ADVANCES IN MATHEMATICS 15, 216–231 (1975)

A Spectral Sequence for Double Complexes of Groups

JAN R. STROOKER*

Mathematisch Instituut der Rijksuniversiteit, Utrecht, The Netherlands

AND

ORLANDO E. VILLAMAYOR*

Departamento de Matemáticas, Universidad de Buenos Aires, Buenos Aires, Argentina

STATEMENT OF AIM

In this paper we set up a spectral sequence for double complexes of arbitrary groups and we derive an analog of the 7-term exact sequence of [1, Lemma 7.5]. Here "exact sequence" has to be suitably interpreted for maps between pointed sets acted upon by groups.

We developed these sequences in order to compare different theories of higher K-functors, but we decided to separate this general material from the body of that paper [3]. For we believe that the results, though naturally more cumbersome than their abelian prototypes, may find other applications in the future.

1. DOUBLE COMPLEXES OF GROUPS

A double complex of groups is a set $\{G_{i,j}\}$ of groups indexed by the integers $\mathbf{Z}$ and equipped with group homomorphisms $d\colon G_{i,j} \to G_{i,j-1}$ and $\delta\colon G_{i,j} \to G_{i-1,j}$ such that $dd = 1$, $\delta\delta = 1$ and $d\delta = \delta d$. We think of i as the row index and j as the column index.

* The authors were partly supported by, respectively, O.A.S. contracts, through the Mathematics Multinational Project, and the Netherlands Organization for the Advancement of Pure Research (Z.W.O.), through a visiting fellowship.

A string of elements $a = \{a_k\}$, $k \in \mathbf{Z}$ such that $a_k \in G_{k,n-k}$ is called homogeneous of total degree n. A string a is called a cycle provided $da_{k-1} = \delta a_k$ for all k. Since d and δ are group homomorphisms the homogeneous cycles of total degree n constitute a subgroup of the group of all strings of that degree. We can then speak of a cycle as a set of homogeneous cycles, one for each total degree, but most questions clearly reduce to the homogeneous components.

A positive complex is one for which $G_{i,j} = (1)$ whenever $i < 0$ or $j < 0$. In this case a homogeneous cycle of total degree n is just a finite string $a = (a_0, ..., a_n)$ with $da_{k-1} = \delta a_k$ for $1 \leqslant k \leqslant n$, since $\delta a_0 = 1 = da_n$.

We now define a boundary operation β which attaches to homogeneous strings b of total degree $n + 1$ an operation $\beta(b)$ on strings a of degree n by putting $\beta(b) \cdot a = a' = \{a_k'\}$, with $a_k' = db_k^{-1} a_k \delta b_{k+1}$, $a_{k+1}' = \delta b_{k+2}^{-1} a_{k+1} db_{k+1}$, when k is even.

This determines an action on the group of strings of degree n. We shall speak of an action even though $\beta(b_1 b_2) \cdot a = \beta(b_2) \cdot (\beta(b_1) \cdot a)$, i.e., $\beta(b_1 b_2) = \beta(b_2) \circ \beta(b_1)$.

Lemma 1. *If a is a homogeneous cycle of degree n, so is $\beta(b) \cdot a$ for every string b homogeneous of degree $n + 1$.*

Proof. It is obvious that $\beta(b) \cdot a$ is homogeneous of the same degree as a, so we only have to prove it is a cycle. Call $a' = \beta(b) \cdot a$, then for k even, $da_k' = da_k d\delta b_{k+1}$ while $\delta a_{k+1}' = \delta a_{k+1} \delta db_{k+1}$, and, since $da_k = \delta a_{k+1}$ then da_k' and $\delta a_{k+1}'$ are equal. For k odd we have $da_k' = d\delta b_{k+1}^{-1} da_k$ and $\delta a_{k+1}' = \delta db_{k+1}^{-1} \delta a_{k+1}$, which are also equal.

Lemma 2. *The boundary operation induces an equivalence relation on the set of all strings and therefore on the set of homogeneous cycles.*

Proof. The action of the group of strings of total degree $n + 1$ divides the set of strings of degree n into orbits. By the preceding lemma, the cycles fill a subclass of these orbits.

Remark. For any homogeneous string b we can define $D(b) = \beta(b) \cdot 1$, so D is a map of sets, homogeneous of degree -1. Then b is a cycle if and only if $D(b) = 1$ and we may call boundaries the elements $D(b)$ for all strings b of degree $n + 1$. The fact that $DD(b) = 1$ for all b shows that the boundaries form a subset of the cycles.

DEFINITION 3. The set of orbits of the homogeneous cycles of degree n under the boundary action β will be called the nth homology of the double complex and it will be denoted by $H_n(G_{i,j})$. It is a pointed set whose base point is the orbit of the boundaries.

Given a single complex of groups $\{G_i, \delta\}$, we consider it as a double complex $\{G_{i,j}, \delta', d\}$ with $G_{i,0} = G_i$, $G_{i,j} = (1)$ for $j \neq 0$, $\delta' = \delta$ in the 0th column and the unique homomorphisms everywhere else, as in the case of d.

2. DEGENERATE COMPLEXES

A positive double complex of groups is called degenerate if all its columns with positive degree are exact, i.e.,

$$G_{i+1,j} \xrightarrow{\delta} G_{i,j} \xrightarrow{\delta} G_{i-1,j}$$

is exact for $j > 0$ and all i. We prove that the homology of such a degenerate complex is isomorphic (as pointed sets) to the homology of the 0th column. This generalizes a well known and useful result in the abelian case. Indeed, the reader will have noticed that for complexes of abelian groups our procedures specialize to the usual ones; we shall not point this out every time.

LEMMA 4. *For a positive degenerate double complex* $\{G_{i,j}, d, \delta\}$ *there exists a canonical map of pointed set* $\Gamma: H_n(G_{i,j}, d, \delta) \to H_n(G_{i,0}, \delta)$ *for each* n.

Proof. Let $a = (a_0, ..., a_n)$ be a homogeneous cycle of degree n in the double complex. Since $a_k \in G_{k,n-k}$, we know that $\delta a_0 = 1$ $(\delta a_0 \in G_{-1,n} = (1))$ so by exactness of the nth column there exists b_1 in $G_{1,n}$ such that $\delta b_1 = a_0$. Consider the string $b^1 = (1, b_1^{-1}, 1, ..., 1)$ of total degree $n + 1$, then $\beta(b^1) \cdot a = a^1 = (1, a_1 d b_1^{-1}, a_2, ..., a_n)$ because $a_0 \delta b_1^{-1} = 1$. Now $\delta(a_1 d b_1^{-1}) = \delta a_1 d \delta b_1^{-1} = \delta a_1 d a_0^{-1} = 1$ because a is a cycle. By exactness of the $(n - 1)$st column we can carry on, eventually arriving at a cycle $(1, ..., 1, c_n)$ which is equivalent to a and in which, since $\delta c_n = d c_{n-1} = 1$, the element c_n is a cycle in $G_{n,0}$ with respect to the column operator δ.

We must of course verify that this association $a \mapsto c_n$ defines a map at the homology level. Clearly, it is enough to show that, if $c = (1, ..., 1, c_n) \sim (1, ..., 1, e_n) = e$ (equivalence in the double complex), then c_n and e_n differ by a boundary in $G_{n,0}$.

The equivalence means that there exists a homogeneous string $f = (f_0, ..., f_{n+1})$ of total degree $n + 1$ such that $\beta(f) \cdot c = e$. Since $\delta f_0 = 1$, there exists $g_1 \in G_{1,n+1}$ such that $\delta g_1 = f_0$. Now the string $f^1 = = (1, dg_1^{-1} f_1, f_2, ..., f_{n+1})$ also satisfies $\beta(f^1) \cdot c = e$, as we can see by direct computation.

Moreover, $\delta(dg_1^{-1} f_1) = 1$, so we can continue by induction to obtain a string $q = (1, ..., 1, q_{n+1})$ such that $\beta(q) \cdot c = e$. But then $e_n = c_n \delta q_{n+1}$ if n is even or $e_n = \delta q_{n+1}^{-1} c_n$ if n is odd. In both cases e_n and c_n belong to the same homology class of the single complex $\{G_{i,0}, \delta\}$. It is clear that the base point in H_n of the double complex is mapped to the base point of H_n of the single complex.

THEOREM 5. *The map* Γ: $H_n(G_{i,j}, d, \delta) \to H_n(G_{i,0}, \delta)$ *is bijective.*

Proof. Surjectivity is obvious, because if c_n is a cycle in the single complex, so is the string $(1, ..., 1, c_n)$ in the double complex.

To establish injectivity, suppose $\Gamma(a) \sim \Gamma(a')$ in the single complex for two homogeneous cycles of total degree n. Let c_n and c_n' be the elements of $G_{n,0}$ obtained from a and a' respectively, by the previous procedure, then our hypothesis implies the existence of $b_{n+1} \in G_{n+1,0}$ such that (assuming n is even) $c_n' = c_n \delta b_{n+1}$. Call $c = (1, ..., 1, c_n)$, $c' = (1, ..., 1, c_n')$, $b = (1, ..., 1, b_{n+1})$. Then $\beta(b) \cdot c = c'$, hence

$$a \sim c \sim c' \sim a'.$$

A similar argument is valid if n is odd.

3. GROUP-DOMINATED SETS

In the next sections, we set up a spectral sequence and go on to define the homology of its E_r-terms. Since we are not working with abelian groups, we must explain what we mean by homology, exactness, etc. We shall not attempt maximum categorical generality, but rather stick closely to our specific needs.

Let G be a group and S a pointed set consisting of equivalence classes of G with respect to some equivalence relation $\sim$. The base point $*$ of S is the class which contains the identity element of G. We say that the group G dominates S and understand that such a pointed set is always given together with its dominating group. We consider morphisms $(G, S) \to (H, T)$ between such objects which consist of a map ϕ: $G \to H$ which induces a map between pointed sets $S \to T$. The map ϕ need not be a group homomorphism, but it should satisty $\phi(1) = 1$. Clearly, the composition of two morphisms is again a morphism.

Let $\phi: (G, S) \to (H, T)$ be a morphism between group-dominated sets. Let $\operatorname{Ker}\phi$ be the subset of S which goes to $*$; the set theoretic image of S in T is called $\operatorname{Im}\phi$. Note that both are just pointed sets. Suppose now

$$(F, R) \xrightarrow{\psi} (G, S) \xrightarrow{\phi} (H, T) \tag{A}$$

is a sequence of morphisms such that the image of R in T is $\{*\}$. In order to introduce a workable notion of homology we need an extra condition on this sequence: we suppose the map ψ induces an action $\bar{\psi}$ of the group F on G satisfying $\bar{\psi}(f) \cdot 1 = \psi(f)$ for every $f \in F$. We call the sequence a complex provided this induces an action of F on the pointed set $\operatorname{Ker}\phi$.

We now define the homology as the orbit space of $\operatorname{Ker}\phi$ under this action and write it as $H(S)$. The base point of $H(S)$ then is $\{\operatorname{Im}\psi\}$, as follows from the identity $\bar{\psi}(f) \cdot 1 = \psi(f)$. In the case of the E_r-terms in our spectral sequence, this homology can again be conceived of as a group-dominated set for $r \geqslant 2$, as we shall see in Section 7. We can view our definitions as generalizing the abelian case, where all groups are abelian, all morphisms group homomorphisms and the equivalence relations mean working modulo subgroups.

There have been various proposals as to what should be called an exact sequence in a nonabelian category such as pointed sets. For our purposes, we take a naive approach and call the sequence (A) exact when $\operatorname{Im}\psi = \operatorname{Ker}\phi$. Some justification is offered by the following.

LEMMA 6. *The sequence* (A) *is exact if and only if it is a complex with* $H(S) = \{*\}$.

Proof. If the sequence is a complex and $H(S) = \{*\}$, then the whole of $\operatorname{Ker}\phi$ is in $\operatorname{Im}\psi$. Conversely, the latter condition means that $\operatorname{Ker}\phi$ is just the orbit of $*$ under the action of F, hence is an invariant subset under this action, which means the sequence is a complex with $H(S) = \{*\}$.

In the following lemma, the proof of which is immediate, we record what remains of some standard reasoning in the abelian case.

LEMMA 7. *If the sequence* (A) *is exact, then* ψ *maps* R *onto* S *if and only if* $\operatorname{Im}\phi = \{*\}$. *If* ϕ *is injective, Ker* $\phi = \{*\}$.

However, from $\operatorname{Ker}\phi = \{*\}$ we cannot, in all cases, conclude that ϕ is injective on the pointed set S. In the exact sequence developed in Section 10 and the one in [3, Section 12], partial conclusions are drawn, but it is not feasible to discuss this in general.

4. Definition of $Z_r^{p,q}$

We shall filter the double complex by the subcomplexes W_s obtained by replacing the groups $G_{i,j}$ by (1) when $i > s$.

Define $Z_r^{p,n-p}$ as the set of homogeneous strings a of total degree n such that $a \in W_p$, $D(a) \in W_{p-r}$ (where $D(a) = \beta(a) \cdot 1$). It is easily seen that $Z_r^{p,n-p}$ is a group, actually a subgroup of the direct product of the groups $G_{i,n-i}$.

Recall that $DD = 1$, hence $D(a)$ is a cycle for any homogeneous string. Every cycle of degree n which lives in W_p belongs to $Z_r^{p,n-p}$ for all r. More generally $Z_r^{p,n-p} \subseteq Z_s^{p,n-p}$ whenever $r \geqslant s$.

The following result will be used in several places:

Lemma 8. *Let a be a string of total degree n, and b of total degree $n+1$. If $D(a) \in W_s$, then $D(\beta(b) \cdot a) \in W_s$.*

Proof. By direct computation one obtains that the ith element of $D(\beta(b) \cdot a)$ is the same as that of $D(a)$ for i odd, or its conjugate by δdb_{i+1} for i is even.

From the definition it follows that $Z_r^{p,n-p}$ is the set of homogeneous strings of total degree n, say $a = \{a_k\}$ such that $a_k = 1$ for $k > p$, $da_p = 1$, and $\delta a_k = da_{k-1}$ for $p \geqslant k \geqslant p - r + 2$.

Lemma 9. *$Z_{r-1}^{p-1,n-p+1}$ is a normal subgroup of $Z_r^{p,n-p}$.*

Proof. Let $a = \{a_k\} \in Z_r^{p,n-p}$, $c = \{c_k\} \in Z_{r-1}^{p-1,n-p+1}$ then $a_k c_k a_k^{-1} = 1$ for $k > p - 1$, hence $aca^{-1} \in W_{p-1}$. Since $d(a_k c_k a_k^{-1}) = \delta(a_{k+1} c_{k+1} a_{k+1}^{-1})$ for $p - 1 \geqslant k \geqslant p - r + 2$, then $aca^{-1} \in Z_{r-1}^{p-1,n-p+1}$.

Remark. The quotient $Z_r^{p,n-p}/Z_{r-1}^{p-1,n-p+1}$ can be represented by the group of r-tuples $(a_{p-r+1}, \ldots, a_p)$ satisfying $da_p = 1$, $da_{k-1} = \delta a_k$ $(p \geqslant k \geqslant p - r + 2)$, modulo those sequences with $a_p = 1$.

Lemma 10. *Given a homogeneous string b of total degree $n + 1$ such* *that $D(b) \in W_p$ and $a \in Z_r^{p,n-p}$, there is a cycle $h \in W_{p-r+1}$ of total degree n such that $D(\beta(b) \cdot a) = \beta(h) \cdot D(a)$.*

Proof. Call $f_k = D(\beta(b) \cdot a)$, then

$$f_k = d\delta b_{k+1}^{-1}\, da_k^{-1}\, \delta a_{k+1}\, \delta db_{k+1} = d\delta b_{k+1}^{-1}(D(a))_k\, d\delta b_{k+1} \qquad (k \text{ even}),$$

$$f_k = (D(a))_k \qquad (k \text{ odd}).$$

Now, consider the string $h = \{h_k\}$ defined by $h_k = \delta b_{k+1}$ for k even and $p - r \geqslant k$, $h_k = db_k$ for k odd and $p - r + 1 \geqslant k$, $h_k = 1$ otherwise. Then check that $D(h) = 1$ and $h \in W_{p-r+1}$. The following lemma is proved by direct computation.

LEMMA 11. *If a, c are strings of total degree n, then $D(ac) = \beta(c) \cdot D(a)$.*

5. DEFINITION OF $E_r^{p,q}$

We shall use the following notation throughout this section: $a = \{a_k\} = (..., a_0, ..., a_n, ...)$ will denote the strings in $Z_r^{p,n-p}$, $b = \{b_k\}$ those in $Z_{r-1}^{p+r-1,n-p-r+2}$, $c = \{c_k\}$ in $Z_{r-1}^{p-1,n-p+1}$.

DEFINITION 12. If a, $a' \in Z_r^{p,n-p}$, they will be called equivalent (notation $a \sim a'$) if there exist $c \in Z_{r-1}^{p-1,n-p+1}$, $b \in Z_{r-1}^{p+r-1,n-p-r+2}$ such that $\beta(b) \cdot (ca) = a'$.

Remark. The string c may multiply either on the left or on the right, since $Z_{r-1}^{p-1,n-p+1}$ is a normal subgroup of $Z_r^{p,n-p}$.

LEMMA 13. *Given a, b, c as before, there exists $c' \in Z_{r-1}^{p-1,n-p+1}$ such that $c(\beta(b) \cdot a) = \beta(b) \cdot (c'a)$.*

Proof. The kth term of the string $c(\beta(b) \cdot a)$ is

$$c_k\, db_k^{-1} a_k\, \delta b_{k+1} \qquad \text{if } k \text{ is even},$$

or

$$c_k\, \delta b_{k+1}^{-1} a_k\, db_k \qquad \text{if } k \text{ is odd}.$$

We can then call $c_k' = db_k c_k db_k^{-1}$ (k even) or $c_k' = \delta b_{k+1} c_k \delta b_{k+1}^{-1}$ (k odd), so the string $c' = \{c_k'\}$ will satisfy $\beta(b) \cdot (c'a) = c(\beta(b) \cdot a)$, and we have to check that $c' \in Z_{r-1}^{p-1,n-p+1}$.

In fact for $k > p - 1$, $c_k = 1$, so $c_k' = 1$, i.e., $c' \in W_{p-1}$. On the other hand, c' has total degree n. We still have to prove $D(c') \in W_{p-r}$.

The kth element of $D(c')$ is $D(c')_k = d(c_k')^{-1}\delta c_{k+1}' = dc_k^{-1}\delta c_{k+1}$ for k even, and $D(c')_k = \delta(c_{k+1}')^{-1} dc_k' = \delta db_{k+1}\delta c_{k+1}^{-1} dc^k d\delta b_{k+1}^{-1}$ for k odd, so the result follows from the fact that $D(c) \in W_{p-r}$.

PROPOSITION 14. *$\sim$ is an equivalence relation.*

Proof. (i) $a \sim a$. Take $b = c = (1)$.

(ii) Let $a \sim a'$, i.e., there are b, c such that $a' = \beta(b) \cdot (ca)$, hence $a = c^{-1}(\beta(b^{-1}) \cdot a')$ and, by the previous lemma, there is a $c' \in Z_{r-1}^{p-1,n-p+1}$ such that $a = \beta(b^{-1}) \cdot (c'a')$, thus $a' \sim a$.

(iii) Let $a \sim a'$, $a' \sim a''$, so we can write $a' = \beta(b) \cdot (ca)$, $a'' = \beta(b') \cdot (c'a')$, hence $a'' = \beta(b') \cdot (c'(\beta(b) \cdot (ca)))$. Again, says the lemma, there is a c'' such that $c'\beta(b) \cdot (ca) = \beta(b) \cdot (c''ca)$, so

$$a'' = \beta(b') \cdot (\beta(b) \cdot (c''ca)) = \beta(bb') \cdot ((c''c)a),$$

with $bb' \in Z_{r-1}^{p+r-1,n-p-r+2}$, $c''c \in Z_{r-1}^{p-1,n-p+1}$, thus $a \sim a''$.

DEFINITION 15. $E_r^{p,n-p}$ is the set of equivalence classes of $Z_r^{p,n-p}$ by the previous equivalence relation.

PROPOSITION 16. *Let $\{G_{i,j}, d, \delta\}$ be a double complex of groups. If $G_{p,q+1} \xrightarrow{d} G_{p,q} \xrightarrow{d} G_{p,q-1}$ is exact, then $E_2^{p,q} = \{*\}$.*

Proof. Let $a = \{a_k\} \in Z_2^{p,q}$, $a_p \in G_{p,q}$, $da_p = 1$, then there is $b_p \in G_{p,q+1}$ such that $db_p = a_p^{(-1)^p}$ and call $b = (\ldots, 1, \ldots, 1, b_p, 1, \ldots)$. Then $b \in Z_1^{p+1,q}$ and

$$a \sim \beta(b) \cdot a = (\ldots, a_0, \ldots, a_{p-2}, a_{p-1}\delta b_p, a_p db_p, 1, \ldots) \qquad \text{or}$$

$$= (\ldots, a_0, \ldots, a_{p-2}, \delta b_p^{-1} a_{p-1}, db_p^{-1} a_p, 1, \ldots)$$

according to p being odd or even, respectively, but in either case the pth term is 1, hence $\beta(b) \cdot a \in Z_1^{p-1,q+1}$, since $d(a_{p-1}\delta b_p) = da_{p-1} d\delta b_p = \delta a_p \delta db_p = \delta(a_p db_p) = 1$ if p is odd, and similarly if p is even.

So for every $a \in Z_2^{p,q}$ we have $a \sim 1$, hence $E_2^{p,q} = \{*\}$.

6. The Map $\Lambda_r{}^p : E_r^{p,n-p} \to E_r^{p-r,n-p+r-1}$ for $r \geqslant 2$

Let $a = \{a_k\} \in Z_r^{p,n-p}$, and call $f = \{f_k\} = D(a)$ then

$$f_k = da_k^{-1}\,\delta a_{k+1} \qquad (k \text{ even})$$

$$f_k = \delta a_{k+1}^{-1}\,da_k \qquad (k \text{ odd})$$

so $f_k = 1$ for $k > p - r$ and, since it is a boundary,

$$f = D(a) \in Z_r^{p-r,n-p+r-1}.$$

To show that D induces a map $E_r^{p,n-p} \to E_r^{p-r,n-p+r-1}$ it will be enough to show that equivalent elements are mapped into equivalent elements. We will check the two elementary cases $a' = \beta(b) \cdot a$ and $a' = ca$ for b and c in the corresponding groups.

(i) Suppose there is a $b \in Z_{r-1}^{p+r-1,n-p-r+2}$, $a, a' \in Z_r^{p,n-p}$ such that $a' = \beta(b) \cdot a$.

By Lemma 10 there is a cycle $h \in W_{p-r+1}$ such that $\beta(h) \cdot D(a) = D(a')$. Now for $r \geqslant 2$, $h \in Z_{r-1}^{p-1,n-p+1}$, hence $D(a) \sim D(a')$.

(ii) If $a' = ca$, then, by Lemma 11, $D(a') = \beta(c) \cdot D(a)$, with $c \in Z_{r-1}^{p-1,n-p+1}$, which is the group acting on $Z_r^{p,n-p}$ by multiplication and also the one acting on $Z_r^{p-r,n-p+r-1}$ by the boundary action, hence $D(a') \sim D(a)$.

This proves the existence of a map $\Lambda_r{}^p: E_r^{p,n-p} \to E_r^{p-r,n-p+r-1}$ induced by D.

Notice that the map is not always defined for $r = 1$, in distinction to the abelian case for which E_2 is the homology of E_1.

7. The Homology of E_r for $r \geqslant 2$

We shall start by considering the sequence of groups

$$Z_r^{p+r,n-p-r+1} \xrightarrow{D} Z_r^{p,n-p} \xrightarrow{D} Z_r^{p-r,n-p+r-1}$$

and the associated sequence of group-dominated sets

$$E_r^{p+r,n-p-r+1} \xrightarrow{\Lambda_r^{p+r}} E_r^{p,n-p} \xrightarrow{\Lambda_r^{p}} E_r^{p-r,n-p+r-1},$$

and we want to prove it is a complex whose homology is $E_{r+1}^{p,n-p}$.

LEMMA 17. *$Z_r^{p,n-p}$ is stable under β-action of $Z_r^{p+r,n-p-r+1}$.*

Proof. Let $a \in Z_r^{p,n-p}$, $b \in Z_r^{p+r,n-p-r+1}$, then $a \in W_p$, $D(a) \in W_{p-r}$, $b \in W_{p+r}$, $D(b) \in W_p$, so $\beta(b) \cdot a \in W_p$ since, for every $s > p$, $a_s = 1$ and then the sth element of $\beta(b) \cdot a$ coincides with that of $D(b)$, but $D(b) \in W_p$, hence such an element is 1. By Lemma 8, $D(\beta(b) \cdot a) \in W_{p-r}$, so $\beta(b) \cdot a \in Z_r^{p,n-p}$.

LEMMA 18. *The action of $Z_r^{p+r,n-p-r+1}$ on $Z_r^{p,n-p}$ via β induces an action on $E_r^{p,n-p}$.*

Proof. We have to show that, given $a, a' \in Z_r^{p,n-p}$, $b \in Z_r^{p+r,n-p-r+1}$, if $a \sim a'$, then $\beta(b) \cdot a \sim \beta(b) \cdot a'$.

Since the equivalence relation is obtained by action of $Z_{r-1}^{p-1,n-p+1}$ and $Z_{r-1}^{p+r-1,n-p-r+2}$, we can divide our proof in two steps, namely for the cases $a' = ca$ or $a' = \beta(b') \cdot a$ with $c \in Z_{r-1}^{p-1,n-p+1}$, $b' \in Z_{r-1}^{p+r-1,n-p-r+2}$.

(i) Let $a' = \beta(b') \cdot a$. We have, first of all, to recall that $Z_{r-1}^{p+r-1,n-p-r+2}$ is a normal subgroup of $Z_r^{p+r,n-p-r+1}$ (see Lemma 9), hence, for b, b' as before, there is a $b'' \in Z_{r-1}^{p+r-1,n-p-r+2}$ such that $b'b = bb''$.

So $\beta(b) \cdot (\beta(b') \cdot a) = \beta(b'b) \cdot a = \beta(bb'') \cdot a = \beta(b'') \cdot (\beta(b) \cdot a)$, hence $\beta(b) \cdot a' \sim \beta(b) \cdot a$.

(ii) Let $a' = ca$. Then $\beta(b) \cdot a'$ is the string whose sth element is

$$db_s^{-1} c_s a_s \, \delta b_{s+1} \qquad (s \text{ even}),$$

$$\delta b_{s+1}^{-1} c_s a_s \, db_s \qquad (s \text{ odd}),$$

hence we can write

$$(\beta(b) \cdot a')_s = (db_s^{-1} c_s \, db_s)(db_s^{-1} a_s \, \delta b_{s+1})$$

or

$$= (\delta b_{s+1}^{-1} c_s \, \delta b_{s+1})(\delta b_{s+1}^{-1} a_s \, db_s)$$

according to the parity of s, where the second parenthesis is $(\beta(b) \cdot a)_s$.

To prove the equivalence, call c' the string whose sth element is the first parenthesis of the previous formulae, so we have to show $c' \in Z_{r-1}^{p-1,n-p+1}$. It is obvious that the total degree is n. Also if $c_s = 1$ the product becomes 1, so $c' \in W_{p-1}$. By direct computation it can be proved that $D(c') \in W_{p-r}$, and we are done.

LEMMA 19. *The inverse image in $E_r^{p,n-p}$ of any element of $E_r^{p-r,n-p+r-1}$ is an invariant subset under the action of $Z_r^{p+r,n-p-r+1}$ for $r \geqslant 1$.*

Proof. Consider $a \in Z_r^{p,n-p}$, $b \in Z_r^{p+r,n-p-r+1}$, then by Lemma 10 we have $D(\beta(b) \cdot a) = \beta(b') \cdot D(a)$ where b' is a cycle contained in W_{p-r+1}. Hence, for $r \geqslant 1$, $b' \in Z_{r-1}^{p+r-1,n-p-r+1}$, which is the group acting on $Z_r^{p-r,n-p+r-1}$, so we obtain the same element in $E_r^{p-r,n-p+r-1}$.

In particular, Ker Λ_r^p is an invariant subset. As explained before, the homology of the sequence of E's is the pointed set of orbits of Ker Λ_r^p under the action of $Z_r^{p+r,n-p-r+1}$, and it will be denoted by $H^{p,n-p}(E_r)$.

Call Γ_r^p the inverse image of Ker Λ_r^p in the group $Z_r^{p,n-p}$ dominating $E_r^{p,n-p}$. Hence Γ_r^p is the set of elements $a \in Z_r^{p,n-p}$ such that $D(a) \sim 1$ in $Z_r^{p-r,n-p+r-1}$. (Use Lemma 11 and the fact that $r \geqslant 2$.)

Now let $a \in \Gamma_r^p$, hence $D(a) \sim 1$, i.e., we obtain 1 in $Z_r^{p-r,n-p+r-1}$ from $D(a)$ by a product with an element of $Z_{r-1}^{p-r-1,n-p+r}$ and by β-action of an element of $Z_{r-1}^{p-1,n-p+1}$.

LEMMA 20. *Γ_r^p is the saturation of $Z_{r-1}^{p,n-p}$ by the equivalence relation defined in $Z_r^{p,n-p}$.*

Proof. To prove that $Z_{r+1}^{p,n-p}$ is contained in Γ_r^p, consider $a \in Z_{r+1}^{p,n-p}$, then $D(a) \in W_{p-r+1}$, i.e., $D(a) \in Z_{r-1}^{p-r-1,n-p+r}$, so $D(a) \sim 1$ in $Z_r^{p-r,n-p+r-1}$.

Now let $a \in \Gamma_r^p$, then $D(a) \sim 1$ in $Z_r^{p-r,n-p+r-1}$, hence there are $c \in Z_{r-1}^{p-r-1,n-p+r}$, $b \in Z_{r-1}^{p-1,n-p+1}$ such that $D(a) = \beta(b) \cdot c$. In $Z_r^{p,n-p}$, $a \sim ab^{-1}$ and $D(ab^{-1}) = c \in W_{p-r-1}$, hence $ab^{-1} \in Z_{r+1}^{p,n-p}$.

This result implies that the image of $Z_{r+1}^{p,n-p}$ into $E_r^{p,n-p}$ given by the inclusion into $Z_r^{p,n-p}$ followed by the quotient map, is equal to Ker Λ_r^p.

THEOREM 21. *The inclusion $Z_{r+1}^{p,n-p} \to Z_r^{p,n-p}$ induces a bijection of group-dominated sets:*

$$E_{r+1}^{p,n-p} \to H^{p,n-p}(E_r).$$

Proof. First of all, we have to show that the map $E_{r+1}^{p,n-p} \to H^{p,n-p}(E_r)$ is well defined.

Recall that $H^{p,n-p}(E_r)$ is the set of orbits of Ker Λ_r^p under the β-action of $Z_r^{p+r,n-p-r+1}$.

On the other hand, $E_{r+1}^{p,n-p}$ is obtained from $Z_{r+1}^{p,n-p}$ by the equivalence relation already defined, now by multiplication with elements of $Z_r^{p-1,n-p+1}$ and the β-action by $Z_r^{p+r,n-p-r+1}$.

Let a and a' be two elements in $Z_{r+1}^{p,n-p}$ which give the same element in $E_{r+1}^{p,n-p}$, i.e., there are $c \in Z_r^{p-1,n-p+1}$, $b \in Z_r^{p+r,n-p-r+1}$ such that $a' = \beta(b) \cdot (ac)$.

Since $Z_r^{p-1,n-p+1} \subseteq Z_{r-1}^{p-1,n-p+1}$, a and ac give the same element in $E_r^{p,n-p}$ and $b \in Z_r^{p+r,n-p-r+1}$ implies a' and ac (hence a' and a) give the same element in $H^{p,n-p}(E_r)$.

Hence the map is well defined. Surjectivity follows from the previous lemma.

Consider now a, $a' \in Z_{r+1}^{p,n-p} \subseteq Z_r^{p,n-p}$ such that both represent the same element in $H^{p,n-p}(E_r)$. So there are $c \in Z_{r-1}^{p-1,n-p+1}$, $b \in Z_r^{p+r,n-p-r+1}$, such that $a' = \beta(b) \cdot (ac)$, or $\beta(b^{-1}) \cdot a' = ac$. But $a'' = \beta(b^{-1}) \cdot a' \in Z_{r+1}^{p,n-p}$ and $D(a'') = D(ac) = \beta(c) \cdot D(a)$. Since $D(a)$ and $D(a'')$ belong to W_{p-r-1}, by direct computation it follows that $D(c) \in W_{p-r-1}$ hence $c \in Z_r^{p-1,n-p+1}$; so a and a' yield the same element in $E_{r+1}^{p,n-p}$ and the map is injective.

8. The Limit Process

Suppose, for the sake of simplicity, that we work with a positive double complex (i.e., $G_{i,j} = (1)$ whenever $i < 0$ or $j < 0$).

For $r > \max(p, n - p + 1)$, $Z_r^{p-r,n-p+r-1} = \{*\}$ and $Z_r^{p+r,n-p-r+1} = Z_{r-1}^{p+r-1,n-p-r+2}$, so the action of $Z_r^{p+r,n-p-r+1}$ on $E_r^{p,n-p}$ is trivial. Hence $E_{r+1}^{p,n-p} = E_r^{p,n-p}$, which means that the spectral sequence becomes stable.

If we consider the filtration induced on cycles of the double complex by the filtration $\{W_p\}$, it induces a filtration on the homology H_n, we can call $\{H_n{}^p\}$. In other words, if we call $C_n{}^p = C_n \cap W_p$ (C_n is the group of cycles of total degree n), then $H_n{}^p$ is the image in H_n of $C_n{}^p$, i.e., the set of equivalence classes of $C_n{}^p$ under the boundary operation by all elements of total degree $n + 1$.

We can define, then, $H_n{}^p/H_n^{p-1}$ as the set of equivalence classes of $C_n{}^p$ using as equivalence relation the product by elements of C_n^{p-1} and the β-action by those elements b of total degree $n + 1$ such that $D(b) \in W_p$.

This last condition is imposed in order to map $C_n{}^p$ into itself by the boundary action. The proof that this is an equivalence relation follows the same lines as in Proposition 14.

For $r > \max(p, n - p + 1)$, $Z_r^{p,n-p} = C_n{}^p$, $Z_r^{p-1,n-p+1} = C_n^{p-1}$ and $Z_{r-1}^{p+r-1,n-p-r+2}$ is the set of all strings b of total degree $n + 1$ such that $D(b) \in W_p$. It follows from the definitions that, for

$$r > \max(p, n - p + 1), \qquad H_n{}^p/H_n^{p-1} = E_r^{p,n-p}.$$

So, in the stable range, the E_r-terms describe the layers of the homology of the double complex.

Remark. The same conclusions may be extended to bounded double complexes as defined in MacLane's book [2] with the only difference that the bounds imposed on r will depend on the bounds of the complex.

9. The Edge Maps

Suppose we have a positive double complex. We shall define maps $H_n \to E_2^{n,0}$ and $E_2^{0,n} \to H_n$ which will be called the edge maps.

$E_2^{n,0}$ is the set of equivalence classes of $Z_2^{n,0}$, i.e., the group of homogeneous strings a of total degree n such that $D(a) \in W_{n-2}$. If we call C_n the group of cycles of degree n, there is an inclusion $\alpha_n\colon C_n \to Z_2^{n,0}$.

$E_2^{n,0}$ is defined by the action on $Z_2^{n,0}$ of the subgroup $Z_1^{n-1,1}$ and the boundary operation of $Z_1^{n+1,0}$. Now $Z_1^{n-1,1} = Z_2^{n,0} \cap W_{n-1}$ and $Z_1^{n+1,0}$ is the group of all strings of total degree $n + 1$. Hence the equivalence classes defining H_n are contained in those giving $E_2^{n,0}$, so α_n induces a map $\bar{\alpha}_n\colon H_n \to E_2^{n,0}$.

On the other hand, $E_2^{0,n}$ is defined from $Z_2^{0,n}$, i.e., strings of total degree n contained in W_0, say $(a_0, 1,\ldots, 1)$ with $da_0 = 1$, so $Z_2^{0,n}$ is a subgroup of C_n. The groups acting on $Z_2^{0,n}$ to define $E_2^{0,n}$ are $Z_2^{-1,n+1} = \{1\}$ and $Z_1^{1,n}$, hence this action is a restriction of the boundary action used to define H_n, hence the inclusion $Z_2^{0,n} \to C_n$ induces a map $E_2^{0,n} \to H_n$. Both maps are morphisms of group-dominated sets.

Remark. The previous considerations are also valid for $E_r^{n,0}$ and $E_r^{0,n}$ for all $r \geqslant 2$, and by using the results of Section 7 we find maps $E_r^{0,n} \to E_{r+1}^{0,n}$, $E_{r+1}^{n,0} \to E_r^{n,0}$ giving two sequences of maps

$$E_2^{n,0} \to \cdots \to E_r^{n,0} \to E_{r+1}^{n,0} \to \cdots \to H_n\,,$$

$$H_n \to \cdots \to E_{r+1}^{0,n} \to E_r^{0,n} \to \cdots \to E_2^{0,n}.$$

Given the group of cycles C_n, call $C_n{}^p = C_n \cap W_p$ and consider the equivalence relation defined in the following way: if a, $a' \in C_n$ say $a \sim a'$ if there exists $c \in C_n{}^p$, b of total degree $n + 1$, such that $a' = \beta(b) \cdot (ca)$. Call F^pH_n the set of equivalence classes.

There is an obvious map $F^pH_n \to F^sH_n$ whenever $s \geqslant p$, and $F^{-1}H_n = H_n$, $F^nH_n = \{*\}$.

LEMMA 22. *$\tilde{\alpha}_n: H_n \to E_2^{n,0}$ induces a map $\bar{\alpha}_n: F^{n-1}H_n \to E_2^{n,0}$.*

Proof. Suppose $a, a' \in C_n$ and there are $c \in C_n^{n-1}$, b of total degree $n+1$, such that $a' = \beta(b) \cdot (ca)$. Since $Z_1^{n-1,1} = Z_2^{n,0} \cap W_{n-1}$, then $c \in Z_1^{n-1,1}$ and $b \in Z_1^{n+1,0}$, so a and a' give the same element in $E_2^{n,0}$.

PROPOSITION 23. *$\bar{\alpha}_n: F^{n-1}H_n \to E_2^{n,0}$ is injective and* $\operatorname{Im} \bar{\alpha}_n = \operatorname{Im} \tilde{\alpha}_n$.

Proof. The second assertion is trivial. For the first, consider $a, a' \in C_n$ such that both give the same element in $E_2^{n,0}$, so there are $c \in Z_1^{n-1,1}$, $b \in Z_1^{n+1,0}$, such that $a' = \beta(b) \cdot (ca)$.

But $D(a) = D(a') = 1$, so, by Lemma 10, there is an h such that

$$1 = D(a') = D(\beta(b) \cdot (ca)) = \beta(h) \cdot D(ac) = \beta(h) \cdot (\beta(c) \cdot D(a)) = \beta(h) \cdot D(c)$$

and, according to the proof of that lemma, $\beta(h)$ is a conjugation by elements $d\delta b_r$, hence $D(c) = 1$.

So c is a cycle contained in W_{n-1}, hence a and a' give the same element of $F^{n-1}H_n$.

Notation. The composite of the canonical map $F^pH_n \to F^{n-1}H_n$ with $\bar{\alpha}_n: F^{n-1}H_n \to E_2^{n,0}$ will be called $\alpha_n{}^p$.

10. THE 7-TERMS EXACT SEQUENCE

Given a positive double complex we shall prove an equivalent of the 7-terms exact sequence studied by Chase and Rosenberg [1, Lemma 7.5] for a first quadrant spectral sequence of abelian groups.

We want, then, to prove the exactness of the sequence

$$E_2^{3,0} \xrightarrow{\Delta_2{}^3} E_2^{1,1} \xrightarrow{\psi_2} F^0H_2 \xrightarrow{\alpha_2{}^0} E_2^{2,0} \xrightarrow{\Delta_2{}^2} E_2^{0,1} \xrightarrow{\psi_1} H_1 \xrightarrow{\tilde{\alpha}_1} E_2^{1,0} \longrightarrow \{*\}$$

where the maps ψ_i will be defined presently.

a. *The maps ψ_i*

ψ_1 is the edge map defined in the previous section.

To define $\psi_2: E_2^{1,1} \to F^0H_2$ consider:

$Z_2^{1,1}$ is the group of strings a of total degree 2 such that $a \in W_1$ and $D(a) \in W_{-1} = (1)$, so $Z_2^{1,1} = C_2 \cap W_1$, while $Z_2^{0,2} = Z_2^{1,1} \cap W_0 = C_2 \cap W_0$.

Hence, the equivalence relation given by multiplication by $Z_2^{0,2}$ and β-action of $Z_1^{2,1}$ is a restriction of the one defining F^0H_2, and the inclusion $Z_2^{1,1} \to C_2$ induces a map ψ_2: $E_2^{1,1} \to F^0H_2$.

Remark. A similar argument shows the existence of a map, also induced by an inclusion, $E_r^{n-1,1} \to F^{n-2}H_n$ for all $r \geqslant n$, which, in the case $n = 1$, $r = 2$ is the map ψ_1: $E_2^{0,1} \to H_1$, and if $n = r = 2$ is ψ_2.

b. *Exactness at* $E_2^{2,0}$

Let a, $a' \in Z_2^{2,0}$ and call $\bar{a}$, $\bar{a}'$ their images in $E_2^{2,0}$. Suppose $\Lambda_2{}^2(\bar{a}) = \Lambda_2{}^2(\bar{a}')$, i.e., $D(a)$ and $D(a')$ represent the same element of $E_2^{0,1}$. The groups acting on $Z_2^{0,1}$ are $Z_1^{-1,2}$ by multiplication and $Z_1^{1,1}$ via β. But $Z_1^{-1,2} = (1)$, so there is an element $b \in Z_1^{1,1}$ such that

$$D(a') = \beta(b) \cdot D(a) = D(ab),$$

but ab and a give the same element in $E_2^{2,0}$. Now, by direct computation, it follows that $D((ab)^{-1}a') = 1$, so $(ab)^{-1}a' \in C_2$.

In particular, if $\bar{a} \in \operatorname{Ker} \Lambda_2{}^2$ we can take $a' = 1$ and $\bar{a}$ is in the image of $\alpha_2{}^0$.

Hence $\operatorname{Ker} \Lambda_2{}^2 \subseteq \operatorname{Im} \alpha_2{}^0$. More generally, $\Lambda_2{}^2(\bar{a}) = \Lambda_2{}^2(\bar{a}')$ implies existence of a representative of $\bar{a}$, say a'', such that $a''^{-1}a' \in \operatorname{Im} \alpha_2{}^0$. On the other hand, if $a \in C_2$, then $D(a) = 1$, so $\operatorname{Im} \alpha_2{}^0 \subseteq \operatorname{Ker} \Lambda_2{}^2$.

c. *Exactness at* $E_2^{1,1}$

The map $\Lambda_2{}^3$: $E_2^{3,0} \to E_1^{1,1}$ is induced by a map D between the dominating Z groups. If $b \in Z_2^{3,0}$, then $D(b)$ represents the base point of H_2, hence of F^0H_2, so $\operatorname{Im} \Lambda_2{}^3 \subseteq \operatorname{Ker} \psi_2$.

Let $a \in Z_2^{1,1}$ and call $\bar{a}$ its image in $E_2^{1,1}$. Suppose $\psi_2(\bar{a})$ is the base point of F^0H_2, i.e., there exist $c \in C_2 \cap W_0$ and b of total degree 3 such that $a = \beta(b) \cdot c$. But $a \in W_1$ and $c \in W_0$ imply $D(b) \in W_1$, then $b \in Z_2^{3,0}$ and a coincides with $D(b)$ up to an element in $C_2 \cap W_0$, so $\bar{a}$ is in the image of $\Lambda_2{}^3$. This proves the exactness.

d. *Exactness at* F^0H_2

$E_2^{2,0}$ was defined as a quotient of $Z_2^{2,0}$ by $Z_1^{1,1}$ with $Z_1^{3,0}$ acting. $E_2^{1,1}$ is a quotient of $Z_2^{1,1}$ and $Z_2^{1,1} \subseteq Z_1^{1,1}$. Since both maps ψ_2 and $\alpha_2{}^0$ are defined by inclusions, it follows that $\operatorname{Im} \psi_2 \subseteq \operatorname{Ker} \alpha_2{}^0$.

Suppose now a, a' are cycles whose images $\bar{a}$, $\bar{a}' \in F^0H_2$ verify $\alpha_2{}^0(\bar{a}) = \alpha_2{}^0(\bar{a}')$. Then there exist $f \in Z_1^{1,1}$, $b \in Z_1^{3,0}$ such that $af = \beta(b) \cdot a'$. Since a, a' are cycles, direct computation shows that f is a cycle, hence $f \in Z_2^{1,1}$.

If $a \in \operatorname{Ker} \alpha_2{}^0$, we can take $a' = 1$ and $af = D(b)$ so $\bar{a}$ coincides with the image of f^{-1}, hence $\operatorname{Ker} \alpha_2{}^0 \subseteq \operatorname{Im} \psi_2$.

On the other hand, since $\beta(b) \cdot a'$ is another element mapping onto $\bar{a}'$, $\alpha_2{}^0(\bar{a}) = \alpha_2{}^0(\bar{a}')$ implies the existence of cycles a, a'' representing $\bar{a}$, $\bar{a}'$ and an $f \in Z_2^{1,1}$ such that $af = a''$.

e. We have proved the exactness of

$$E_2^{3,0} \xrightarrow{\Delta_2{}^3} E_2^{1,1} \xrightarrow{\psi_2} F^0H_2 \xrightarrow{\alpha_2{}^0} E_2^{2,0} \xrightarrow{\Delta_2{}^2} E_2^{0,1}.$$

By similar arguments and the fact that $E^{-1,1} = \{*\}$, we obtain the exactness of

$$E_2^{2,0} \xrightarrow{\Delta_2{}^2} E_2^{0,1} \xrightarrow{\psi_1} H_1 \xrightarrow{\bar{\alpha}_1} E_2^{1,0} \longrightarrow \{*\},$$

hence, of the 7-terms exact sequence of group-dominated sets.

References

1. S. U. Chase and A. Rosenberg, Amitsur cohomology and the Brauer group, *Mem. Amer. Math. Soc.* **52** (1965), 34–79.
2. S. MacLane, "Homology," Grundlehren Math. Wiss. 114, Springer, Berlin, 1963.
3. J. R. Strooker and O. E. Villamayor, Building *K*-theories, *Advances in Math.* **15** (1975), 232–268.