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A SET OF OSCILLATORS IN A COMMON BATH

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The well-known model of a heavy oscillator interacting with a bath of light oscillators is extended to several heavy oscillators. New features appear when they are in resonance. These are worked out for the case of two identical oscillators, with the following conclusions. If initially one of them is excited its damping is slowed down by the presence of the other. If initially both are excited their emitted field exhibits second order coherence and therefore "interference between photons". If initially the bath is in thermal equilibrium the oscillators end up by being thermally excited and uncorrelated. Finally the formulation in terms of a Langevin equation is discussed.

1. Survey of the problem

The classical equations of motion of a set of linearly coupled harmonic oscillators can be solved explicitly; one only needs to find the normal modes, which amounts to diagonalizing a matrix[1]). This is even true for an arbitrarily large number of oscillators, provided that the matrix is sufficiently regular to be diagonalizable. An example is provided by the theory of lattice vibrations[2]). In the years around 1960 the idea arose that such an explicitly solvable system could be used to investigate the assumptions that are usually made for many-body systems in order to obtain such phenomena as dissipation and fluctuations. Mazur and Montroll[3]) and Ford, Kac and Mazur[4]) and others[5]) studied arrays of particles of equal mass, Mazur and Braun[6]) and others[7–9]) considered the fate of one heavy particle embedded in such an array, and Ullersma[10]) more generally treated a heavy oscillator in a bath of independent oscillators. Ford, Kac and Mazur also pointed out that the solution remains valid in quantum mechanics, thanks to the linearity of the equations of motion, see also refs. 11 and 12.

Incidentally, the fact that such a system is integrable implies that it is not ergodic, not chaotic, and therefore does not have the properties that are often regarded as prerequisite for stochastic behavior. This confirms the view that such mathematical prerequisites are too strong; it may be true that they are required in order that the system behaves irreversibly and stochastic in *all* its aspects, but all one needs to know is that it behaves so in its *physically observed* aspects. Thus, the energies of the normal modes may well be

conserved, but the velocity of the heavy particle is damped. The fact that ergodicity and chaos are not needed is fortunate, because these concepts do not carry over to quantum mechanics[13]).

The present article is concerned with the case of *several* heavy oscillators coupled to one bath of light oscillators. For this purpose a very useful theorem is established (section 2), which makes the calculations possible (section 3). When the proper frequencies of the heavy oscillators are far apart they behave roughly as if they were independent oscillators, each with its own bath. If, however, two frequencies are near each other (within the level broadening caused by the interaction with the bath), the two oscillators are no longer independent since both interact with the same set of bath oscillators. For simplicity we shall study in detail only the case of strict resonance, both oscillators having the same proper frequency and also the same coupling strength with the bath (section 4).

The purpose is to illustrate various phenomena in quantum optics by explicit calculation. Consider two identical atoms, located at two different points in space, not too far apart. If initially one of them is in an excited state, it will emit radiation; but it turns out that the emission rate is less than if this atom were alone. The reason is that the emitted energy is not free to disappear right away into infinity but tarries in the second atom (section 5).

Again, if initially both atoms are excited, they will emit radiation into the same normal modes of the field, which gives rise to a kind of interference (section 6). The fact that the light from two independent sources may interfere is well established[14]). Yet it is sometimes felt to be incompatible with the basic tenets of quantum mechanics, as discussed by Bohr[15]) and expressed in Dirac's dictum: "Interference between different photons never occurs"[16]). Experiments that failed to show correlations between photons have been hailed as evidence for the correctness of quantum mechanics[17]).

The paradox is caused by the ambiguity of the word "photon". If one means by it an excitation of one *normal mode* of the electromagnetic field, it cannot interfere with an excitation of another mode, because different modes are orthogonal and their creation and annihilation operators commute. If, however, it means a *wave packet*, i.e., the excitation of a linear combination of these modes (such as emitted by an atom), then it may well interfere with a similar wave packet, because they both involve the same normal modes. This kind of interference does not appear in the intensity of the field at any one point, but in the correlation of the intensities at two points in space or time[18,19]).

Finally, if initially the bath oscillators are in thermal equilibrium, the two heavy oscillators will – after an initial transient period of the order of the emission time – also be thermally excited, with the bath temperature and

uncorrelated to each other (section 7). Whether or not this situation can be described by a quantum mechanical Langevin force is an interesting question[20,21]) which is briefly discussed in section 8.

2. The model

Our model consists of a set of heavy oscillators, labeled r, with frequencies Ω_r and annihilation and creation operators A_r, $A_r^\dagger$. Moreover there is a bath of light oscillators, labeled $\boldsymbol{k}$, with frequencies k and operators a_k, $a_k^\dagger$. The symbol $\boldsymbol{k}$ is a combination of the frequency k and all additional parameters needed to specify the small oscillator. (If the bath consists of the electromagnetic field, these parameters are the direction and polarization.) For the interaction we take the simplest possible form, given by the Hamiltonian

$$H = \sum_r \Omega_r A_r^\dagger A_r + \sum_k k a_k^\dagger a_k + \sum_{rk} (v_{rk} A_r^\dagger a_k + v_{rk}^* A_r a_k^\dagger) . \tag{1}$$

As long as we are playing with a mathematical model the coupling constants v_{rk} may be chosen freely, subject to some general restrictions imposed later. When (1) is applied to the emission of light by atoms the v_{rk} have definite values involving (apart from normalization factors) the dipole moment of the atom, while its position $\boldsymbol{R}_r$ enters into v_{rk} through a factor $\exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_r]$. Of course this application is somewhat precarious since it replaces each atom with a harmonic oscillator and moreover makes the rotating wave approximation, with the result that the total number of quanta is a conserved quantity.

From (1) one obtains for the Heisenberg operators the equations of motion

$$\mathrm{i}\dot{A}_r = \Omega_r A_r + \sum_k v_{rk} a_k , \tag{2a}$$

$$\mathrm{i}\dot{a}_k = k a_k + \sum_r v_{rk}^* A_r . \tag{2b}$$

This is a set of linear equations for the variables $\{A_r(t), a_k(t)\}$ with the hermitian matrix

$$H = \begin{pmatrix} \Omega_r \delta_{rr'} & v_{rk'} \\ v_{r'k}^* & k\delta_{k,k'} \end{pmatrix} . \tag{3}$$

The primed subscripts label the columns. For the purpose of solving these equations the fact that A_r, a_k are operators is irrelevant and they may be treated as scalar variables.

The general solution with given initial values at $t = 0$ can be written formally

$$\begin{pmatrix} A_r(t) \\ a_k(t) \end{pmatrix} = \frac{1}{2\pi i} \int_L e^{-i\lambda t}\, d\lambda [\lambda - H]^{-1} \begin{pmatrix} A_{r'}(0) \\ a_{k'}(0) \end{pmatrix} . \tag{4}$$

Here $[\lambda - H]^{-1}$ is the resolvent operator, and the contour L surrounds all eigenvalues. Alternatively one may write the solution in terms of an evolution matrix or propagator,

$$A_r(t) = \sum_{r'} U_{rr'}(t) A_{r'}(0) + \sum_{k'} V_{rk'}(t) a_{k'}(0) , \tag{5}$$

$$a_k(t) = \sum_{r'} W_{kr'}(t) A_{r'}(0) + \sum_{k'} S_{kk'}(t) a_{k'}(0) . \tag{6}$$

The evolution matrix

$$\begin{pmatrix} U(t) & V(t) \\ W(t) & S(t) \end{pmatrix} \tag{7}$$

is unitary.

For an actual solution, however, one still has to find the eigenvectors and eigenvalues ω of (3) from

$$\Omega_r A_r + \sum_{k'} v_{rk'} a_{k'} = \omega A_r , \tag{8}$$

$$k a_k + \sum_{r'} v^*_{r'k} A_{r'} = \omega a_k . \tag{9}$$

For the case that there is a single heavy oscillator this problem has been solved many times. One obtains from (9), omitting the now superfluous index r,

$$a_k = \frac{v^*_k}{\omega - k} A \tag{10}$$

and substitution in (8) yields the eigenvalue equation

$$\omega = \Omega + \sum_k \frac{|v_k|^2}{\omega - k} \equiv K(\omega) . \tag{11}$$

If one plots the function $K(\omega)$ one sees immediately that there is one solution ω_ν between any two successive values of k. Each of them gives an eigenvector, whose first component A^ν is arbitrary and whose other components a^ν_k are given by (10). The normalization then determines A^ν (which may be taken real):

$$|A^{\nu}|^2\left[1+\sum_k \frac{|v_k|^2}{(\omega_\nu - k)^2}\right]=1\,. \tag{12}$$

Having found the eigenvalues and normalized eigenvectors one can express the evolution operator in terms of them. For instance,

$$U(t)=\sum_{\nu} A^{\nu}\,\mathrm{e}^{-\mathrm{i}\omega_\nu t}(A^{\nu})^*\,. \tag{13}$$

This may again be written as a contour integral,

$$U(t)=\frac{1}{2\pi\mathrm{i}}\int_L \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\lambda[\lambda - K(\lambda)]^{-1}\,. \tag{14}$$

It is evident that the poles are the eigenvalues. What is remarkable, however, is that the residues reproduce the required normalization, thanks to the fact that the factor [ ] in (12) happens to be the derivative of $\omega - K(\omega)$ at $\omega = \omega_\nu$.

The theorem that will enable us to treat the case of more heavy oscillators is the generalization of this fact. Again solve (9) to obtain

$$a_k=\frac{1}{\omega - k}\sum_{r'} v^*_{r'k}A_{r'} \tag{15}$$

and substitute in (8):

$$\omega A_r=\Omega_r A_r+\sum_{r'}\left[\sum_k \frac{v_{rk}v^*_{r'k}}{\omega - k}\right]A_{r'}\equiv\sum_{r'} K_{rr'}(\omega)A_{r'}\,. \tag{16}$$

Our basic theorem states *that the same equation (14) remains true as an equation expressing the matrix U(t) in the matrix K(λ).* This is proved in the appendix. There also the other blocks of the evolution matrix (7) are found:

$$V_{rk}(t)=\sum_{r'}\frac{1}{2\pi\mathrm{i}}\int_L \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\lambda\,\frac{1}{\lambda - k}\left[\frac{1}{\lambda - K(\lambda)}\right]_{rr'}v_{r'k}\,, \tag{17}$$

$$W_{kr}(t)=\sum_{r'} v^*_{r'k}\frac{1}{2\pi\mathrm{i}}\int_L \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\lambda\,\frac{1}{\lambda - k}\left[\frac{1}{\lambda - K(\lambda)}\right]_{r'r}\,, \tag{18}$$

$$S_{kk'}(t)=\frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\lambda}{\lambda - k}\left\{\delta_{kk'}+\sum_{rr'} v^*_{rk}\left[\frac{1}{\lambda - K(\lambda)}\right]_{rr'}v_{r'k'}\,\frac{1}{\lambda - k'}\right\}. \tag{19}$$

3. The dense spectrum approximation

We first compute the matrix $U_{rr'}(t)$ given by (14). Since the eigenvalues ω_ν are real the integration path L consists of two parallel lines above and below the real axis. For $t>0$ the contribution of the latter vanishes, so that

$$U(t) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega [\omega - K(\omega + i\varepsilon)]^{-1} . \tag{20}$$

According to (16) the matrix K is

$$K_{rr'}(\omega + i\varepsilon) = \Omega_r \delta_{rr'} + \sum_k \frac{v_{rk} v^*_{r'k}}{\omega + i\varepsilon - k} . \tag{21}$$

Now assume that *the k-values lie very dense*[22]) and define the strength function $\sigma_{rr'}(k)$ by

$$\sigma_{rr'}(k)\, \Delta k = \sum_{k<|k|<k+\Delta k} v_{rk} v^*_{r'k} . \tag{22}$$

Then (21) may be written

$$K_{rr'}(\omega + i\varepsilon) = \Omega_r \delta_{rr'} + \int \frac{\sigma_{rr'}(k)}{\omega + i\varepsilon - k}\, dk . \tag{23}$$

This approximation requires ε to be large compared to the interval between successive k-values. If at the same time ε is small compared to a typical k-interval over which $\sigma_{rr'}(k)$ varies appreciably, the integral is the sum of a principal-value integral and half the residue at the pole:

$$K_{rr'}(\omega + i\varepsilon) = \Omega_r \delta_{rr'} + \int\!\!\!\!\!\!- \frac{\sigma_{rr'}(k)}{\omega - k}\, dk - \pi i \sigma_{rr'}(\omega) . \tag{24}$$

The dense spectrum approximation consists in the assumption that there exists an ε that obeys both requirements. It is obviously fulfilled when the bath is the electromagnetic field – unless that field is enclosed in a cavity, as in a laser.

For $r = r'$ the principal-value integral constitutes a small shift of the frequency Ω_r which may be neglected. For $r \neq r'$ it constitutes a coupling between the oscillators r and r'. Yet it is small compared to the residue term, provided that $\sigma_{rr'}(k)$ varies smoothly with k. In the case of atoms in the electromagnetic field $\sigma_{rr'}(k)$ contains the factor $\exp[i\boldsymbol{k}\cdot(\boldsymbol{R}_r - \boldsymbol{R}_{r'})]$; our proviso requires that the distance $|\boldsymbol{R}_r - \boldsymbol{R}_{r'}|$ times the range of relevant k – values, i.e., the natural line width, is small. In other words the distance must be small compared to the coherence length of the emitted light.

To proceed with the computation of (20) we have to find the reciprocal of the matrix

$$(\omega - \Omega_r)\delta_{rr'} + \pi i \sigma_{rr'}(\omega) . \tag{25}$$

Now suppose that *the coupling with the bath is weak*; more precisely, that the σ are small on the scale of the Ω_r. Alternatively one may say that the oscillators r must be heavy, because the σ are proportional to their reciprocal masses. At any rate the determinant of (25) is, to first order in σ,

$$D = \prod_r [\omega - \Omega_r + \pi i \sigma_{rr}(\omega)] . \tag{26}$$

To the same order I may insert $\sigma_{rr}(\omega) = \sigma_{rr}(\Omega_r)$. This determinant vanishes at the points $\Omega_r - \pi i \sigma_{rr}(\Omega_r)$, which are the original frequencies with damping added. As a consequence the integral (20) consists of a sum of contributions from these poles,

$$U(t) = \sum_r C_r \exp[-i\Omega t - \pi \sigma_{rr}(\Omega_r)t] . \tag{27}$$

Each term is the same as one would find if only that particular oscillator were present. In the next order also off-diagonal $\sigma_{rr'}$ enter, so that some interaction between the oscillators appears. More interesting, however, is the case that two of the Ω_r are equal or almost equal. If their difference is not large compared to σ the two oscillators strongly interact and cannot be separated as in (27).

4. Resonance

Suppose that there are just two heavy oscillators, which we take identical,

$$\Omega_1 = \Omega_2 \equiv \Omega , \quad \pi\sigma_{11}(\Omega) = \pi\sigma_{22}(\Omega) \equiv \Gamma . \tag{28}$$

Moreover we set

$$\pi\sigma_{12} = \Gamma' e^{i\gamma} , \quad \pi\sigma_{21} = \Gamma' e^{-i\gamma} , \quad \Gamma' > 0 . \tag{29}$$

The matrix (25) takes the form

$$\omega - K = \begin{pmatrix} \omega - \Omega + i\Gamma & i\Gamma' e^{i\gamma} \\ i\Gamma' e^{-i\gamma} & \omega - \Omega + i\Gamma \end{pmatrix} . \tag{30}$$

Its determinant is

$$D = (\omega - \Omega + \mathrm{i}\Gamma - \mathrm{i}\Gamma')(\omega - \Omega + \mathrm{i}\Gamma + \mathrm{i}\Gamma')$$
$$\equiv (\omega - \Lambda_1)(\omega - \Lambda_2)\,. \tag{31}$$

The common frequency Ω turns out to be split into two frequencies Λ_1, Λ_2, which have the same real part. Of course $\operatorname{Im}\Lambda_2 = -(\Gamma + \Gamma') < 0$, and it is also true that $\operatorname{Im}\Lambda_1 = -(\Gamma - \Gamma') < 0$, because $|\sigma_{12}|^2 < \sigma_{11}\sigma_{22}$. (This would not be true if the bath oscillators were uniquely characterized by their frequency k alone, that is, if the label $\boldsymbol{k}$ contained no other parameters specifying the direction or the polarization.)

The reciprocal of (30) is

$$[\omega - K]^{-1} = \frac{1}{D(\omega)}\begin{pmatrix} \omega - \Omega + \mathrm{i}\Gamma & -\mathrm{i}\Gamma'\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\mathrm{i}\Gamma'\,\mathrm{e}^{-\mathrm{i}\gamma} & \omega - \Omega + \mathrm{i}\Gamma \end{pmatrix}. \tag{32}$$

The residue at the pole Λ_1 is

$$\frac{1}{\Lambda_1 - \Lambda_2}\begin{pmatrix} \mathrm{i}\Gamma' & -\mathrm{i}\Gamma'\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\mathrm{i}\Gamma'\,\mathrm{e}^{-\mathrm{i}\gamma} & \mathrm{i}\Gamma' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{e}^{\mathrm{i}\gamma} \\ -\mathrm{e}^{-\mathrm{i}\gamma} & 1 \end{pmatrix}. \tag{33}$$

The residue at Λ_2 is the same without the minus signs in front of the off-diagonal elements. Hence

$$U(t) = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\Lambda_1 t}\begin{pmatrix} 1 & -\mathrm{e}^{\mathrm{i}\gamma} \\ -\mathrm{e}^{-\mathrm{i}\gamma} & 1 \end{pmatrix} + \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\Lambda_2 t}\begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\gamma} \\ \mathrm{e}^{-\mathrm{i}\gamma} & 1 \end{pmatrix}$$
$$= \mathrm{e}^{-\mathrm{i}\Omega t - \Gamma t}\begin{pmatrix} \cosh\Gamma' t & -\mathrm{e}^{\mathrm{i}\gamma}\sinh\Gamma' t \\ -\mathrm{e}^{-\mathrm{i}\gamma}\sinh\Gamma' t & \cosh\Gamma' t \end{pmatrix}. \tag{34}$$

Next we compute $V_{r\boldsymbol{k}}(t)$, given by (17),

$$V_{r\boldsymbol{k}}(t) = -\sum_{r'}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\,\frac{1}{\omega + \mathrm{i}\varepsilon - k}\left[\frac{1}{\omega - K(\omega + \mathrm{i}\varepsilon)}\right]_{rr'} v_{r'\boldsymbol{k}}\,. \tag{35}$$

There are two contributions from the poles Λ_1, Λ_2 and one from the pole at k.

$$\begin{pmatrix} V_{1\boldsymbol{k}}(t) \\ V_{2\boldsymbol{k}}(t) \end{pmatrix} = \frac{1}{2}\,\frac{\mathrm{e}^{-\mathrm{i}\Lambda_1 t}}{\Lambda_1 - k}\begin{pmatrix} v_{1\boldsymbol{k}} - \mathrm{e}^{\mathrm{i}\gamma}v_{2\boldsymbol{k}} \\ -\mathrm{e}^{-\mathrm{i}\gamma}v_{1\boldsymbol{k}} + v_{2\boldsymbol{k}} \end{pmatrix} + \frac{1}{2}\,\frac{\mathrm{e}^{-\mathrm{i}\Lambda_2 t}}{\Lambda_2 - k}\begin{pmatrix} v_{1\boldsymbol{k}} + \mathrm{e}^{\mathrm{i}\gamma}v_{2\boldsymbol{k}} \\ \mathrm{e}^{-\mathrm{i}\gamma}v_{1\boldsymbol{k}} + v_{2\boldsymbol{k}} \end{pmatrix}$$
$$+ \frac{\mathrm{e}^{-\mathrm{i}kt}}{(k - \Lambda_1)(k - \Lambda_2)}\begin{pmatrix} (k - \Omega + \mathrm{i}\Gamma)v_{1\boldsymbol{k}} - \mathrm{i}\Gamma'\,\mathrm{e}^{\mathrm{i}\gamma}v_{2\boldsymbol{k}} \\ -\mathrm{i}\Gamma'\,\mathrm{e}^{-\mathrm{i}\gamma}v_{1\boldsymbol{k}} + (k - \Omega + \mathrm{i}\Gamma)v_{2\boldsymbol{k}} \end{pmatrix}. \tag{36}$$

Similarly one finds for (18)

$$W_{k1}(t) = \frac{1}{2}\frac{e^{-i\Lambda_1 t}}{\Lambda_1 - k}(v_{1k}^* - e^{-i\gamma}v_{2k}^*) + \frac{1}{2}\frac{e^{-i\Lambda_2 t}}{\Lambda_2 - k}(v_{1k}^* + e^{-i\gamma}v_{2k}^*)$$
$$+ \frac{e^{-ikt}}{(k-\Lambda_1)(k-\Lambda_2)}\{(k - \Omega + i\Gamma)v_{1k}^* - i\Gamma' e^{-i\gamma}v_{2k}^*\}, \tag{37a}$$

$$W_{k2}(t) = \frac{1}{2}\frac{e^{-i\Lambda_1 t}}{\Lambda_1 - k}(-e^{i\gamma}v_{1k}^* + v_{2k}^*) + \frac{1}{2}\frac{e^{-i\Lambda_2 t}}{\Lambda_2 - k}(e^{i\gamma}v_{1k}^* + v_{2k}^*)$$
$$+ \frac{e^{-ikt}}{(k-\Lambda_1)(k-\Lambda_2)}\{-i\Gamma' e^{i\gamma}v_{1k}^* + (k - \Omega + i\Gamma)v_{2k}^*\}. \tag{37b}$$

For completeness we add the formula for $S_{kk'}$, although we shall have no need of it.

$$S_{kk'}(t) = e^{-ikt}\delta_{kk'}$$
$$+ \frac{e^{-ikt}}{k-k'}\frac{(k - \Omega + i\Gamma)P_{kk'} - i\Gamma' Q_{kk'}}{(k-\Lambda_1)(k-\Lambda_2)}$$
$$+ \frac{e^{-ik't}}{k'-k}\frac{(k' - \Omega + i\Gamma)P_{kk'} - i\Gamma' Q_{kk'}}{(k'-\Lambda_1)(k'-\Lambda_2)}$$
$$+ \frac{e^{-i\Lambda_1 t}}{2}\frac{P_{kk'} - Q_{kk'}}{(k-\Lambda_1)(k'-\Lambda_1)} + \frac{e^{-i\Lambda_2 t}}{2}\frac{P_{kk'} + Q_{kk'}}{(k-\Lambda_2)(k'-\Lambda_2)}, \tag{38}$$

where we have used the abbreviations

$$P_{kk'} = v_{1k}^* v_{1k'} + v_{2k}^* v_{2k'}, \tag{39a}$$

$$Q_{kk'} = e^{i\gamma}v_{1k}^* v_{2k'} + e^{-i\gamma}v_{2k}^* v_{1k'}. \tag{39b}$$

5. Emission from one oscillator

First suppose that initially oscillator 1 is in its first excited state, while oscillator 2 and all bath oscillators are in their ground state:

$$|\Phi\rangle = A_1^\dagger|0\rangle, \tag{40}$$

where $|0\rangle$ denotes the ground state of the entire system. The expectation value of the energy of that oscillator is at $t > 0$

$$\begin{aligned}\langle \Phi|\Omega A_1^\dagger(t)A_1(t)|\Phi\rangle &= \Omega\langle 0|A_1A_1^\dagger(t)A_1(t)A_1^\dagger|0\rangle \\ &= \Omega\sum_{rr'} U_{1r}^*(t)U_{1r'}(t)\langle 0|A_1A_r^\dagger A_{r'}A_1^\dagger|0\rangle \\ &= \Omega|U_{11}(t)|^2 \\ &= \tfrac{1}{4}\Omega\,\mathrm{e}^{-2\Gamma t}(\mathrm{e}^{2\Gamma' t}+2+\mathrm{e}^{-2\Gamma' t})\,. \end{aligned} \tag{41}$$

The term with $\mathrm{e}^{2\Gamma' t}$ shows that the presence of the second oscillator *reduces* the rate at which the first one looses its energy. During the emission the energy partly resides in the second oscillator:

$$\begin{aligned}\langle \Phi|\Omega A_2^+(t)A_2(t)|\Phi\rangle &= \Omega|U_{21}(t)|^2 \\ &= \tfrac{1}{4}\Omega\,\mathrm{e}^{-2(\Gamma-\Gamma')t} - \tfrac{1}{2}\Omega\,\mathrm{e}^{-2\Gamma t} + \tfrac{1}{4}\Omega\,\mathrm{e}^{-2(\Gamma+\Gamma')t}\,. \end{aligned} \tag{42}$$

In the slowest mode the energy is equally divided between both oscillators.

In conventional quantum mechanical perturbation theory the situation looks as follows. The unperturbed Hamiltonian has two degenerate eigenstates, corresponding to either the one or the other oscillator being excited. The interaction through the field selects two linear combinations of them, symmetric and antisymmetric. The amplitude of the former decays like $\mathrm{e}^{-(\Gamma+\Gamma')t}$ and that of the latter like $\mathrm{e}^{-(\Gamma-\Gamma')t}$. The fast decay is called "superradiance".

It is also possible to find the higher moments of the energy, and in fact its entire probability distribution at any time t. We omit the factor Ω; the remaining operator is the occupation number $N_r = A_r^\dagger A_r$ rather than the energy. The characteristic function of the distribution of N_1 is

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}qN_1(t)}\rangle &= \langle \Phi|\exp[\mathrm{i}qA_1^\dagger(t)A_1(t)]|\Phi\rangle \\ &= 1+\sum_{n=1}^{\infty}\frac{(\mathrm{i}q)^n}{n!}\langle \Phi|A_1^\dagger(t)A_1(t)\cdots A_1^\dagger(t)\underline{A_1(t)A_1^\dagger(t)}A_1(t)|\Phi\rangle\,. \end{aligned} \tag{43}$$

If one commutes the two underlined factors the two annihilation operators acting on Φ give zero. Hence only the commutator survives, which equals unity. In this way one obtains

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}qN_1}\rangle &= 1+\sum_{n=1}^{\infty}\frac{(\mathrm{i}q)^n}{n!}\langle \Phi|A_1^\dagger(t)A_1(t)|\Phi\rangle \\ &= 1+(\mathrm{e}^{\mathrm{i}q}-1)|U_{11}|^2\,. \end{aligned} \tag{44}$$

This is the characteristic function of a random variable N_1 with possible values 0, 1 and corresponding probabilities $1-|U_{11}|^2$ and $|U_{11}|^2$. Yet *one cannot interpret the process as the random emission of a photon*, because that would require $|U_{11}(t)|^2$ to be a single exponential rather than a combination of three.

The joint probability distribution $P_{N_1N_2}(t)$ can be found from the joint characteristic function

$$\langle e^{iq_1N_1+iq_2N_2}\rangle = 1+\sum_{n=1}^{\infty}\frac{i^n}{n!}\langle\Phi|\{q_1A_1^\dagger(t)A_1(t)+q_2A_2^\dagger(t)A_2(t)\}^n|\Phi\rangle \tag{45}$$

in a similar way. The result is

$$1+(e^{iq_1}-1)|U_{11}|^2+(e^{iq_2}-1)|U_{21}|^2\,. \tag{46}$$

The only non-vanishing probabilities are

$$p_{10}(t)=|U_{11}|^2\,,\quad p_{01}(t)=|U_{21}|^2\,,\quad p_{00}(t)=1-|U_{11}|^2-|U_{21}|^2\,. \tag{47}$$

Again this result *cannot* be reproduced by some master equation for these three probabilities, because such an equation cannot give rise to more than two decaying exponentials.

Incidentally, as in this case the total number of quanta is 1, the higher levels of the oscillators never enter into the calculation; consequently the solution remains valid when the harmonic oscillators are replaced with two-level atoms. That is no longer true in the case treated in the next section.

6. Emission from both oscillators

Consider the initial state in which *both* heavy oscillators are excited, while the bath is still in its ground state,

$$|\Psi\rangle = A_1^\dagger A_2^\dagger|0\rangle\,. \tag{48}$$

It is again possible to find the probability distribution of the occupation numbers, but in this case we are more interested in the emitted field. Suppose the small oscillators $\boldsymbol{k}$ are the normal modes of the electromagnetic field. Then the field strength at some point in space is a linear combination of the $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$:

$$E=\sum_{\boldsymbol{k}}(e_{\boldsymbol{k}}a_{\boldsymbol{k}}+e_{\boldsymbol{k}}^*a_{\boldsymbol{k}}^\dagger)=E^{(+)}+E^{(-)}\,. \tag{49}$$

We need not specify the e_k. The two terms $E^{(+)}$ and $E^{(-)}$ comprise the terms with e^{-ikt} ("positive frequency") and those with e^{ikt} ("negative frequency") respectively.

In this section we put $\Omega = 1$ so that it is not necessary to distinguish between energy and excitation number. According to Glauber[18]) the intensity observed by a detector is at time t

$$\begin{aligned} I &= \langle \Psi | E^{(-)} E^{(+)} | \Psi \rangle \\ &= \sum_{kk'} e_k^* e_{k'} \langle \Psi | a_k^\dagger(t) a_{k'}(t) | \Psi \rangle \\ &= \sum_{kk'} \sum_{rr'} e_k^* e_{k'} W_{kr}^* W_{k'r'} \langle 0 | A_1 A_2 A_r^\dagger A_{r'} A_1^\dagger A_2^\dagger | 0 \rangle \\ &= \left| \sum_k e_k W_{k1} \right|^2 + \left| \sum_k e_k W_{k2} \right|^2 . \end{aligned} \tag{50}$$

We find that the intensity produced by both sources is the sum of the intensities due to the separate sources. No interference is observed at a single point in space, the light emitted by both oscillators has no first order coherence.

Now take two points a, b in space and the fields E_a, E_b in these points. The correlation between the intensities is

$$\langle I_a I_b \rangle - \langle I_a \rangle \langle I_b \rangle = \langle E_a^{(-)} E_b^{(-)} E_b^{(+)} E_a^{(+)} \rangle - \langle E_a^{(-)} E_a^{(+)} \rangle \langle E_b^{(-)} E_b^{(+)} \rangle . \tag{51}$$

The computation yields

$$- \left| \left(\sum_k e_{ak} W_{k1} \right)^* \left(\sum_k e_{bk'} W_{k'1} \right) - \left(\sum_k e_{ak} W_{k2} \right)^* \left(\sum_{k'} e_{bk'} W_{k'2} \right) \right|^2 . \tag{52}$$

This correlation is *negative* because there are only two quanta: if one is observed at a there is less chance to observe one at b as well.

This is *not* the intensity interference found by Hanbury Brown and Twiss[14,19]), because they dealt with thermally excited sources. That situation can be described in the present model by taking the initial state

$$|\Xi\rangle = (A_1^\dagger)^{N_1} (A_2^\dagger)^{N_2} |0\rangle \tag{53}$$

and afterwards averaging over N_1, N_2 with weight

$$e^{-\beta(N_1+N_2)} \Big/ \sum_{N_1 N_2} e^{-\beta(N_1+N_2)} . \tag{54}$$

For each of the intensities I_a, I_b one finds again (50), multiplied with the average excitation number $(e^{\beta}-1)^{-1}$. However, one now obtains a *positive* correlation:

$$(e^{\beta}-1)^{-2}\left|\left(\sum_k e^*_{ak}W^*_{k1}\right)\left(\sum_{k'} e_{bk'}W_{k'1}\right)+\left(\sum_k e^*_{ak}W^*_{k2}\right)\left(\sum_k e_{bk'}W_{k'2}\right)\right|^2 . \quad (55)$$

This agrees with the standard result[19]). In the case of a single oscillator it vanishes, although that is not obvious from this expression.

7. Thermal fluctuations

Now suppose that initially the heavy oscillators are in their ground states while the bath oscillators are thermally excited. Thus the initial state of the bath is a statistical mixture of wave functions described by the density operator

$$\rho = Z^{-1}\exp\left[-\beta\sum_k k a^\dagger_k a_k\right], \quad (56a)$$

$$Z = \mathrm{Tr}\exp\left[-\beta\sum_k k a^\dagger_k a_k\right] = \prod_k (1-e^{-\beta k})^{-1} . \quad (56b)$$

The average energy of heavy oscillator 1 at $t \geqslant 0$ is

$$\begin{aligned}\mathrm{Tr}\,\rho\Omega A^\dagger_1(t)A_1(t) &= \Omega\sum_{kk'} V^*_{1k}V_{1k'}\,\mathrm{Tr}\,\rho a^\dagger_k a_{k'} \\ &= \Omega\sum_k |V_{1k}|^2(e^{\beta k}-1)^{-1} . \end{aligned} \quad (57)$$

If one substitutes V_{1k} from (36) there are a number of terms that decay exponentially. They describe how the memory of the special initial state fades away. Afterwards only the last term of (36) survives:

$$|V_{1k}(t)|^2 = \frac{|(k-\Omega+i\Gamma)v_{1k} - i\Gamma' e^{i\gamma}v_{2k}|^2}{|k-\Lambda_1|^2|k-\Lambda_2|^2} . \quad (58)$$

Substitution in (57) yields

$$\Omega\int dk\,\frac{[(k-\Omega)^2+\Gamma^2]\sigma_{11}(k)+\Gamma'^2\sigma_{22}(k)+2\Gamma'\,\mathrm{Im}(k-\Omega-i\Gamma)\sigma_{21}(k)\,e^{i\gamma}}{[(k-\Omega)^2+(\Gamma-\Gamma')^2][(k-\Omega)^2+(\Gamma+\Gamma')^2](e^{\beta k}-1)} . \quad (59)$$

The main contribution originates from the vicinity of Ω; hence we may insert

for the σ's the values (28), (29). For the same reason we may replace $e^{\beta k}$ with $e^{\beta\Omega}$; this assumes that the temperature is large compared to the line width: $\beta\Gamma \ll 1$, which is again a weak interaction assumption. Then (59) becomes

$$\frac{\Omega}{e^{\beta\Omega}-1}\,\frac{1}{\pi}\int dx\,\frac{(x^2+\Gamma^2)\Gamma+\Gamma'^2\Gamma-2\Gamma'^2\Gamma}{[x^2+(\Gamma-\Gamma')^2][x^2+(\Gamma+\Gamma')^2]}=\frac{\Omega}{e^{\beta\Omega}-1}\,. \tag{60}$$

Thus, when the transient has disappeared, the heavy oscillators have the average energy that corresponds with the bath temperature, as was to be expected when the interaction is weak.

In the same way one may compute the variances and the covariance of the occupation numbers. The result is equally unexciting.

$$\begin{aligned}\langle N_r N_s\rangle &= \operatorname{Tr}\rho A_r^\dagger(t)A_r(t)A_s^\dagger(t)A_s(t)\\ &= \operatorname{Tr}\rho A_r^\dagger(t)A_s^\dagger(t)A_r(t)A_s(t)+\delta_{rs}\operatorname{Tr}\rho A_r^\dagger(t)A_r(t)\,.\end{aligned} \tag{61}$$

Express again the operators in terms of the initial a_k, $a_k^\dagger$, take the average over the initial distribution (56), and use the expression (60) for the averages $\langle N_r\rangle$:

$$\langle N_r N_s\rangle-\langle N_r\rangle\langle N_s\rangle=\delta_{rs}(\langle N_r\rangle+\langle N_r\rangle^2)\,. \tag{62}$$

Thus, once equilibrium has been reached, N_1 and N_2 are uncorrelated and each of them has the variance that belongs to the Bose distribution[23]).

8. The Langevin force

The approximations in sections 3 and 4 have reduced the matrix $K(\omega)$ given in (21) to

$$K=\begin{pmatrix}\Omega-i\Gamma & -i\Gamma' e^{i\gamma}\\ -i\Gamma' e^{-i\gamma} & \Omega-i\Gamma\end{pmatrix}, \tag{63}$$

which no longer depends on ω. As a consequence $U(t)$, being the Laplace transform of $(\lambda-K)^{-1}$, obeys

$$\partial_t U(t)=-iKU(t)\,. \tag{64}$$

This is easily checked using the explicit expression (34) for $U(t)$. Thus

$$\left[\partial_t+i\Omega+\begin{pmatrix}\Gamma & \Gamma' e^{i\gamma}\\ \Gamma' e^{-i\gamma} & \Gamma\end{pmatrix}\right]U(t)=0\,. \tag{65}$$

When the same operation [ ] is applied to $V_k(t)$, see (36), it also annihilates the two terms originating from the poles Λ_1, Λ_2 and a single term survives:

$$\left[\partial_t + \mathrm{i}\Omega + \begin{pmatrix} \Gamma & \Gamma' \mathrm{e}^{\mathrm{i}\Gamma} \\ \Gamma' \mathrm{e}^{-\mathrm{i}\gamma} & \Gamma \end{pmatrix}\right] V_k(t) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}kt}\begin{pmatrix} v_{1k} \\ v_{2k} \end{pmatrix}. \tag{66}$$

Hence, on applying this operation to (5), one gets

$$\partial_t A_1(t) = (-\mathrm{i}\Omega - \Gamma)A_1(t) - \Gamma' \mathrm{e}^{\mathrm{i}\gamma} A_2(t) + f_1(t)\,, \tag{67a}$$

$$\partial_t A_2(t) = (-\mathrm{i}\Omega - \Gamma)A_2(t) - \Gamma' \mathrm{e}^{-\mathrm{i}\gamma} A_1(t) + f_2(t)\,, \tag{67b}$$

$$f_r(t) = -\mathrm{i}\sum_k \mathrm{e}^{-\mathrm{i}kt} v_{rk} a_k(0)\,. \tag{68}$$

These equations exhibit the damping Γ of each oscillator, the damping $\Gamma' \mathrm{e}^{\pm\mathrm{i}\gamma}$ that couples the oscillators, and additional forces $f_r(t)$ determined by the initial state of the bath. If the bath is initially in thermal equilibrium one may take averages:

$$\langle f_r(t)\rangle = 0\,, \quad \langle f_r(t) f_s(t')\rangle = 0\,, \tag{69}$$

$$\begin{aligned}\langle f_r^\dagger(t) f_s(t')\rangle &= \sum_k \mathrm{e}^{\mathrm{i}k(t-t')} v_{rk}^* v_{sk} (\mathrm{e}^{\beta k} - 1)^{-1} \\ &= \int \mathrm{e}^{\mathrm{i}k(t-t')} \sigma_{sr}(k)(\mathrm{e}^{\beta k} - 1)^{-1}\,\mathrm{d}k\,.\end{aligned} \tag{70}$$

Thus the f_r are *not* delta-correlated in time for two reasons: the $\sigma_{rs}(k)$ are not constant (although slowly varying over a range $\Delta k \sim \Omega$), and the high frequencies are cut off by the Planck factor. The first one can be partially remedied by setting $A_r(t) = \tilde{A}_r(t)\,\mathrm{e}^{-\mathrm{i}\Omega t}$, so that

$$\partial_t \tilde{A}_1 = -\Gamma \tilde{A}_1 - \Gamma' \mathrm{e}^{\mathrm{i}\gamma} \tilde{A}_2 + \tilde{f}_1\,, \tag{71a}$$

$$\partial_t \tilde{A}_2 = -\Gamma \tilde{A}_2 - \Gamma' \mathrm{e}^{-\mathrm{i}\gamma} \tilde{A}_1 + \tilde{f}_2\,, \tag{71b}$$

$$\tilde{f}_r(t) = -\mathrm{i}\sum_k \mathrm{e}^{\mathrm{i}(\Omega - k)t} v_{rk} a_k(0)\,. \tag{72}$$

The new forces $\tilde{f}_r$ obey again (69) but (70) changes into

$$\langle \tilde{f}_r^\dagger(t) \tilde{f}_s(t')\rangle = \int \mathrm{e}^{\mathrm{i}(k-\Omega)(t-t')} \sigma_{sr}(k)(\mathrm{e}^{\beta k} - 1)^{-1}\,\mathrm{d}k\,. \tag{73}$$

Here σ can be replaced by its value at Ω provided that

$$\Omega|t-t'| \gg 1 . \tag{74}$$

Thus (73) is at best a peak of width $\Delta t \sim \Omega^{-1}$.

If $|t-t'|$ satisfies (74) one has

$$\langle \tilde{f}_1^\dagger(t)\tilde{f}_1(t')\rangle = \langle \tilde{f}_2^\dagger(t)\tilde{f}_2(t')\rangle = \frac{\Gamma}{\pi}\int e^{i(k-\Omega)(t-t')}(e^{\beta k}-1)^{-1}\,dk , \tag{75a}$$

$$\langle \tilde{f}_1^\dagger(t)\tilde{f}_2(t')\rangle = \frac{\Gamma'}{\pi}e^{-i\gamma}\int e^{i(k-\Omega)(t-t')}(e^{\beta k}-1)^{-1}\,dk . \tag{75b}$$

The broadening due to the Planck factor is of order $\Delta t \sim \beta = (k_B T)^{-1}$. This condition is weaker than (74) when $k_B T > \Omega$, that is, in the classical case. Otherwise one has to take the effect of the Planck factor on the correlation functions of the f_r into account, as was done in detail by Haake and Reibold[12]) for the case of a single oscillator.

Is it legitimate to regard these additional forces as Langevin forces? First the condition (74) shows that, even for high temperature, the $\tilde{f}_r$ are delta-correlated only with respect to the slow motion determined by the damping Γ, but not with respect to the free oscillation. That is the reason why in (67) the f_r cannot be treated as rapidly fluctuating.

Secondly, in classical Brownian movement, the damping and the Langevin force serve as a substitute for the bath. The bath variables are eliminated and replaced by these two terms in the equation for the heavy particle. In the present quantum mechanical case, however, the bath has *not* been eliminated because the $\tilde{A}_r$ and $\tilde{f}_r$ are still operators in the entire Hilbert space. That fact is also the reason why (71) cannot be used to compute averages of products such as $\langle A_r^\dagger A_s\rangle$, which is one of the virtues of the classical Langevin equation[21]).

Thirdly, the motivation for writing a Langevin equation for a system whose equations of motion are already explicitly solved, is that one hopes that the same Langevin force may be applied to other, non-linear systems as well. That can only be expected, however, if the expression for the Langevin force does not involve properties of the special system considered. However, the $\tilde{f}_r$ involve not only Ω (in contrast to the f_r) but also know that our system is degenerate. This is exhibited by their mutual correlation; if the two frequencies were far apart the two Langevin forces would be uncorrelated.

I conclude that eqs. (71) do not have the virtues that make Langevin equations expedient. The same objections do not hold with respect to a formulation in density matrices rather than in field amplitudes.

Appendix

Consider a set of linear equations for a vector $x(t)$,

$$\dot{x} = -\mathrm{i}Hx\,. \tag{76}$$

where H is a constant matrix, not necessarily hermitian. The solution with given initial values at $t = 0$ is

$$x(t) = \frac{1}{2\pi\mathrm{i}} \int_L \mathrm{e}^{-\mathrm{i}t\lambda}\, \mathrm{d}\lambda\, (\lambda - H)^{-1} x(0)\,, \tag{77}$$

where the contour L surrounds the spectrum of H. Let the components of x be partitioned into two blocks y, z. Then eq. (76) takes the form

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = -\mathrm{i}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}. \tag{78}$$

In order to find $(\lambda - H)^{-1}$ in the corresponding block form one has to solve

$$(\lambda - A)y - Bz = p\,, \tag{79a}$$

$$-Cy + (\lambda - D)z = q\,, \tag{79b}$$

so as to express y, z as linear combinations of p, q.

First one obtains from (79b)

$$z = (\lambda - D)^{-1} q + (\lambda - D)^{-1} Cy\,. \tag{80}$$

Substitute in (79a),

$$[\lambda - A - B(\lambda - D)^{-1} C]y = p + B(\lambda - D)^{-1} q\,. \tag{81}$$

With the abbreviation

$$A + B(\lambda - D)^{-1} C = K(\lambda) \tag{82}$$

one has therefore

$$y = [\lambda - K(\lambda)]^{-1} p + [\lambda - K(\lambda)]^{-1} B(\lambda - D)^{-1} q\,. \tag{83}$$

On substituting this result in (80) one also finds z:

$$z = (\lambda - D)^{-1}C(\lambda - K)^{-1}p + (\lambda - D)^{-1}\{1 + C(\lambda - K)^{-1}B(\lambda - D)^{-1}\}q\,. \tag{84}$$

Thus we have found $(\lambda - H)^{-1}$ in block form and the equivalent of (77) is

$$y(t) = \frac{1}{2\pi i}\int_L e^{-it\lambda}\,d\lambda\,\{[(\lambda - K(\lambda)]^{-1}y(0) + [\lambda - K(\lambda)]^{-1}B(\lambda - D)^{-1}z(0)\}\,, \tag{85a}$$

$$z(t) = \frac{1}{2\pi i}\int_L e^{-it\lambda}\,dy\,\{(\lambda - D)^{-1}C[\lambda - K(\lambda)]^{-1}y(0) + (\lambda - D)^{-1}\{1 + C[\lambda - K(\lambda)]^{-1}B(\lambda - D)^{-1}\}z(0)\}\,. \tag{85b}$$

These are the equations (14), (17), (18), (19).

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