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REDUCTION OF THE SEMISIMPLE 1:1 RESONANCE

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The method of "averaging" is often used in Hamiltonian systems of two degrees of freedom to find periodic orbits. Such periodic orbits can be reconstructed from the critical points of an associated "reduced" Hamiltonian on a "reduced space". This paper details the construction of the reduced space and the reduced Hamiltonian for the semisimple 1:1 resonance case. The reduced space will be a 2-sphere in R^3, and the reduced differential equations will be Euler's equations restricted to this sphere. The orbit projection from the energy surface in phase space to this sphere will be the Hopf map. The results of the paper are related to problems in physics on "degeneracies" due to symmetries of classical two-dimensional harmonic oscillators and their quantum analogues for the hydrogen atom.

1. Introduction

It is the purpose of this paper to detail the construction of the reduced space and the reduced Hamiltonian for the two degree of freedom semisimple 1:1 resonance case (the terminology will be explained below). This is one of the simplest nontrivial examples where reduction can be explicitly carried out with applications to such problems as the Hénon–Heiles Hamiltonian (see [5], [13, 14], and [21]). A special case of such a system is the two-dimensional isotropic harmonic oscillator. Jauch and Hill [12] observed that the quantum mechanical degeneracies of this harmonic oscillator were due to its SU(2) dynamical symmetry group (see also [6]). Later Dulock and McIntosh [7] found that for the harmonic oscillator the space of all orbits of a given energy (which is topologically a 3-sphere in R^4) could be mapped by the Hopf mapping to the standard 2-sphere in R^3 (see also [9, p. 169] and [13–15]). The Hopf mapping (given in (1.5) below) maps distinct orbits of the oscillator (which are great circles on the 3-sphere) to distinct points of the 2-sphere which is then the reduced space (orbit space) for this oscillator. Finally it was realized by [2], [20, p. 120], and [17] that the dynamics of the harmonic oscillator induced dynamics on the reduced phase space which are given by Euler's equations for a rigid body (see also [13, p. 58]). For some of the background to [12] and [7] we refer to [8] and [3].

An important use of the reduced space is for finding periodic orbits of real analytic Hamiltonian systems of differential equations with two degrees of freedom. This involves "averaging" the Hamiltonian H out to some

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order and truncating to obtain a polynomial Hamiltonian in "normal form" as in (1.2) and (1.3) below. If the truncated Hamiltonian system has periodic orbits that are actually reparametrized orbits of the flow generated by the quadratic terms of H, then these will show up as critical points of an associated reduced Hamiltonian on the reduced space [1, p. 306]. The explicit construction of the reduced Hamiltonian for our case turns out to be trivial and is detailed in (2.24)–(2.26) below. Having located such periodic orbits for the truncated Hamiltonian, one can try to carry them over to the original, full Hamiltonian. In the case that the periodic orbits originate as the energy increases from an equilibrium point of the Hamiltonian at energy zero, this can often be done by some perturbation scheme that then gives their existence in the original Hamiltonian flow at low positive energies (see for example [19]). Such a program is carried out in detail in [5], [13, 14], and [21].

Let $(\mathrm{R}^4, \omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$ be a symplectic vector space with global coordinates (x_1, x_2, y_1, y_2). We define the linear map ω^b by $\omega^b(\partial/\partial x_i) = \mathrm{d}y_i$ and $\omega^b(\partial/\partial y_i) = -\mathrm{d}x_i$ for $i = 1, 2$, and set $\omega^{\#} = (\omega^b)^{-1}$. By the semisimple 1:1 resonance case we mean that the quadratic terms H_2 of our Hamiltonian have the form

$$H_2(x_1, x_2, y_1, y_2) = \tfrac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2). \tag{1.1}$$

H_2 is semisimple because the linear Hamiltonian vector field $X_2 = \omega^{\#}(\mathrm{d}H_2)$ is diagonalizable over the complex numbers (see (3.5) and (3.6) below). Moreover, since i is a double eigenvalue of X_2 at the origin (which is an equilibrium point of X_2), we say X_2 (or H_2 itself) is in 1:1 resonance.

We will consider polynomial Hamiltonian functions of the form

$$H = \sum_{i=1}^{n} H_{2i}, \tag{1.2}$$

where the H_{2i} are homogeneous polynomials of degree $(2i)$ in the variables (x_1, x_2, y_1, y_2). We will assume that H is in normal form w.r.t. H_2; that is, the Lie derivative of H w.r.t. the Hamiltonian vector field X_2 vanishes. In standard notation [1, p. 79 and p. 192] we have

$$L_{X_2}(H) = X_2(H) = \{H, H_2\} = 0, \tag{1.3}$$

where $\{\,,\}$ denotes Poisson brackets defined w.r.t. the symplectic form ω above. We show in section 3 that (1.3) is equivalent to

$$H = H(W_1, W_2, W_3, W_4), \tag{1.4}$$

where the W_j, $j = 1, 2, 3, 4$, are quadratic polynomials in the variables (x_1, x_2, y_1, y_2) given in (2.10) below. The W_j are referred to as "Hopf variables" and the map

$$(x_1, x_2, y_1, y_2) \to (W_1, W_2, W_3) \tag{1.5}$$

is the standard "Hopf map" when restricted to the 3-sphere in R^4. Thus H as in (1.2) being in normal form w.r.t. H_2 means that H can be expressed as a polynomial (1.4) in these Hopf variables.

We mention two standard ways of computing the normal form w.r.t. H_2 of a given Hamiltonian up to some order, so that on truncating the higher order terms we have (1.2) with (1.3). The original scheme for computing the Birkhoff normal form of a Hamiltonian was modified by Gustavson in [10] to apply to resonance cases such as (1.1). An account of this method can also be found in [18, pp. 10–13]. An alternate technique uses the method of Lie series (see [1, pp. 500–502] and the references therein). This latter method was applied to the Hénon–Heiles Hamiltonian in [5].

We can now summarize the results of this paper. Let S^n_r denote the standard n-sphere in R^{n+1} centered at the origin with radius r. In section 2 we will carry out the reduction process on the Hamiltonian vector field $X_H = \omega^{\#}(\mathrm{d}H)$ restricted to $M = S^3_{(2h)^{1/2}} = H_2^{-1}(h) \subset \mathrm{R}^4$ (see [1, p.

299 and p. 304]). We will find that the reduced space $M_R = M_R(h)$ can be identified as the sphere $S^2_{(2h)} \subset R^3$ by factoring the canonical projection $\pi : M \to M_R$ through the following commutative diagram:

$$\begin{array}{ccc} M = S^3_{(2h)^{1/2}} & \xrightarrow{J} & su(2)^* \\ \pi \downarrow & & \downarrow \gamma^\# \\ M_R \simeq S^2_{(2h)} & \xleftarrow{\lambda^{-1}} & su(2) \end{array} \qquad (1.6)$$

Here J is the Ad*-equivariant momentum map associated to a linear SU(2) action on R^4, γ is the standard Killing form on the Lie algebra su(2) with induced linear map $\gamma^b : su(2) \to su(2)^*$ and $\gamma^\# = (\gamma^b)^{-1}$, and $(2\lambda) : R^3 \to su(2)$ is a natural identification map. The orbit map π sends distinct periodic orbits of the flow of X_2 in M to distinct points in M_R. The factor of 2 in the identification map (2λ) has been put in so that π will be precisely the standard Hopf mapping [11, p. 654], and $(\gamma^b \circ \lambda)(M_R)$ will be an orbit in su(2)* of the coadjoint action of SU(2) on the dual of its Lie algebra.

The factoring of π in (1.6) allows us to calculate explicitly the induced symplectic form ω_R on M_R satisfying the pullback relation $\pi^*\omega_R = i^*\omega$, where $i : M \to R^4$ is inclusion. We can then find the reduced Hamiltonian K that is associated to H by $K \circ \pi = H \circ i$, and the reduced vector field $X_K = (\omega_R)^\#(dK)$ on M_R. In (2.29) we show that this reduced vector field is just Euler's equations restricted to the sphere M_R (see [13, p. 58]). That the orbit projection π is the Hopf map was expressed in [7].

We will use the notation of [1] throughout (see also [16, Lectures 3 and 4]).

2. Reduction of X_H

We begin the reduction of the Hamiltonian vector field X_H by showing that the hypotheses of the reduction theorem [1, p. 299 and p. 304] are satisfied for the S^1-action on R^4 generated by X_2. Since H is in normal form, $0 = L_{X_2}(H) = -L_{X_H}(H_2)$, implying that H_2 is an integral of X_H. Thus for $h > 0$ the flow of X_H leaves the 3-sphere $M = H_2^{-1}(h)$ invariant, and X_H restricts to M. The flow of the linear Hamiltonian vector field X_2 defines an S^1-action $\Phi : [0, 2\pi) \times R^4 \to R^4$ by

$$\Phi(t, x) = \Phi_t(x) = \begin{pmatrix} (\cos t)I_2 & (\sin t)I_2 \\ -(\sin t)I_2 & (\cos t)I_2 \end{pmatrix} x, \qquad (2.1)$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $x = (x_1, x_2, y_1, y_2)$. Φ_t is a linear symplectic mapping of (R^4, ω), and the S^1-momentum mapping associated to Φ is just H_2 itself which is trivially Ad*-equivariant. Since Φ restricts to a free and proper S^1-action on M, the reduction process in [1, p. 299], applied to the momentum map H_2, yields a smooth orbit manifold $M_R = M/S^1$. The smooth surjective orbit map $\pi : M \to M_R$ together with ω determines a unique symplectic form ω_R on M_R by the pullback relation $\pi^*\omega_R = i^*\omega$, where $i : M \to R^4$ is inclusion. Now (1.3) implies that H is invariant under the action of Φ_t, hence by [1, p. 304] there is an induced Hamiltonian K on M_R such that $K \circ \pi = H \circ i$ and

$$\pi_* X_H(x) = X_K(\pi(x)), \quad \text{for all } x \in M, \qquad (2.2)$$

where π_* is the tangent map of π.

We now determine a model for (M_R, ω_R, π) by factoring π through a momentum mapping associated to a linear SU(2) action on (R^4, ω). One can easily show that any two models for the reduced space are symplectomorphic.

With $A, B \in gl(2, R)$ we let

$$\begin{aligned} U(2) &= Sp(4, R) \cap SO(4, R) \\ &= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\middle|\, (AA^t + BB^t) = I_2,\ AB^t = BA^t \right\} \end{aligned} \qquad (2.3)$$

be the unitary group with Lie algebra

$$\begin{aligned} u(2) &= sp(4, R) \cap so(4, R) \\ &= \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a = -a^t,\ b = b^t \right\}. \end{aligned} \qquad (2.4)$$

(We remark that in many books on Lie groups Sp(4, R) is denoted by Sp(2, R), etc.) Then

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{U}(2) \mid \det(A + \mathrm{i}B) = 1 \right\} \quad (2.5)$$

is the special unitary group with Lie algebra

$$\mathrm{su}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{u}(2) \mid \mathrm{Tr}(b) = 0 \right\}. \quad (2.6)$$

Let

$$e_1 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right), \quad e_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right),$$

$$e_3 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \quad e_4 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right). \quad (2.7)$$

Then $\{e_1, e_2, e_3, e_4\}$ is a basis for u(2), and $\{e_1, e_2, e_3\}$ is a basis for su(2).

For $V \in \mathrm{SU}(2)$ and $x \in \mathrm{R}^4$ the linear symplectic action $(V, x) \to V \cdot x$ of SU(2) on (R^4, ω) has an associated momentum mapping $J: \mathrm{R}^4 \to \mathrm{su}(2)^*$ given by (see [1, pp. 287–288(iv)])

$$J(x) \cdot e = \tfrac{1}{2}\omega(ex, x), \quad (2.8)$$

where $e = (a_1e_1 + a_2e_2 + a_3e_3) \in \mathrm{su}(2)$ and the a_j, $j = 1, 2, 3$, are real coefficients.

With $\langle\, , \rangle$ as the standard inner product on R^4, we have

$$\begin{aligned} J(x) \cdot e &= (1/2)\langle ex, e_4x\rangle \\ &= (1/2)[a_1W_1 + a_2W_2 + a_3W_3], \end{aligned} \quad (2.9)$$

where the W_j are the Hopf variables (see [11, p. 654])

$$W_1 = 2(x_1x_2 + y_1y_2),$$

$$W_2 = 2(x_2y_1 - x_1y_2),$$

$$W_3 = (x_1^2 + y_1^2 - x_2^2 - y_2^2),$$

$$W_4 = (x_1^2 + x_2^2 + y_1^2 + y_2^2) = 2H_2. \quad (2.10)$$

The $W_j(x) = \langle e_jx, e_4x\rangle$ for $j = 1, 2, 3, 4$, satisfy the identity

$$(W_1^2 + W_2^2 + W_3^2) = W_4^2. \quad (2.11)$$

The mapping J can then be written

$$J(x) = \tfrac{1}{2}[W_1e_1^* + W_2e_2^* + W_3e_3^*], \quad (2.12)$$

where $\{e_i^*\}$ is the basis of su(2)* dual to $\{e_i\}$.

We now show that $J(M)$ is a coadjoint orbit in su(2)*. For $V \in \mathrm{SU}(2)$ and $e \in \mathrm{su}(2)$ we have for every $x \in \mathrm{R}^4$

$$\begin{aligned} J(Vx) \cdot e &= \tfrac{1}{2}\omega(eVx, Vx) = \tfrac{1}{2}\omega(V^{-1}eVx, x) \\ &= J(x) \cdot V^{-1}eV, \end{aligned} \quad (2.13)$$

or

$$J(Vx) = \mathrm{Ad}^*_{V^{-1}} \circ J(x). \quad (2.14)$$

Thus, J is an Ad*-equivariant momentum mapping for the linear symplectic action of SU(2) on (R^4, ω). Since SU(2) gives a *transitive* action on the 3-sphere M, we have for $x_0 = ((2h)^{1/2}, 0, 0, 0)$ that

$$\begin{aligned} J(M) &= J(\mathrm{SU}(2) \cdot x_0) \\ &= \{\mathrm{Ad}^*_{V^{-1}} \circ J(x_0) \mid V \in \mathrm{SU}(2)\} \subset \mathrm{su}(2)^*, \end{aligned} \quad (2.15)$$

where $J(x_0) = he_3^*$. Thus, in particular, the coadjoint orbit $J(M)$ is even dimensional and has a standard symplectic form given by the Kirillov–Kostant–Souriau theorem [1, p. 302]. We now determine a symplectic form Ω on this coajoint orbit so that $J^*\Omega = i^*\omega$ where $i: M \to \mathrm{R}^4$ is inclusion.

Let $x \in \mathrm{R}^4 = P$ and $J(x) = \mu \in \mathrm{su}(2)^* = Q$. Let

$\xi_P(x)$ denote the infinitesimal generator at x corresponding to $\xi \in su(2)$ of the above linear action of SU(2) on P, and $\xi_Q(\mu)$ the corresponding entity for the Ad*-action of SU(2) on Q (see [1, p. 267] for definitions). Letting J_* denote the tangent map of J, the equivariance relation (2.14) implies $J_*\xi_P(x) = \xi_Q(\mu)$ (see [1, p. 270]). Let $\{\,,\}$ denote the Poisson brackets given by the standard symplectic form ω on R^4 (see [1, p. 192]). On infinitesimalizing the Ad*-equivariance of J in formula (2.14), we obtain for $\xi, \eta \in su(2)$ on setting $J^*\Omega = i^*\omega$ (see [1, p. 276 and p. 281] for notation):

$$\begin{aligned}\Omega(\xi_Q, \eta_Q)(\mu) &= \Omega(J_*\xi_P, J_*\eta_P)(Jx)\\ &= (J^*\Omega)(\xi_P, \eta_P)(x)\\ &= \omega(\xi_P, \eta_P)(x)\\ &= \omega(X_{\hat{J}(\xi)}, X_{\hat{J}(\eta)})(x)\\ &= \{\hat{J}(\xi), \hat{J}(\eta)\}(x)\\ &= \hat{J}[\xi, \eta](x)\\ &= J(x)\cdot[\xi, \eta]\\ &= \mu\cdot[\xi, \eta]. \end{aligned} \tag{2.16}$$

Thus Ω is the Kirillov–Konstant–Souriau form on the coadjoint orbit $J(M)$ (modulo signum; see [1, p. 303]).

We will eventually realize the reduced space M_R in R^3, and for this purpose we must first pass from su(2)* to su(2). Let $e = \Sigma_{i=1}^3 a_i e_i$ and $e' = \Sigma_{i=1}^3 a_i' e_i$ be in su(2). We define a positive definite inner product γ on su(2) by

$$\gamma(e, e') = -\tfrac{1}{4}\mathrm{Tr}(e\cdot e') = [a_1a_1' + a_2a_2' + a_3a_3']. \tag{2.17}$$

Then γ is a Killing form on su(2) that induces a linear map $\gamma^b : su(2) \to su(2)^*$ by $\gamma^b(e)\cdot e' = \gamma(e, e')$. We set $\gamma^\# = (\gamma^b)^{-1}$. A direct computation then shows that the following diagram is commutative:

$$\begin{array}{ccc} su(2) & \xrightarrow{\gamma^b} & su(2)^* \\ \downarrow{\scriptstyle Ad_V} & & \downarrow{\scriptstyle Ad^*_{V^{-1}}} \\ su(2) & \xrightarrow{\gamma^b} & su(2)^* \end{array} \tag{2.18}$$

It follows that $(\gamma^\# \circ J)(M)$ is an orbit of the adjoint action of SU(2) on its Lie algebra. Infinitesimalizing the equivariance relation (2.18) gives for $\xi \in su(2) = T$ and $su(2)^* = Q$ (see [1, p.270])

$$\gamma^b(\xi_T) = \xi_Q \circ \gamma^b, \tag{2.19}$$

since $(\gamma^b)_* = \gamma^b$ as γ^b is linear. A computation similar to (2.16) then shows that the symplectic form $\tilde{\Omega} = (\gamma^b)^*\Omega$ on this adjoint orbit satisfies

$$\tilde{\Omega}(\xi_T, \eta_T)(\nu) = \gamma(\nu, [\xi, \eta]) \tag{2.20}$$

for $\xi, \eta, \nu \in su(2)$ where $\xi_T(\nu) = [\xi, \nu]$ (see [1, pp. 267–268(b)]).

The last stage in realizing the reduced space M_R as a subset of R^3 requires an identification of su(2) with R^3. Let $\{f_1, f_2, f_3\}$ be the standard orthonormal basis of R^3. Define a linear map $\lambda : R^3 \to su(2)$ by $\lambda(f_i) = \frac{1}{2}e_i$ for $i = 1, 2, 3$. With Γ as the standard inner product on R^3, note that (2λ) is an isometry of (R^3, Γ) with $(su(2), \gamma)$; that is, $(2\lambda)^*\gamma = \Gamma$.

We compute from (2.7) the su(2) Lie bracket $[e_1, e_2] = (e_1e_2 - e_2e_1) = -2e_3$. Hence $[\lambda f_1, \lambda f_2] = -\lambda(f_1 \times f_2)$ where $\times$ is the standard cross product on R^3. There are corresponding identities on cyclic permutation of the indices. Thus $\lambda : R^3 \to su(2)$ is a Lie algebra isomorphism relative to $(-1)\times$ as the bracket operation on R^3.

Letting $\tilde{J} = (\lambda^{-1} \circ \gamma^\# \circ J)$, we derive from (2.12) that

$$\tilde{J}(x) = [W_1f_1 + W_2f_2 + W_3f_3] = (W_1, W_2, W_3) \in R^3. \tag{2.21}$$

Restricting $x \in M$, the identity (2.11) implies

$$\tilde{J}(M) = S^2_{(2h)} \subset R^3. \tag{2.22}$$

Thus the coadjoint orbit in su(2)* has now been mapped onto a sphere in R^3. We can now compute the symplectic form ω_R on this sphere satisfying $\omega_R = \lambda^*\tilde{\Omega}$. Let $\tilde{\xi} \times \tilde{\nu}$ and $\tilde{\eta} \times \tilde{\nu}$ be tangent vectors at $\tilde{\nu} \in S^2_{(2h)}$. Let $\lambda(\tilde{\xi}) = \xi$, $\lambda(\tilde{\eta}) = \eta$, $\lambda(\tilde{\nu}) = \nu$ be in su(2), and recall that $\lambda(\tilde{\xi} \times \tilde{\nu}) = -[\xi, \nu]$, etc. Then using (2.20) we have

$$\begin{aligned}\omega_R(\tilde{\xi} \times \tilde{\nu}, \tilde{\eta} \times \tilde{\nu})(\tilde{\nu}) &= (\lambda^*\tilde{\Omega})(\tilde{\xi} \times \tilde{\nu}, \tilde{\eta} \times \tilde{\nu})(\tilde{\nu}) \\ &= \tilde{\Omega}(\lambda(\tilde{\xi} \times \tilde{\nu}), \lambda(\tilde{\eta} \times \tilde{\nu}))(\nu) \\ &= \tilde{\Omega}([\xi, \nu], [\eta, \nu])(\nu) \\ &= \gamma(\nu, [\xi, \eta]) \\ &= -\tfrac{1}{4}\gamma((2\lambda)(\tilde{\nu}), (2\lambda)(\tilde{\xi} \times \tilde{\eta})) \\ &= -\tfrac{1}{4}[(2\lambda)^*\gamma](\tilde{\nu}, \tilde{\xi} \times \tilde{\eta}) \\ &= -\tfrac{1}{4}\Gamma(\tilde{\nu}, \tilde{\xi} \times \tilde{\eta}), \end{aligned} \qquad (2.23)$$

since (2λ) is an isometry. Note that the result is independent of the representation of the tangent vectors $\tilde{\xi} \times \tilde{\nu} = (\tilde{\xi} + r\tilde{\nu}) \times \tilde{\nu}$ where $r \in R^1$.

For $h = \frac{1}{2}$ formulas (2.21) and (2.22) show that the map $\tilde{J}$ is the Hopf map of S^3 onto S^2 (see [11, p. 654]). The Hopf map is a fibration of S^3 over S^2 with fiber S^1 and Hopf invariant +1. One easily calculates from (2.10) that $L_{X_2}(W_i) = 0$ for $i = 1, 2, 3, 4$, and hence the components of $\tilde{J}$ are integrals of X_2. Thus each fiber $(\tilde{J})^{-1}(w)$ for $w \in S^2_{(2h)}$ is an integral manifold of X_2 that is diffeomorphic to S^1 and hence is a unique Φ orbit in M. We choose $(M_R, \omega_R, \pi) = (S^2_{(2h)}, \omega_R, \tilde{J})$ as a model for the reduced space and note that by our construction $\tilde{J} = \pi : M \to M_R$ satisfies $\pi^*\omega_R = i^*\omega$ where ω_R is given by (2.23). We can now compute the induced differential equations on M_R.

In section 3 below we show that (1.3) implies that H as in (1.2) satisfies

$$H_{2i}(x_1, x_2, y_1, y_2) = K_i(W_1, W_2, W_3, W_4), \qquad (2.24)$$

where K_i is a homogeneous polynomial of degree i in the Hopf variables (2.10). Letting $w = (w_1, w_2, w_3)$ be global coordinates on R^3, we define

$$K^e(w) = \sum_{i=1}^{n} K_i(w_1, w_2, w_3, 2h), \qquad (2.25)$$

and set

$$K = K^e \mid M_R. \qquad (2.26)$$

Clearly $K \circ \pi = H \circ i$, so that K is the reduced Hamiltonian associated to H (see [1, p. 304]). By (2.23) the Hamiltonian vector field $X_K(w) = A \times w$ satisfies for an arbitrary tangent vector $B \times w$ at $w \in M_R$:

$$\begin{aligned}\Gamma(w \times \nabla K^e(w), B) &= \Gamma(\nabla K^e(w), B \times w) \\ &= dK(w) \cdot (B \times w) \\ &= \omega_R(X_K(w), B \times w)(w) \\ &= \omega_R(A \times w, B \times w)(w) \\ &= -\tfrac{1}{4}\Gamma(w, A \times B) \\ &= \tfrac{1}{4}\Gamma(X_K(w), B). \end{aligned} \qquad (2.27)$$

Again, the result is independent of the representation of the tangent vector $B \times w = (B + rw) \times w$ where $r \in R^1$. Since B is arbitrary, the reduced vector field X_K is just Euler's equations (modulo the factor 4) restricted to M_R:

$$\dot{w} = X_K(w) = 4w \times \nabla K^e(w), \qquad (2.28)$$

or

$$(X_K \circ \pi)(x) = 4\pi(x) \times [(\nabla K^e) \circ \pi(x)]. \qquad (2.29)$$

Note that by (2.29) the critical points of K on M_R are just the points where $\nabla K^e(w)$ is normal to the sphere M_R at w. If $X_K(\pi(p)) = 0$, then p is called a relative equilibrium for H. There is then an orbit of X_H through p that is a reparametrized orbit of X_{H_2} (see [1, p. 306]). Under suitable conditions this will be a periodic orbit of X_H. This approach was used in [5] and [13, 14] to find periodic orbits at low positive energies in the Hénon–Heiles and related Hamiltonians. For computational purposes involving stability questions about the periodic orbit, it is often convenient to rotate the critical point to one of the poles $(0, 0, \pm 2h)$. This can be done by a rotation of M_R that is induced by a linear symplectic change of coordinates in R^4 given by a suitable $V \in SU(2)$. This construction and the applications to stability analysis of the periodic orbits are given in [5] and [13, 14] (see also [4, pp. 36–38]).

3. Normal form of the semisimple 1:1 resonance

We now verify the relation given in formula (2.24). For an alternate proof see [5, lemma 4.1]. Recall that the Hopf variables of (2.10) satisfy $L_{X_2}(W_i)=0$ for $i=1,2,3,4$, and that $(W_1^2+W_2^2+W_3^2)=W_4^2$. Let $P_n(x_1,x_2,y_1,y_2)$ (resp. $Q_m(W_1,W_2,W_3,W_4)$) be the vector space of homogeneous polynomials with real coefficients in the specified variables of degree n (resp. m). Because L_{X_2} is a derivation we have

$$Q_m(W_1,W_2,W_3,W_4) \subseteq \operatorname{Ker}(L_{X_2} \mid P_{2m}(x_1,x_2,y_1,y_2)). \tag{3.1}$$

We wish to show that equality holds in (3.1). This will be done by computing the dimensions of the vector spaces on both sides of (3.1). Using the above relation between the Hopf variables we see that as vector spaces we have

$$Q_m(W_1,W_2,W_3,W_4)=Q_m(W_1,W_2,W_3) + W_4\cdot Q_{m-1}(W_1,W_2,W_3). \tag{3.2}$$

Now the dimension of the right-hand side of (3.2) is

$$\tfrac{1}{2}(m+2)(m+1)+\tfrac{1}{2}(m+1)(m)=(m+1)^2. \tag{3.3}$$

To compute the dimension of $\operatorname{Ker}(L_{X_2} \mid P_n)$ we introduce complex conjugate variables ($k=1,2$)

$$\begin{aligned} z_k &= x_k+\mathrm{i}y_k, \quad \partial/\partial z_k=\tfrac{1}{2}[\partial/\partial x_k-\mathrm{i}\partial/\partial y_k],\\ \bar z_k &= x_k-\mathrm{i}y_k, \quad \partial/\partial \bar z_k=\tfrac{1}{2}[\partial/\partial x_k+\mathrm{i}\partial/\partial y_k]. \end{aligned} \tag{3.4}$$

This coordinate change diagonalizes

$$X_2=[y_1(\partial/\partial x_1)+y_2(\partial/\partial x_2)-x_1(\partial/\partial y_1)-x_2(\partial/\partial y_2)]. \tag{3.5}$$

to

$$\tilde X_2=-\mathrm{i}[z_1(\partial/\partial z_1)+z_2(\partial/\partial z_2) - \bar z_1(\partial/\partial \bar z_1)-\bar z_2(\partial/\partial \bar z_2)], \tag{3.6}$$

and transforms $P_n(x_1,x_2,y_1,y_2)$ to

$$\begin{aligned} &HP_n(z_1,z_2,\bar z_1,\bar z_2)\\ &=\Big\{\sum c(i_1,i_2,j_1,j_2)z_1^{i_1}z_2^{i_2}\bar z_1^{j_1}\bar z_2^{j_2} \mid (i_1+i_2+j_1+j_2)=n,\\ &\text{and}\\ &c(i_1,i_2,j_1,j_2)=\bar c(j_1,j_2,i_1,i_2)\Big\}. \end{aligned} \tag{3.7}$$

Since any polynomial in HP_n lying in $\operatorname{Ker}(L_{\tilde X_2} \mid HP_n)$ corresponds to a unique polynomial in P_n lying in $\operatorname{Ker}(L_{X2} \mid P_n)$ under the above coordinate change (3.4), we have

$$\dim \operatorname{Ker}(L_{X_2} \mid P_n)=\dim \operatorname{Ker}(L_{\tilde X_2} \mid HP_n). \tag{3.8}$$

Now,

$$L_{\tilde X_2}(z_1^{i_1}z_2^{i_2}\bar z_1^{j_1}\bar z_2^{j_2})=-\mathrm{i}(i_1+i_2-j_1-j_2)(z_1^{i_1}z_2^{i_2}\bar z_1^{j_1}\bar z_2^{j_2}) \tag{3.9}$$

implies that $\dim \operatorname{Ker}(L_{\tilde X_2} \mid HP_n)$ is the number of non-negative integer solutions of

$$\begin{aligned} &(i_1+i_2-j_1-j_2)=0,\\ &(i_1+i_2+j_1+j_2)=n, \end{aligned} \tag{3.10}$$

which is equivalent to

$$2(i_1+i_2)=n=2(j_1+j_2). \tag{3.11}$$

Thus $n=2m$ for some m in order that $\operatorname{Ker}(L_{X_2} \mid P_n)$ be non-zero. The number of non-negative integer solutions of (3.11) when $n=2m$ is $(m+1)^2$, and hence equality holds in (3.1).

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