Skyrmions in a ferromagnetic Bose-Einstein condensate

U. Al Khawaja and H. T. C. Stoof

Institute for Theoretical Physics, University of Utrecht, Princetonplein 5,3584 CC Utrecht, The Netherlands

(April 20, 2001)

The recently realized multicomponent Bose-Einstein condensates [1–3] provide opportunities to explore the rich physics brought about by the spin degrees of freedom. For instance, we can study spin waves and phase separation [4], macroscopic quantum tunneling [5], Rabi oscillations [6], the coupling between spin gradients and superfluid flow, squeezed spin states, vortices [7] and other topological excitations. Theoretically, there have been already some studies of the ground-state properties [8–11] of these systems and their line-like vortex excitations [8,12,13]. In analogy with nuclear physics or the quantum Hall effect, we explore here the possibility of observing point-like topological excitations or skyrmions [14,15]. These are nontrivial spin textures that in principle can exist in a spinor Bose-Einstein condensate. In particular, we investigate the stability of skyrmions in a fictitious spin-1/2 condensate of ^{87}Rb atoms. We find that skyrmions can exist in this case only as a metastable state, but with a lifetime of the order of, or even longer than, the typical lifetime of the condensate itself. In addition to determining the size and the lifetime of the skyrmion, we also present its spin texture and finally briefly consider its dynamical properties.

PACS numbers: 03.75.Fi, 03.65.Db, 05.30.Jp, 32.80.Pj

An essential feature of a spinor Bose-Einstein condensate is that two or more hyperfine states of the atoms in the condensate have almost the same energy. As a result this spin degree of freedom becomes a relevant dynamical variable, which gives rise to new excitations that are not present in the usual single-component Bose-Einstein condensates, where the spins are effectively frozen. Among these excitations is the skyrmion, which is a topological spin texture. Roughly speaking, the skyrmion is a point-like object that can be created out of the ground state, in which all the spins are aligned, by reversing the average spin in a finite region of space. Although topological considerations indeed allow for these excitations, the fundamental question that needs to be answered is whether such a configuration is also energetically stable. A simple physical argument in favor of the stability of the skyrmion excitation is that, although gradients of the average local spin increase the kinetic energy of the condensate, they also result in a reduction of the density of the gas and therefore in a reduction of the interaction energy. So we may anticipate that stability occurs when these two competing factors are of the same order of magnitude. However, when solving the appropriate Gross-Pitaevskii equation, we find that in principle this reduction of the interaction energy is always insufficient to prevent the skyrmion from collapsing to zero size. Fortunately, it turns out that for sufficiently small sizes of the skyrmion another stability mechanism starts to work. A number of atoms in the centre of the skyrmion, which we denote from now on as the core atoms, will be trapped by an effective three-dimensional potential barrier, *i.e.*, a repulsive shell with a finite radius that is induced by the gradients in the spin texture of the skyrmion itself. As the skyrmion shrinks in size, the barrier height of the repulsive shell increases and the radius decreases. This leads to a squeezing of the core atoms and thus to an increase in their energy, which ultimately stabilizes the skyrmion. Of course, this is not an equilibrium state of the condensate since the core atoms will tunnel over the barrier and give the skyrmion a finite lifetime. Before discussing this in detail, we first need to point out that we perform our calculations for a uniform condensate. In spite of this, our results are also valid for trapped gases, since the size of the skyrmion always turns out to be of the order of the correlation length of the condensate, which is typically much smaller than the size of the condensate. Moreover, in order to produce realistic estimates that can possibly be compared with future experiments, we discuss only the case of a spin-1/2 ^{87}Rb condensate, since the spin-1 ^{23}Na condensate that has also been realized experimentally has an antiferromagnetic ground state. Considering other spin states or other species is in principle straightforward and will be presented elsewhere. We thus consider a uniform condensate of constant density n and constant spinor $\zeta_z = (1, 0)$, with all spins being oriented along the z-axis. This represents the ground state of the gas. For the skyrmion, however, both n and ζ are position-dependent. A convenient way to take the position dependence of $\zeta(\mathbf{r})$ into account is to express the spinor as $\zeta(\mathbf{r}) = \exp\left\{-2i\Omega(\mathbf{r}) \cdot \hat{\mathbf{S}}\right\}\zeta_z$, where $\hat{\mathbf{S}}$ are the usual angular momentum operators for spin-1/2. The physical meaning of this formula is that the average spin at a position $\mathbf{r}$ is rotated by an angle $2\Omega(\mathbf{r})$ from its initial orientation, which is in the positive z-direction, with $\Omega(\mathbf{r})/\Omega(\mathbf{r})$ the axis of rotation. An explicit form of $\Omega(\mathbf{r})$ determines now a specific texture of the skyrmion. We consider only the most symmetric shape of skyrmion, because on general grounds this is expected to have the lowest energy. We thus take $\Omega(\mathbf{r}) = \omega(r)\mathbf{r}/r$. The boundary conditions at $r = 0$ and $r \to \infty$ that the function $\omega(r)$ should satisfy are the following. First, at $r \to \infty$ all spins must be oriented as in the ground state, since otherwise it requires an infinite amount of energy to create the skyrmion. This implies $\lim_{r\to\infty}\omega(r) = 0$. Along the z-axis the spins are also not rotated by our ansatz for

$\Omega(\mathbf{r})$, so in order to have a nonsingular texture of the spinor with a nonzero winding number, we must require that $\omega(0) = 2\pi$. Finally, to avoid a singular behavior of $\Omega(\mathbf{r})$ itself, we take only functions $\omega(r)$ with zero slope at the origin. In summary, $\omega(r)$ is therefore a monotonically decreasing function that starts from 2π at the origin and reaches zero when $r \to \infty$. The specific functional form of $\omega(r)$ at intermediate distances between zero and infinity is not crucial for the stability of the skyrmion. Only the above boundary conditions are important for that. From a quantum mechanical point of view, the condensate is described by a macroscopic wave function, or order parameter, $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\zeta(\mathbf{r})$. Furthermore, the grand-canonical energy of the gas can in the usual mean-field approximation be expressed as a functional of $n(\mathbf{r})$ and $\zeta(\mathbf{r})$ [8]. In detail we have

$$E[n(\mathbf{r}),\zeta(\mathbf{r})] \equiv \int d(\mathbf{r}) \left[\frac{\hbar^2}{2m}\left(\nabla\sqrt{n(\mathbf{r})}\right)^2 - \mu n(\mathbf{r}) + \frac{\hbar^2}{2m} n(\mathbf{r})|\nabla\zeta(\mathbf{r})|^2 + \frac{1}{2}T^{2B} n^2(\mathbf{r})\right], \qquad (1)$$

where m is the mass of the atoms, μ is their chemical potential and $T^{2B} = 4\pi\hbar^2/2m$ is the appropriate coupling constant that represents the strength of the inter-atomic interactions in terms of the positive scattering length a. It is clear from the above expression that the gradients in the spin texture lead to an energy contribution, which is proportional to $|\nabla\zeta(\mathbf{r})|^2$. Using the above-mentioned form of $\zeta(\mathbf{r})$ and inserting the explicit forms of the spin-1/2 matrices, the square of the spinor gradient can be written explicitly as

$$|\nabla\zeta(\mathbf{r})|^2 = 2\left(\sin\frac{(\omega(\mathbf{r}))}{r}\right)^2 + \left(\frac{d\omega}{dr}\right)^2. \qquad (2)$$

In principle, both $n(r)$ and $\omega(r)$ can be calculated exactly by minimizing the energy functional in Eq. (2) with respect to arbitrary functions $n(r)$ and $\omega(r)$ that satisfy the boundary conditions. However, the resulting nonlinear and coupled equations are quite difficult to solve. Therefore, we use here a variational approach and take for the simple ansatz $\omega(r) = 4cot^{-2}\left[(r/\lambda)^2\right]$, where the variational parameter λ physically corresponds to the size of the skyrmion. This ansatz is chosen here because it automatically incorporates the correct boundary conditions and leads to a minimal skyrmion energy as compared to various other ansatzes that we have considered. Having specified $\omega(r)$, we then calculate $n(r)$ exactly by solving numerically the differential equation for $n(r)$ obtained by varying $E[n(\mathbf{r}),\zeta(\mathbf{r})]$ with respect to $n(\mathbf{r})$. Substituting this density profile back into the energy functional, the energy of the skyrmion becomes a function of λ only and the equilibrium properties can be obtained by minimizing this energy with respect to λ. Inserting our ansatz for $\omega(r)$ in Eq. (2), $\hbar^2|\nabla\zeta(\mathbf{r})|^2/2m$ takes the shape of an off-centered potential barrier with a maximum of $24.3\hbar^2/2m\lambda^2$ located at 0.68λ. We now distinguish between two cases. The first case occurs when the height of the barrier is lower than the chemical potential μ of the atoms. In this case, atoms can move freely across the barrier. In the second case, the barrier height is higher than the chemical potential and thus atoms become trapped behind the barrier near the centre of the skyrmion. Equating $\hbar^2|\nabla\zeta(\mathbf{r})|^2/2m$ and μ gives therefore the maximum value of $\lambda_{max} \approx 5\xi$ below which trapping takes place. Here ξ is the correlation length given by $\xi = 1/\sqrt{8\pi a n}$, and n is the density at $r \gg \lambda$. It should be noted that to obtain this value for λ_{max}, we used the fact that at sufficiently large radial distances, gradient terms in the equation of motion derived from the energy functional vanish, and we obtain $\mu = T^{2B}n$. For $\lambda < \lambda_{max}$ we calculate first the equilibrium size of the skyrmion by minimizing for a fixed number of core atoms the total energy of the condensate. Next we also calculate the tunneling rate for the core atoms to escape to the outer region (see Appendix). The results of these calculations are presented in Fig. 1, where we plot the equilibrium size of the skyrmion as a function of the number of the core atoms trapped by the texture barrier, together with the corresponding tunneling rate. As mentioned previously, we use the parameters of ^{87}Rb with a density of 10^{13} cm^{-3}. We see from this figure that only a few atoms in the core of the skyrmion are needed to stabilize it and to give it a sufficiently long lifetime. We note that the tunneling rate is considerably less than the decay rate of the condensate, which is due to two-body collisions and equal to [16] Gn with a measured value of [1] $G \approx 2.2 \times 10^{-14}$cm^{-3}/s. We also mention that the lifetime becomes even much larger for slightly smaller values of n. Finally, we present some of the interesting properties of the skyrmion, which are its texture and its dynamics. The texture can be best presented in terms of the three average spin components $\langle S_x\rangle(\mathbf{r}) \equiv \dagger\zeta(\mathbf{r})\hat{S}_x\zeta(\mathbf{r})$, $\langle S_y\rangle(\mathbf{r})$, and $\langle S_z\rangle(\mathbf{r})$ in the three Cartesian planes. These are shown in Fig. 2. The most interesting dynamical property of the skyrmion comes from the fact that if all spins of the texture are rotated around the z-axis by a constant angle θ, *i.e.*, $\zeta(\mathbf{r}) \to \exp(-i\theta S_z)\zeta(\mathbf{r})$, the resulting skyrmion will have the same energy. As a result the angle θ undergoes phase diffusion. It also leads to Josephson-like coupling between two skyrmions, which will have important consequences for the physics of a skyrmion lattice [17]. Another dynamical property is the center of mass motion of the skyrmion, which can be shown to be identical to that of a free particle. Both dynamical properties will be discussed in detail in a future publication (U. and H. T. C., in preparation).

APPENDIX

To calculate the energy of the skyrmion we need to solve the equation for the density profile that is obtained from minimizing the energy functional in Eq. (2). We solve this equation numerically for the region outside the core. Inside the core we solve the equation analytically by using the Thomas-Fermi approximation, which amounts to neglecting the gradients of the density profile. Using our ansatz for $\omega(r)$, the texture gradient potential $V(r) = \hbar^2|\nabla\zeta(\mathbf{r})|^2/2m$ reads

$$V(r) = \frac{32\hbar^2}{2m\lambda^2}\frac{(r/\lambda)^2\left[3+2(r/\lambda)^4+3(r/\lambda)^8\right]}{[1+(r/\lambda)^4]^4}. \tag{3}$$

For small r/λ this potential can be approximated by a harmonic potential with a characteristic frequency $\omega_0 = \sqrt{96\hbar^2/2m^2\lambda^4}$ and width $l = \lambda/\sqrt{96}$, as shown in the dotted curve in Fig. 3. The use of a Thomas-Fermi approximation is justified when the mean-field interaction energy is larger than the spacing between the lowest energy levels of the harmonic trap. Specifically, the ratio $2Na/l = 2\sqrt{96}Na/\lambda$ should be bigger than 1. From Fig. 1, we observe that this ratio equals approximately 1 for $N = 4$ and it increases for larger N. The lifetime of the skyrmion is estimated by calculating the tunneling rate from the core to the outer region over the barrier $V(r)$. To this end we employ the following WKB expression for the tunneling rate [18]

$$\Gamma \approx \frac{\omega_0}{2\pi}\exp\left[-2\int_{r_1}^{r_2} dr\sqrt{\frac{2m}{\hbar^2}\left(V(r)-\mu_{core}\right)}\right], \tag{4}$$

where μ_{core} is the chemical potential of the core atoms. The radial points r_1 and r_2 are the points where $v(r)$ and μ_{core} intersect as shown in Fig. 3. The chemical potential μ_{core} is calculated by differentiating the total energy of the core with respect to the number of core atoms.

ACKNOWLEDGEMENTS

The authors would like to thank Michiel Bijlsma for help in the numerical calculations and for the helpful remarks. We would also like to thank James Anglin, Gerard ‘t Hooft, David Olive, and Jan Smit for useful discussions. This work is supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM), which is financially supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).

[1] Myatt, C. J., Burt, E. A.,Ghrist, R. W., Cornell, E. A., and Wieman, C. E. Production of two overlapping Bose-Einstein condensates by sympathetic cooling. *Phys. Rev. Lett.* **78**, 586-589 (1997).

[2] Stamper-Kurn, D. M. et al. Optical confinment of a Bose-Einstein condensate. *Phys. Rev. Lett.* **80**, 2027-2030 (1998).

[3] Stenger, J. et al. Spin domains in ground-state Bose-Einstein condensates. *Nature* **396**, 345-348(1998).

[4] Hall, D. S., Matthews, M. R., Ensher, J. R., Wieman, C. E. and Cornell, E. A. Dynamics of component separation in a binary mixture of Bose-Einstein condensates. *Phys. Rev. Lett.* **81**, 1539-1542 (1998).

[5] Stamper-Kurn, D. M., et al. Quantum tunneling across spin domains in a Bose condensate. *Phys. Rev. Lett.* **83**, 661-664 (1999).

[6] Matthews, M. R., et al. Watching a superfluid untwist itself: Recurrence of Rabi oscillations in a Bose-Einstein condensaste. *Phys. Rev. Lett.* **83**, 3358-3361 (1999).

[7] Matthews, M. R., et al. Vortices in a Bose-Einstein condensate. *Phys. Rev. Lett.* **83**, 2498-2501 (1999).

[8] Ho, T.-L. Spinor Bose condensates in optical traps. *Phys. Rev. Lett.* **81**, 742-745 (1998).

[9] Ohmi, T. and Machida, K. Bose-Einstein condensation with internal degrees of freedom in alkali atom gases. *J. Phys. Soc. Jpn.* **67**, 1822-1825 (1998).

[10] Law, C. K., Pu, H and Bigelow, N. B. Quantum spins mixing in spin Bose-Einstein condensates. *Phys. Rev. Lett.* **81**, 5257-5261 (1998).

[11] Ho, T.-L. and Yip, S.-K. Fragmented and single condensate ground states of spin-1 Bose gas. *Phys. Rev. Lett.* **84**, 4031-4034 (2000).

[12] Yip, S.-K. Internal vortex structure of a trapped spinor Bose-Einstein condensate. *Phys. Rev. Lett.* 83, 4677-4681 (1999).

[13] Williams, J. E. and Holland, M. J. Preparing topological states of a Bose-Einstein condensate. *Nature* **401**, 568-572 (1999).

[14] Skyrme, T. H. R. A non-linear field theory. *Proc. Soc. A* **260**, 127-138 (1961)

[15] Skyrme, T. H. R. A unified field theory of mesons and baryons. *Nucl. Phys.* **31**, 556-569 (1962).

[16] Julienne, P. S., Mies, F. H., Tiesinga, E. and Williams, C. J. Collisional stability of double Bose condensates. *Phys. Rev. Lett.* **78**, 1880-1883 (1997).

[17] Ct, R. et al. Collective Excitations, NMR, and phase transitions in Skyrme Crystals. *Phys. Rev. Lett.* **78**, 4825-4828 (1997).

[18] Stoof, H. T. C. Macroscopic quantum tunneling of a Bose-Einstein condensate. *J. Stat. Phys.* **87**, 1353-1366 (1997).

FIGURE CAPTIONS

FIG. 1. Decay rate and size of the skyrmion as a function of the number of atoms trapped in the core. The two curves are calculated for the parameters of a ^{87}Rb spinor condensate with a density of $10^{13}\mathrm{cm}^{-3}$. The size is calculated in units of the correlation length , which for these parameters equals 0.85μm.

FIG. 2. The average spin texture of the skyrmion. Shown are the components $\langle S_z(\mathbf{r})\rangle$ and $\langle S_y(\mathbf{r})\rangle$ in the three Cartesian planes. A value of $\lambda \approx \xi$, which corresponds to 20 core atoms, was chosen to compute these quantities. The distances are in units of the coherence length ξ. The $\langle S_x(\mathbf{r})\rangle$ component can be obtained from the $\langle S_y(\mathbf{r})\rangle$ component, by using the cylindrical symmetry of the skyrmion. Fig. 1a shows most clearly that our skyrmion is composed of two coaxial tori. The two pink rings represent the cores of these two tori where $\langle S_z(\mathbf{r})\rangle$ reaches its maximum negative value -1/2. Fig. 1b shows the cross sections of the two tori.

FIG. 3. Figure 3 The potential barrier produced by the skyrmion texture. The value of λ used for this figure is again approximately xi, which corresponds to 20 core atoms. The lifetime of the skyrmion is determined by the tunneling of core atoms with a chemical potential μ_{core} to the right of the barrier. The square root of the shaded area is the integrand of Eq. (3). The dotted curve represents the harmonic approximation to $V(r)$ for small r/λ. The inset shows $\omega(r)$ for the same value of λ.