INFORMATION AND CONTROL 27, 109–125 (1975)

TOL Schemes and Control Sets*

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Suppose given two of the following: a set L_1 of start words, a set L_2 of target words, and a control set $\mathscr{C}$ of finite sequences of applications of a given finite set of homomorphisms (or finite substitutions) which map L_1 into L_2. Using notions from OL systems, the present paper investigates what can be said about the remaining set in case the given sets are regular. When the start and target sets are regular, the set of all control words turns out to be regular. (This is true even when the regularity assumption on the start set is removed.) When a regular target set L_2 and a regular control set $\mathscr{C}$ are given, the set of all words mapped into L_2 by $\mathscr{C}$ is regular. (This result remains true even when the regularity assumption on $\mathscr{C}$ is removed.) When a regular start set L and a regular control set $\mathscr{C}$ are given, the set $\mathscr{C}(L)$ is an ETOL language. In fact, this characterizes ETOL languages. Finally, it is shown that the set $\mathscr{H}(\Sigma)$ of all possible homomorphisms (or the set $\mathscr{S}(\Sigma)$ of all finite substitutions) from a given alphabet Σ into itself cannot be a control set. In other words, neither of the semigroups $\mathscr{H}(\Sigma)$ or $\mathscr{S}(\Sigma)$ is finitely generated.

INTRODUCTION

Recently, there has been a flock of papers on developmental systems.[1] (Developmental systems are formal structures which model the development of certain biological organisms. They may also be regarded as elementary models of parallel processes (Rozenberg, *b*).) Most of them have been

* This research was supported in part by the National Science Foundation under Grant GJ-28787.

[1] For example, see Lindenmayer and Rozenberg (1972), Rozenberg (1973a), and their references.

concerned with developmental systems possessing one "environment." (Mathematically speaking, an environment may be regarded as a homomorphism in the case of deterministic behavior, and a finite substitution in the case of nondeterministic behavior.) A few of them, however, have dealt with a finite number of environments.[2] The present document is concerned with changes of environment in the multienvironment case. Specifically, the notion of a control set from AFL theory (Ginsburg and Spanier, 1968) is used to describe the changes of environment. From a purely mathematical viewpoint, this may be thought of as the effect of certain sets of finite sequences of homomorphisms or finite substitutions.

The basic situation under examination consists of being given two of the following: a set L_1 of start words, a set L_2 of target words, and a control set $\mathscr{C}$ of finite sequences of applications of a given finite set of homomorphisms (in the deterministic case) or finite substitutions (in the nondeterministic case) which map L_1 into L_2. The problem is to ascertain information about the remaining set. A logical place to begin, which is also physically well motivated, is to assume a single element start set and a finite target set. (Finite start and target sets are natural since these correspond to having a finite number of observations of whatever biological phenomenon is under study.) It can be shown that in this case the control set is regular. The same result occurs if the target set is extended to a regular set (which, in language theory, is a natural generalization of a finite set). From these, and other considerations, it seems reasonable to limit ourselves to the situation where each of the components $\mathscr{C}$, L_1, and L_2 is a regular set. (From the biological viewpoint, regular control sets have been used almost exclusively. For example, the effect of cyclic changes of environment, as in Jerebzoff (1965), is a phenomenon extensively studied in developmental biology. The alternation of light and darkness is an obvious regular control set L affecting plant development.) This we have done.

The paper itself is divided into five sections. Section 1 reviews concepts relating to TOL schemes and regular sets. Section 2 treats the case when a regular start set and a regular target set are given. It is shown that in this situation the set of all control words is regular. (Surprisingly, it turns out that this result is true even when the regularity assumption on the start set is removed, i.e., the start set can be an arbitrary set.) Moreover, given any regular set $\mathscr{C}$, one can find a start word w_1, a target word w_2, and underlying homomorphisms (finite substitutions) so that the set of all control words mapping w_1 into w_2 is $\mathscr{C}$.

[2] For example, see Lindenmayer, Rozenberg (1973a), Rozenberg (a), and Rozenberg (1973b).

Section 3 examines the situation when a regular target set L_2 and a regular control set $\mathscr{C}$ are given. It is shown that the set of all words mapped into L_2 by $\mathscr{C}$ is regular. (As in Section 2, this result remains true even when the regularity assumption on $\mathscr{C}$ is removed, i.e., $\mathscr{C}$ can be an arbitrary set.) Also, for every regular set L_1, there exists a regular control set $\mathscr{C}$ and a regular target set L_2 such that the set of all words which are mapped into L_2 by $\mathscr{C}$ is L_1. The abovementioned results of Sections 2 and 3 thus reinforce our earlier contention that in the study of control sets applied to developmental systems, regular sets are reasonable objects.

Section 4 considers the case when a regular start set and a regular control set are given. It is shown that the family of languages obtained in this way coincide with the family of ETOL languages.

In Section 5, each control word is regarded as a homomorphism (or a finite substitution). The question then arises as to whether the set $\mathscr{H}(\Sigma)$ of all possible homomorphisms (or the set $\mathscr{S}(\Sigma)$ of all finite substitutions) from a given alphabet Σ into itself can be a control set. It is shown that the answer is no. However, Σ can be enlarged to a finite set $\bar{\Sigma}$ and a control set $\mathscr{C}$, with respect to $\bar{\Sigma}$, can be obtained so that each elements in $\mathscr{H}(\Sigma)(\mathscr{S}(\Sigma))$ coincides with some element of $\mathscr{C}$ appropriately restricted.

Throughout the paper, we shall assume a familiarity with the rudiments of formal languages and automata theory, especially regular sets and finite state acceptors. The reader is referred to Ginsburg (1966) for all unexplained notation and terminology.

1. Preliminaries

In this section we present some ideas, symbolism, and terminology needed in the paper. We start with TOL schemes, and follow that with nondeterministic finite state acceptors and regular sets.

Definition. A TOL *scheme* is an ordered pair $S = (\Sigma, \mathscr{P})$, where

(i) Σ is a finite nonempty set (the *alphabet*)[3] and

(ii) $\mathscr{P}$ (the set of *tables*) is a finite nonempty set such that for each element T in $\mathscr{P}$, (a) T is a finite subset of $\Sigma \times \Sigma^*$, and (b) for each a in Σ, (a, α) is in T for at least one element α in Σ^*.

Thus each table is a subset of $\Sigma \times \Sigma^*$ in which the projection onto the

[3] In general, Σ will always denote a finite nonempty set of symbols.

first coordinate is Σ. Since $\mathscr{P}$ is a set, no two tables T_1 and T_2 in $\mathscr{P}$ are the same subset of $\Sigma \times \Sigma^*$. A companion treatment, with identical results, can be given if a table is regarded as a subset of $\Sigma \times \Sigma^*$ together with a label. In the latter case, two distinctly labeled tables can have the same subset of $\Sigma \times \Sigma^*$.

Motivationwise, a set of tables, as contrasted with the more studied situation of one table, represents a set of environments, each of which can affect the development of an organism. The reader is referred to Rozenberg (a) for a more elaborate discussion.

A TOL scheme is a rewriting system in the following sense.

Notation. Let $S = (\Sigma, \mathscr{P})$ be a TOL scheme. Let $\Rightarrow_S$ be the relation on Σ^* defined by $x \Rightarrow_S y$ if there exist T in $\mathscr{P}$, $n \geqq 1$, and $a_1, \ldots, a_n$, $\alpha_1, \ldots, \alpha_n$ such that $x = a_1 \cdots a_n$, $y = \alpha_1 \cdots \alpha_n$, and $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$ are in T. (This is also denoted by $x \Rightarrow y$ or $x \Rightarrow_T y$.) Let $\overset{*}{\Rightarrow}_S$, or $\overset{*}{\Rightarrow}$ when S is understood, be the reflexive transitive extension of $\Rightarrow$.

Tables serve as functions on Σ^* in the following way.

Notation. Let $(\Sigma, \mathscr{P})$ be a TOL scheme. For each T in $\mathscr{P}$ and each word x in Σ^+, let $T(x) = \{y/x \Rightarrow_T y\}$. Let $T(\epsilon) = \{\epsilon\}$. For each u in $\mathscr{P}^*$ and x in Σ^*, let $u(x) = \{x\}$ if $u = \epsilon$, and $u(x) = T_m \cdots T_1(x)$ if $u = T_1 \cdots T_m$, $m \geqq 1$ and each T_i in $\mathscr{P}$. For $\mathscr{C} \subseteq \mathscr{P}^*$ and $L \subseteq \Sigma^*$, let $\mathscr{C}(L) = \{u(x)/u$ in $\mathscr{C}$ and x in $L\}$.

From the mathematical point of view each table represents a finite substitution, or, if $T(x)$ has exactly one element for each x in Σ, a homomorphism. (Thus a table may also be viewed as a subset of $\Sigma^* \times \Sigma^*$.) A sequence of tables $T_{i_1} \cdots T_{i_p}$, written without the commas, in a TOL scheme $(\Sigma, \mathscr{P})$ may be considered as either the word $T_{i_1} \cdots T_{i_p}$ over the alphabet $\mathscr{P}$ or as the function which is the composition of the functions $T_{i_1}, \ldots, T_{i_p}$. Both considerations will be used in the sequel.

As noted in the introduction, regular sets will play a vital role in our study. The main tool used here for handling them is that of a nondeterministic finite state acceptor with multiple start states. For completeness, we now recall this concept and its connection with regular sets.

DEFINITION. *A nondeterministic finite state acceptor with multiple start states*, abbreviated *nfsa with multiple start states*, is a 5-tuple $A = (K, \Sigma, \delta, I, F)$, where (i) K and Σ are finite sets (of *states* and *inputs*, respectively); (ii) δ (the *transition* function) is a mapping from $K \times \Sigma$ into subsets of K; (iii) $I \subseteq K$ (the set of *initial* or *start* states); and (iv) $F \subseteq K$ (the set of *accepting* states). If I contains exactly one element, say p_0, then $(K, \Sigma, \delta, I, F)$ is written as

$(K, \Sigma, \delta, p_0, F)$ and is called an *nfsa*. If $A = (K, \Sigma, \delta, p_0, F)$ is an nfsa and $\delta(p, a)$ consists of exactly one element for each p in K and a in Σ (so that δ may be regarded as a function from $K \times \Sigma$ into K), then A is called a *deterministic fsa* or *fsa*.

The nfsa with multiple starts moves as follows:

Notation. Let $\vdash$ be the relation on $K \times \Sigma^*$ defined by $(p, w) \vdash (p', w')$ if $w = aw'$ and p' is in $\delta(p, a)$ for some p in K and a in Σ. Let $\overset{*}{\vdash}$ be the reflexive transitive extension of $\vdash$.

The nfsa with multiple start states determines a set of words in the following manner:

DEFINITION. For each nfsa $A = (K, \Sigma, \delta, I, F)$ with multiple start states let $W(A) = \{w \text{ in } \Sigma^*/(p, w) \overset{*}{\vdash} (p', \epsilon) \text{ for some } p \text{ in } I \text{ and } p' \text{ in } F\}$. Each word in $W(A)$ is said to be *accepted* by A. A language L is said to be *regular* if $L = W(A)$ for some nfsa A with multiple start states.

It is well known (Ginsburg, 1966) that a language L is regular if and only if $L = W(A)$ for some fsa. Thus a language L is regular if and only if $L = W(A)$ for some nfsa. Also, if $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$ are regular sets (and effectively given), then so are $L_1 \cup L_2$, $L_1 - L_2$, and $\Sigma^* - L_1$ (and each is effectively computable) (Ginsburg, 1966).

2. CONTROL SETS WITH REGULAR TARGETS

As mentioned in the Introduction we deal with the following question: Given an underlying TOL scheme $(\Sigma, \mathscr{P})$, what relations exist between a set L_1 of start words over Σ, a set L_2 of target words over Σ, and a set $\mathscr{C} \subseteq \mathscr{P}^*$ such that $\mathscr{C}(L_1) \subseteq L_2$? (We shall informally refer to a set $\mathscr{C} \subseteq \mathscr{P}^*$ as a "control set" or "control language," and a word in $\mathscr{C}$ as a "control word.") In the present section we inquire into the nature of a largest possible $\mathscr{C}$, given L_1 and L_2, with L_2 regular.

In order to ascertain the form of a largest possible $\mathscr{C}$, we introduce two notions of control languages with respect to given L_1 and L_2, namely a "weak" and a "strong" language. While primary interest is in the strong version, it turns out that the weak one is easier to handle mathematically and provides an access to the strong one.

We now formalize the idea of a weak control language with respect to a given start and a given target language. The adjective "weak" is omitted to simplify the terminology.

Definition. *Let* $S = (\Sigma, \mathscr{P})$ *be a* TOL *scheme*, $L_1 \subseteq \Sigma^*$, *and* $L_2 \subseteq \Sigma^*$. *The* (S, L_1, L_2)*-control language is the set*

$$\mathscr{C}(S, L_1, L_2) = \{u \text{ in } \mathscr{P}^*/u(L_1) \cap L_2 \neq \varnothing\}.$$

A language L is called a *control language with a regular target* if $L = \mathscr{C}(S, L_1, L_2)$ for some TOL scheme S, some language L_1, and some regular set L_2.

The definition of an (S, L_1, L_2)-control language is "existential" in the sense that a control word u is in $\mathscr{C}(S, L_1, L_2)$ if there exists x in L_1 and y in L_2 such that x is transformed into y by u even though there may be some word in L_1 transformed by u into a word not in L_2. The situation where each word in $\mathscr{C}(S, L_1, L_2)$ maps each word in L_1 only into words in L_2 leads to the strong version of a control language, and will be treated later.

For our first result, we shall show that a language L may be regarded as a control language with a regular target if and only if L is regular. This requires three lemmas as well as certain nfsa related to a given one in a particular way.

Notation. For each nfsa $A = (K, \Sigma, \delta, p_0, F)$ let $\mathscr{N}_A$ be the set of all nfsa $B = (K, \Sigma, \delta_B, p_0, F)$.

Thus $\mathscr{N}_A$ consists of all nfsa which have the same set of states, inputs, start state, and accepting states as A. Only the transition function of B can differ from that of A. Clearly $\mathscr{N}_A$ is finite.

With each table in a TOL scheme and each nfsa A, we associate a specific nfsa in $\mathscr{N}_A$.

Notation. Let $(\Sigma, \mathscr{P})$ be a TOL scheme and $A = (K, \Sigma, \delta, p_0, F)$ an nfsa. For each T in $\mathscr{P}$ let $T^{-1}(A)$ be the nfsa $(K, \Sigma, \delta_{T^{-1}}, p_0, F)$, where $\delta_{T^{-1}}$ is the function defined by

$$\delta_{T^{-1}}(p, a) = \{q \text{ in } K/(p, x) \overset{*}{\vdash} (q, \epsilon) \text{ for some } x \text{ in } T(a)\}$$

for each p in K and a in Σ.

Thus $\delta_{T^{-1}}(p, a)$ consists of the states p is led to by δ via words in $T(a)$.

We now construct an nfsa with multiple start states which plays an important role in Sections 2 and 3.

Notation. Let A be an nfsa over Σ and $S = (\Sigma, \mathscr{P})$ a TOL scheme. Let $\mathscr{A}(A) = \bigcup_{i \geq 0} \mathscr{A}_i(A)$, where $\mathscr{A}_0(A) = \{A\}$ and, by induction,

$$\mathscr{A}_{i+1}(A) = \mathscr{A}_i(A) \cup \{T^{-1}(B')/T \text{ in } \mathscr{P}, B' \text{ in } \mathscr{A}_i(A)\}$$

for each $i \geqq 0$. [Observe that $\mathscr{A}(A) \subseteq \mathscr{N}(A)$ and thus is finite.] For each $L \subseteq \Sigma^*$ let $C(S, L, A)$ be the nfsa $(\mathscr{A}(A), \mathscr{P}, \delta', Q_0, \{A\})$ with multiple start states, where δ' is the function defined by $\delta'(B, T) = \{B' \text{ in } \mathcal{O}(A) / T^{-1}(B') = B\}$ for all B in $\mathcal{O}(A)$ and T in $\mathcal{P}$, and $Q_0 = \{B \text{ in } \mathcal{O}(A) / W(B) \cap L \neq \varnothing\}$.

Note that $\mathcal{O}_i(A) \subseteq \mathcal{O}_j(A)$ for all $i \leqq j$. Since $\mathcal{O}_j(A) \subseteq \mathcal{O}(A)$ and $\mathcal{O}(A)$ is finite, there exists some i_0 such that $\mathcal{O}_i(A) = \mathcal{O}_{i_0}(A)$ for all $i \geqq i_0$. Clearly $\mathcal{O}(A)$ is effectively computable.

The following result, the proof of which is left to the reader, is easily established.

LEMMA 2.1. *Let A be an nfsa over Σ, $S = (\Sigma, \mathcal{P})$ be a* TOL *scheme, and $L \subseteq \Sigma^*$. Let $T_1, \ldots, T_n$, $n \geqq 0$, be a sequence of elements from $\mathcal{P}$, w_1 be a word in L, and w_2 be a word in $W(A)$. Then w_2 is in $T_n \cdots T_1(w_1)$ if and only if there exists B in $\mathcal{O}(A)$ such that w_1 is in $W(B)$ and $(B, T_1 \cdots T_n) \overset{*}{\vdash}_{C(S,L,A)} (A, \epsilon)$.*

Using Lemma 2.1 we have

LEMMA 2.2. *Each control language with a regular target is regular.*

Proof. Let $\mathscr{C}(S, L, W(A))$ be an arbitrary control language with a regular target. From Lemma 2.1, it follows that $\mathscr{C}(S, L, W(A)) = W(C(S, L, A))$. Thus $\mathscr{C}(S, L, W(A))$ is regular.

We now turn to a modified version of the converse of Lemma 2.2. Specifically, we show that each regular set may be relabeled so as to be the control language with respect to some TOL scheme, start language, and regular target language.

LEMMA 2.3. *For each regular set $L \subseteq \Sigma^*$, there exist a* TOL *scheme $S = (V, \mathcal{P})$, a one-to-one homomorphism h from*[4] *$\mathcal{P}^*$ onto Σ^*, and symbols x, y in V such that $h(\mathscr{C}(S, \{x\}, \{y\})) = L$.*

Proof. Since L is regular, there exists an nfsa $A = (K, \Sigma, \delta, p_0, F)$ such that $W(A) = L$. Without loss of generality, we may assume that F contains exactly one element, say $F = \{p_f\}$, and that $\delta(p, a) \neq \varnothing$ for each p in K and a in Σ. We may also assume that

(*) for all a and b in Σ, $a \neq b$, there exist p in K such that $\delta(p, a) \neq \delta(p, b)$.

Let $S = (K, \mathcal{P})$ be the TOL scheme in which $\mathcal{P} = \{T_a / a \text{ in } \Sigma\}$, where, for each a in Σ, T_a is defined by $T_a(q) = \delta(q, a)$ for every q in K. By (*), $T_a \neq T_b$ for $a \neq b$. Let h be the homomorphism from $\mathcal{P}^*$ onto Σ^* defined by $h(T_a) = a$ for each T_a in $\mathcal{P}$. Clearly h is one-to-one and $h(\mathscr{C}(S, \{p_0\}, \{p_f\})) = L$.

Since a one-to-one homomorphism maps a regular set onto a regular set, from Lemmas 2.2 and 2.3 there follows:

[4] Elements of $\mathcal{P}^*$ are here regarded as words over the alphabet $\mathcal{P}$.

THEOREM 2.1. *A language L is the image, under a one-to-one homomorphism, of a control language with a regular target if and only if L is a regular set.*

We are now ready to consider strong control languages with respect to a given start language and a given target language.

DEFINITION. Let $S = (\Sigma, \mathscr{P})$ be a TOL scheme, $L_1 \subseteq \Sigma^*$, and $L_2 \subseteq \Sigma^*$. The *strong* (S, L_1, L_2)*-control language* is the set

$$\hat{\mathscr{C}}(S, L_1, L_2) = \{u \text{ in } \mathscr{P}^*/u(L_1) \subseteq L_2\}.$$

A language L is called a *strong control language with a regular target* if $L = \hat{\mathscr{C}}(S, L_1, L_2)$ for some TOL scheme S, some language L_1, and some regular set L_2.

We shall show that the analog to Theorem 2.1 holds, so that the family of control languages with a regular target 'essentially' coincides with the family of strong control languages with a regular target.

LEMMA 2.4. *Each strong control language with a regular target is regular.*

Proof. Let $S = (\Sigma, \mathscr{P})$, $L_1 \subseteq \Sigma^*$, and $L_2 \subseteq \Sigma^*$, with L_2 regular. Consider $\hat{\mathscr{C}}(S, L_1, L_2)$. Let $L_2' = \Sigma^* - L_2$. Since L_2 is regular, so is L_2'. By Lemma 2.2, $\mathscr{C}(S, L_1, L_2)$ and $\mathscr{C}(S, L_1, L_2')$ are regular. It is easily seen that

$$\hat{\mathscr{C}}(S, L_1, L_2) = \mathscr{C}(S, L_1, L_2) - \mathscr{C}(S, L_1, L_2').$$

Since the difference of two regular sets is regular, $\hat{\mathscr{C}}(S, L_1, L_2)$ is regular. Hence Lemma 2.4 holds.

Remark. There exists an algorithm which, given a TOL scheme S, an nfsa A, and a language L such that it is decidable whether or not the intersection of L with an arbitrary regular set is empty, will produce nfsa B_1 and B_2 with multiple start states such that $W(B_1) = \mathscr{C}(S, L, W(A))$ and $W(B_2) = \hat{\mathscr{C}}(S, L, W(A))$. This result follows from Lemmas 2.2 and 2.4 and their proofs by noting that intersection with a regular set is used in effectively determining the set of start states in $C(S, L, A)$.

In general, Lemma 2.3 does not hold for the case of a strong control language. However, the following slight variation is valid.

LEMMA 2.5. *For each regular set $L \subseteq \Sigma^*$ there exists a* TOL *scheme $S = (V, \mathscr{P})$, a one-to-one homomorphism h from $\mathscr{P}^*$ onto Σ^*, a symbol x in V, and a subset V' of V such that $h(\hat{\mathscr{C}}(S, \{x\}, V')) = L$.*

Proof. The proof is a slight variation of that for Lemma 2.3. Let $A = (K, \Sigma, \delta, p_0, F)$ be an fsa such that $W(A) = L$. Without loss of generality we may assume that for all a and b in Σ, $a \neq b$, there exist p in K such that $\delta(p, a) \neq \delta(p, b)$. Let $S = (K, \mathscr{P})$ be the TOL scheme in which $\mathscr{P} = \{T_a/a \text{ in } \Sigma\}$, where, for each a in Σ, T_a is defined by $T_a(q) = \delta(q, a)$ for every q in K. Let h be the homomorphism from $\mathscr{P}^*$ onto Σ^* defined by $h(T_a) = a$ for each T_a in $\mathscr{P}$. Clearly h is one-to-one and $h(\hat{\mathscr{C}}(S, \{p_0\}, F)) = L$.

From Lemmas 2.4 and 2.5 we get

THEOREM 2.2. *A language L is the image, under a one-to-one homomorphism, of a strong control language with a regular target if and only if L is a regular set.*

From Theorems 2.1 and 2.2 we get

COROLLARY. *A language is the image, under a one-to-one homomorphism, of a control language with a regular target if and only if it is the image, under a one-to-one homomorphism, of a strong control language with a regular target.*

3. START SETS WITH REGULAR TARGETS

We now examine the nature of sets of the form $\{x/\mathscr{C}(x) \subseteq L_2\}$ for TOL schemes $S = (\Sigma, \mathscr{P})$, regular sets L_2, and control sets $\mathscr{C} \subseteq \mathscr{P}^*$. (As in Section 2, the key to the solution lies in studying the existential counterpart, i.e., in considering sets of the form $\{x/\mathscr{C}(x) \cap L_2 \neq \varnothing\}$.)

DEFINITION. Let $S = (\Sigma, \mathscr{P})$ be a TOL scheme, $\mathscr{C} \subseteq \mathscr{P}^*$, and $L_2 \subseteq \Sigma^*$. The [*strong*] $(S, \mathscr{C}, L_2)$-*full start language* is the set

$$\mathscr{S}(S, \mathscr{C}, L_2) = \{x \text{ in } \Sigma^*/\mathscr{C}(x) \cap L_2 \neq \varnothing\}$$

$$[\hat{\mathscr{S}}(S, \mathscr{C}, L_2) = \{x \text{ in } \Sigma^*/\mathscr{C}(x) \subseteq L_2\}].$$

A language L is called a [*strong*] *full start language with a regular target* if $L = \mathscr{S}(S, \mathscr{C}, L_2)$ for some S, $\mathscr{C}$ and L_2, with L_2 regular.

LEMMA 3.1. *Let* $S = (\Sigma, \mathscr{P})$ *be a* TOL *scheme,* $A = (K, \Sigma, \delta, p_0, F)$ *an* nfsa, *and*[5] $C(S, \Sigma^*, A) = (\mathfrak{A}(A), \mathscr{P}, \delta', Q_0, \{A\})$. *Let* $\mathscr{C} \subseteq \mathscr{P}^*$ *and* $Q_{\mathscr{C}} =$

[5] The nfsa $C(C, \Sigma^*, A)$ with multiple start states is defined in Section 2. Here $Q_0 = \{B \text{ in } Q/W(B) \neq \varnothing\}$.

$\{B$ in $\mathcal{O}\mathcal{l}(A)/(B, w) \vdash^{*}_{C(S,\Sigma^*,A)} (A, \epsilon)$ for some w in $\mathscr{C}\}$. Then a word x is in $\mathscr{S}(S, \mathscr{C}, W(A))$ if and only if x is in $W(B)$ for some B in $Q_{\mathscr{C}}$.

Proof. This follows directly from the construction of $C(S, \Sigma^*, A)$.

LEMMA 3.2. (a) *Each full start language with a regular target is a regular set.*

(b) *Each strong full start language with a regular target is a regular set.*

Proof. (a) Let $\mathscr{S}(S, \mathscr{C}, L_2)$ be a full start language with a regular target. Let $S = (\Sigma, \mathscr{P})$, A an fsa such that $L_2 = W(A)$, $C(S, \Sigma^*, A) = (\mathcal{O}\mathcal{l}(A), \mathscr{P}, \delta', Q_0, \{A\})$, and $Q_{\mathscr{C}} = \{B$ in $Q/(B, w) \vdash^{*}_{C(S,\Sigma^*,A)} (A, \epsilon)$ for some w in $\mathscr{C}\}$. By Lemma 3.1, x is in $\mathscr{S}(S, \mathscr{C}, L_2)$ if and only if x is in B for some B in $Q_{\mathscr{C}}$. Thus $\mathscr{S}(S, \mathscr{C}, L_2) = \bigcup_{B \text{ in } Q_{\mathscr{C}}} W(B)$, i.e., $\mathscr{S}(S, \mathscr{C}, L_2)$ is a finite union of regular sets. Thus $\mathscr{S}(S, \mathscr{C}, L_2)$ is regular.

(b) Let $\hat{\mathscr{S}}(S, \mathscr{C}, L_2)$ be a strong full start language, with $S = (\Sigma, \mathscr{P})$ and L_2 regular. Let $L_2' = \Sigma^* - L_2$. Since

$$\hat{\mathscr{S}}(S, \mathscr{C}, L_2) = \mathscr{S}(S, \mathscr{C}, L_2) - \mathscr{S}(S, \mathscr{C}, L_2'),$$

$\hat{\mathscr{S}}(S, \mathscr{C}, L_2)$ is regular by (a) and the fact that the difference of regular sets is regular.

Remark. From the proof of Lemma 3.2, using the fact that B is in $Q_{\mathscr{C}}$ if and only if $\mathscr{C} \cap W(\Delta(B)) \neq \varnothing$, where $\Delta(B)$ is the nfsa $(\mathcal{O}\mathcal{l}(A), \mathscr{P}, \delta', B, \{A\})$ with $C(S, \Sigma^*, A) = (\mathcal{O}\mathcal{l}(A), \mathscr{P}, \delta', Q_0, \{A\})$, we get the following: There exists an algorithm which, given a TOL scheme $S = (\Sigma, \mathscr{P})$, an nfsa A, and a language $\mathscr{C} \subseteq \mathscr{P}^*$ such that it is decidable whether or not the intersection of $\mathscr{C}$ with an arbitrary regular set is empty, will produce fsa B_1 and B_2 such that $W(B_1) = \mathscr{S}(S, \mathscr{C}, W(A))$ and $W(B_2) = \hat{\mathscr{S}}(S, \mathscr{C}, W(A))$.

The converse to Lemma 3.2 is trivially true, as the next lemma shows.

LEMMA 3.3. *Each regular set is a [strong] full start language with a regular target.*

Proof. Let $L \subseteq \Sigma^*$ be a regular set. Let $S = (\Sigma, \mathscr{P})$ be the TOL scheme in which $\mathscr{P} = \{T\}$, where $T = \{(a, a)/a$ in $\Sigma\}$ and $\mathscr{C} = \{T\}$. Clearly $\mathscr{S}(S, \mathscr{C}, L) = \hat{\mathscr{S}}(S, \mathscr{C}, L) = L$. Since L is regular, Lemma 3.3 holds.

Combining Lemmas 3.2 and 3.3 we have

THEOREM 3.1. *A language L is a [strong] full start language with a regular target if and only if L is regular.*

4. Target Sets with Regular Start and Regular Control

In this section we consider the form of $\mathscr{C}(L)$ for L a regular start set and C a regular control set. We shall show that the family of all such $\mathscr{C}(L)$ is the family of all ETOL languages considered in Rozenberg (1973b).

Definition. Let $S = (\Sigma, \mathscr{P})$ be a TOL scheme, $\mathscr{C} \subseteq \mathscr{P}^*$, and $L_1 \subseteq \Sigma^*$. Then $\mathscr{C}(L_1)$ is said to be the $(S, L_1, \mathscr{C})$-*target language*. A language L is called a *target language with regular start and regular control* if L is a $(S, L_1, \mathscr{C})$-target language for some S, some regular set L_1, and some regular control set $\mathscr{C}$.

We now recall (see Rozenberg, 1973b) the notions of an ETOL system and language.

Definition. An ETOL *system* is a 4-tuple $G = (V, \mathscr{P}, \omega, \Sigma)$, where $(V, \mathscr{P})$ is a TOL scheme, ω is in V^+, and $\Sigma \subseteq V$. If $V = \Sigma$ then G is called a TOL system. Let $L(G) = \{x \text{ in } \Sigma^*/\omega \overset{*}{\Rightarrow}_{(V,\mathscr{P})} x\}$. A language L is called an ETOL *language* (TOL *language*) if $L = L(G)$ for some ETOL (TOL) system G.

Lemma 4.1. *Each target language with regular start and regular control is an* ETOL *language.*

Proof. Let $L = \mathscr{C}(L_1)$, where $S = (\Sigma, \mathscr{P})$, $L_1 \subseteq \Sigma^*$ is regular, and $\mathscr{C} \subseteq \mathscr{P}^*$ is regular. Since each regular set is an ETOL language (Rozenberg, 1973b) and L_1 and $\mathscr{C}$ are regular, $L_1 = L(G_1)$ and $\mathscr{C} = W(A)$ for some ETOL system $G_1 = (V_1, \mathscr{P}_1, \omega_1, \Sigma)$ and fsa $A = (K, \mathscr{P}, \delta, p_0, F)$. Clearly we may assume that V_1, $\mathscr{P}_1$, $\mathscr{P}$, and K are pairwise disjoint. Let X and Y be new symbols and for each element x in V_1 let x' be a new symbol. Let h be the homomorphism on V_1^* defined by $h(x) = x'$ for each element x in V_1. Let $U = h(V_1) \cup \Sigma \cup K \cup \{X, Y\}$. For each table T in $\mathscr{P}_1$ let

$$T' = \{(h(u), h(v))/(u, v) \text{ in } T\} \cup \{(p_0, p_0), (X, X), (Y, Y)\} \cup \{(x, X)/x \text{ in } \Sigma \cup (K - \{p_0\})\}.$$

For each table T in $\mathscr{P}$ let

$$T' = T \cup \{(p, q)/p, q \text{ in } K \text{ and } \delta(p, T) = q\} \cup \{(x, x)/x \text{ in } U - (\Sigma \cup K)\}.$$

Let

$$T_t = \{(x', x)/x \text{ in } \Sigma\} \cup \{(p_0, p_0)\} \cup \{(Y, \epsilon), (y, X)/y \text{ in } U - (\{p_0, Y\} \cup \{x'/x \text{ in } \Sigma\})\}$$

and

$$T_f = \{(p, \epsilon)/p \text{ in } F\} \cup \{(x, x)/x \text{ in } U - F\}.$$

Now let H be the ETOL system $(U, \mathscr{P}_2, Yp_0h(\omega_1), \Sigma)$, where

$$\mathscr{P}_2 = \{T_t, T_f\} \cup \{T'/T \text{ in } \mathscr{P}_1 \cup \mathscr{P}\}.$$

It is easily seen that $L = L(H)$, whence the lemma. [Intuitively, note that X can never be ultimately converted to a symbol in Σ. Now if some T', for T in $\mathscr{P}$, is applied before T_t, then Y is converted to X. If some T', T in $\mathscr{P}_1$, is applied after T_t, then each symbol in Σ is converted to X. Thus, in order to get a word w in $L(H)$, it is necessary and sufficient that tables in $\{T'/T \text{ in } \mathscr{P}_1\}$ are first applied until a word in $Yp_0h(\Sigma^*)$ is obtained, followed by T_t, followed by tables in $\{T'/T \text{ in } \mathscr{P}\}$ until p_0 goes to an accepting state, followed by T_f. This holds if and only if w is in $\mathscr{C}(L_1)$.]

LEMMA 4.2. *Each* ETOL *language is a target language with regular start and regular control.*

Proof. Let L be an ETOL language. It is known (Ehrenfeucht and Rozenberg, 1974) that each ETOL language is the image under a letter-to-letter homomorphism of some TOL language. Thus $L = h(L(G))$ for some TOL system $G = (\Sigma, \mathscr{P}, \omega, \Sigma)$ and some letter-to-letter homomorphism h on Σ^*. Without loss of generality we may assume that $\Sigma \cap h(\Sigma) = \varnothing$. Let

$$T_h = \{(a, h(a))/a \text{ in } \Sigma\} \cup \{(x, x)/x \text{ in } h(\Sigma)\}.$$

For each T in P, let $T' = T \cup \{(x, x)/x \text{ in } h(\Sigma)\}$ and let $\mathscr{P}' = \{T'/T \text{ in } \mathscr{P}\}$. Let S be the TOL scheme $(\Sigma \cup h(\Sigma), \mathscr{P}' \cup \{T_h\})$. Let p_0, p_1, and p_2 be new symbols and A the fsa $(\{p_0, p_1, p_2\}, \mathscr{P}' \cup \{T_h\}, \delta, p_0, \{p_1\})$, where δ is defined by $\delta(p_0, x) = p_0$ for each element x in $\mathscr{P}'$, $\delta(p_0, T_h) = p_1$, and $\delta(p_1, y) = p_2 = \delta(p_2, y)$ for each element y in $\mathscr{P}' \cup \{T_h\}$. It is easily seen that $L = \mathscr{C}(\{\omega\})$, where $\mathscr{C} = W(A)$. Since $\{\omega\}$ and $\mathscr{C}$ are regular, Lemma 4.2 holds.

Combining Lemmas 4.1 and 4.2 we get

THEOREM 4.1. *A language L is a target language with regular start and regular control if and only if it is an* ETOL *language.*

The above theorem permits us to obtain some information about target languages with regular start and regular control by making use of the literature on ETOL systems. For example, the class of ETOL languages is properly contained in the class of context-free ϵ-free programmed grammars of Rozenkrantz (1969) which, in turn, is known (Rozenberg, 1973b) to be included in the class of extended context-sensitive languages, i.e., the family $\{L, L \cup \{\epsilon\}/L$ context sensitive$\}$.

Remark. There exists an algorithm which, given a TOL scheme S, an fsa A, and an fsa B, produces an ETOL system G such that $L(G) = \mathscr{C}(W(A))$, where $\mathscr{C} = W(B)$. Also, there exists an algorithm which, given an ETOL system G, produces a TOL scheme S, a singleton $\{x\}$, and an fsa A such that $\mathscr{C}(\{x\}) = L(G)$, where $\mathscr{C} = W(A)$.

5. Universality of TOL Schemes

In this section we view each table T in $\mathscr{P}$, thus each element in $\mathscr{P}^*$, as a finite nonempty substitution on Σ^*. (In case $(\Sigma, \mathscr{P})$ is deterministic, i.e., each $T(x)$ is a set of exactly one element for each T in $\mathscr{P}$ and x in Σ, each table, thus each control word, is regarded as a homomorphism.) The questions considered here are whether there exists some $\mathscr{P}$ such that $\mathscr{P}^*$ is the set of all finite nonempty substitutions on Σ^* and whether there exists some $\mathscr{P}$ such that $\mathscr{P}^*$ is the set of all homomorphisms on Σ^*. The answer is no for both questions as is now shown.

Notation. For each finite alphabet Σ let $\mathscr{H}(\Sigma)$ be the set of all homomorphisms from Σ^* into Σ^*, and $\mathscr{S}(\Sigma)$ the set of all substitutions from Σ^* into finite nonempty subsets of Σ^*.

THEOREM 5.1. *For each finite alphabet Σ there is no* TOL *scheme $S = (\Sigma, \mathscr{P})$ such that $\mathscr{H}(\Sigma) \subseteq \mathscr{P}^*(\mathscr{S}(\Sigma) \subseteq \mathscr{P}^*)$.*

Proof. Let $S = (\Sigma, \mathscr{P})$ be a TOL scheme. It suffices to show there exists a homomorphism from Σ^* into Σ^* which is not in $\mathscr{P}^*$. To this end let[6] $t = \max\{|\alpha|/(a, \alpha)$ in T for some a in Σ, T in $\mathscr{P}\}$ and let $p > t$ be some fixed prime number.

First suppose Σ has just one element, say $\Sigma = \{a\}$. Let h be the homomorphism on a^* defined by $h(a) = \{a^p\}$. Suppose there exist $n \geqq 0$ and $T_1, \ldots, T_n$ in $\mathscr{P}$ such that $T_1 \cdots T_n = h$. Since $h(a) = \{a^p\}$, for no i does $T_i(a)$ contain ϵ. Thus $T_i(a)$ contains a word a^{s_i}, $s_i \geqq 1$, for each i, $1 \leqq i \leqq n$. Then $T_n \cdots T_1(a) = \{a^{s_1 \cdots s_n}\} = \{a^p\}$. Since $s_i < p$ for each i, p is not prime, a contradiction.

[6] For each word w, $|w|$ denotes its length.

Now suppose Σ has at least two elements, say $\Sigma = \{a_1, \ldots, a_m\}$, $m \geqq 2$. Let h be the homomorphism on Σ^* defined by $h(a_1) = \{a_1{}^p\}$ and $h(a_i) = \{a_i\}$ for all i, $2 \leqq i \leqq m$. Suppose there exist $n \geqq 0$ and $T_1, \ldots, T_n$ in $\mathscr{P}$ such that $h = T_1 \cdots T_n$. Since $p > t$, $n > 0$ and $h \neq T_1$. Thus $n \geqq 2$. Let $\alpha_{11}, \ldots, \alpha_{1n}, \alpha_{21}, \ldots, \alpha_{2n}, \ldots, \alpha_{m1}, \ldots, \alpha_{mn}$ be a sequence of words such that $\alpha_{l1} = a_l$, $\alpha_{l(j+1)}$ is in $T_j(\alpha_{lj})$ for each l, $1 \leqq l \leqq m$, and each j, $1 \leqq j < n$, $a_1{}^p$ is in $T_n(\alpha_{1n})$, and a_l is in $T_n(\alpha_{ln})$ for all l, $2 \leqq l \leqq m$. Since $T_1 \cdots T_n = h$, such a sequence exists. Call a symbol a in α_{lj} *productive* (*in* α_{lj}) if ϵ is not in $T_n \cdots T_j(a)$. The following hold:

(1) Each α_{lj} contains a productive symbol. [Otherwise ϵ is in $T_n \cdots T_j(\alpha)$, so that ϵ is in $T_n \cdots T_1(a_l) = h(a_l)$, a contradiction.]

(2) There is no symbol which is simultaneously productive in both α_{lj} and α_{kj} for some $k \neq l$ and some j. [For otherwise,[7] $\varnothing = \operatorname{Min}(h(a_l)) \cap \operatorname{Min}(h(a_k)) \supseteq \operatorname{Min}(T_n \cdots T_j(\alpha_{lj})) \cap \operatorname{Min}(T_n \cdots T_j(\alpha_{kj})) \neq \varnothing$.]

(3) If a symbol a is productive in α_{lj} and occurs in α_{kj}, then a is productive in α_{kj}, so that $k = l$.

Since there are exactly m symbols in Σ, it follows from (1)–(3) that for each j, each symbol in Σ is productive in exactly one word α_{lj} and that α_{lj} is a power of that symbol. This implies that

(4) $\bigcup_{w \text{ in } T_n \cdots T_j(a_r)} \operatorname{Min}(w)$ is a set containing exactly one symbol, and that $\bigcup_{w \text{ in } T_n \cdots T_j(a_r)} \operatorname{Min}(w) \neq \bigcup_{w \text{ in } T_n \cdots T_j(a_s)} \operatorname{Min}(w)$ for $r \neq s$.

Suppose that for some j and l, $T_j(\alpha_{lj})$ contains at least two words, say w_1 and w_2. Clearly $j < n$ since $T_n(\alpha_{ln}) = h(a_l)$ contains exactly one word. From (4), it follows that w_1 and w_2 are powers of the same symbol, say $w_1 = a_k^{t_1}$ and $w_2 = a_k^{t_2}$. Since $T_n \cdots T_{j+1}(a_k)$ is nonempty and does not contain ϵ, it contains a non-ϵ word y. Then y^{t_1} and y^{t_2} are two distinct words in $T_n \cdots T_j(\alpha_{lj}) = h(a_j)$, a contradiction. Thus $T_j(\alpha_{lj})$ contains exactly one word for each j and l. Therefore $T_1(a_1) = \{a_{i_1}^{v_1}\}$, $T_2T_1(a_1) = \{a_{i_2}^{v_2}\}, \ldots$, $T_{n-1} \cdots T_1(a_1) = \{a_{i_{n-1}}^{v_{n-1}}\}$, and $T_n \cdots T_1(a_1) = \{a_{i_n}^{p}\}$, $i_n = 1$, for some $i_1, \ldots, i_{n-1}$ in $\{1, \ldots, m\}$ and nondecreasing integers $v_1, \ldots, v_{n-1}$. Hence, for each j in $\{1, \ldots, n-1\}$, $T_{j+1}(a_{i_j}) = \{a_{i_{j+1}}^{s_{j+1}}\}$ for some $s_{j+1} \geqq 1$. By definition of p, $v_1 < p$ and $s_{j+1} < p$ for each j. Now

$$a_1^{v_1 s_2 \cdots s_n} = T_n \cdots T_1(a_1) = \{a_1{}^p\}.$$

Thus $v_1 s_2 \cdots s_n = p$. Since $v_1 < p$ and $s_{j+1} < p$ for each j, p cannot be prime, a contradiction. Thus $T_1, \ldots, T_n$ do not exist and the theorem is proved.

[7] For each word w, $\operatorname{Min}(w)$ is the smallest set of symbols Σ_1 such that w is in $\Sigma_1{}^*$.

Theorem 5.1 may be rephrased in terms of semigroups as follows: "*For no* Σ *is either* $\mathscr{H}(\Sigma)$ *or* $\mathscr{S}(\Sigma)$ *finitely generated.*"

Although Theorem 5.1 says that for a given alphabet Σ there is no TOL scheme (over Σ) whose control set contains $\mathscr{H}(\Sigma)(\mathscr{S}(\Sigma))$ as a subset, there is a way to represent $\mathscr{H}(\Sigma)(\mathscr{S}(\Sigma))$ by a TOL scheme. The method consists of enlarging Σ and then considering the restriction to Σ^* of TOL schemes over the enlarged alphabet. Specifically, we have the following.

THEOREM 5.2. *For each alphabet* Σ,

(a) *there exists an alphabet* $\bar{\Sigma}$ *containing* Σ *and a* TOL *scheme* $S = (\bar{\Sigma}, \mathscr{P})$ *such that*[8]

$$\{u \mid \Sigma^*/u \text{ in } \mathscr{P}^*,\ u(x) \text{ in } \Sigma^* \text{ for each } x \text{ in } \Sigma^*\} = \mathscr{H}(\Sigma).$$

(b) *there exists an alphabet* $\bar{\Sigma}$ *containing* Σ *and a* TOL *scheme* $S = (\bar{\Sigma}, \mathscr{P})$ *such that*[9]

$$\{u \cap (\Sigma^* \times \Sigma^*)/u \text{ in } \mathscr{P}^*,\ u(x) \cap \Sigma^* \neq \varnothing \text{ for each } x \text{ in } \Sigma^*\} = \mathscr{S}(\Sigma).$$

Proof. Let $\Sigma = \{a_1, ..., a_m\}$ and for each a_i in Σ let b_i and c_i be new symbols.

(a) *Let* $S = (\bar{\Sigma}, \mathscr{P})$ be the TOL scheme where $\bar{\Sigma} = \{a_i, b_i/1 \leqq i \leqq m\}$ and $\mathscr{P}$ consists of the following tables:

(1) $T_0 = \{(a_i, b_i), (b_i, b_i)/1 \leqq i \leqq m\}$.

(2) $T_i = \{(b_i, \epsilon), (x, x)/x \text{ in } \bar{\Sigma} - \{b_i\}\}$, for each i, $1 \leqq i \leqq m$.

(3) $T_{i,j} = \{(b_i, a_j b_i), (x, x)/x \text{ in } \bar{\Sigma} - \{b_i\}\}$, for all i, j, $1 \leqq i, j \leqq m$.

Let $\Omega = \{u \mid \Sigma^*/u \text{ in } \mathscr{P}^*,\ u(x) \text{ in } \Sigma^* \text{ for each } x \text{ in } \Sigma^*\}$. Obviously each element in Ω is in $\mathscr{H}(\Sigma)$. To prove that $\Omega = \mathscr{H}(\Sigma)$, it thus suffices to show that each element in $\mathscr{H}(\Sigma)$ is in Ω. Therefore let h be in $\mathscr{H}(\Sigma)$, i.e., h is a homomorphism on Σ^*, with $h(a)$ in Σ^* for each a in Σ. For each i, $1 \leqq i \leqq m$, let $h(a_i) = a_{f(i,1)} \cdots a_{f(i,t(i))}$, with $t(i) \geqq 0$ and each $a_{f(i,j)}$ in Σ; and let $\alpha_i = T_{i,f(i,1)} T_{i,f(i,2)} \cdots T_{i,f(i,t(i))} T_i$. Let $u_h = T_0 \alpha_1 \cdots \alpha_m$. It is easily seen that $u_h \mid \Sigma^*$ is in Ω and that $u_h \mid \Sigma^* = h$. (T_0 converts each a_i into b_i. Each α_i then changes b_i into $h(a_i)$.)

[8] For each u in $\mathscr{P}^*$, $u \mid \Sigma^*$ is the function which is the restriction of the domain of u to Σ^*, i.e., is the set of all pairs (x, y) in u such that x is in Σ^*.

[9] Here an element u of $\mathscr{P}^*$ is regarded as the set of all pairs (x, y) in $\bar{\Sigma}^* \times \bar{\Sigma}^*$ such that y is in $u(x)$.

(b) Let $S = (\bar{\Sigma}, \mathscr{P})$ be the TOL scheme where $\bar{\Sigma} = \{a_i, b_i, c_i/1 \leqq i \leqq m\}$ and $\mathscr{P}$ consists of the following tables:

(1) $T_0 = \{(a_i, b_i), (b_i, b_i), (c_i, c_i)/1 \leqq i \leqq m\}$.

(2) $T_i = \{(b_i, \epsilon), (x, x)/x \text{ in } \bar{\Sigma} - \{b_i\}\}$, for each i, $1 \leqq i \leqq m$.

(3) $\bar{T}_i = \{(b_i, c_i), (x, x)/x \text{ in } \bar{\Sigma}\}$, for each i, $1 \leqq i \leqq m$.

(4) $\hat{T}_i = \{(c_i, b_i), (x, x)/x \text{ in } \bar{\Sigma} - \{c_i\}\}$, for each i, $1 \leqq i \leqq m$.

(5) $T_{i,j} = \{(b_i, a_j b_i), (x, x)/x \text{ in } \bar{\Sigma} - \{b_i\}\}$ for all i, j, $1 \leqq i, j \leqq m$.

Let $\Omega = \{u \cap (\Sigma^* \times \Sigma^*)/u \text{ in } \mathscr{P}^*, u(x) \cap \Sigma^* \neq \varnothing \text{ for each } x \text{ in } \Sigma^*\}$, with each element in Ω regarded as a function over Σ^*. Obviously each element in Ω is in $\mathscr{S}(\Sigma)$. To prove that $\Omega = \mathscr{S}(\Sigma)$, it thus suffices to show that each element in $\mathscr{S}(\Sigma)$ is in Ω. Therefore let τ be in $\mathscr{S}(\Sigma)$, i.e., let τ be a substitution on Σ^*, with $\tau(a)$ a finite nonempty subset of Σ^* for each a in Σ. Then for each i, $1 \leqq i \leqq m$, $\tau(a_i) = \{w_{i1}, \ldots, w_{ir(i)}\}$, where $r(i) \geqq 1$ and each w_{ik} is in Σ^*. For all i and k, $1 \leqq i \leqq m$ and $1 \leqq k \leqq r(i)$, let $w_{ik} = a_{f(i,k,1)} \cdots a_{f(i,k,t(i,k))}$, with each $a_{(fi,k,l)}$ in Σ, and

$$\alpha_{ik} = \bar{T}_i T_{i,f(i,k,1)} \cdots T_{i,f(i,k,t(i,k))} T_i \hat{T}_i .$$

Let

$$u_\tau = T_0 \alpha_{11} \cdots \alpha_{1r(1)} \alpha_{21} \cdots \alpha_{2r(2)} \cdots \alpha_{m1} \cdots \alpha_{mr(m)} .$$

Then u_τ is in $\mathscr{P}^*$ and, as is easily seen, $u_\tau \cap (\Sigma^* \times \Sigma^*) = \tau$. (Initially, T_0 converts each a_i into b_i. Each α_{ik} converts b_i into the set $\{b_i, w_{ik}\}$, using c_i to temporarily store b_i, so that $\alpha_{i1} \cdots \alpha_{ir(i)}$ changes b_i into the set $\{b_i, w_{i1}, \ldots, w_{ir(i)}\}$. Intersecting u_τ with $\Sigma^* \times \Sigma^*$ deletes the elements (a_i, b_i), $1 \leqq i \leqq m$, as well as restricting the domain to Σ^*.) Since $u_\tau(x) \cap \bar{\Sigma}^* = \tau(x)$ for each x in Σ^*, τ is in Ω and the proof of (b) is complete.

Remark. The proof of (b) actually shows that $\{u \cap (\Sigma^* \times \Sigma^*/u \text{ in } \mathscr{P}^*\} = \mathscr{S}'(\Sigma)$, where $\mathscr{S}'(\Sigma)$ is the set of all finite, possibly empty, substitutions of Σ^*.

RECEIVED: August 27, 1973; REVISED: June 21, 1974

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