 <_A a1 \Rightarrow \sigma(a0) < \sigma(a1) \quad .$$

By symmetry, the same formula holds with $a0$ and $a1$ interchanged. Now we can (finally!) use the linearity of $\leq_A$ to obtain

$$a0 \leq_A a1 \equiv \sigma(a0) \leq \sigma(a1)$$

and also

$$a0 \neq a1 \Rightarrow \sigma(a0) \neq \sigma(a1) \quad ,$$

in other words, σ is injective. Since domain and codomain of σ have the same finite number of elements, σ is in fact a bijection.

We have now proved that σ is an isomorphism from A onto $\{1, 2, ..., n\}$ with the usual ordering. Since σ^{-1} must then also be an isomorphism, the definition of equivalence gives $f \approx f \circ \sigma^{-1}$, hence,

$C(f)$
$= \{f \approx f \circ \sigma^{-1}$, definition of C $\}$
$C(f \circ \sigma^{-1})$
$= \{$surjectivity of f , definition of $\underline{C}$ $\}$
$(\underline{C}\ i: 1 \leq i \leq n: f(\sigma^{-1}(i))$
$= \{$definition of $<...>$ $\}$
$<f(\sigma^{-1}(1)), f(\sigma^{-1}(2)), ..., f(\sigma^{-1}(n))>$.

It remains to show uniqueness. Suppose that also $C(f) = <x_1, x_2, ..., x_n>$. Then there exists an isomorphism τ from $\{1, 2, ..., n\}$ with the usual ordering onto itself, such that for every i ,

$$x_i = f(\sigma^{-1}(\tau(i))) \quad .$$

However, such a τ can only be the identity mapping, as we shall now prove. Consider a minimal i with $\tau(i) \neq i$. Then

$\tau(i) \neq i$
$= \{$bijectivity of τ $\}$
$\tau(\tau(i)) \neq \tau(i) \wedge \tau^{-1}(\tau(i)) \neq \tau^{-1}(i)$
$= \{\}$
$\tau(\tau(i)) \neq \tau(i) \wedge \tau(\tau^{-1}(i)) \neq \tau^{-1}(i)$
$> \{$minimality of i $\}$
$\tau(i) \geq i \wedge \tau^{-1}(i) \geq i$
$= \{$ τ is an isomorphism$\}$
$\tau(i) \geq i \wedge i \geq \tau(i)$
$= \{$reflexivity and antisymmetry of $\leq$ $\}$
$\tau(i) = i$,

so no such i exists. Hence τ must be the identity mapping.

EXAMPLE 1. By means of dummy transformations and filtering, the same finite sequence can be written in a variety of ways. For instance,

$(\underline{C}\ i{:}\ 0 \le i \le 2{:}\ i^2 + 2)$,

$(\underline{C}\ i{:}\ 3 \le i \le 5{:}\ i^2 - 6i + 11)$,

$(\underline{C}\ i{:}\ 2 \le i \le 6 \wedge i \ne 4 \wedge i \ne 5{:}\ i)$,

$(\underline{C}\ i{:}\ 2 \le i \le 3 \vee i = 6{:}\ i)$,

$(\underline{C}\ i{:}\ i > 1 \wedge 6\ \underline{\text{mod}}\ i = 0{:}\ i)$

all denote the sequence <2, 3, 6> .

DEFINITION. An <u>infinite sequence</u> is a construct, say C(f) with $f \in A \to X$, such that f is surjective and A is isomorphic to the natural numbers with their usual ordering.

EXAMPLE 2. The constructs

$(\underline{C}\ i{::}\ 2i + 1)$

and

$(\underline{C}\ i{:}\ i\ \underline{\text{mod}}\ 2 \ne 0{:}\ i)$

both denote the same infinite sequence, the one whose terms are the odd natural numbers in their usual order.

REMARK. It is easy to define the <u>reverse</u> of a construct, say C(f) with $f \in A \to X$: simply replace the ordering $\le_A$ by its inverse relation $\ge_A$. This corresponds with the well-known reversal operation on finite sequences. If we wish to have the set of all sequences closed under reversal, we must also admit left-infinite sequences.

<u>Bags</u>

EXAMPLE 3. Let A denote the set {0, 1, 2}, ordered discretely: $a0 \le_A a1 \equiv a0 = a1$. Define $f \in A \to \{7, 10\}$ by $f(0) = 7$, $f(1) = 10$, $f(2) = 7$. A mapping $g \in B \to Y$ satisfies $g \approx f$ if and only if $Y = \{7, 10\}$ and V_B consists of three elements, ordered discretely, such that g maps two of these onto 7 and one onto 10. What these different g

have in common is that they take the value 7 twice and the value 10 once. Therefore we wish to identify C(f) with the bag [7, 7, 10] . Below we shall give a definition of bags that achieves precisely this effect.

DEFINITION. A _bag_ is a construct, say C(f) with $f \in A \to X$, such that f is surjective and $\leq_A$ is discrete (i.e., $a0 \leq_A a1 \equiv a0 = a1$).

A bag C(f) , where $f \in A \to X$, is called _finite_ if V_A is finite. In particular, a construct of the form

$$(\underline{C}\ i: i \in (\mathbf{N}, =) \wedge 1 \leq i \leq n: x_i)$$

is a finite bag, one that may also be denoted by $[x_1, x_2, ..., x_n]$.

THEOREM 3. Every finite bag can be written as $[x_1, x_2, ..., x_n]$. The elements $x_1, x_2, ..., x_n$ are uniquely determined but for permutations.

PROOF. Let C(f) be a finite bag, where $f \in A \to X$. Let n denote the number of elements of V_A . This means that there exists a bijection σ from {1, 2, ..., n} onto V_A ; such a σ is also an isomorphism from {1, 2, ..., n} with the discrete order onto A . By the definition of equivalence, $f \approx f \circ \sigma$, hence

$$
\begin{aligned}
& C(f) \\
= & \{f \approx f \circ \sigma \text{ , definition of C } \} \\
& C(f \circ \sigma) \\
= & \{\text{surjectivity of } f \text{ , definition of } \underline{C} \ \} \\
& (\underline{C}\ i: i \in (\mathbf{N}, =) \wedge 1 \leq i \leq n: f(\sigma(i))) \\
= & \{\text{definition of } [...] \ \} \\
& [f(\sigma(1)), f(\sigma(2)), ..., f(\sigma(n))] \quad .
\end{aligned}
$$

It remains to show uniqueness but for permutations. Suppose that also $C(f) = [x_1, x_2, ..., x_n]$. Then there exists an isomorphism τ from {1, 2, ..., n} with the discrete ordering onto itself, i.e., a permutation of these numbers, such that

$$x_i = f(\sigma(\tau(i))) \quad .$$

We shall now prove that our definition of bags is equivalent to the usual one.

DEFINITION. For a bag B, say $B = C(f)$ with $f \in A \to X$, and for $x \in X$, the number $\#(B, x)$ is defined as

$$\#(B, x) = (\underline{N}\ i: i \in A: f(i) = x) \quad .$$

THEOREM 4. For any two bags $B0$, $B1$, say $B0 = C(f)$ and $B1 = C(g)$ with $f \in A \to X$ and $g \in B \to Y$,

(0) $$B0 = B1 \equiv X = Y \wedge (\underline{A}\ x: x \in X: \#(B0, x) = \#(B1, x)) \quad .$$

PROOF. According to the definition of C, the bags $C(f)$ and $C(g)$ are equal if and only if there exists an isomorphism σ from A onto B such that $f = g \circ \sigma$. As A and B are discretely ordered, such an isomorphism is nothing but a bijection from V_A onto V_B. From the existence of such a bijection, the first conjunct of (0) clearly follows; we now prove that the second one does too. For any x in X,

$$\begin{aligned}
&\#(B0, x)\\
= &\ \{\text{definition of } \#\}\\
&(\underline{N}\ i: i \in A: f(i) = x)\\
= &\ \{f = g \circ \sigma\}\\
&(\underline{N}\ i: i \in A: g(\sigma(i)) = x)\\
= &\ \{\text{dummy transformation, } j := \sigma(i)\}\\
&(\underline{N}\ j: j \in B: g(j) = x)\\
= &\ \{\text{definition of } \#\}\\
&\#(B1, x) \quad .
\end{aligned}$$

On the other hand, assume that $\#(B0, x) = \#(B1, x)$ for every x in X. Then there exists, for every x in X, a bijection σ_x from $f^{-1}[x]$ onto $g^{-1}[x]$. Define $\sigma \in V_A \to V_B$ by

$$\sigma(a) = \sigma_{f(a)}(a) \quad .$$

We prove the injectivity of σ . For all a0 , a1 in V_A ,

$\sigma(a0) = \sigma(a1)$

= {definition of σ }

$\sigma_{f(a0)}(a0) = \sigma_{f(a1)}(a1)$

= {application of g }

$g(\sigma_{f(a0)}(a0)) = g(\sigma_{f(a1)}(a1)) \wedge \sigma_{f(a0)}(a0) = \sigma_{f(a1)}(a1)$

= {definition of σ_x implies $g(\sigma_x(y)) = x$ }

$f(a0) = f(a1) \wedge \sigma_{f(a0)}(a0) = \sigma_{f(a1)}(a1)$

$\Rightarrow$ {}

$\sigma_{f(a0)}(a0) = \sigma_{f(a0)}(a1)$

= {injectivity of $\sigma_{f(a0)}$ }

a0 = a1 .

Moreover, σ is surjective: for any b in V_B we have $b \in g^{-1}[g(b)]$, so by the bijectivity of the σ_x , there exists an a in $f^{-1}[g(b)]$ with $\sigma_{g(b)}(a) = b$. Since $a \in f^{-1}[g(b)]$, we have $f(a) = g(b)$, so $\sigma(a) = \sigma_{f(a)}(a) = \sigma_{g(b)}(a) = b$.

We have now established that σ is a bijection from V_A onto V_B . For any a in V_A , $g(\sigma(a)) = g(\sigma_{f(a)}(a)) = f(a)$. Together with X = Y this yields $f = g \circ \sigma$.

Theorem 4 shows that a bag B may equally well be identified with the function #(B,) X – **N** ; that is the usual definition.

Concatenation and summation

DEFINITION. Let A and B be ordered sets. The concatenation of A and B , denoted A ++ B , is the ordered set $(V_{A++B}, \leq_{A++B})$, where

$$V_{A++B} = \{(0, a) \mid a \in V_A\} \cup \{(1, b) \mid b \in V_B\}$$

and

$$(i0,\ c0) \le_{A{+}{+}B} (i1,\ c1)$$
$$\equiv (i0 = 0 \wedge i1 = 1) \vee (i0 = i1 = 0 \wedge c0 \le_A c1) \vee (i0 = i1 = 1 \wedge c0 \le_B c1) \quad .$$

DEFINITION. Consider constructs $C(f)$ and $C(g)$, where $f \in A \to X$ and $g \in B \to Y$. The <u>concatenation</u> of $C(f)$ and $C(g)$, denoted $C(f) \mathbin{++} C(g)$, is the construct $C(h)$, where $h \in (A \mathbin{++} B) \to X \cup Y$ is defined by

(1) $h(0,\ a) = f(a) \wedge h(1,\ b) = g(b)$

for $a \in A$, $b \in B$.

THEOREM 5 ("domain split law"). For f defined on $A \mathbin{++} B$,

$$(\underline{C}\ i\colon i \in A \mathbin{++} B\colon f(i))$$
$$= (\underline{C}\ i\colon i \in A\colon f(0,\ i)) \mathbin{++} (\underline{C}\ i\colon i \in B\colon f(1,\ i)) \quad .$$

EXAMPLE 4.

(a) $\langle 1,\ 2,\ 5\rangle \mathbin{++} \langle 5,\ 3,\ 0\rangle = \langle 1,\ 2,\ 5,\ 5,\ 3,\ 0\rangle$.

(b) $(\underline{C}\ i\colon 0 \le i < m\colon f(i)) \mathbin{++} (\underline{C}\ i\colon 0 \le i < n\colon g(i))$

$= (\underline{C}\ i\colon 0 \le i < m{+}n\colon \underline{\text{if}}\ i < m \to f(i) \ \square\ i \ge m \to g(i{-}m)\ \underline{\text{fi}})$.

(c) $(\underline{C}\ i\colon i \in (\mathbf{N},\ \ge)\colon f(i)) \mathbin{++} (\underline{C}\ i\colon i \ge 1\colon f(i))$

$= (\underline{C}\ i\colon i \in (\mathbf{Z},\ \le)\colon f(|i|))$.

(d) $(\underline{C}\ i\colon\colon f(i)) \mathbin{++} \langle 2\rangle$

$= (\underline{C}\ i\colon i \in (\boldsymbol{\omega} + 1,\ \le)\colon \underline{\text{if}}\ i < \boldsymbol{\omega} \to f(i) \ \square\ i = \boldsymbol{\omega} \to 2\ \underline{\text{fi}})$.

(e) $\langle 1,\ 2,\ 5\rangle \mathbin{++} [2,\ 2]$ is the partially ordered construct

```
      1
      |
      2
      |
      5
     / \
    2   2
```

Concatenation of constructs reduces to the normal concatenation operator when the constructs are sequences. There is also an operation on constructs that generalizes the normal operator on bags:

DEFINITION. Let A and B be ordered sets. The <u>sum</u> of A and B , denoted A + B , is the ordered set $(V_{A+B}, \leq_{A+B})$, where $V_{A+B} = V_{A++B}$ and

$$(i0,\ c0) \leq_{A+B} (i1,\ ci)$$
$$\equiv (i0 = i1 = 0 \wedge c0 \leq_A c1) \vee (i0 = i1 = 1 \wedge c0 \leq_B c1) \quad .$$

DEFINITION. Consider constructs C(f) and C(g) , where $f \in A \to X$ and $g \in B \to Y$. The <u>sum</u> of C(f) and C(g) , denoted C(f) + C(g) , is the construct C(h) , where $h \in (A + B) \to X \cup Y$ is defined by (1).

THEOREM 6 ("domain split law"). For f defined on A + B ,

$$(\underline{C}\ i\colon i \in A + B\colon f(i))$$
$$= (\underline{C}\ i\colon i \in A\colon f(0,\ i)) + (\underline{C}\ i\colon i \in B\colon f(1,\ i)) \quad .$$

In the final part of this section, we explore the relationship between ++ and + .

DEFINITION. Let A be an ordered set. Consider $(\leq_A, f) \in A \to X$. Define

$$Bag(\leq_A,\ f) = C(=,\ f) \quad ,$$

i.e., the construct that corresponds to f considered as a mapping defined on a discretely ordered set.

THEOREM 7. For f , g in F ,

$$C(f) = C(g) \Rightarrow Bag(f) = Bag(g) \quad .$$

PROOF. Let σ be an isomorphism from A onto B . Then σ is injective, so for a0 , a1 in A ,

$$a0 = a1 \equiv \sigma(a0) = \sigma(a1) \quad .$$

Therefore σ is an isomorphism from $(V_A, =)$ onto $(V_B, =)$.

COROLLARY. We can define BAG(C0) for every construct C0 in the following way: choose an f such that C0 = C(f) and let BAG(C0) = Bag(f). The theorem ensures that the result is independent of the choice of f . Every bag B can be obtained in this way, since BAG(B) = B.

THEOREM 8. For constructs C0 , C1 ,

BAG(C0 ++ C1) = BAG(C0) + BAG(C1) .

PROOF. Write C0 = $C(\le_A, f)$, where $f \in V_A \to X$, and C1 = $C(\le_B, g)$, where $g \in B \to Y$. According to the definition of ++ , we have C0 ++ C1 = $C(\le_{A++B}, h)$, where h is defined by (1). Therefore

BAG(C0 ++ C1)
= {definition of BAG }
$Bag(\le_{A++B}, h)$
= {definition of Bag }
C(=, h)
= {definition of + }
C(=, f) + C(=, g)
= {definition of Bag }
$Bag(\le_A, f) + Bag(\le_B, g)$
= {definition of BAG }
BAG(C0) + BAG(C1) .

Canonical decomposition

We shall call a construct C(f) , with $f \in A \to X$, empty if A is empty. It follows that there is only one empty construct, namely {∅} .

We shall call a construct irreducible if it cannot be written as the concatenation or sum of nonempty constructs. Clearly, every finite construct may be built from irreducible constructs with the operators ++ and + . The question naturally occurs whether this decomposition is in some sense unique. Obviously, it cannot be completely unique, since both ++ and + are associative and + is commutative (as operators between constructs; they are so up to isomorphism as operators between ordered sets). Therefore, we must allow rearrangement of terms and introduction of parentheses that exploit these properties; for instance, [2] + ([1] + [0]) = [0] + [1] + [2] . However, apart from this the decomposition turns out to be unique. The next three theorems have the purpose of establishing this.

THEOREM 9. There are no nonempty constructs C0 , C1 , C2 , C3 with

$$C0 \mathbin{++} C1 = C2 + C3 \quad .$$

PROOF. For elements a and b of an ordered set A we define

$$a \leftrightarrow_A b \equiv (\underline{E}\ x\colon x \in A\colon (x \le_A a \lor x \ge_A a) \land (x \le_A b \lor x \ge_A b)) \quad .$$

Now ++ has the property that for nonempty ordered sets A and B ,

$$(\underline{A}\ a, b\colon a \in A \mathbin{++} B \land b \in A \mathbin{++} B\colon a \leftrightarrow_A b) \quad .$$

This can be seen in the following way. If a and b are both in the part of A ++ B that corresponds to A , every element x of the part that corresponds to B is comparable to both a and b . Similarly if they are both in the other part. If they are in different parts, they are themselves comparable and we may take the x in the definition of $\leftrightarrow_A$ to be a or b .

On the other hand, + has the property that for nonempty ordered sets A and B ,

$$(\underline{E}\ a, b: a \in A + B \wedge b \in A + B: \neg\ a \leftrightarrow_A b) \quad .$$

To see this, it suffices to take a in the part of A + B that corresponds to A and b in the part that corresponds to B .

Therefore no ordered set can be written nontrivially as both a concatenation and a sum; it follows that no construct can either.

THEOREM 10. If C0 , C1 , C2 , C3 are constructs such that

$$C0 \mathbin{++} C1 = C2 \mathbin{++} C3 \quad ,$$

there exists a construct C4 such that

$$C0 = C2 \mathbin{++} C4 \vee C2 = C0 \mathbin{++} C4 \quad .$$

PROOF. Let C0 ++ C1 be C(f) with $f \in A \to X$. Let A0 , A1 , A2 , A3 denote the parts of A corresponding to C0 , C1 , C2, C3 respectively. For $a \in A0 \setminus A2$ and $b \in A2 \setminus A0$ we have $a <_A b$ since C(f) = C0 ++ C1 , and $a >_A b$ since C(f) = C2 ++ C3 . These are mutually contradictory, so

$$A0 \setminus A2 = \emptyset \vee A2 \setminus A0 = \emptyset \quad .$$

We only consider the case $A0 \setminus A2 = \emptyset$; the other one is symmetric. Since in this case $A0 = A0 \cap A2$, it follows that $C2 = C0 \mathbin{++} C(f \mid A2 \setminus A0)$.

THEOREM 11. Suppose that C0 is a nonempty construct that is not the sum of two nonempty constructs. If C1 , C2 , C3 are constructs such that

$$C0 + C1 = C2 + C3 \quad ,$$

there exists a construct C4 such that

$$C2 = C0 + C4 \vee C3 = C0 + C4 \quad .$$

PROOF. Let $A0$ through $A3$ be defined as in the preceding proof. Obviously,

$$A0 \cap A2 \neq \emptyset \vee A0 \cap A3 \neq \emptyset \quad .$$

We assume the first disjunct; the other one is treated by exchanging the roles of $C2$ and $C3$.

If $A0 \setminus A2 \neq \emptyset$, we have $C0 = C(f \mid A0 \cap A2) + C(f \mid A0 \setminus A2)$, where both terms are nonempty. Since it is given that $C0$ is not such a sum, it follows that $A0 \setminus A2 = \emptyset$. Therefore $A0 = A0 \cap A2$, so $C2 = C0 + C(f \mid A2 \setminus A0)$.

COROLLARY. Decomposition of a finite construct into irreducible constructs by means of the operators ++ and + is unique except for the trivial variations that correspond to the associativity of both operators and the commutativity of + .

<u>Trees</u>

EXAMPLE 5. Let A denote the set of words of length at most 2 over the alphabet $\{0, 1, 2\}$. For words t and u we let $t \leq_A u$ mean that t is a prefix of u . Define f on A as the sum of the numerical values of the symbols in a word. We claim that $C(f)$ is a proper abstraction for the ternary tree

```
                 0
            /    |    \
        0        1        2
      / | \    / | \    / | \
     0  1  2  1  2  3  2  3  4
```

Note that the left-right ordering of the successors of any given node is purely arbitrary: interchanging, for instance, the leftmost 0 and 1 on the bottom row would not change $C(f)$.

It is our intention to give the name "tree" to a construct of the kind considered in Example 5. We shall now proceed to formalize the definition and explore the relationship between those constructs and the concept of a tree as it is known from graph theory.

DEFINITION. Let A be an ordered set. The relation $(<_A)$ is defined on A by

$$a\ (<_A)\ b \equiv a <_A b \lor \neg (\underline{E}\ x{:}\ x \in A{:}\ a <_A x <_A b) \quad .$$

If a and b satisfy $a\ (<_A)\ b$, we say that a is a <u>predecessor</u> of b , and that b is a <u>successor</u> of a .

DEFINITION. An ordered set A is called <u>tree-like</u> if A is finite and there exists an r in A such that for all a in A ,

$$(\underline{N}\ x{:}\ x \in A{:}\ x\ (<_A)\ a) = \underline{\text{if}}\ a = r \rightarrow 0\ \square\ a \neq r \rightarrow 1\ \underline{\text{fi}} \quad .$$

If such an r exists, it is obviously unique; we call it the <u>root</u> of A .

THEOREM 12. For every tree-like ordered set A ,

$$(\underline{N}\ x, y{:}\ x \in A \land y \in A{:}\ x\ (<_A)\ y) = |V_A| - 1 \quad .$$

PROOF. Let A be tree-like with root r . Then

$(\underline{N}\ x, y{:}\ x \in A \land y \in A{:}\ x\ (<_A)\ y)$

= {domain split}

$(\underline{S}\ y{:}\ y \in A{:}\ (\underline{N}\ x{:}\ x \in A{:}\ x\ (<_A)\ y))$

= { A is tree-like with root r }

$(\underline{S}\ y{:}\ y \in A{:}\ \underline{\text{if}}\ y = r \rightarrow 0\ \square\ y \neq r \rightarrow 1\ \underline{\text{fi}})$

= {domain split}

$(\underline{S}\ y{:}\ y \in A \land y \neq r{:}\ 1)$

= {definition of $\underline{N}$ }

$(\underline{N}\ y{:}\ y \in A{:}\ y \neq r)$

= { $r \in A$ }

$|V_A| - 1$.

THEOREM 13. In a tree-like ordered set, the root is the least element.

PROOF. Let A be tree-like with root r . Define $d \in A - \mathbf{N}$ by

$$d(a) = (\underline{N}\ x{:}\ x \in A{:}\ x <_A a) \quad .$$

The claim that every a in A satisfies $r \leq_A a$ will be proved by induction on $d(a)$. If $d(a) = 0$, we deduce from $x \; (<_A) \; a \Rightarrow x <_A a$ that $a = r$ and therefore, by reflexivity, $r \leq_A a$. If $d(a) > 0$, there is an element b in A with $b <_A a$. Now

$$
\begin{aligned}
& d(a) \\
= \; & \{\text{definition of } d\} \\
& (\underline{N}\, x: x \in A: x <_A a) \\
= \; & \{\text{domain split}\} \\
& (\underline{N}\, x: x \in A \wedge x <_A b: x <_A a) + (\underline{N}\, x: x \in A \wedge \neg\, x <_A b: x <_A a) \\
= \; & \{\, b <_A a \text{ , so } x <_A b \Rightarrow x <_A a \,\} \\
& (\underline{N}\, x: x \in A: x <_A b) + (\underline{N}\, x: x \in A \wedge \neg\, x <_A b: x <_A a) \\
> \; & \{\, \neg\, b <_A b \text{ and } b <_A a \,\} \\
& (\underline{N}\, x: x \in A: x <_A b) \\
= \; & \{\text{definition of } d\} \\
& d(b) \quad .
\end{aligned}
$$

Applying the induction hypothesis to b , we obtain $r \leq_A b$. Hence, by transitivity, $r \leq_A a$.

DEFINITION. For any tree-like ordered set A with root r , the predecessor function p_A is determined by $p_A \in A \setminus \{r\} \to A$ and $p_A(a) \; (<_A) \; a$ for all a in $A \setminus \{r\}$.

THEOREM 14. For all a and b in a tree-like ordered set A ,

$$a \leq_A b \equiv (\underline{E}\; i: i \in \mathbf{N}: a = p_A^{\,i}(b)) \quad .$$

PROOF. Define $d \in A \times A \to \mathbf{N}$ by

$$d(a, b) = (\underline{N}\, x: x \in A: a <_A x <_A b) \quad .$$

We shall prove the implication from left to right by induction on $d(a, b)$. The theorem then follows by observing that the other implication is a direct consequence of the transitivity of $\leq_A$.

Base: assume $a \leq_A b$. Then

$d(a, b) = 0$

$=$ {definition of d }

$\neg (\underline{E}\ x: x \in A: a <_A x <_A b)$

$=$ $\{a \leq_A b \equiv a = b \vee a <_A b\}$

$a = b \vee a\ (<_A)\ b$

$=$ {definition of p_A }

$a = b \vee a = p_A(b)$

$>$ {instantiation at $i = 0$ and $i = 1$ respectively}

$(\underline{E}\ i: i \in \mathbf{N}: a = p_A^i(b))$.

Step: suppose $d(a, b) > 0$. Then there exists an element c of A with $a <_A c <_A b$. As $d(a, c) < d(a, b)$, the induction hypothesis gives an i with $a = p_A^i(c)$. As also $d(c, b) < d(a, b)$, the induction hypothesis gives a j with $c = p_A^j(b)$. Therefore $a = p_A^{i+j}(b)$.

Given an ordered set A , we may construct an undirected graph G (to be referred to as "the graph of A") as follows. Vertices of G are the elements of V_A ; there is an edge between a and b iff $a\ (<_A)\ b \vee b\ (<_A)\ a$. If A is tree-like, Theorems 13 and 14 together imply that G is connected (since any two points are connected by way of the root). Moreover, by Theorem 12, the number of edges of G is 1 less than the number of vertices. This establishes that G is a tree in the sense of graph theory. The next theorem shows that every graph-theoretical tree can be obtained in this way.

THEOREM 15. Let G be a connected undirected graph of which the number of edges is 1 less than the number of vertices. Then there exists a tree-like ordered set A such that G is the graph of A .

PROOF. For V_A , we take the collection of vertices of G . A root r is chosen arbitrarily in V_A . It is known from graph theory (e.g. [3], Theorem 4.1) that G is free of cycles; therefore, for every a in V_A there exists precisely one path from a to r . Let $a \leq_A b$ mean that a occurs on the path from b to r . Then $A = (V_A, \leq_A)$ is an ordered set.

Moreover, $a\ (<_A)\ b$ now has the meaning that a is the first node after b on the path from b to r ; therefore $a\ (<_A)\ b \vee b\ (<_A)\ a$ implies the existence of an edge between a

and b .

Conversely, let a and b be connected by an edge of G . If $\neg\, a \leq_A b$, the node a does not occur on the path from b to r . Then the edge from a to b , followed by the path from b to r , constitutes a path from a to r on which b occurs as the first node after a . Hence $b\ (<_A)\ a$. We have now proved that

$$a \leq_A b \vee b\ (<_A)\ a \quad .$$

By symmetry, the same formula holds with a and b interchanged. Distributing $\wedge$ over $\vee$ yields

$$a\ (<_A)\ b \vee b\ (<_A)\ a \quad .$$

The last two paragraphs together show that two nodes are connected by an edge iff one precedes the other in the ordering of A . Therefore G is the graph of A .

REMARK. The difference in style between this proof and the preceding ones is one of the reasons why we have chosen to define constructs in the language of ordered sets rather than that of graphs.

The idea of a tree as we shall now define it is slightly different from the concept in graph theory. By a tree, we shall mean an arrangement of (not necessarily distinct) values; isomorphic trees that have the same values in corresponding places will be identified. The following definition is intended to reflect this point of view.

DEFINITION. A <u>tree</u> is a construct $C(f)$, say with $f \in A \rightarrow X$, such that A is tree-like and f is surjective.

EXAMPLE 6. In Example 5, a tree was introduced as the construct $C(f)$ of an explicitly given function f . However, it could also have been described with the aid of the operators ++ and + , since the tree involved is exactly equal to

```
[0] ++ (  ([0] ++ ([0] + [1] + [2]))
        + ([1] ++ ([1] + [2] + [3]))
        + ([2] ++ ([2] + [3] + [4]))
       )   .
```

This is no accident: in the rest of this section we shall show that every tree can be decomposed in this way.

THEOREM 16. Every tree may be built from singleton bags with the operators ++ and + .

PROOF. Consider a tree $C(f)$, where $f \in A \to X$. Let r be the root of A . We prove the statement by induction on the number of elements of A . If $|A| = 1$, we have $C(f) = [f(r)]$. If $|A| > 1$, consider

$$B = \{b \in A \mid r\ (<_A)\ b\} \quad .$$

We shall prove that

(2) $$C(f) = [f(r)] \mathbin{++} (\underline{S}\ b\colon b \in B\colon (\underline{C}\ a\colon a \in T(b)\colon f(a))) \quad ,$$

where $\underline{S}$ denotes the quantifier corresponding to the operator + and where

$$T(b) = (\{a \in A \mid b \le_A a\}, \le_A) \quad .$$

Note that

$T(b)$ is tree-like with root b

= {definition of tree-like}

$(\underline{A}\ a\colon a \in T(b)\colon (\underline{N}\ x\colon x \in T(b)\colon x\ (<_A)\ a) = \underline{\text{if}}\ a = b \to 0\ \square\ a \ne b \to 1\ \underline{\text{fi}}\ )$

= {definition of p_A }

$(\underline{A}\ a\colon a \in T(b)\colon (\underline{N}\ x\colon x \in T(b)\colon x = p_A(a)) = \underline{\text{if}}\ a = b \to 0\ \square\ a \ne b \to 1\ \underline{\text{fi}}$

$)$

$= \{$one-point rule for $\underline{N}$ $\}$

$(\underline{A}\ a:\ a \in T(b):\ p_A(a) \in T(b) \equiv a \neq b)$

$= \{$definition of T $\}$

$(\underline{A}\ a:\ b \leq_A a:\ b \leq_A p_A(a) \equiv a \neq b)$

$= \{$domain split$\}$

$(\underline{A}\ a:\ b <_A a:\ b \leq_A p_A(a)) \land \neg\, b \leq_A p_A(b)$

$= \{\ p_A(b) <_A b$, antisymmetry$\}$

$(\underline{A}\ a:\ b <_A a:\ b \leq_A p_A(a))$

$= \{$Theorem 14$\}$

$(\underline{A}\ a:\ (\underline{E}\ i:\ i > 0:\ b = p_A^{\,i}(a)):\ (\underline{E}\ i:\ i \geq 0:\ b = p_A^{\,i}(p_A(a)))$

$= \{\}$

true .

As $|T(b)| < |A|$ for all b in B , the theorem then follows by induction.

It remains to prove (2). Due to Theorem 13, it is sufficient to prove that for $a0$, $a1$ in $A \setminus \{r\}$ and $b0$, $b1$ in B ,

(3) $a0 \leq_A a1 \Rightarrow (\underline{E}\ b:\ b \in B:\ a0 \in T(b) \land a1 \in T(b))$,

(4) $b0 \neq b1 \Rightarrow T(b0) \cap T(b1) = \emptyset$.

Proof of (3):

$a0 \leq_A a1$

$= \{$Theorem 13, $a0 \neq r$ $\}$

$a0 \leq_A a1 \land r <_A a0$

$= \{$Theorem 14$\}$

$a0 \leq_A a1 \land (\underline{E}\ i:\ i > 0:\ r = p_A^{\,i}(a0))$

$= \{$dummy transformation, $b := p_A^{\,i-1}(a0)\}$

$a0 \leq_A a1 \land (\underline{E}\ b:\ b \in A:\ r\ (<_A)\ b \land b \leq_A a0)$

$> \{$transitivity$\}$

$(\underline{E}\ b:\ b \in A:\ r\ (<_A)\ b \land b \leq_A a0 \land b \leq_A a1)$

$= \{$definitions of B and T $\}$

$(\underline{E}\ b: b \in B: a0 \in T(b) \wedge a1 \in T(b))$.

Proof of (4):

$a \in T(b0) \wedge a \in T(b1)$

$= \{\text{definition of } T\}$

$b0 \leq_A a \wedge b1 \leq_A a$

$= \{\text{Theorem 14}\}$

$(\underline{E}\ i, j: i \in \mathbf{N} \wedge j \in \mathbf{N}: b0 = p_A{}^i(a) \wedge b1 = p_A{}^j(a))$

$> \{k := |j-i|\}$

$(\underline{E}\ k: k \in \mathbf{N}: b0 = p_A{}^k(b1) \vee b1 = p_A{}^k(b0))$

$= \{ p_A(b0) = r , p_A(b1) = r , r \text{ not in the domain of } p_A \}$

$b0 = b1 \vee b0 = r \vee b1 = r$

$= \{ b0 \in B , \text{ so } b0 \neq r ; b1 \in B , \text{ so } b1 \neq r \}$

$b0 = b1$.

Theorem 16 establishes that every tree may be decomposed into singleton bags. Since these are obviously irreducible, the corollary to Theorem 15 shows that this decomposition is essentially unique.

REMARK. It is not true that every finite construct can be decomposed into singleton bags. As a counterexample, consider the ordered set A consisting of {2, 3, 6, 9} with the divisibility ordering. If f is the identity function on A , then C(f) is the construct

```
2   3
|  /  |
6   9
```

This is not a concatenation, since we have, in the notation of the proof of Theorem 9, $\neg\ 2 \leftrightarrow_A 9$. Neither is it a sum, since its graph is connected.

Consider the class of all nonempty constructs that can be built from singleton bags by means of ++ and + . By Theorem 16, this class contains all trees; it obviously contains all finite nonempty sequences and bags as well. Because of the essential uniqueness of the canonical decomposition, functions on this class may be defined by describing their values on singleton bags and their behaviour under ++ and + . It is this observation that makes the use of constructs a suitable technique for those who wish to treat the Bird-Meertens formalism [2, 4] while avoiding polymorphism. Details are given in [1].

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