THE APPROACH TO EQUILIBRIUM
OF A MULTIPLE-COMPONENT MAGNETIC SYSTEM

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Synopsis

A magnetic system is considered, which consists of domains with submagnetizations m_r $(r = 1, 2, \ldots, M)$. Equations describing the evolution of $m_r(t)$ are studied and it is shown that there is a rapid transition towards a "scaling state", followed by a slow approach to equilibrium. Near the critical temperature these effects are most pronounced.

In a previous paper[1]) we considered the equilibrium properties of an Ising ferromagnet consisting of M chains, with every spin interacting only with all spins in the neighboring chains. For this one-dimensional array of chains, or domains, we studied the exact equations of state

$$m_r = \tanh \left[\alpha \, (m_{r-1} + m_{r+1}) \right] \qquad (r = 1, 2, \ldots, M),$$

$$m_0 = m_{M+1} = 0, \qquad \alpha = \beta J / M > 0, \tag{1}$$

where m_r is the average magnetization of a spin in the rth chain, when the system is in thermal equilibrium. Eq. (1) is always satisfied by the trivial solution (all $m_r = 0$), but below the critical temperature, *i.e.*, for $\alpha > \alpha_c = [2 \cos \{\pi/(M+1)\}]^{-1}$ there are other solutions, two of which are of the ferromagnetic type, meaning that all m_r have the same sign. It was shown in I that all solutions of eq. (1) can be found as extrema of the free-energy functional

$$\phi = \frac{1}{M} \sum_{1 \leqslant r \leqslant M} \left[\left(\frac{1 + m_r}{2} \right) \ln \left(\frac{1 + m_r}{2} \right) \right.$$

$$\left. + \left(\frac{1 - m_r}{2} \right) \ln \left(\frac{1 - m_r}{2} \right) - \alpha m_r m_{r+1} \right]. \tag{2}$$

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It was also made plausible and it was proved for $M \leqslant 6$, that the only stable solutions are the ferromagnetic ones. These have the symmetry $m_r = m_{M+1-r}$. For these cases the function ϕ therefore attains its absolute minimum and the other solutions are of no significance at thermal equilibrium[2]). However, when we investigate how an arbitrary initial domain magnetization $m_r(0)$, approaches the equilibrium value, the motion of $m_r(t)$ will be strongly influenced by the presence of the other extrema of ϕ.

For the time evolution of the functions $m_r(t)$ we adopt the following set of equations

$$\frac{\mathrm{d}m_r}{\mathrm{d}t} = -Mg_r \frac{\partial \phi}{\partial m_r} = \tanh\left[\alpha\left(m_{r-1} + m_{r+1}\right)\right] - m_r, \tag{3}$$

where g_r is given by eq. (2.10) of I and is always positive[+].

From the above equation it follows that

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \sum_{1 \leqslant r \leqslant M} \frac{\partial \phi}{\partial m_r} \frac{\mathrm{d}m_r}{\mathrm{d}t} = -M \sum_r g_r \left(\frac{\partial \phi}{\partial m_r}\right)^2 \leqslant 0, \tag{4}$$

where the inequality is satisfied as an equality if and only if an equilibrium of eq. (3) has been reached. Thus, ϕ is essentially a Lyapunov function which contains the entire stability structure of the dynamical system and implies that there are no periodic solutions and that the system will always approach one of the solutions of eq. (1). This must necessarily be the stable, $i.e.$, the ferromagnetic solution, unless the initial state has a certain symmetry, which prohibits the approach to the symmetric ferromagnetic solution. By observing that the eqs. (3) are first order in time and that the right-hand side is an odd function of its arguments it is easy to see that an initial state with $m_r = -m_{M+1-r}$ will never loose this symmetry property. This symmetry condition is analogous to the condition that the system move along a trajectory which terminates at a saddle point. The set of initial points which are on such a trajectory is obviously a set of measure zero in the full space.

In order to further study the solutions of eq. (3) we have made some numerical investigations.

For $M = 8$ and $M = 10$ we have shown for a number of arbitrary initial states of no special symmetry that they all approach one of the two ferromagnetic states, both of which are symmetric. This again supports the idea that this behavior will persist for arbitrary integer $M > 0$.

[+] Equation (3) was motivated by the phenomenological idea that the negative free-energy gradient provides the driving force causing $m_r(t)$ to tend to equilibrium; however, similar equations have been obtained from Glauber's master equation for the Ising model. See $e.g.$ ref. 3.

For $M = 4$ we considered first symmetric solutions $m_1 = m_4$ and $m_2 = m_3$, so that eq. (3) reduced to

$$\frac{dm_1}{dt} = -m_1 + \tanh \alpha m_2,$$

$$\frac{dm_2}{dt} = -m_2 + \tanh \alpha \, (m_1 + m_2).$$

(5)

In fig. 1 the equilibrium values of m_1 and m_2 are plotted as functions of the reduced temperature $T^* = \alpha_c/\alpha$, with $\alpha_c = (2 \cos \tfrac{1}{5}\pi)^{-1} = \tfrac{1}{2}(\sqrt{5} - 1) = 0.618034$. In fig. 2 we show a number of trajectories $m_1(t)$ and $m_2(t)$ in the m_1–m_2 plane for half the critical temperature, *i.e.*, for $\alpha = 2\alpha_c$. The tick marks indicate equal time intervals. It is seen that all trajectories approach either the equilibrium E_+ or the equilibrium E_-. The separatrix (full line) divides the m_1–m_2 plane into sets of points, P_+ and P_-, such that any trajectory in $P_+(P_-)$ approaches $E_+(E_-)$.
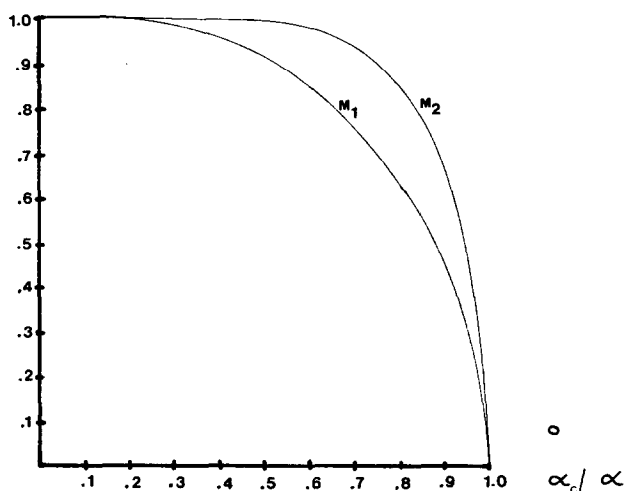

Fig. 1. The equilibrium values of m_1 and m_2 as functions of the reduced temperature $T^* = \alpha_c/\alpha$.

For $\alpha > \alpha_c$ the origin is a nonstable equilibrium point; however, points on the separatrix, itself a trajectory, cannot approach E_+ or E_-, and must therefore approach the origin. Any trajectory satisfies the differential equation

$$\frac{dm_2}{dm_1} = \frac{\tanh \alpha \, (m_1 + m_2) - m_2}{\tanh \alpha m_2 - m_1}$$
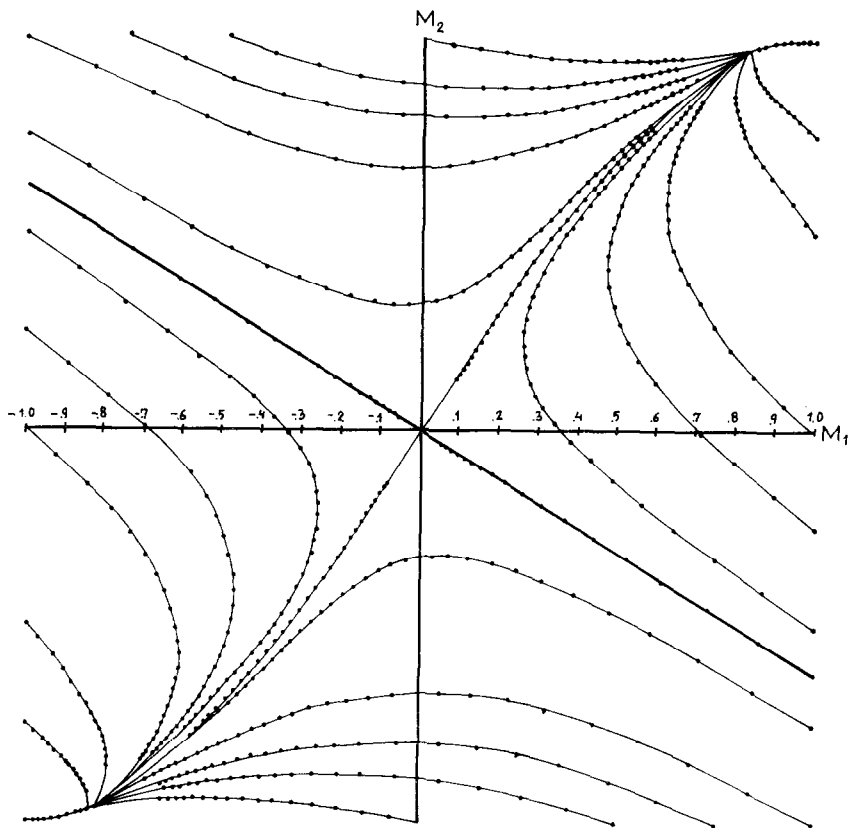
(6)

Fig. 2. Trajectories $m_1(t)$ and $m_2(t)$ in the m_1-m_2 plane for half the critical temperature. Tick marks indicate equal time intervals.

and the separatrix is singled out by requiring $m_2 = am_1$ for small m_1 and m_2. By substituting this linearity condition into eq. (6) and linearizing the right-hand side of eq. (6) we obtain for the coefficient a the equation $a^2 - a - 1 = 0$, which is independent of α, that is, of the temperature. The solution $a_- = -\frac{1}{2}(\sqrt{5} - 1) = -\alpha_c$ specifies the separatrix. The other solution $a_+ = \frac{1}{2}(\sqrt{5} + 1) = 1/\alpha_c$ specifies another trajectory, which we call the ortho-separatrix, because near the origin it is orthogonal to the separatrix. Above the critical temperature all trajectories approach the origin along the ortho-separatrix. Below the critical temperature all trajectories approach either E_+ or E_- also along the ortho-separatrix. This has been confirmed by numerical calculation and is shown in figs. 2 and 3 for $\alpha = 2\alpha_c$ and $\alpha = 1.01\alpha_c$, respectively. To understand this, linearize eq. (3) near an equilibrium point. Due to the stability of the equilibrium (see I) the

matrix A in the linearized equation

$$\frac{d}{dt} |m - m_{eq}\rangle = A |m - m_{eq}\rangle$$

is positive definite. This implies that the two eigenvalues have the same sign so that the equilibrium is a nodal point[4]) and in this point all trajectories are indeed tangent to the ortho-separatrix.

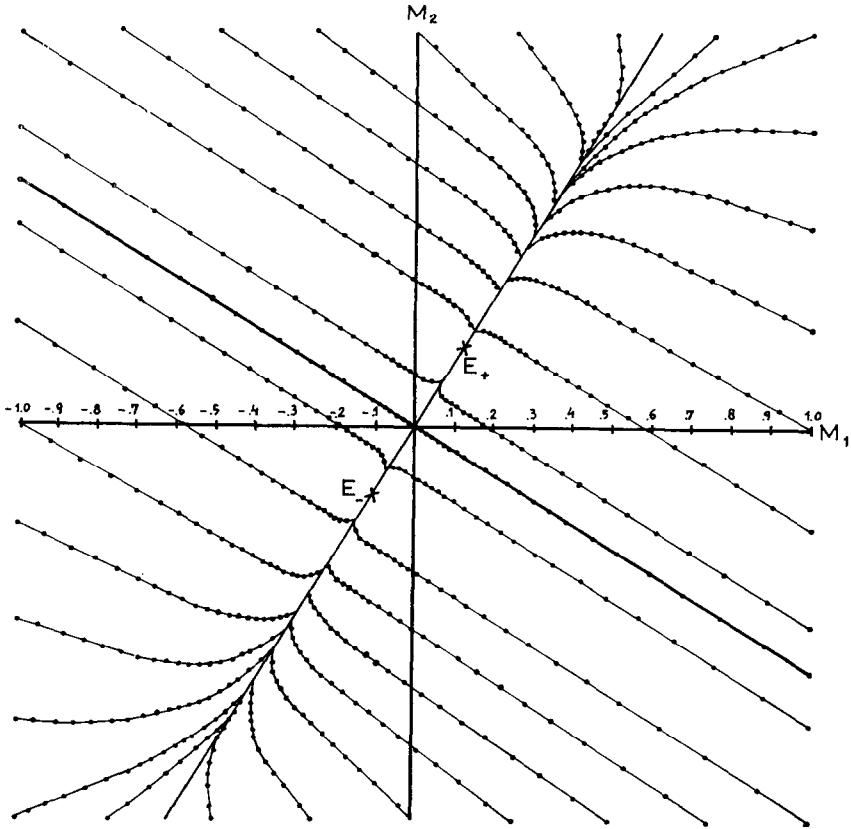For any temperature the equilibrium will therefore always be approached with a fixed value of the ratio m_2/m_1.

Fig. 3. The same as fig. 2 for T one percent below the critical temperature. The two equilibrium points are indicated by crosses.

It is not difficult to see that for arbitrary M the ferromagnetic equilibrium is approached along a line in the M-dimensional space spanned by $m_1, \ldots, m_M$. Essentially the argument hinges on the fact that the eigenvalues of A are distinct[5])

and non-negative (due to the stability). Consequently, the direction of the eigen-vector corresponding to the smallest eigenvalue determines the approach to equi-librium. For $M = 6$ and with a number of initial states this behavior has been checked by numerical calculation. It was found to be most pronounced near the critical temperature (see fig. 3 for $M = 4$). This is due, of course, to the fact that at this temperature the smallest eigenvalue becomes zero. In this case the ratio's m_n/m_{n+s} very quickly reach the values given by eq. (4.3) of I, after which a very slow approach to equilibrium remains.

During this slow stage we can say that thermal equilibrium has already been reached, except for a scale factor. The only process taking place in this last stage is the adjustment of the total magnetization to its true equilibrium value.

The question whether a similar behavior occurs in more realistic many-com-ponent systems lies beyond the scope of this note.

REFERENCES

1) Falk, H. and Ruijgrok, Th. W., Physica **78** (1974) 73. To be referred to as I.
2) This has now been proved generally by C. J. Thompson, Physica, to be published.
3) Suzuki, M. and Kubo, R., J. Phys. Soc. (Japan) **24** (1968) 51.
   Ruijgrok, Th. W. and Tjon, J. A., Physica **65** (1973) 539.
4) Hurewicz, W., Lectures on Ordinary Differential Equations, M.I.T. Press (Cambridge, Mass., 1958) p. 77, figs. 1 and 2.
5) Wilkinson, J. H., The Algebraic Eigenvalue Problem, Clarendon Press (Oxford, 1965) § 37, p. 300.