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VAUGHT'S THEOREM ON
AXIOMATIZABILITY BY A SCHEME

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ABSTRACT. In this paper we give an improvement of Vaught's Theorem on Axiomatizability by a Scheme. We show that any RE theory that directly interprets the theory of non-surjective unordered pairing, where pairing need not be functional, is axiomatizable by a scheme.

1. INTRODUCTION

Craig's well known trick shows us that every RE theory in predicate logic (of finite signature) has a decidable axiomatization. Inspection of the argument shows that we even get a p-time decidable axiomatization. It is less well known that there is a substantial class of theories that can be axiomatized by a scheme. This was shown by Vaught in his [Vau67]. Here is a precise formulation of the result. Consider the following axioms (in the language with $=$ and $\in$):

$\mathsf{vs}_n \vdash \forall x_0, \ldots, x_{n-1} \, \exists y \, \forall u \, (u \in y \leftrightarrow \bigvee_{i<n} u = x_i)$.

Note that vs_0 provides one or more empty sets, and that, for $0 < k \leqslant n$, we have $\mathsf{vs}_n \vdash \mathsf{vs}_k$. The theory VS_n is axiomatized by $\mathsf{vs}_0, \ldots, \mathsf{vs}_n$, or, more economically, just by vs_0 plus vs_n. Vaught's Set Theory VS_ω, or simply VS, is the theory axiomatized by all the vs_n.

We say that U *directly interprets* V iff there is an interpretation K of V in U, that is *direct*, i.e., that is not relativized and that translates identity to identity. We call a theory that directly interprets VS a *Vaught theory*. So, a Vaught theory contains a formula Axy that satisfies all axioms of VS.

Vaught's theorem tells us that:

> *All Vaught theories are axiomatizable by a scheme.*

Here are examples of Vaught theories of which it is not immediately evident how to axiomatize them by a scheme:

- $I\Delta_0 + \{\Omega_n \mid n \in \omega\}$.
- $\mathsf{EA} + \{\mathsf{con}_n(\mathsf{EA}) \mid n \in \omega\}$,
  where EA is Elementary Arithmetic, aka Elementary Function Arithmetic or EFA, aka $I\Delta_0 + \mathsf{Exp}$, and where '$\mathsf{con}_n(\mathsf{EA})$' stands for consistency only involving proofs of complexity below n.
- $\mathsf{EA} + \{\mathsf{con}^n(\mathsf{EA}) \mid n \in \omega\}$,
  where $\mathsf{con}^0(\mathsf{EA}) := \top$, $\mathsf{con}^{n+1}(\mathsf{EA}) := \mathsf{con}(\mathsf{EA} + \mathsf{con}^n(\mathsf{EA}))$.
- PRA (in a variant with finite signature).

We do not consider schemes with substitutions from restricted classes. Thus, $I\Delta_0$ and $\mathsf{S}_2 = I\Delta_0 + \Omega_1$ are also examples of theories that are not evidently axiomatizable by a scheme.

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It is not difficult to see that VS interprets Tarski, Mostowski and Robinson's theory R. Hence, VS and therewith every consistent Vaught theory is essentially undecidable.

Note that VS_2 is the theory of, not necessarily functional, non-surjective, unordered pairing. We call a theory that directly interprets VS_2: *a pair theory*. The main theorem of this paper is that:

> *All pair theories are axiomatizable by a scheme.*

There are decidable pair theories. This is an immediate consequence of the results on ordered pairing by Richard Tenney ([Ten]), reported in [FR79], p162. A beautiful special case is the decidability of the true theory of the Cantor Pairing Function C with Successor, $Th(\mathbb{N}, C, S)$, which is proved in [CR01]. Ferrante and Rackoff show that pair theories for functional ordered pairing are non-elementary.[1] See [FR79]. The existence of decidable pair theories shows that our main theorem is a true improvement on Vaught's result. Ferrante and Rackoff's result shows that *to be a pair theory* is still a substantial restriction on theories.

Vaught provides several examples of RE theories that are not axiomatizable by a scheme. E.g., let X be an RE, non-recursive set. Let $E!_n$ be the statement, in the language of identity, that there are precisely n elements. Let U be the theory in the language of identity, axiomatized by the $\neg E!_n$, for $n \in X$. If U were axiomatizable by a scheme, then the class of finite models (say, as coded in the natural numbers in some standard way) would be decidable, since it is decidable whether a finite model satisfies a given scheme. Quod non.

At the end of our paper, we briefly discuss the possibility of compressing schemes to single sentences in the presence of comprehension principles.

Vaught's Paper. The present paper is in many respects a re-presentation or remake of Section 3 of Vaught's beautiful seven page paper [Vau67]. Vaught writes the following in the beginning of Section 3 about the improvement his Theorem 2, that states that any RE Vaught theory is axiomatizable by a scheme, makes over his Theorem 1. Vaught's theory S is our theory VS. Vaught's theory T_1 is a theory of sequences and numbers.

> Theorem 2 improves on Theorem 1 in two ways. For one thing, S appears to be somewhat weaker than T_1 (as well as more elegant). More significant, however, is the fact that Theorem 2 states that T is outright axiomatizable by a schema, rather than by a schema plus the axioms of S.
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> Nevertheless, there do not appear to be any very interesting theories whose schematic axiomatizability is obtainable from Theorem 2 but not from Theorem 1. Consequently, we shall give a brief sketch of the proof.

It seems to me that the observation that Vaught's theorem extends to all theories that directly interpret a weak theory of pairing does extend the scope of the theorem sufficiently substantially to justify a more extended presentation.

Our proof follows Vaught's rather closely. Thus, it can be viewed for a large part as a careful exposition of Vaught's work. Here is the main difference. Vaught uses a very clever way of schematically axiomatizing expansions of VS with new predicates to insure that we need only one scheme without an infinity of auxiliary axioms. We, on the other hand, employ the fact that the theory of pairing *locally*

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[1]I think that their result must also work in the non-extensional case, but did not try to prove this. Their result trivially transfers to functional unordered pairing.

directly interprets the theory VS. Thus, we work with a varying interpretation of larger and larger parts of VS, where Vaught employs a fixed interpretation.

2. PRELIMINARIES

In this paper, we consider RE theories of finite signature. These theories will one-sorted in most of the paper. The exception is Section 7, where we study theories with a second sort of classes. We will assume that our theories are *officially* relational, However, we will often pretend there are function symbols present. As is well known, the function symbols can be eliminated by an efficient translation.

Interpretations in this paper will be multi-dimensional relative interpretations with parameters. An interpretation K of U in V is based on a translation τ of the language of U to the language of V, such that, for all U-sentences A, we have $U \vdash A$ implies $V \vdash A^\tau$. See e.g. [Vis06] or [Vis09b] or [Vis08] for careful definitions. An interpretation is *direct* iff it is unrelativized and does translate identity into identity. Note that direct interpretations are automatically one-dimensional. A theory V *locally interprets* a theory U iff V interprets every finite subtheory of U. We remind the reader of the notations:

- $V \rhd U$ for: V interprets U.
- $U \lhd V$ for: U is interpretable in V.

Further determinations of the kind of interpretability intended will be added as subscript. E.g., $V \rhd_{\mathsf{dir}} U$ will mean: V directly interprets U.

A scheme $\mathcal{S}$ for a theory U is obtained as follows. We extend the signature, say Σ_U of U with a finite number of predicate symbols to a new signature Σ_U^+. We assume that the new symbols are marked in Σ_U^+. A scheme is a sentence in the extended language. Let's say that an *s-translation* is a translation of the language given by Σ_U^+ into the language of U that is constant on the non-marked or old vocabulary; the predicate symbols of the new vocabulary are replaced by formulas of the old vocabulary. We do not have domain restriction in s-translations. The scheme $\mathcal{S}$ is valid in U if, for all s-translations τ, $U \vdash \mathcal{S}^\tau$. Our formulations allow two readings: do we allow parameters or not in the translations τ. I will assume that the default reading of scheme is *scheme with parameters*. This means that the s-translations in the definition of validity range over translations with parameters. These parameters are unrestricted, i.e., they range over the full domain of U, if we do not allow parameters, we will speak about a *closed scheme*.

3. THEORIES OF NUMBER(S)

We formulate theories that can be a both theories of *a number* and of *all numbers*. The theory TN_0 is intended to be possibly the theory of a number. It is given as follows. We have, apart from equality, the following symbols in the signature: a constant 0, two binary relation symbols S and $\leqslant$, and two ternary relations A and M. The theory is axiomatized as follows.

tn1 $\vdash (\mathsf{S}xy \wedge \mathsf{S}uv) \rightarrow ((x = u \wedge y = v) \vee (x \neq u \wedge y \neq v))$,
tn2 $\vdash \neg \mathsf{S}x0$,
tn3 $\vdash x = 0 \vee \exists y\, \mathsf{S}yx$
tn4 $\vdash (\mathsf{A}xyu \wedge \mathsf{A}xyv) \rightarrow u = v$,
tn5 $\vdash \mathsf{A}x0x$,
tn6 $\vdash \exists u\, (\mathsf{S}yu \wedge \mathsf{A}xuv) \leftrightarrow \exists z\, (\mathsf{S}zv \wedge \mathsf{A}xyz)$,

tn7 $\vdash (\mathsf{M}xyu \wedge \mathsf{M}xyv) \to u = v$,

tn8 $\vdash \mathsf{M}x00 \wedge \mathsf{M}0x0$,

tn9 $\vdash x \neq 0 \to (\exists u\,(\mathsf{S}yu \wedge \mathsf{M}xuv) \leftrightarrow \exists z\,(\mathsf{A}zxv \wedge \mathsf{M}xyz))$,

tn10 $\vdash x \leqslant y \leftrightarrow \exists z\,\mathsf{A}zxy$.

The assumption of Axiom tn9 is essential. To see that, suppose we drop it. Since we have A000 and M0y0, it would follow that, for some u, Syu. This frustrates our intention to formulate a theory that can also be the theory of *a number*. Note that if we would both drop the assumption of Axiom tn9 and the second conjunct of tn8, we still would be able to prove the undesirable: $\exists x_1, \ldots, x_n\,(S0x_1 \wedge \ldots \wedge Sx_{n-1}x_n)$, for every $n > 0$.

We will use partial function notation. Partial terms will be eliminated via the algorithm $(\cdot)^\star$. We adopt the small scope convention and we employ weak evaluation for multiplication with 0. We first translate formulas of the form $t = x$, where x does not occur in t.

- $(y = x)^\star := (y = x)$,
- $(0 = x)^\star := (0 = x)$,
- $(\mathsf{S}t = x)^\star := \exists x_0\,((t = x_0)^\star \wedge \mathsf{S}x_0x)$,
- $(t_0 + t_1 = x)^\star := \exists x_0\,\exists z\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge \mathsf{A}x_0x_1x)$,
- $(t_0 \times t_1 = x)^\star = (\exists x_0\,((t_0 = x_0)^\star \wedge x_0 = 0 \wedge x = 0) \vee$
  $\qquad\qquad (\exists x_1\,((t_1 = x_1)^\star \wedge x_1 = 0 \wedge x = 0) \vee$
  $\qquad\qquad \exists x_0\,\exists x_1\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge \mathsf{M}x_0x_1x))$.

We now define the full translation for all formulas not of the form $t = x$, where x is not in t, as follows:

- $(t_0 = t_1)^\star := (\exists x_0\,\exists x_1\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge x_0 = x_1))$,
- $(t_0 \leqslant t_1)^\star := (\exists x_0\,\exists x_1\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge x_0 \leqslant x_1))$,
- $(\cdot)^\star$ commutes with the propositional connectives and quantifiers.

We use $t \simeq u$ for: either t and u are both defined and their values are equal, or both are undefined. The expression $t \downarrow$ will mean: t is defined. The expression $t \uparrow$ will mean: t is undefined. We will use $+$ for A and $\times$ or $\cdot$ for M. In these notations we have the following consequences of our axioms:

tnf1 $\vdash t = u \to t \downarrow \wedge u \downarrow$

tnf2 $\vdash 0 \downarrow$

tnf3 $\vdash \mathsf{S}t \downarrow \to t \downarrow$

tnf4 $\vdash \mathsf{S}x = \mathsf{S}u \to x = u$

tnf5 $\vdash \mathsf{S}x \neq 0$,

tnf6 $\vdash x = 0 \vee \exists y\,\mathsf{S}y = x$

tnf7 $\vdash t + u \downarrow \to (t \downarrow \wedge u \downarrow)$

tnf8 $\vdash x + 0 = x$,

tnf9 $\vdash x + \mathsf{S}y \simeq \mathsf{S}(x + y)$,

tnf10 $\vdash t \times u \downarrow \to (t = 0 \vee u = 0 \vee (t \downarrow \wedge u \downarrow))$

tnf11 $\vdash x \times 0 = 0 \wedge 0 \times x = 0$

tnf12 $\vdash x \times \mathsf{S}y \simeq (x \times y) + x$

tnf13 $\vdash x \leqslant y \leftrightarrow \exists z\,z + x = y$

It is easy to see that these principles follow indeed from the axioms. Note that to prove tnf12, we need to split into the cases whether x is or is not 0 and whether y is or is not 0. The antecedent of tn9 disappears in tnf12, since our conventions do

not demand the successor to be defined in that case. Note that, conversely, we can derive our original axioms from the new ones except tn8 and part of tn9.

The disappearance of the assumption of tn9 is the main point of the use of the 'weak convention'. It allows us to compute with undefined terms in the usual way, giving us e.g. Lemma 3.1(4).

We write $\underline{n}$ for $\mathsf{S}^n 0$. The theory TN_n is $\mathsf{TN}_0 + \underline{n}{\downarrow}$. The theory TN_ω is the union of the TN_n.

Consider any natural number n. We define 0, S, A, M, $\leqslant$ on $\{0, \cdots, n\}$ in the obvious way, e.g., $\mathsf{A}xyz :\leftrightarrow x + y = z$ and $x, y, z \leqslant n$. We call this model $\mathbb{N}_n$. We have $\mathbb{N}_n \models \mathsf{TN}_n$. Of course, we also have $\mathbb{N} \models \mathsf{TN}_\omega$.

Here is a lemma collecting some basic facts about the TN_n.

Lemma 3.1.   *(1) The relation $\simeq$ is TN_0-provably an equivalence relation between terms.*
  (2) Suppose t and t' are substitutable for x in Ax, then:
    $\mathsf{TN}_0 \vdash t \simeq t' \rightarrow (At \leftrightarrow At')$.
  (3) Suppose $t\vec{m} = n$. Then, $\mathsf{TN}_0 \vdash t\underline{\vec{m}} \simeq \underline{n}$.
  (4) Suppose $t\vec{m} = t'\vec{m}$. Then, $\mathsf{TN}_0 \vdash t\underline{m} \simeq t'\underline{\vec{m}}$.
  (5) Suppose $t\vec{m} \leqslant t'\vec{m}$. Then, $\mathsf{TN}_0 \vdash t'\underline{\vec{m}} {\downarrow} \rightarrow t\underline{m} \leqslant t'\underline{\vec{m}}$.
  (6) If $t\vec{m} \neq t'\vec{m}$, then $\mathsf{TN}_0 \vdash t\underline{\vec{m}} \neq t'\underline{\vec{m}}$.
  (7) If $t'\vec{m} < t\vec{m}$, then $\mathsf{TN}_0 \vdash \neg\, t\underline{\vec{m}} \leqslant t'\underline{\vec{m}}$.
  (8) Suppose $t\vec{m} = t'\vec{m} \leqslant n$. Then, $\mathsf{TN}_n \vdash t\underline{\vec{m}} = t'\underline{\vec{m}}$.
  (9) Suppose $t\vec{m} \leqslant t'\vec{m} \leqslant n$. Then $\mathsf{TN}_n \vdash t\underline{\vec{m}} \leqslant t'\underline{\vec{m}}$.
  (10) $\mathsf{TN}_0 \vdash \bigvee_{m<n} x = \underline{m} \vee \underline{n} \leqslant x$.
  (11) $\mathsf{TN}_0 \vdash (\underline{n}{\uparrow} \vee x \leqslant \underline{n}) \leftrightarrow \bigvee_{m\leqslant n} x = \underline{m}$.
  (12) $\mathsf{TN}_n \vdash x \leqslant \underline{n} \vee \underline{n} \leqslant x$.
  (13) $\mathsf{TN}_0 \vdash (x \leqslant \underline{n} \wedge \underline{n} \leqslant x) \rightarrow x = \underline{n}$.
  (14) $\mathsf{TN}_0 \vdash (\underline{n} \leqslant x \wedge x \leqslant y) \rightarrow \underline{n} \leqslant y$.
  (15) $\mathsf{TN}_0 \vdash (x \leqslant \underline{n} \wedge \underline{n} \leqslant y) \rightarrow x \leqslant y$.
  (16) $\mathsf{TN}_0 \vdash (x \leqslant y \wedge y \leqslant \underline{n}) \rightarrow x \leqslant \underline{n}$.

Proof. We prove (10) by induction on n. The case that $n = 0$ is easy. Suppose $n = k + 1$ and we have $\mathsf{TN}_0 \vdash \bigvee_{m<k} x = \underline{m} \vee \underline{k} \leqslant x$. Reason in TN_0. If we have $\bigvee_{m<k} x = \underline{m}$, then we also have $\bigvee_{m<n} x = \underline{m}$ and we are done. Suppose we have $\underline{k} \leqslant x$. So, for some u, $u + \underline{k} = x$. In case $u = 0$, we find $0 + \underline{k} = x$ and hence, by an easy argument, $\underline{k} = x$ and, thus, $\bigvee_{m<n} x = \underline{m}$. In case $u = \mathsf{S}v$, we find $\mathsf{S}v + \underline{k} = x$. By an easy argument this gives $v + \underline{n} = x$, so $\underline{n} \leqslant x$. ❑

Lemma 3.2. *Suppose $A\vec{x}$ is a Δ_0-formula. There is a term $t_A\vec{x}$, which majorizes every x in $\vec{x}$, such that, for every n and for every $\vec{m}$ with $t_A\vec{m} \leqslant n$, we have:*

  (1) If $A\vec{m}$, then $\mathsf{TN}_n \vdash A\underline{\vec{m}}$,
  (2) If $\neg\, A\vec{m}$, then $\mathsf{TN}_n \vdash \neg\, A\underline{\vec{m}}$.

Proof. The proof is by induction, simultaneous in both cases, on the complexity of A. Suppose A is $t\vec{x} = t'\vec{x}$. Then, we can take $t_A\vec{x} := t\vec{x} + t'\vec{x}$. The cases of the Boolean connectives are obvious.

Suppose $A\vec{x}$ is $\forall y{\leqslant}t\vec{x}\, B\vec{x}y$. We take $t_A\vec{x} := t\vec{x} + t_B(\vec{x}, t\vec{x})$. Suppose $t_A\vec{m} \leqslant n$. Let $t\vec{m} = k$. Since, $t\vec{m} \leqslant n$, we find, by Lemma 3.1(8), $\mathsf{TN}_n \vdash t\underline{\vec{m}} = \underline{k}$. By

Lemma 3.1(11), we find:

$$\mathsf{TN}_n \vdash A\vec{m} \leftrightarrow \bigwedge_{p \leqslant k} B(\vec{\underline{m}}, \underline{p}).$$

Since, $t_B(\vec{m}, p) \leqslant t_B(\vec{m}, k) \leqslant t_A(\vec{m}) \leqslant n$, the desired result is immediate from the Induction Hypothesis. The case of the bounded existential quantifier is similar. ❑

Suppose $S\vec{x}$ is a Σ_1^0-formula. Say, $S\vec{x} = \exists \vec{y}\, S_0\vec{x}\vec{y}$, where $S_0\vec{x}\vec{y}$ is Δ_0. We define $S^z\vec{x}$, by $\exists \vec{y}{\leqslant}z\, (t_{S_0}\vec{x}\vec{y} \leqslant z \wedge S_0\vec{x}\vec{y})$. Here t_{S_0} is the term provided by Lemma 3.2.

Lemma 3.3. *Let $S\vec{x}$ be a Σ_0-formula.*

(1) *We have: $S\vec{m}$ iff, for some n, $S^n\vec{m}$.*
(2) *Suppose $S^n\vec{m}$. Then, $\mathsf{TN}_n \vdash S^n\vec{\underline{m}}$.*
(3) *Suppose $\neg\, S^n\vec{m}$. Then, $\mathsf{TN}_0 \vdash \neg\, S^n\vec{\underline{m}}$.*

Proof. We treat (2). Suppose $S^n\vec{m}$. So, for some $\vec{k}$, we have $t_{S_0}\vec{m}\vec{k} \leqslant n$ and $S_0\vec{m}\vec{k}$. By Lemma 3.1(9), $\mathsf{TN}_n \vdash t_{S_0}(\vec{\underline{m}}, \vec{\underline{k}}) \leqslant \underline{n}$. By Lemma 3.2, we have $\mathsf{TN}_n \vdash S_0(\vec{\underline{m}}, \vec{\underline{k}})$. Hence, $\mathsf{TN}_n \vdash S^n\vec{\underline{m}}$.

We treat (3). Suppose $\neg\, S^n\vec{m}$. Let $U := \mathsf{TN}_0 + S^n\vec{\underline{m}}$. Note that U extends TN_n. By Lemma 3.2, we find, for $\vec{k}$ such that $t_{S_0}\vec{m}\vec{k} \leqslant n$, that $U \vdash \neg\, S_0(\vec{\underline{m}}, \vec{\underline{k}})$. Also, by Lemma 3.1(7), if $n < t_{S_0}\vec{m}\vec{k} \leqslant n$, then $U \vdash \neg\, t_{S_0}(\vec{\underline{m}}, \vec{\underline{k}}) \leqslant \underline{n}$. By Lemma 3.1(11), we find $U \vdash \bigvee_{\vec{k}<n}(t_{S_0}(\vec{\underline{m}}, \vec{\underline{k}}) \leqslant \underline{n} \wedge S_0(\vec{\underline{m}}, \vec{\underline{k}}))$. Combining, we find $U \vdash \bot$. So, we may conclude that $\mathsf{TN}_0 \vdash \neg\, S^n\vec{m}$. ❑

Lemma 3.4. *Suppose $S(\vec{x}, y)$ is a Σ_1^0-formula representing a function. Then there is a Σ_1^0-formula $S^\circ(\vec{x}, y)$, such that, for all $\vec{m}$, if $S(\vec{m}, n)$, then, for some k, $\mathsf{TN}_k \vdash S^\circ(\vec{\underline{m}}, \underline{n}) \wedge \forall y\, (S^\circ(\vec{\underline{m}}, y) \to \underline{n} = y)$.*

Proof. Suppose $S(\vec{x}, y)$, is $\exists \vec{z}\, S_0\vec{x}y\vec{z}$, where S_0 is in Δ_0. Let:

- $S_0^\circ(\vec{x}, y, z) :\leftrightarrow y \leqslant z \wedge \exists \vec{u}{\leqslant}z\, S_0\vec{x}y\vec{u} \wedge \forall \vec{v}, w{\leqslant}z\, (S_0\vec{x}w\vec{v} \to w = y)$.
- $S^\circ(\vec{x}, y) :\leftrightarrow \exists z\, S_0\vec{x}yz$.

Suppose $S\vec{m}n$. By the functionality of S, we can clearly find a q such that $S_0^\circ\vec{m}nq$. Let $k \geqslant t_{S_0^\circ}\vec{m}nq$. By Lemma 3.2, we have $\mathsf{TN}_k \vdash S_0^\circ(\vec{\underline{m}}, \underline{n}, \underline{q})$.

We reason in TN_k. Suppose $S_0^\circ(\vec{\underline{m}}, y, z)$. By Lemma 3.1(12), either $z \leqslant \underline{q}$ or $\underline{q} \leqslant z$.

Suppose $z \leqslant \underline{q}$. Since $S_0^\circ(\vec{m}, n, \underline{q})$, for every $\vec{v}, w{\leqslant}\underline{q}$, if $S_0(\vec{\underline{m}}, w, \vec{v})$, then $w = \underline{n}$. On the other hand, by $S_0^\circ(\vec{\underline{m}}, y, z)$, we have $y \leqslant z$ and, for some $\vec{u} \leqslant z$, $S_0(\vec{\underline{m}}, y, \vec{u})$. By Lemma 3.1(16), we find $\vec{u}, y \leqslant \underline{q}$. So, it follows that $y = \underline{n}$.

Suppose $\underline{q} \leqslant z$. Since $S_0^\circ(\vec{\underline{m}}, y, z)$, for every $\vec{v}, w{\leqslant}z$, if $S_0(\vec{\underline{m}}, w, \vec{v})$, then $w = y$. On the other hand, by $S_0^\circ(\vec{\underline{m}}, \underline{n}, \underline{q})$, we have $\underline{n} \leqslant \underline{q}$ and, for some $\vec{u}{\leqslant}\underline{q}$, $S_0(\vec{\underline{m}}, n, \vec{u})$. By Lemma 3.1(15), we find $\vec{u}, \underline{n} \leqslant z$. So, $\underline{n} = y$. ❑

Lemma 3.5 (Gödel Fixed Point Lemma). *Suppose that K is an interpretation of TN_0 in U. Let $Ax\vec{y}$ be a formula in the language of U. Suppose that:*

$$U \vdash Ax\vec{y} \to \delta_K(x) \text{ and } U \vdash x_0 =_K x_1 \to (Ax_0\vec{y} \leftrightarrow Ax_1\vec{y}).$$

Then, there is a formula $B\vec{v}$ and an n, such that

$$U + (\underline{n}\downarrow)^K \vdash B\vec{z} \leftrightarrow \exists x\, ((x = \ulcorner B\vec{v}\urcorner)^K \wedge Ax\vec{z}).$$

Proof. Suppose sub is a Σ_1^0-representation of the Gödel substitution function or the language of U, such that $\mathsf{sub}(n, \ulcorner Cx\vec{y}\urcorner, \ulcorner \exists x{:}\delta_K ((x = \underline{n})^K \wedge Cx\vec{y}\urcorner$, where x is a designated variable. Let:

- $A^* y\vec{v} :\leftrightarrow \exists x{:}\delta_K ((\mathsf{sub}^\circ(y, y, x))^K \wedge Ax\vec{v})$,
- $m := \ulcorner A^* u\vec{v}\urcorner$,
- $B\vec{v} :\leftrightarrow \exists y{:}\delta_K ((\underline{m} = x)^K \wedge A^* y\vec{v})$.
- $k := \ulcorner B\vec{v}\urcorner$.

By Lemma 3.4, we can find n such that:

$$\mathsf{TN}_n \vdash \mathsf{sub}^\circ(\underline{m}, \underline{m}, \underline{k}) \wedge \forall y\, (\mathsf{sub}^\circ(\underline{m}, \underline{m}, y) \to \underline{k} = y).$$

Thus, we have, in $U + (\underline{n} \downarrow)^K$:

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\begin{aligned}
B\vec{z} \quad &\leftrightarrow \quad \exists y{:}\delta_K ((\underline{m} = y)^K \wedge \exists x{:}\delta_K ((\mathsf{sub}^\circ(y, y, x))^K \wedge Ax\vec{z})) \\
&\leftrightarrow \quad \exists x, y{:}\delta_K ((\underline{m} = y \wedge \mathsf{sub}^\circ(y, y, x))^K \wedge Ax\vec{z}) \\
&\leftrightarrow \quad \exists x{:}\delta_K ((\underline{k} = x)^K \wedge Ax\vec{z}) \qquad\qquad \square
\end{aligned}
$$

4. Sets

We define a theory of sets TS_0 as follows. Our signature consists of two unary symbols N, Z, five binary symbols S, $\leqslant$, E, $\in$, η, and two ternary symbols A and M. The intended interpretation of η is the relation between a set and its cardinality. However, our axioms do not go far in constraining η to be so.

We let $a, b, c, \ldots$ range over N. This means that if e.g. 'a' occurs freely in a formula, then we assume that $\mathsf{N}(a)$ is present in the context. The formula $\exists a\, A$ means $\exists a\, (\mathsf{N}a \wedge A)$, and similarly for the universal quantifier. We use $x, y, z, w, \ldots$ to range over all objects.

ts1 Axioms stating that Z, E, S, $\leqslant$, A, M are relations on N. E.g.,
$\vdash \mathsf{A}xyz \to (\mathsf{N}x \wedge \mathsf{N}y \wedge \mathsf{N}z)$.
An axiom stating that η is a relation on N in the second component, i.e.,
$\vdash x\, \eta\, y \to \mathsf{N}y$,

ts2 Axioms stating that E is an equivalence relation of N which is a congruence w.r.t. Z, S, $\leqslant$, A, M. An axiom stating that E is a congruence w.r.t η in the second component, i.e., $\vdash (x\, \eta\, a \wedge a\, \mathsf{E}\, b) \to x\, \eta\, b$,

ts3 $\vdash \exists a\, \mathsf{Z}a$,

ts4 $\vdash (\mathsf{Z}a \wedge \mathsf{Z}b) \to a\, \mathsf{E}\, b$,

ts5 $\vdash (\mathsf{S}ab \wedge \mathsf{S}cd) \to ((a\, \mathsf{E}\, c \wedge b\, \mathsf{E}\, d) \vee (\neg a\, \mathsf{E}\, c \wedge \neg b\, \mathsf{E}\, d))$,

ts6 $\vdash \neg (\mathsf{S}ab \wedge \mathsf{Z}b)$,

ts7 $\vdash \mathsf{Z}a \vee \exists b\, \mathsf{S}ba$

ts8 $\vdash (\mathsf{A}abc \wedge \mathsf{A}abd) \to c\, \mathsf{E}\, d$,

ts9 $\vdash \mathsf{Z}b \to \mathsf{A}aba$,

ts10 $\vdash \exists d\, (\mathsf{S}bd \wedge \mathsf{A}ade) \leftrightarrow \exists c\, (\mathsf{S}ce \wedge \mathsf{A}abc)$,

ts11 $\vdash (\mathsf{M}abc \wedge \mathsf{M}abd) \to c\, \mathsf{E}\, d$,

ts12 $\vdash \mathsf{Z}b \to (\mathsf{M}abb \wedge \mathsf{M}bab)$,

ts13 $\vdash \neg \mathsf{Z}a \to (\exists d\, (\mathsf{S}bd \wedge \mathsf{M}ade) \leftrightarrow \exists c\, (\mathsf{A}cae \wedge \mathsf{M}abc))$,

ts14 $\vdash a \leqslant b \leftrightarrow \exists c\, \mathsf{A}cab$.

ts15 $\vdash \mathsf{Z}a \to \exists x\, (x\, \eta\, a \wedge \forall y\, y \notin x)$,

ts16 $\vdash (x\, \eta\, a \wedge \mathsf{S}ab) \to \forall u\, \exists y\, (y\, \eta\, b \wedge \forall v\, (v \in y \leftrightarrow (v \in x \vee v = u)))$.

Note that our first 14 axioms are just translations of the TN_0 axioms. So, modulo translation, our new theory is the extension of TN_0 with two axioms. Let us define:

- $\mathsf{empty}(x) :\leftrightarrow \forall y\, y \notin x$,
- $\mathsf{adj}_u(x,y) :\leftrightarrow \forall v\, (v \in y \leftrightarrow (v \in x \vee v = u))$,

With these definitions, our new axioms read:

- $\vdash \mathsf{Z}a \to \exists x\, (x\, \eta\, a \wedge \mathsf{empty}(x))$,
- $\vdash (x\, \eta\, a \wedge \mathsf{S}ab) \to \forall u\, \exists y\, (y\, \eta\, b \wedge \mathsf{adj}_u(x,y))$.

Par abus de langage, we suppress the standard interpretation of TN_0 in TS_0. We define $\mathsf{TS}_n := \mathsf{TS}_0 + \underline{n}\!\downarrow$

Theorem 4.1. *The theory TS_n is directly interpretable in VS_2 via a one dimensional interpretation without parameters. If follows that TS_ω is locally directly interpretable in VS_2.*

Proof. We fix some k be such that $2^k \geqslant n$. We define:

- $x \in^0 y :\leftrightarrow x = y$, $x \in^{j+1} y :\leftrightarrow \exists z\, (x \in z \wedge z \in^j y)$.
- $\mathsf{set}(y) :\leftrightarrow \forall u_0, \ldots, u_n\, (\bigwedge_{j \leqslant n} u_i \in^k y \to \bigvee_{i < j \leqslant n} u_i = u_j)$.
- For $j \leqslant n$,
  $$j(x) :\leftrightarrow \mathsf{set}(x) \wedge \exists u_0, \ldots u_{j-1}\, (\bigwedge_{i<j} u_i \in x \wedge$$
  $$\bigwedge_{s<i<j} u_s \neq u_i \wedge \forall v \in y\, \bigvee_{i<j} v = u_i).$$

Let k be such that $2^k \geqslant n$. We define a direct translation τ as follows.

- $\mathsf{N}_\tau(x) :\leftrightarrow x = x$,
- $\mathsf{Z}_\tau x :\leftrightarrow 0(x)$,
- $\mathsf{S}_\tau xy :\leftrightarrow \bigvee_{j<n} (j(x) \wedge (j+1)y)$,
- $x \leqslant_\tau y :\leftrightarrow \bigvee_{i \leqslant j \leqslant n} (i(x) \wedge j(y))$,
- $x\, \mathsf{E}_\tau\, y :\leftrightarrow \bigvee_{i \leqslant n} (i(x) \wedge i(y))$,
- $x \in_\tau y :\leftrightarrow x \in^k y \wedge \mathsf{set}(y)$,
- $x\, \eta_\tau\, y :\leftrightarrow x \in_\tau y$,
- $\mathsf{A}_\tau xyz :\leftrightarrow \bigvee_{s+i=j\ \&\ j \leqslant n} (s(x) \wedge i(y) \wedge j(z))$,
- $\mathsf{M}_\tau xyz :\leftrightarrow \bigvee_{s \times i=j\ \&\ s,i,j \leqslant n} (s(x) \wedge i(y) \wedge j(z))$,

Note that in VS_2 we can prove that there are infinitely many objects, by considering iterated singletons on empty sets. So we can show in VS_2 that n (as interpreted by τ) exists. E.g., in case $n = 7$, we produce singletons $a_0, \ldots, a_6$, where $a_i \in a_{i+1}$. We may show that these are pairwise distinct. Next we find $b_0, \ldots, b_3$, where b_0, b_1, b_2 are unordered pairs, b_3 is a singleton and $a_0, a_1 \in b_0$, $a_2, a_3 \in b_1$, $a_4, a_5 \in b_2$ and $a_6 \in b_3$. Note that the b_i are pairwise distinct. Next we find unordered pairs c_0 and c_1 such that $b_0, b_1 \in c_0$ and $b_2, b_3 \in c_1$. Note that $c_0 \neq c_1$, Finally, we find an unordered pair d with $c_0, c_1 \in d$. The object d will be a set with 7 elements under τ. Using these ideas we may easily show that τ yields an interpretation of TS_n in VS_2.   ❑

Remark 4.2. The interpretation constructed in the preceding proof is highly inefficient in terms of the size of the interpreting formulas as a function of n. Employing the usual definitions of set theory we can interpret TS_n in far more efficient ways. Also, we can do much better in defining $\in^k$. Such efficiency could be necessary if we want to verify our theorem in S_2^1. Also, it would be quite interesting to try to adapt our methods to reprove the lower bound result of Ferrante and Rackoff (see

[FR79]). This would also ask for more efficient interpretations. However, for our present purposes such efficiency is not needed. ❑

5. Satisfaction

Consider any theory U. We extend the language of U with the vocabulary of TS_0 plus a new binary predicate sat. We define the formula SAT that says roughly that 'below c' the predicate sat behaves as a satisfaction predicate with the first argument in the role of assignment and the second argument in the role of formula. To keep things readable we assume that the language of U has a unary predicate P, a binary predicate Q and that the only propositional connectives are the 0-ary connectives $\top$ and $\bot$, negation and conjunction and that the only quantifier is universal quantification.

We may find Σ_1^0-formulas defining some useful predicates:

- $\mathsf{var}(b)$: 'b codes a variable'.
- $\mathsf{fv}(b, a)$: 'a codes a formula and b codes a free variable occurring in a.'
- $[\![P]\!](b, a)$: 'b codes a variable 'v' and a codes 'Pv'.' Similarly for $[\![Q]\!](b, a)$.
- $[\![\top]\!](a)$: 'a codes '$\top$''. Similarly for $\bot$.
- $[\![\neg]\!](b, a)$: 'b codes a formula A and a codes $\neg A$.'
- The other cases are as expected.

We need some further definitions:

- $x \subseteq y :\leftrightarrow \forall z\ (z \in x \to z \in y)$,
- $\mathsf{pair}(x, y, z) :\leftrightarrow \exists u, v\ \forall w\ ((w \in z \leftrightarrow (w = u \lor w = v)) \land$
  $\qquad\qquad (w \in u \leftrightarrow w = x) \land (w \in v \leftrightarrow (w = x \lor w = y)))$
- $\mathsf{rel}(x) :\leftrightarrow \forall y {\in} x\ \exists u, v\ \mathsf{pair}(u, v, y)$
- $\mathsf{fun}(x) :\leftrightarrow \forall y, z {\in} x\ \forall u, v, w\ ((\mathsf{pair}(u, v, y) \land \mathsf{pair}(u, w, y)) \to y = z).$[2]
- $x(y) = z :\leftrightarrow \mathsf{fun}(x) \land \exists u {\in} x\ \mathsf{pair}(y, z, u)$
- $\mathsf{dom}(y, x) :\leftrightarrow \exists z\ x(y) = z$
- $\mathsf{funn}(x) :\leftrightarrow \mathsf{fun}(x) \land \forall y\ (\mathsf{dom}(y, x) \to \mathsf{N}(y)) \land$
  $\qquad \forall a, b, u, v\ ((a \mathrel{\mathsf{E}} b \land x(a) = u \land x(b) = v) \to (a = b \land u = v))$
- $x \star a = y :\leftrightarrow \mathsf{funn}(x) \land \exists b\ (a \mathrel{\mathsf{E}} b \land x(b) = y)$
- $\mathsf{ass}(c, x, a) :\leftrightarrow \mathsf{funn}(x) \land \forall b\ (\mathsf{dom}(b, x) \leftrightarrow \mathsf{var}^c(b))$
- $x[a]y :\leftrightarrow \mathsf{funn}(x) \land \mathsf{funn}(y) \land \forall b\ (\neg a \mathrel{\mathsf{E}} b \to x \star b \simeq y \star b) \land y \star a \downarrow)$

The definition of the 'this is a satisfaction predicate below c'-predicate SAT is given in Figure 5.

Theorem 5.1. *Suppose we have an interpretation $K : U \rhd_{\mathsf{dir}} \mathsf{TS}_0$. We will suppress '$K$' notationally. Let A be $A(v_0, \dots v_{k-1})$. Suppose we have $\mathsf{form}^n(A)$. We assume that $n \geqslant 2$. We have:*

$$U + \underline{n} \leqslant b + \mathsf{SAT}(b, B) + \mathsf{ass}(b, x, \ulcorner A \urcorner) \vdash A(x \star \ulcorner v_0 \urcorner, \dots, x \star \ulcorner v_i \urcorner) \leftrightarrow B(x, \ulcorner A \urcorner).$$

Proof. We proceed by induction on A. We treat the case that $A = \forall v\ C$, where C is $C(v, v_0, \dots, v_{k-1})$.

We have $\mathsf{form}^n(\ulcorner A \urcorner)$ and, hence, $\mathsf{form}^n(\ulcorner C \urcorner)$, $[\![\forall]\!]^n(\ulcorner v \urcorner, \ulcorner C \urcorner, \ulcorner A \urcorner)$ and $\mathsf{fv}^n(\ulcorner x \urcorner)$. Let V be the theory $U + \underline{n} \leqslant b + \mathsf{SAT}(b, B) + \mathsf{ass}(b, x, \ulcorner A \urcorner)$. In V, we have $\mathsf{form}^b(\ulcorner A \urcorner)$,

---

[2]We have '$y = z$' rather than '$v = w$' in the consequent, because our pairing is non-functional. To have lots of copies of the same pair in the representation of a function seems a bit wasteful.

$$\begin{aligned}
\mathsf{SAT}(c,\mathsf{sat}) :&\leftrightarrow \forall x\, \forall a{\leqslant}c\, [\, (\mathsf{form}^c(a) \wedge \mathsf{ass}(c,x,a)) \rightarrow \\
&\quad \{\, \mathsf{sat}(x,a) \leftrightarrow \langle\, \exists b{<}c\, (\,[\![P]\!]^{\,c}(b,a) \wedge P(x(b)) \vee \\
&\qquad\qquad\qquad \exists b_0,b_1{<}c\, (\,[\![Q]\!]^{\,c}(b_0,b_1,a) \wedge Q(x(b_0),x(b_1))) \vee \\
&\qquad\qquad\qquad [\![\top]\!]^{\,c}(a) \vee [\![\bot]\!]^{\,c}(a) \vee \\
&\qquad\qquad\qquad \exists b{<}c\, (\,[\![\neg]\!]^{\,c}(b,a) \wedge \neg\,\mathsf{sat}(x,b)) \vee \\
&\qquad\qquad\qquad \exists b_0,b_1{<}c\, \exists x_0,x_1\, (x_0 \subseteq x \wedge x_1 \subseteq x \wedge \mathsf{ass}(c,x_0,b_0) \wedge \\
&\qquad\qquad\qquad\qquad\qquad \mathsf{ass}(c,x_1,b_1) \wedge (\,[\![\wedge]\!]^{\,c}(b_0,b_1,a) \wedge \\
&\qquad\qquad\qquad\qquad\qquad \mathsf{sat}(x_0,b_0) \wedge \mathsf{sat}(x_1,b_1)) \vee \\
&\qquad\qquad\qquad \exists b_0,b_1{<}a\, (\,[\![\forall]\!]^{\,c}(b_0,b_1,a) \wedge \\
&\qquad\qquad\qquad\qquad \forall w\, (x[b_0]w \rightarrow \mathsf{sat}(w,b_1)))\rangle\,\}\,]
\end{aligned}$$

FIGURE 1. Definition of "sat is a satisfaction predicate below c".

$\mathsf{form}^b(\ulcorner C\urcorner)$, $[\![\forall]\!]^{\,b}(\ulcorner v\urcorner,\ulcorner C\urcorner,\ulcorner A\urcorner)$ and $\mathsf{fv}^b(\ulcorner v\urcorner)$. The induction hypothesis gives us:

$$(\dagger)\quad V \vdash \mathsf{ass}(b,w,\ulcorner C\urcorner) \rightarrow (B(b,w,\ulcorner C\urcorner) \leftrightarrow C(w \star \ulcorner v\urcorner, w \star \ulcorner v_0\urcorner, \ldots, w \star \ulcorner v_{k-1}\urcorner)).$$

We work in V. Suppose $B(b,x,\ulcorner A\urcorner)$. It follows that $\forall w\; x[\ulcorner v\urcorner]w\;\; B(b,w,\ulcorner C\urcorner)$. Consider any z. Since $n > 2$, we have that $\underline{2}$ exists. So we have unordered pairing. Hence, we can construct an ordered pair p such that $\mathsf{pair}(\ulcorner v\urcorner z, p)$. Since the number of free variables in C is smaller than $\ulcorner A\urcorner$ and $\ulcorner A\urcorner$ exists, we can, using repeated adjunction, find a y such that on $\ulcorner v_i\urcorner$, for $i < k$, we have $y \star \ulcorner v_i\urcorner = x \star \ulcorner v_i\urcorner$, and $y \star \ulcorner v\urcorner = z$. It is easy to see that $\mathsf{ass}(b,y,\ulcorner C\urcorner)$ and $x[\ulcorner v\urcorner]y$. By $(\dagger)$, we find $C(y \star \ulcorner v\urcorner, y \star \ulcorner v_0\urcorner, \ldots, y \star \ulcorner v_{k-1}\urcorner)$. We may conclude that

$$C(z, x \star \ulcorner v_0\urcorner, \ldots, x \star \ulcorner v_{k-1}\urcorner).$$

Since z was arbitrary, it follows that $\forall z\; C(z, x \star \ulcorner v_0\urcorner, \ldots, x \star \ulcorner v_{k-1}\urcorner)$, in other words, $A(x \star \ulcorner v_0\urcorner, \ldots, x \star \ulcorner v_{k-1}\urcorner)$.

Conversely, suppose $A(x \star \ulcorner v_0\urcorner, \ldots, x \star \ulcorner v_{k-1}\urcorner)$. This means:

$$\forall z\; C(z, x \star \ulcorner v_0\urcorner, \ldots, x \star \ulcorner v_{k-1}\urcorner).$$

Consider a y with $x[\ulcorner v\urcorner]y$. We find: $C(y \star \ulcorner v\urcorner, y \star \ulcorner v_0\urcorner, \ldots, y \star \ulcorner v_{k-1}\urcorner)$. Hence, by $(\dagger)$, $B(b,y,\ulcorner C\urcorner)$. Thus, we find that, for all y with $x[\ulcorner v\urcorner]y$, we have $B(b,y,\ulcorner C\urcorner)$. It follows that $B(b,x,\ulcorner A\urcorner)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 5.2. *Let K be any direct interpretation of TS_0 in U, possibly with free parameters $z_0, \ldots, z_{p-1}$. Fix any U-formula $A(b,x,a,\vec{z})$. Then, for some n, we have: $U + \mathsf{SAT}(b,A) \vdash \bigvee_{i\leqslant n} b = \underline{i}$.*

Proof. By Lemma 3.5, for some n, and for some formula $B\vec{v}$ with Gödel number m:

$$U + \underline{n}\downarrow \vdash B\vec{y} \leftrightarrow \exists a, s\, (\mathsf{ass}(b,s,a) \wedge \underline{m} = a \wedge \bigwedge_{i<p} s \star \ulcorner z_i\urcorner = z_i \wedge \neg A(s,a,\vec{z})).$$

We take n so big that also $V + \underline{n}\downarrow \vdash \mathsf{form}^n(\underline{m})$.

Reason in U. Suppose $\mathsf{SAT}(b, A(x, a, \vec{z}))$. Suppose $\underline{n} \leqslant b$. Then we have, by Theorem 5.1,

$$
\begin{aligned}
B\vec{z} \quad &\leftrightarrow \quad \exists a, s \, ((\underline{m} = a)^M \wedge \bigwedge_{i < p} s(\ulcorner z_i \urcorner) = z_i \wedge \neg A(s, a, \vec{z})) \\
&\leftrightarrow \quad \neg B\vec{z}.
\end{aligned}
$$

We may conclude, by Lemma 3.1(10), that $U + \mathsf{SAT}(b, A) \vdash \bigvee_{i < n} b = \underline{i}$. ❑

6. PROOF OF THE MAIN RESULT

Consider any theory U. Let α be a Σ_1^0-formula defining the set of Gödelnumbers of the axioms of U. We assume that $\alpha^n(m)$ implies $\mathsf{form}^n(m)$. We define our scheme as follows:

- $\mathcal{S}_\alpha := \bigwedge \mathsf{TS}_0 \to \forall a, b \, ((\alpha^b(a) \wedge \mathsf{SAT}(b, \mathsf{sat})) \to \exists s \, (\mathsf{empty}(s) \wedge \mathsf{sat}(b, s, a))).$

We find that U satisfies the scheme.

Theorem 6.1. *Suppose the axiom set of U is given by the Σ_1^0-formula α. Then, U validates $\mathcal{S}_\alpha$.*

Proof. Consider any direct interpretation K (possibly with free parameters $\vec{z}$) for the language of TS_0. In the following we will suppress K. Let $A := A(b, s, a, \vec{z})$ be any formula. Let $V := U + \mathsf{TS}_0 + \alpha^b(a) + \mathsf{SAT}(b, A)$. By Theorem 5.2, we find, for some n, $V \vdash \bigvee_{j < n} b = \underline{j}$. It follows that $V \vdash \bigvee_{i < n} a = \underline{i}$. Fix $i, j < n$. If we have not $\alpha^j(i)$. Then, by Lemma 3.3, we find $V \vdash \neg \alpha^{\underline{j}}(\underline{i})$. So, it follows that $V \vdash \neg (a = \underline{i} \wedge b = \underline{j})$ and a fortiori, $V + a = \underline{i} + b = \underline{j} \vdash \exists s \, (\mathsf{empty}(s) \wedge \mathsf{A}(b, s, a)))$. If we have $\alpha^i(j)$. Then we find, by Lemma 3.3, $V \vdash \alpha^{\underline{j}}(\underline{i})$. Suppose i is the Gödelnumber of C. Since, $U \vdash C$, we find, by Theorem 5.1, $V + a = \underline{i} + b = \underline{j} \vdash \exists s \, (\mathsf{empty}(s) \wedge A(b, s, a)))$. So, we obtain $V \vdash \exists s \, (\mathsf{empty}(s) \wedge A(b, s, a)))$. ❑

Theorem 6.2. *Consider a theory U whose axiom set is given by the Σ_1^0-formula α. We suppose that we have a direct interpretation M, of VS_2 in U. We write $[\mathcal{S}_\alpha]$, for the set of instances of $\mathcal{S}_\alpha$ in the signature of U. Then, $\mathsf{VS}_2^M + [\mathcal{S}_\alpha]$ proves the same theorems as U.*

Proof. The fact that U proves all principles of $V := \mathsf{VS}_2^M + [\mathcal{S}_\alpha]$ is immediate by Theorem 6.1. We prove the converse.

Consider an axiom A of U and suppose $\alpha^n(\ulcorner A \urcorner)$. Theorem 4.1 provides a direct interpretation K of TS_n in VS_2. So, $L := K \circ M$ is a direct interpretation of TS_n in U. We define a satisfaction predicate $B(x, c)$, for all formulas C, such that $\mathsf{form}^n(C)$, just by making it the disjunction of all formulas:

- $(c = \ulcorner C \urcorner) \wedge \mathsf{ass}(\underline{n}, x, \ulcorner C \urcorner) \wedge C(x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner),$
  where $C = C(v_0, \ldots, v_{k-1})$ and $\mathsf{form}^n(\ulcorner C \urcorner)$.[3]

We find in V that, under interpretation L, we have $\mathsf{SAT}(\underline{n}, \ulcorner B \urcorner)$. Let s be any empty set. By $\mathcal{S}_\alpha$, instantiated via L for the signature of VS_n and by substituting B for sat, we find: $V \vdash B(\underline{n}, s, \ulcorner A \urcorner)$. Hence, $V \vdash A$. ❑

---

[3]Of course, this is a very inefficient way, in terms of size, to make such a predicate. One can clearly do much better.

We note that the result works both for the case that we consider closed schemes or schemes with parameters.

Inspection of the methods used shows that the result is verifiable in EA. It follows that $\mathsf{EA} \vdash \mathsf{con}(\alpha) \leftrightarrow \mathsf{con}(\mathsf{VS}_2^M + [\mathcal{S}_\alpha])$.

7. Comprehension Principles

Schemes can be condensed into single sentences in the presence of suitable comprehension principles. We briefly consider what happens in two cases: First Order Comprehension or FOC and Predicative Comprehension or PC.

7.1. **First Order Comprehension.** Let us start with a pair theory U which is axiomatized by a scheme $\mathcal{S}_0$. We first show that $\mathcal{S}_0$ can be replaced by a unary scheme $\mathcal{S}$, i.e., a scheme in which only unary schematic predicates occur. We note that in VS_2 we can define Wiener-Kuratowski pairing. Let $\mathsf{pair}(x, y, z)$ be:

$$\exists u, v \ (\ \forall w \ (w \in z \leftrightarrow (w = u \vee w = v)) \wedge$$
$$\forall w' \ (w' \in u \leftrightarrow w' = x) \wedge \forall w'' \ (w'' \in v \leftrightarrow (w'' = x \vee w'' = y)) \ ).$$

Using pair we can define n-tuple for standard n. Also we can define the associated projection functions. We can verify the desired properties in VS_2. We assign to every schematic predicate P a unary schematic predicate X_P. Consider an occurrence of a schematic predicate $P(x_0, \ldots, x_{n-1})$ in $\mathcal{S}_0$. We replace this occurrence by $\exists s \ (n\text{-}\mathsf{tuple}(s) \wedge (s)_0 = x_0 \wedge \ldots \wedge (s)_{n-1} = x_{n-1} \wedge s \in X_P)$. Similarly, for all other occurrences of schematic predicates in $\mathcal{S}_0$.[4] Suppose in the original scheme we would substitute $A\vec{x}$ for $P\vec{x}$. Clearly, we will obtain the same effect by substituting $(n\text{-}\mathsf{tuple}(s) \wedge \exists x_0, \ldots, x_{n-1} \ ((s)_0 = x_0 \wedge \ldots \wedge (s)_{n-1} = x_{n-1} \wedge A\vec{x}))$ for X_P.

We suppose that U is axiomatized by a unary scheme $\mathcal{S}$. We extend U with a new sort of classes $\mathfrak{c}$, identity on classes and a binary relation $\in$ of type $\mathfrak{oc}$. Here $\mathfrak{o}$ is the sort of the original objects of U. We use the variables $X, Y, Z, \ldots$ to range over classes. The theory $\mathcal{S}^{\mathsf{foc}}$ in the extended signature is defined as follows.

$\mathcal{S}^{\mathsf{foc}}1. \ \vdash \forall \vec{y} \exists X \forall x \ (x \in X \leftrightarrow Ax\vec{y})$,
    where A is a formula from the original language of U.
$\mathcal{S}^{\mathsf{foc}}2. \ \vdash \forall X, Y \ (X = Y \leftrightarrow \forall z \ (z \in X \leftrightarrow z \in Y))$.
$\mathcal{S}^{\mathsf{foc}}3. \ \vdash \forall \vec{X} \ \mathcal{S}^\star \vec{X}$,
    where $\mathcal{S}^\star$ is the result of replacing all occurrences of schematic predicates in $\mathcal{S}$ by class variables in the obvious way, e.g. Xx by $x \in X$.

The next theorem gives us the obvious connection between U and $\mathcal{S}^{\mathsf{foc}}$.

Theorem 7.1. *Suppose U is a pair theory, axiomatized by a unary scheme $\mathcal{S}$. Clearly, $\mathcal{S}^{\mathsf{foc}}$ is a conservative extension of U. Moroever U locally $\mathfrak{o}$-directly interprets $\mathcal{S}^{\mathsf{foc}}$.*

Proof. We work in U. Suppose we want to have comprehension for the formulas $A_i x\vec{y}_i$, for $i = 0, \ldots, k - 1$, where the length of $\vec{y}_i$ is n_i. We define:

- $\mathbf{i}(v) :\leftrightarrow \exists v_0, \ldots, v_{i-1} \ (\ \forall w \ w \notin v_0 \wedge \forall w \ (w \in v_1 \leftrightarrow w = v_0) \wedge \ldots$
$\forall w \ (w \in v \leftrightarrow w = v_{i-1}) \ ).$
- $\mathsf{class}_i(v) :\leftrightarrow (n_i + 1)\text{-}\mathsf{tuple}(v) \wedge \mathbf{i}((v)_0).$

---

[4]If a schematic predicate in $\mathcal{S}_0$ is 0-ary, we introduce a dummy variable.

- $\mathsf{class}(v) :\leftrightarrow \bigvee_{i<k} \mathsf{class}_i(v)$.
- $x \in v :\leftrightarrow \mathsf{class}(v) \wedge \bigvee_{i<k} (\ \mathbf{i}((v)_0) \wedge A_i(x, (v)_1, \ldots, (v)_{n_i})\ )$.
- $v =_{\mathfrak{c}} w :\leftrightarrow \mathsf{class}(v) \wedge \mathsf{class}(w) \wedge \forall x\ (x \in v \leftrightarrow x \in w)$.

We take class as the domain for the sort of classes, $\in$ in the obvious role and $=_{\mathfrak{c}}$ as identity of classes. It is easy to check that this interpretation has the desired properties. ❏

An interpretation is $\mathfrak{o}$-direct if it interprets the sort $\mathfrak{o}$ as the sort $\mathfrak{o}$ and if it is unrelativized for $\mathfrak{o}$ and sends $\mathfrak{o}$-identity to itself. We have an interesting connection between $\mathsf{VS}_2^{\mathsf{foc}}$ and VS.

Theorem 7.2. *We have* $\mathsf{VS}_2^{\mathsf{foc}} \rhd_{\mathfrak{o}\text{-dir}} \mathsf{VS}$.

Proof. We work in $\mathsf{VS}_2^{\mathsf{foc}}$. We first define Wiener-Kuratowski pairing. We work with ordered pairs. To give the heuristic, let's ignore for a moment the fact that our ordered pairing is not necessarily functional. The basic idea is to code e.g. the set consisting of a, b, c as $\langle\langle\langle 0, a\rangle, b\rangle, c\rangle$, where 0 is a non-pair. Now forget about functionality again. We define:

- $\mathsf{dc}(Y) :\leftrightarrow \forall u, v, p\ ((\mathsf{pair}(u, v, p) \wedge p \in Y) \to u \in Y)$,
  (We will also write Y:dc for $\mathsf{dc}(Y)$.)
- $x \in^\star y :\leftrightarrow \forall Y{:}\mathsf{dc}\ (y \in Y \to \exists w, q\ (\mathsf{pair}(w, x, q) \wedge q \in Y))$.

Consider any non-pair z. We clearly have $\{z\}$:dc. If we would have $x \in z$, then, for some pair q, we would have that q is in the class $\{z\}$, quod non. So z is an empty $\star$-set. This gives us VS_0.

We have the following insight (†): Consider any x and y. Pick any z with $\mathsf{pair}(y, x, z)$. Clearly, $x \in^\star z$. Moreover, we find $y \subseteq^\star z$. For, suppose $u \in^\star y$, $\mathsf{dc}(Y)$ and $z \in Y$. We find $y \in Y$, and, hence, for some w and q, $\mathsf{pair}(w, u, q)$ and $q \in Y$. So $u \in z$.

We now show VS_n be induction on n. Let $x_0, \ldots, x_{n-1}$ be given. We construct a sequence of $\star$-sets $z_0, \ldots, z_n$ as follows. Let z_0 be any non-pair. We take z_{i+1} to be an object such that $\mathsf{pair}(z_i, x_{i-1}, z_{i+1})$. We note, by (†), that $x_j \in^\star z_i$, for all $j < i$.

Let Y_i be the class $\{z_0, \ldots, z_i\}$. We easily verify Y_i is downwards closed. Clearly for any u not among $x_0, \ldots x_{i-1}$, we find that no v such that $\mathsf{pair}(w, u, v)$, for some w, is in Y_i. On the other hand z_i is in Y_i. It follows that $u \not\in^\star z_i$. Ergo, for all u, $u \in^\star z_i$ iff $u = x_0$ or $\ldots$ or $u = x_{i-1}$. ❏

Theorem 7.2 suggests the following questions.

Open Question 7.3. Suppose U is a pair theory axiomatized by a scheme $\mathcal{S}$. Do we have, for any W with $W \rhd U \rhd_{\mathsf{loc}} W$, that $\mathcal{S}^{\mathsf{foc}} \rhd W$?[5] ❏

Open Question 7.4. Do we have $\mathsf{VS} \rhd \mathsf{VS}_2^{\mathsf{loc}}$? ❏

7.2. **Predicative Comprehension.** In our paper [Vis09a], we elaborate the equation "consistency = predicative comprehension". In that context, Vaught's result provides on way of articulating what predicative comprehension for infinitely axiomatized RE pair theories means.

Consider a pair theory U that is axiomatized by a scheme $\mathcal{S}$. As before we may assume that $\mathcal{S}$ only contains unary predicate variables. We extend the language

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[5]We know that this holds when U is a *sequential theory*.

of U with class variables. The theory $\mathcal{S}^{\mathsf{pc}}$ in the extended signature is defined as follows.

$\mathcal{S}^{\mathsf{pc}}1.$ $\vdash \forall \vec{y} \exists X \forall x \ (x \in X \leftrightarrow Ax\vec{y}\vec{Y})$, where A is a formula that contains no bound class variables and that does not contain X.

$\mathcal{S}^{\mathsf{pc}}2.$ $\vdash \forall X, Y \ (X = Y \leftrightarrow \forall z \ (z \in X \leftrightarrow z \in Y))$.

$\mathcal{S}^{\mathsf{pc}}3.$ $\vdash \forall \vec{X} \ \mathcal{S}^\star \vec{X}$, where $\mathcal{S}^\star$ is the result of replacing all occurrences of schematic predicates in $\mathcal{S}$ by class variables.

Clearly, $\mathcal{S}^{\mathsf{pc}}$ is a conservative extension of U. One can show that $\mathcal{S}^{\mathsf{pc}}$ is finitely axiomatizable. In case that U is sequential one can show that $(\mathsf{Q} + \mathsf{con}([\mathcal{S}])) \equiv \mathcal{S}^{\mathsf{pc}}$. The details of this are explained in [Vis09b] and [Vis09a]. So, we note that, by the Second Incompleteness Theorem, $\mathcal{S}^{\mathsf{pc}}$ is really stronger than U.

We may apply Vaught's result, verifiably in EA, to any consistent sequential theory RE theory U, say axiomatized using α. This gives us:

$$\mathsf{EA} \vdash (\mathsf{Q} + \mathsf{con}(\alpha)) \equiv \mathcal{S}_\alpha{}^{\mathsf{pc}}.$$

However it is easy to see that, for any true Π^0_1-sentence P, we can find an axiomatization β of U such that $(\mathsf{Q} + \mathsf{con}(\beta)) \rhd (\mathsf{Q} + P)$. Thus, $\mathcal{S}_\beta{}^{\mathsf{pc}}$ can be made arbitrarily strong, when β ranges over schemes for U. This illustrates that a scheme is 'intensional' from the point of view of consistency strength.

We end with a question.

Open Question 7.5. Do we have $(\mathsf{Q} + \mathsf{con}([\mathcal{S}])) \equiv \mathcal{S}^{\mathsf{pc}}$, for pair theories U axiomatized by $\mathcal{S}$?   ◘

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