MATHEMATICS

ALGEBRAS OF HOLOMORPHIC FUNCTIONS AND THEIR MAXIMAL IDEAL SPACES

BY

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§ 1. INTRODUCTION

In [13] ROSSI proved the following theorem: Let X be an analytic space and A a closed, not necessarily point separating subalgebra of Hol (X) containing the constant functions and such that X is A-convex. Then the maximal ideal space ΔA of A can be given the structure of a Stein space such that A is isomorphic to Hol (ΔA). ΔA with the usual topology turns out to be homeomorphic to X/R when provided with the quotient topology; here R is the following equivalence relation on $X \times X$: xRy if and only if $f(x)=f(y)$ for all $f \in A$.

Now suppose A is a closed subalgebra of the algebra, B, of all holomorphic functions on a Stein space, X. Then the natural map $\pi: X \to Y = \Delta A$ is a holomorphic, surjective and proper map with finite fibers. We prove in section 3 that a compact subset L of Y is holomorphically convex if and only if $\pi^{-1} L$ is holomorphically convex. The same statement is true for A-convex subsets of Y: L is A-convex if and only if $\pi^{-1} L$ is B-convex.

In section 4 we look at the algebras rather than their maximal ideal spaces and show that under certain additional assumptions $A = B$ and that for a compact subset K of X, $(A|K)^{-} = (B|K)^{-}$ is a local property.

Section two is devoted to notation and terminology. Also various well-known results that will be used in subsequent sections are stated in section 2.

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§ 2. PRELIMINARIES

Let K be a compact subset of a Stein space (X, O_X) which is always assumed to be reduced, and let $H(K)$ be the uniform closure in $C(K)$ of the algebra of functions holomorphic in some neighborhood of K.

K is holomorphically convex if K can be identified with the maximal ideal space $\Delta H(K)$ of the Banach Algebra $H(K)$ [7]. In [7] it is proved that in the case of a Stein manifold X, K is holomorphically convex if and only if the cohomology groups $H^q(K, F)$ are zero for all integers $q \geqslant 1$ and all coherent analytic sheaves F on K.

The generalization of this fact to Stein spaces is in [14]. This characterization of compact holomorphically convex sets will be used to prove the main theorem in the next section.

Now let X and Y be analytic spaces with structure sheaves O_X and O_Y respectively and π: $X \to Y$ a holomorphic map.

For an analytic sheaf F on X and integers $q \geqslant 0$ define the sheaf $\pi_q F$ on Y to be the sheaf associated to the preasheaf $U \to H^q(\pi^{-1} U, F)$. If F is an analytic sheaf on Y we define the topological pullback $\pi^{-1} F$ on X by: $(\pi^{-1} F)_x = F_{\pi x}$ while the topology on $\pi^{-1} F$ is the coarsest topology rendering the canonical maps $\pi^{-1} F \to X$ and $\pi^{-1} F \to F$ continuous. The analytic pullback $\pi^* F$ of F is the sheaf, on X, associated to the preasheaf $U \to H^0(U, \pi^{-1} F) \otimes_{H^0(U, \pi^{-1} O_Y)} H^0(U, O_X)$. With the above notation we get

2.1 If F is a coherent analytic sheaf on Y, then $\pi^* F$ is a coherent analytic sheaf on X [4].

2.2 If F is an analytic sheaf on X and $\pi_q F = 0$ for $q \geqslant 1$, then $H^p(X, F) = H^p(Y, \pi F)$ for $p \geqslant 0$ [4].

2.3 If F is a coherent analytic sheaf on X and π a proper holomorphic map with discrete fibers, then $\pi_0 F$ is coherent and $\pi_q F = 0$ for $q \geqslant 1$ [5].

§ 3. In this section we prove using techniques from [9] and [1] the two theorems announced in the introduction. We begin with a special case.

LEMMA 3.1. Let X and N be analytic spaces, N normal and π: $X \to N$ a proper holomorphic map with finite fibers from X onto N. L is a compact subset of N and $K = \pi^{-1} L$. Then $H(K)$ is integral over $H(L)$.

PROOF: There exists a subvariety V of X such that $X \backslash V$ is an unramified p-sheeted covering of $N \backslash \pi V$ for some p.

We may assume $\pi V \supset S(N)$, the singular set of N. For $f \in \mathrm{Hol}\,(U)$, where U is a saturated neighborhood of K i.e. $\pi^{-1} \pi U = U$, and $x \in \pi U \backslash \pi V$ define $a_i(x)$ to be the i^{th} elementary symmetric polynomial in the values of f at the points in the fiber $\pi^{-1}(x)$. Then a_i is a holomorphic function on $\pi U \backslash \pi V$ which is clearly locally bounded on πU. Since πU is a normal analytic space and πV a subvariety of πU, a_i extends to a holomorphic function on πU. Also $f^p(x) + \sum_{i=0}^{p-1} a_i(\pi(x)) f^i(x) = 0$ for all $x \in U$. Now take $f \in H(K)$ and let $f = \lim_{n \to \infty} f_n$ where $f_n \in \mathrm{Hol}\,(K)$. For each n there exist a_{in}, constructed as above, such that $f_n^p(x) + \sum_{i=0}^{p-1} a_{in}(\pi(x)) f_n^i(x) = 0$ for all x in some saturated neighborhood of K. Since the sequence $\{f_n\}_n$ converges on K to f, the sequences $\{a_{in}\}_n$ converge on L to an element of $H(L)$, say a_i.

Clearly $f^p(x) + \sum_{i=0}^{p-1} a_i(\pi(x)) f^i(x) = 0$ for all $x \in K$.

Thus f is integral over $H(L)$.

COROLLARY 3.2. If K is holomorphically convex, then L is holomorphically convex.

PROOF: Since $H(K)$ is integral over $H(L)$ every complex homomorphism x of $H(L)$ extends to a complex homomorphism y of $H(K)$, which is point evaluation at some point in K. Clearly $\pi y = x$. Thus $x \in L$.

LEMMA 3.3. Let K be a holomorphically convex subset of a Stein space X and V a subvariety of X. Then $K \cap V$ is a holomorphically convex subset of V.

PROOF: From Cartan's theorem A (or from [11]) it easily follows that $H(K)-\text{hull}\,(K \cap V) \equiv \{x \in K \mid |f(x)| \leq \|f\|_{K \cap V} \text{ for all } f \in H(K)\} = K \cap V$. Hence $\Delta[H(K)|K \cap V]^{-} = K \cap V$.

Since every function holomorphic in a neighborhood of $K \cap V$ in V is locally approximable on $K \cap V$ by functions in $H(K)$ it follows from [12] that $K \cap V$ is the maximal ideal space of $H^0(K \cap V, O_V)^{-}$, where O_V is the structure sheaf of the variety V.

Thus $K \cap V$ is holomorphically convex.

THEOREM 3.4. Let (X, O_X) and (Y, O_Y) be Stein spaces, π a proper holomorphic map from X onto Y, L a compact subset of Y and $K = \pi^{-1}L$.

Then K is holomorphically convex if and only if L is holomorphically convex.

PROOF: We begin with the easy part: let L be holomorphically convex. Suppose F is a coherent analytic sheaf on K. We have to prove $H^q(K, F) = 0$ for $q \geq 1$. We may assume that F is defined on a neighborhood U of K [2]. Let $\{U_\alpha\}$ be a fundamental system of saturated neighborhoods of K such that $U_\alpha \subset U$ for all α. $\{\pi U_\alpha\}$ is then a fundamental system of neighborhoods of $\pi K = L$. It follows that

$$
\begin{aligned}
H^q(K, F) &= \varinjlim_{\alpha} H^q(U_\alpha, F) \\
&= \varinjlim_{\alpha} H^q(\pi U_\alpha, \pi_0 F) \quad \text{according to 2.2 and 2.3} \\
&= H^q(L, \pi_0 F) \\
&= 0 \quad \text{for } q \geq 1
\end{aligned}
$$

since L is holomorphically convex and $\pi_0 F$ coherent.

Thus K is holomorphically convex.

Conversely, assume K is holomorphically convex.

The proof is by induction on $\dim X$. Without loss of generality we may suppose $\dim X < \infty$ since only finitely many irreducible branches of X intersect K. Also $\dim X = \dim Y$ since π has finite fibers.

If $\dim X = 0$, then there is nothing to prove.

So assume the theorem is true in case the dimension of the spaces is less than or equal to $n-1$ and consider the theorem as stated with $\dim X = n$.

Let (M, ϕ) and (N, ψ) be the normalization of X and Y respectively. Then there exists a proper holomorphic map f from M onto N such that $\pi \circ \phi = \psi \circ f$. Since M is a Stein space f has finite fibers. By the one half of the theorem we already proved $\phi^{-1} K$ is a holomorphically convex subset of M and since $f^{-1}(f\phi^{-1} K) = \phi^{-1} K$ it follows from 3.2 that $P \equiv f(\phi^{-1} K)$ is a holomorphically convex subset of N.

Let F be a coherent analytic sheaf on L. We are going to prove $H^q(L, F) = 0$ for $q > 0$. Since F is arbitrary this will imply that L is holomorphically convex.

Again F is defined on a neighborhood U of L; for $\mathfrak{A}$ the sheaf of germs of universal denominators [10] define

$$F^* = \psi^{-1}(\mathfrak{A} \cdot F) \otimes_{\psi^{-1} O_Y} O_N.$$

F^* is a coherent analytic sheaf on $\pi^{-1} U$.

$F_1 \equiv \psi_0 F^*$ is a coherent analytic subsheaf of F and for $x \in U \backslash S(Y)$ we have $F_{1x} = F_x$.

From the short exact sequence of sheaves

$$0 \to F_1 \to F \to F/F_1 \to 0$$

we get the long exact sequence of groups

$$\dots \to H^q(L, F_1) \to H^q(L, F) \to H^q(L, F/F_1) \to H^{q+1}(L, F_1) \to \dots$$

But $\psi^{-1} L = P$ is holomorphically convex and it follows as in the proof of the first half of the theorem that

$$H^q(L, F_1) = H^q(\psi^{-1} L, F^*) = H^q(P, F^*) = 0 \text{ for } q > 0.$$

Therefore $H^q(L, F) \cong H^q(L, F/F_1)$.

By lemma 3.3, $P \cap \psi^{-1} S(Y)$ is a holomorphically convex subset of $\psi^{-1} S(Y)$ which is a Stein space of dimension less than or equal to $n-1$. By the induction assumption $\psi(P \cap \psi^{-1} S(Y)) = L \cap S(Y)$ is a holomorphically convex subset of $S(Y)$.

Since F/F_1 is supported on $S(Y)$ we get

$$H^q(L, F/F_1) \cong H^q(L \cap S(Y), F/F_1) = 0 \text{ for } q > 0.$$

Thus $H^q(L, F) = 0$ for $q > 0$.

Before we prove the next theorem we need some additional notation and terminology.

Let A be a function algebra with maximal ideal space ΔA and Silov boundary ∂A. Let U be a subset of ΔA.

A complex valued function f is A-holomorphic on U if for every $x \in U$

there exists a neighborhood V of x such that $f|(U \cap V) \in (A|U \cap V)^{-}$. Thus f is locally approximable on U by functions in A.

The local maximum modulus theorem for A-holomorphic functions says [11]: if f is A-holomorphic on an open subset U of $\Delta A \backslash \partial A$, then $\|f\|_U = \|f\|_{bdU}$.

Finally, a compact subset K of a Stein space X is said to be Hol (X)-convex if $K = \text{Hol}\,(X) - \text{hull}\,K \equiv \{x \in X|\ |f(x)| \leqslant \|f\|_K \text{ for all } f \in \text{Hol}\,(X)\}$.

THEOREM 3.5. Let X, Y be Stein spaces, π a proper holomorphic map from X onto Y, L a compact subset of Y and $K = \pi^{-1}L$. Then K is Hol (X)-convex if and only if L is Hol (Y)-convex.

PROOF: If L is Hol (Y)-convex, then trivially K is Hol (X)-convex. Conversely, assume K is Hol (X)-convex.

Since only finitely many irreducible branches of X intersect K we may assume that $\dim X = n < \infty$.

Define inductively a sequence $\{V_i\}$ of subvarieties of X as follows: $V_0 = X$ and for $i > 0$, $V_i = \pi^{-1}\pi\{x \in V_{i-1}|$ there does not exist a neighborhood U of x such that $\pi|U$ is a biholomorphism onto $\pi U\}$. Then V_{i+1} is a closed subvariety of V_i and $\pi|(V_i \backslash V_{i+1})$ has locally a holomorphic inverse. Clearly $\dim V_i \leqslant n - i$ so that $V_{n+1} = \emptyset$.

Now let $M = \pi^{-1}(\text{Hol}\,(Y) - \text{hull}\,L)$. We have to show $M = K$. Define $A = H(\text{Hol}\,(Y) - \text{hull}\,L)$, $B = H(M)$, $W_i = M \cap V_i$ and $P = B - \text{hull}\,(\pi^{-1}\partial A)$. Then $\Delta A = \text{Hol}\,(Y) - \text{hull}\,L$, $\Delta B = M$ and $\partial A \subset L$. We are going to show $\partial B \subset P \cup W_1$.

Suppose there exists $x \in \partial B \backslash (P \cup W_1)$. Choose a neighborhood U of x in M such that $\pi|\overline{U}$ is a homeomorphism onto $\pi\overline{U} \subset \Delta A \backslash \partial A$ and $\overline{U} \cap (P \cap W_1) = \emptyset$ and such that $\overline{U}$ is component of $\pi^{-1}\pi\overline{U}$; this is possible by the definition of W_1 and the fact that π is proper with finite fibers. Let $y \in U$ be a peakpoint for the algebra B and $f \in B$ a peakfunction. Then $f \circ (\pi|\overline{U})^{-1}$ is on πU locally approximable by functions in A and is continuous on $\overline{\pi U}$. Thus by the local maximum modulus theorem for A-holomorphic functions

$$\|f \circ (\pi|\overline{U})^{-1}\|_{\pi U} = \|f \circ (\pi|\overline{U})^{-1}\|_{bd_{\Delta A}\pi U}.$$

But this contradicts the fact that $f \circ (\pi|\overline{U})^{-1}$ peaks on $\overline{\pi U}$ at πy.

Thus $\partial B \subset P \cup W_1$.

Next we show $M \backslash P \subset W_1$.

First we observe that $B - \text{hull}\,(W_1 \cup P) = W_1 \cup P$ since for $x \notin W_1 \cup P$ there exists $f \in B$ with $f(x) = 1$ and $f|W_1 = 0$ (Cartan's theorem A or [11]) and $g \in B$ with $g(x) = 1 > \|g\|_P$ by the definition of P. Hence for n sufficiently large $fg^n(x) = 1 > \|fg^n\|_{W_1 \cup P}$. Thus $x \notin B - \text{hull}\,(W_1 \cup P)$. The other inclusion is trivial. We finally get $M = B - \text{hull}\,(\partial B) \subset B - \text{hull}\,(W_1 \cup P) = W_1 \cup P$. Thus $M \backslash P \subset W_1$.

Similarly $\partial B \subset P \cup W_2$: let $w \in \partial B \backslash (P \cup W_2) \subset W_1 \backslash (P \cup W_2)$, an open subset of M. Choose a neighborhood V of w in M such that $\pi|\overline{V}$ is a homeomorphism onto $\pi\overline{V} \subset \Delta A \backslash \partial A$ and $\overline{V} \cap (P \cup W_2) = \emptyset$.

Let $z \in V$ be a peakpoint for B and f a peakfunction. Then $f \circ (\pi|\overline{V})^{-1}$ peaks on $\pi\overline{V}$ at πz contradicting the localmaximum modulus theorem for A-holomorphic functions.

Thus $\partial B \subset P \cup W_2$ and this implies again $M \backslash P \subset W_2$.

Continuing in this way we finally get

$$\partial B \subset P \cup W_{n+1} = P = B-\text{hull}\,(\pi^{-1}\partial A) \subset B-\text{hull}\,(K) = K$$

contradicting the fact that ∂B intersects $M \backslash K$. Thus $M = K$ and hence L is Hol$\,(Y)$-convex.

§ 4. In the previous section we studied maximal ideal spaces of algebras of holomorphic functions. In this section we turn our attention to the algebras themselves.

The situation we consider is the following:

$(X, \mathfrak{B})$ is a Stein space and A a closed pointseparating subalgebra of $B = H^0(X, \mathfrak{B})$ containing the constants and such that X is A-convex. Then there exists a subsheaf $\mathfrak{A}$ of $\mathfrak{B}$ such that $(X, \mathfrak{A})$ is a Stein space and $A = H^0(X, \mathfrak{A})$.

The identity $i : H^0(X, \mathfrak{B}) \to H^0(X, \mathfrak{A})$ is then a holomorphic homeomorphism.

For a compact subset K of X we denote by B_K the uniform closure in $C(K)$ of the restriction algebra $B|K$. Similarly A_K is defined. With this notation we prove the following theorem.

Theorem 4.1. Let K be a B-convex compact subset of X and assume that every point $p \in K$ has a compact neighborhood U_p in X with the property $B_{U_p} = A_{U_p}$.

Then $B_K = A_K$.

Proof: Let $p \in K$, V a neighborhood of p and $f \in H^0(V, \mathfrak{B})$. Choose a B-convex compact neighborhood W of p contained in $V \cap U_p$. By the Oka-Weil theorem there exists a sequence $\{f_n\}_n$ in B such that $\|f - f_n\|_W \to 0$. For each n there exists a sequence $\{f_{nj}\}_j$ in A such that $\|f_n - f_{nj}\|_{U_p} \to 0$.

Hence $\|f - f_{nn}\|_W \to 0$. Thus $f \in A_W$.

Therefore $f \in H^0(\text{int } W, \mathfrak{A})$. Since f was arbitrary this implies $\mathfrak{B}_p = \mathfrak{A}_p$, for all $p \in K$.

In other words: the identity $i : (X, \mathfrak{B}) \to (X, \mathfrak{A})$ is biholomorphic in a neighborhood of K. Thus $H^0(K, \mathfrak{B}) = H^0(K, \mathfrak{A})$.

Since K is both B-convex and A-convex by theorem 3.5, the Oka-Weil theorem implies $B_K = A_K$.

If we assume in addition to the usual hypothesis that $(X, \mathfrak{B})$ is in fact a domain of holomorphy (or a Stein manifold), $(X, \mathfrak{B})$ is the normalization of $(X, \mathfrak{A})$ and the notion of universal denominator [10] becomes relevant as we will see in the next theorem.

Let $z_1, \ldots, z_n$ be coordinate functions for $\mathbb{C}^n$ (or for a Stein manifold spread over $\mathbb{C}^n$). An algebra A of holomorphic functions on a domain in $\mathbb{C}^n$ is called p-differentiably stable if for all multi indices α with $|\alpha|=p$ and for all $f\in A$ also $\partial^\alpha f/\partial z^\alpha \in A$.

THEOREM 4.2. Let X be a domain of holomorphy in $\mathbb{C}^n$ (or a Stein manifold spread over $\mathbb{C}^n$), $B=H^0(X,\mathfrak{B})$. A a closed point separating subalgebra of B such that $1\in A$, X is A-convex and A is p-differentiably stable with respect to some system of coordinates for $\mathbb{C}^n$ for some natural number p. Then $A=B$.

PROOF: Let $\mathfrak{A}$ be the sheaf on X such that $A=H^0(X,\mathfrak{A})$. We prove that $\mathfrak{A}=\mathfrak{B}$ by showing that there are coordinate functions $z_1, \ldots, z_n$ for $\mathbb{C}^n$ such that the germs at x of $z_1, \ldots, z_n$ are in $\mathfrak{A}_x$ for all $x\in X$.

So, assume $x=0\in X$.

Let h be a universal denominator of $(X, \mathfrak{A})$ in a neighborhood of the origin. After a suitable linear change of coordinates we get coordinates $z_1, \ldots, z_n$ for $\mathbb{C}^n$ such that h is regular of order $k>0$ in z_n [6]. Since the change of coordinates is linear A is also p-differentiably stable with respect to $z_1, \ldots, z_n$.

By the Weierstrasz preparation theorem [6] we may write $h=ug$ in a polydisc U centered at the origin where u does not vanish on U and g is a Weierstrasz polynomial of order k; i.e.:

$$g(z_1, \ldots, z_n)=z_n^k+\sum_{i=0}^{k-1} a_i(z_1, \ldots, z_{n-1})\, z_n^i.$$

Clearly g is also a universal denominator on U. Thus $z_i^j g\in H^0(U,\mathfrak{A})$ for all j and i.

Since the polydisc U is B-convex, U is also A-convex by theorem 3.5. Thus $A|U$ is dense in $H^0(U,\mathfrak{A})$.

But the operator $f\mapsto \partial^\alpha f/\partial z^\alpha$ is continuous and therefore $H^0(U,\mathfrak{A})$ is p-differentiably stable.

Now choose an integer $q\geq 0$ such that $k+q=sp+1$ for some positive integer s. Then

$$\frac{\partial^{sp}}{\partial z_n^{sp}}(z_n^q g)=(sp+1)!\, z_n+(sp)!\, a_{k-1} \qquad \in H^0(U,\mathfrak{A})$$

$$\frac{\partial^{(s+1)p}}{\partial z_n^{(s+1)p}}(z_n^{q+p} g)=((s+1)p+1)!\, z_n+((s+1)p)!\, a_{k-1}\in H^0(U,\mathfrak{A})$$

and for $j\neq n$

$$\frac{\partial^{sp}}{\partial z_n^{sp}}(z_n^{q-1} z_j\, g)=(sp)!\, z_j \qquad \in H^0(U,\mathfrak{A}).$$

Thus $z_1, \ldots, z_n\in H^0(U,\mathfrak{A})$ which implies $\mathfrak{A}_0=\mathfrak{B}_0$. Thus the sheaves $\mathfrak{A}$ and $\mathfrak{B}$ are equal and therefore $A=B$.

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