

AN EXACTLY SOLVABLE MODEL FOR BROWNIAN MOTION

II. DERIVATION OF THE FOKKER-PLANCK EQUATION AND THE MASTER EQUATION

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Synopsis

As in a previous paper¹⁾ an elastically bound particle, linearly coupled with a bath of small oscillators, is considered. At the initial time the bath is chosen in thermal equilibrium with temperature T . In the classical case the distribution function for the momentum and displacement of the particle is calculated exactly. It turns out that under the same assumptions needed for the derivation of the Langevin equation¹⁾, this distribution function satisfies a Fokker-Planck equation for times large compared to a transient time τ_t .

In the quantummechanical case the probability to find the particle at time t in an eigenstate of its unperturbed Hamiltonian is also calculated exactly. If in addition to the above mentioned assumptions the phases of the initial state of the particle are chosen randomly, this probability satisfies a master equation for times large compared to τ_t and $\tau_q = \hbar/kT$, the quantumstatistical transient time.

The main point in this treatment is that perturbation theory is not applied and that therefore averaging over the bath of small oscillators and over the phases of the state of the particle is only needed at the initial time.

1. *Introduction.* In a previous paper¹⁾ (quoted in the following as I) a model was treated which consists of a harmonic oscillator, linearly coupled with a bath of small oscillators. It was shown that the motion of that oscillator was governed by a Langevin equation, provided some assumptions and restrictions about the interaction and the time region were made. In the present paper we will calculate exactly the distribution function of the momentum and coordinate of the oscillator, assuming that the bath of small oscillators is in thermal equilibrium at the initial time (sect. 2). By means of this distribution function we obtain exact expressions for the higher moments of the momentum and coordinate (sect. 3). Furthermore, it will be shown that this distribution function satisfies a Fokker-Planck equation if we make the same approximations and restrictions as in I sect. 11.

If one writes the Langevin equation of the central oscillator as two first

order differential equations

$$\begin{aligned}\dot{Q} &= P, \\ \dot{P} &= -\Omega_1^2 Q - 2\Gamma P + F(t),\end{aligned}\tag{1}$$

it is clear that the corresponding Markoff process is two-dimensional. Therefore we have to look for a distribution function of the two variables P and Q , $f(P, Q)$.

Since in quantum mechanics P and Q are canonically conjugate variables one cannot choose for P and Q independent initial values. Therefore one has to take an appropriate initial wave packet for the central oscillator. This has been discussed by Braun Guitler²⁾ for his model. We will restrict ourselves with respect to the F.-P. equation to the classical case (sect. 4).

In the quantummechanical case, however, it is more natural to look for a distribution function of the eigenstates of the central oscillator. This will be calculated exactly in sect. 6. Making the same additional restrictions as in I sect. 11 one deduces that this distribution function satisfies a master equation. We will find again a quantumstatistical transient time. Furthermore, this master equation reduces in the classical limit to the F.-P. equation for the distribution function of the energy of the central oscillator (sect. 7).

As known from conventional theories (cf. e.g. Uhlenbeck and Ornstein³⁾, and Kramers⁴⁾) one can deduce the F.-P. equation from the Langevin equation and vice versa. In the present model the main point is that both these equations are derived by first deducing exact expressions for the equation of motion and the distribution function and finally making some approximations and restrictions to obtain the familiar equations. Similarly, in the derivation of the master equation we don't apply perturbation theory, but make only approximations in the exact expression for the probability of state.

2. Distribution function. First we consider again a system consisting of N harmonic oscillators of which the Hamiltonian is given by eq. (I.1). We choose the initial momentum and coordinate of the central oscillator fixed, P_0 and Q_0 , and the bath of small oscillators at $t = 0$ in thermal equilibrium with temperature T . Accordingly, the initial distribution of the whole system is given by

$$\begin{aligned}f(P_0^0, Q_0^0; \{p_n, q_n\}; 0) &= \\ &= \mathfrak{N} \delta(P_0^0 - P_0) \delta(Q_0^0 - Q_0) \prod_n \exp \left[-\frac{(p_n^2 + \omega_n^2 q_n^2)}{2kT} \right],\end{aligned}\tag{2}$$

where \mathfrak{N} represents the normalization constant

$$\mathfrak{N} = \prod_n \frac{\omega_n}{2kT}. \quad (3)$$

Now the conditional probability function that the momentum and coordinate of the central oscillator have the values P and Q at time t with fixed values P_0 and Q_0 at time zero, is given by

$$\begin{aligned} f(P, Q, t | P_0, Q_0) &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dP_0^0 dQ_0^0 dp_1 dq_1 \dots dp_n dq_n \times \\ &\quad \times \delta(P - AP_0^0 - \dot{A}Q_0^0 - \sum_n \{\dot{A}_n p_n + \ddot{A}_n q_n\}) \times \\ &\quad \times \delta(Q - AP_0^0 - \dot{A}Q_0^0 - \sum_n \{A_n p_n + \dot{A}_n q_n\}) \times \\ &\quad \times f(P_0^0, Q_0^0; \{p_n, q_n\}; 0). \end{aligned} \quad (4)$$

Here δ represents the delta function. Furthermore, we used the solutions of the equations of motion (I.7). After inserting eq. (2) into eq. (4) one integrates over P_0^0 and Q_0^0 . Subsequently we replace the remaining delta functions by their Fourier transforms; e.g.

$$\begin{aligned} \delta(P - AP_0 - \dot{A}Q_0 - \sum_n \{\dot{A}_n p_n + \ddot{A}_n q_n\}) &= \\ &= (2\pi)^{-1} \int_{-\infty}^{+\infty} dr \exp[ir(P - AP_0 - \dot{A}Q_0)] \times \\ &\quad \times \exp[-ir \sum_n \{\dot{A}_n p_n + \ddot{A}_n q_n\}]. \end{aligned} \quad (5)$$

Now we define

$$\tilde{P} = P - AP_0 - \dot{A}Q_0, \quad \tilde{Q} = Q - AP_0 - \dot{A}Q_0, \quad (6)$$

$$\begin{aligned} \sigma_1 &= \sum_n (A_n^2 + \omega_n^{-2} \dot{A}_n^2), \quad \sigma_2 = \sum_n (\dot{A}_n^2 + \omega_n^{-2} \ddot{A}_n^2), \\ \sigma_3 &= \frac{1}{2} \dot{\sigma}_1. \end{aligned} \quad (7)$$

Straightforward calculation yields for the conditional probability

$$\begin{aligned} f(P, Q, t | P_0, Q_0) &= \frac{1}{2\pi kT(\sigma_1 \sigma_2 - \sigma_3^2)^{\frac{1}{2}}} \cdot \\ &\quad \cdot \exp \left[-\frac{1}{2kT(\sigma_1 \sigma_2 - \sigma_3^2)} (\sigma_1 \tilde{P}^2 + 2\sigma_3 \tilde{P} \tilde{Q} + \sigma_2 \tilde{Q}^2) \right]. \end{aligned} \quad (8)$$

This represents a Gaussian distribution of P and Q . In app. A.e. it is shown that

$$\sigma_1(t) = \Omega_1^{-2} - A^2(t) - \Omega_1^2 \dot{A}^2(t), \quad (9)$$

$$\sigma_2(t) = 1 - \dot{A}^2(t) - \Omega_1^2 A^2(t). \quad (10)$$

Here Ω_1 is defined by eq. (A.5) and $\hat{A}(t)$ by (cf. eqs. (I.9))

$$\hat{A}(t) = - \sum_{\nu} X_{0\nu}^2 s_{\nu}^{-2} \cos s_{\nu} t. \quad (11)$$

For the existence of eq. (8) it is needed that

$$\sigma_1 \sigma_2 - \sigma_3^2 > 0, \quad (12)$$

for $t > 0$. This is shown in app. A.f.

Eq. (8) is an exact expression. Now we make the same assumptions as in I sect. 5; i.e.

1. the frequencies of the bath of small oscillators are dense ($N \rightarrow \infty$), and
2. the strength function satisfies the Hölder condition and varies slowly in the neighbourhood of Ω_0 .

In that case \hat{A} , A and \dot{A} contain a factor $\exp[-2\Gamma t]$ for $t \gg \tau_t$ (cf. eq. (I.33)). Substituting these expressions for \hat{A} , A and \dot{A} into the eqs. (9) and (10) one obtains to first order in Γ/Ω for the distribution function,

$$f(P, Q, t | P_0, Q_0) = \frac{\Omega}{2\pi kT(1 - e^{-2\Gamma t})^{\frac{1}{2}}} \times \\ \times \exp \left[- \frac{[1 - e^{-2\Gamma t} (1 + \Gamma\Omega^{-1} \sin 2\Omega t)] \hat{P}^2 + 4\Gamma\hat{P}\hat{Q}e^{-2\Gamma t} \sin^2 \Omega t + \Omega^2 [1 - e^{-2\Gamma t} (1 - \Gamma\Omega^{-1} \sin 2\Omega t)] \hat{Q}^2}{2kT(1 - e^{-2\Gamma t})} \right]. \quad (13)$$

Clearly, one obtains for $t \rightarrow \infty$ the Maxwell-Boltzmann distribution of the harmonic oscillator. If one takes into account higher order terms in Γ/Ω one finds a more complicated expression which, however, does not exhibit an essential different behaviour. In that case $f(P, Q, t | P_0, Q_0)$ tends for $t \rightarrow \infty$ to the Maxwell-Boltzmann distribution of a harmonic oscillator with the shifted frequency Ω_1 (cf. eq. (I.31)), where

$$\Omega_1^2 = \Omega_0^2 - \int_0^{\infty} \omega^{-2} \gamma(\omega) d\omega. \quad (14)$$

If one chooses at $t = 0$ an arbitrary distribution function, $g(P_0, Q_0)$, for the central oscillator, the distribution at time t is given by

$$f(P, Q, t | g) = \int dP_0 dQ_0 f(P, Q, t | P_0, Q_0) g(P_0, Q_0), \quad (15)$$

which tends to the same equilibrium distribution as $f(P, Q, t | P_0, Q_0)$.

In the present theory the bath of small oscillators is chosen at $t = 0$ in thermal equilibrium with temperature T , corresponding to a canonical ensemble. Obviously, one finds the same results if one chooses a micro-canonical ensemble with energy E . Then in the results kT must be replaced by $\varepsilon = E/N$.

3. *Averages and fluctuations.* From the distribution function (8) follow immediately the averages and fluctuations of P and Q ; here we restrict ourselves to the case that at $t = 0$ the distribution function is a delta function. One verifies readily (cf. also eq. (I.36))

$$\langle P \rangle = \dot{A}P_0 + \ddot{A}Q_0, \quad \langle Q \rangle = AP_0 + \dot{A}Q_0, \quad (16)$$

$$\langle Q^2 \rangle - \langle Q \rangle^2 = kT\sigma_1, \quad \langle P^2 \rangle - \langle P \rangle^2 = kT\sigma_2, \quad (17)$$

$$\langle PQ \rangle - \langle P \rangle \langle Q \rangle = kT\sigma_3.$$

If the assumptions, mentioned in sect. 2, are valid one gets for infinitely large times

$$\begin{aligned} \langle P \rangle &= 0, \quad \langle Q \rangle = 0, \quad \langle PQ \rangle = 0, \\ \langle P^2 \rangle &= kT, \quad \Omega_1^2 \langle Q^2 \rangle = kT. \end{aligned} \quad (18)$$

This corresponds to equilibrium and the correlation between momentum and displacement is lost.

4. *Fokker-Planck equation.* In the following we assume again that the spectrum of the small oscillators is dense and that the strength function varies smoothly in the neighbourhood of Ω_0 . This implies the validity of (cf. I sect. 5)

$$\ddot{A} + 2\Gamma\dot{A} + \Omega_1^2 A = 0, \quad t \gg \tau_t. \quad (19)$$

Using this relation one verifies easily that the time derivatives of the first and second moments of P and Q , given by the relations (16) and (17), are identical to the corresponding quantities, which one derives from the equation*)

$$\frac{\partial f'}{\partial t} = \Omega_1^2 Q \frac{\partial f'}{\partial P} - P \frac{\partial f'}{\partial Q} + 2\Gamma \frac{\partial}{\partial P} P f' + 2\Gamma kT \frac{\partial^2 f'}{\partial P^2}, \quad t \gg \tau_t. \quad (20)$$

Since $f(P, Q, t | P_0, Q_0)$ is a Gaussian distribution it is fixed by its first and second moments (16) and (17). As is known, the solution of eq. (20) is a Gaussian distribution in P and Q if $f'(P, Q, t = 0) = \delta(P - P_0) \delta(Q - Q_0)$, so that we may conclude that $f(P, Q, t | P_0, Q_0)$ itself satisfies eq. (20).

Rearranging the terms one obtains

$$\frac{\partial f}{\partial t} - \Omega_1^2 Q \frac{\partial f}{\partial P} + P \frac{\partial f}{\partial Q} = 2\Gamma \frac{\partial}{\partial P} P f + 2\Gamma kT \frac{\partial^2 f}{\partial P^2}, \quad t \gg \tau_t, \quad (21)$$

*) In fact on the right hand side appears also a non-dissipative term

$$kT \{ \Omega_1^{-2} (\Omega^2 + \Gamma^2) - 1 \} \frac{\partial^2 f}{\partial P \partial Q}.$$

Clearly in our approximation this term vanishes because of eq. (I.31).

where the left hand side represents the conservative part and the right hand side the dissipative part of the equation. On the right hand side the first term corresponds to the systematic term and the second term to the stochastic term in the Langevin equation. If there is no interaction between the central oscillator and the bath of small oscillators, eq. (21) reduces to the Liouville equation of the harmonic oscillator, where the right hand side vanishes.

Instead of starting with the initial distribution $f(P, Q, 0|P_0, Q_0) = \delta(P - P_0) \delta(Q - Q_0)$ one can choose an arbitrary initial distribution, e.g. $g(P_0, Q_0)$. This leads to the time dependent distribution (15) which evidently under the same assumptions satisfies eq. (21), because

$$f(P, Q, t|P_0, Q_0)$$

does.

Summarizing we conclude that the distribution function of momentum and displacement of the central oscillator obeys the F.-P. equation (21) if one makes the same assumption as was needed for the derivation of the Langevin equation (cf. I sects. 9, 10 and 11).

5. *Derivation of the master equation by perturbation theory.* In contrast to the F.-P. equation the master equation will be derived on the present model for the quantummechanical case. First we will demonstrate the conventional method of first order time dependent perturbation theory. The Hamiltonian (I.1) is written in creation and annihilation operators, which satisfy the ordinary commutation relations. The Hamiltonian is split into an unperturbed part, H'_0 , and a perturbation term, W ,

$$H'_0 = \hbar\Omega_0(a^\dagger a + \frac{1}{2}) + \sum_n \hbar\omega_n(a_n^\dagger a_n + \frac{1}{2}),$$

$$W = \frac{1}{2}\hbar\Omega_0^{-\frac{1}{2}}(a^\dagger + a) \sum_n \varepsilon_n \omega_n^{-\frac{1}{2}}(a_n^\dagger + a),$$

$$a = (2\hbar\Omega_0)^{-\frac{1}{2}}(\Omega_0 Q + iP), \quad a^\dagger = (2\hbar\Omega_0)^{-\frac{1}{2}}(\Omega_0 Q - iP), \quad (22)$$

$$a_n = (2\hbar\omega_n)^{-\frac{1}{2}}(\omega_n q_n + ip_n), \quad a_n^\dagger = (2\hbar\omega_n)^{-\frac{1}{2}}(\omega_n q_n - ip_n),$$

$$[a, a^\dagger] = 1, \quad [a_n, a_m^\dagger] = \delta_{n,m},$$

and all other commutators vanish. The states will be written in the representation of the eigenstates of the unperturbed Hamiltonian H'_0 : $|M, m_1, \dots, m_N\rangle$, where M indicates the state of the central oscillator and m_1, \dots, m_N the states of the small oscillators.

The term W causes transitions between the several states $|M, m_1 \dots m_N\rangle$. To obtain the transition probability first the matrix element

$$\langle M, m_1 \dots m_N | W | L, l_1 \dots l_N \rangle$$

is calculated. This element differs only from zero if $|L - M| = 1$ and all pairs l_i, m_i are equal except one pair which must satisfy also $|l_j - m_j| = 1$. This shows that only transitions exist for which one quantum of energy is exchanged between the central oscillator and the bath.

Time dependent perturbation theory yields for the non vanishing transition probabilities between different states per unit of time (dropping the equal quantum numbers)

$$\begin{aligned} W_{M+1, m_j-1; M, m_j} &= \frac{1}{2} \pi \varepsilon_j^2 \Omega_0^{-1} \omega_j^{-1} (M+1) m_j \delta(\Omega_0 - \omega_j), \\ W_{M-1, m_j+1; M, m_j} &= \frac{1}{2} \pi \varepsilon_j^2 \Omega_0^{-1} \omega_j^{-1} M (m_j+1) \delta(\Omega_0 - \omega_j). \end{aligned} \quad (23)$$

To get the total transition probability we have to take into account that the bath is in thermal equilibrium. The transition probabilities between the states of the central oscillator are then given by ($\beta = 1/kT$)

$$W_{M+1; M} = \sum_j \frac{\sum_{m_j} W_{M+1, m_j-1; M, m_j} e^{-m_j \beta \hbar \omega_j}}{\sum_{m_j} e^{-m_j \beta \hbar \omega_j}}, \quad (24)$$

$$W_{M-1; M} = \sum_j \frac{\sum_{m_j} W_{M-1, m_j+1; M, m_j} e^{-m_j \beta \hbar \omega_j}}{\sum_{m_j} e^{-m_j \beta \hbar \omega_j}}. \quad (25)$$

Replacing the summation over j by integration and using the abbreviation Γ (cf. eq. (I.31)) one gets

$$W_{M+1; M} = 2\Gamma(M+1) \frac{e^{-\beta \hbar \Omega_0}}{1 - e^{-\beta \hbar \Omega_0}}, \quad (26)$$

$$W_{M-1; M} = 2\Gamma M \frac{1}{1 - e^{-\beta \hbar \Omega_0}}. \quad (27)$$

If one postulates the existence of a master equation

$$\frac{dP_M}{dt} = \sum_{M'} (W_{M'; M} P_{M'} - W_{M; M'} P_M), \quad (28)$$

one finds for the probability P_M , that the central oscillator at time t is in the state M , the equation

$$\begin{aligned} \frac{dP_M}{dt} &= \frac{2\Gamma}{1 - e^{-\beta \hbar \Omega_0}} \{ (M+1) P_{M+1} - [(M+1) e^{-\beta \hbar \Omega_0} + M] P_M + \\ &\quad + M e^{-\beta \hbar \Omega_0} P_{M-1} \}. \end{aligned} \quad (29)$$

In the following this equation will be derived by calculating P_M as a function of time and verifying that it obeys eq. (29).

6. *Calculation of the characteristic function of $P_L^M(t)$.* In the following again a system is considered, described by the Hamiltonian (I.1). At $t = 0$

the bath of small oscillators is supposed to be in thermal equilibrium with temperature T ($\beta = 1/kT$), and the central oscillator in the M -th eigenstate of the Hamiltonian H_0 . We are interested in the probability to find the central oscillator at time t in the L -th state of H_0 , which we call $P_L^M(t)$. This is given by

$$P_L^M(t) = \prod_n (e^{\beta \hbar \omega_n/2} - e^{-\beta \hbar \omega_n/2}) \times \\ \times \sum_{\substack{m_1, \dots, m_N \\ l_1, \dots, l_N}} | \langle L, l_1, \dots, l_N | e^{-iHt/\hbar} | M, m_1, \dots, m_N \rangle |^2 \times \\ \times \exp -\beta \{ (m_1 + \frac{1}{2}) \hbar \omega_1 + \dots + (m_N + \frac{1}{2}) \hbar \omega_N \}. \quad (30)$$

The calculation is simplified by introducing a double characteristic function $G(\beta', \beta'', t)$, defined by

$$G(\beta', \beta'', t) = \sum_{M, L} e^{-\beta' M \hbar \Omega_0} e^{-\beta'' L \hbar \Omega_0} P_L^M(t). \quad (31)$$

Substituting eq. (30) into this expression one obtains

$$G(\beta', \beta'', t) = e^{(\beta' + \beta'') \hbar \Omega_0/2} \prod_n (e^{\beta \hbar \omega_n/2} - e^{-\beta \hbar \omega_n/2}) \times \\ \times \sum_{L, l_1, \dots, l_N} \langle L, l_1, \dots, l_N | e^{-\beta'' H_0} e^{-iHt/\hbar} e^{-\beta' H_0} e^{-\beta H_1} \dots \\ \dots e^{-\beta H_N} e^{iHt/\hbar} | L, l_1, \dots, l_N \rangle, \quad (32)$$

(H_0 and H_n represent the unperturbed Hamiltonians of the oscillators) or

$$G(\beta', \beta'', t) = e^{(\beta' + \beta'') \hbar \Omega_0/2} \prod_n (e^{\beta \hbar \omega_n/2} - e^{-\beta \hbar \omega_n/2}) \times \\ \times \text{Tr}(e^{-\beta' H_0} e^{-\beta H_1} \dots e^{-\beta H_N} e^{iHt/\hbar} e^{-\beta'' H_0} e^{-iHt/\hbar}). \quad (33)$$

Under the trace all factors commute except the last three. As we have solved explicitly the equations of motion (cf. I sect. 3) we can calculate $e^{iHt/\hbar} e^{-\beta'' H_0} e^{-iHt/\hbar} = e^{-\beta'' H_0(t)}$. First we expand the operator $e^{-\beta'' H_0}$ in a Fourier integral. Next we apply the time evolution operators $e^{iHt/\hbar}$ and $e^{-iHt/\hbar}$. The advantage is that the resulting operator can be resolved in commuting operators. We use the identity (cf. Groenewold⁵)

$$a(P, Q) = \int_{-\infty}^{+\infty} \int dx dy \alpha(x, y) e^{(i/\hbar)(xP + yQ)}, \quad (34)$$

with

$$\alpha(x, y) = \frac{1}{2\pi\hbar} \text{Tr}\{a(P, Q) e^{-(i/\hbar)(xP + yQ)}\}. \quad (35)$$

Inserting $a(P, Q) = e^{-\beta'' H_0}$ we have to calculate

$$\alpha(x, y) = \frac{1}{2\pi\hbar} \text{Tr}(e^{-\beta'' H_0} e^{-(i/\hbar)(xP + yQ)}). \quad (36)$$

By means of the theorem of Bloch (cf. Messiah⁶) one finds

$$\alpha(x, y) = \frac{1}{2\pi\hbar} (e^{\beta\hbar\Omega_0/2} - e^{-\beta\hbar\Omega_0/2})^{-1} \cdot \exp\left[-\frac{1}{4\hbar\Omega_0} (y^2 + \Omega_0^2 x^2) \coth \frac{\hbar\Omega_0\beta''}{2}\right], \quad (37)$$

and subsequently

$$e^{-\beta''H_0} = \frac{1}{2\pi\hbar} (e^{\beta''\hbar\Omega_0/2} - e^{-\beta''\hbar\Omega_0/2})^{-1} \int_{-\infty}^{+\infty} dx dy e^{(i/\hbar)(xP+yQ)} \times \exp\left[-\frac{1}{4\hbar\Omega_0} (y^2 + \Omega_0^2 x^2) \coth \frac{\hbar\Omega_0\beta''}{2}\right]. \quad (38)$$

Now the problem is reduced to the calculation of

$$\frac{1}{2\pi\hbar} (e^{\beta''\hbar\Omega_0/2} - e^{-\beta''\hbar\Omega_0/2})^{-1} \int_{-\infty}^{+\infty} dx dy \cdot \exp\left[-\frac{\coth \frac{\hbar\Omega_0\beta''}{2}}{4\hbar\Omega_0} (y^2 + \Omega_0^2 x^2)\right] \times \text{Tr}(e^{-\beta'H_0} e^{-\beta H_1} \dots e^{-\beta H_N} e^{+iHt/\hbar} e^{(i/\hbar)(xP+yQ)} e^{-iHt/\hbar}). \quad (39)$$

The trace is also

$$\text{Tr}(e^{-\beta'H_0} e^{-\beta H_1} \dots e^{-\beta H_N} e^{(i/\hbar)(xP(t)+yQ(t))}). \quad (40)$$

Using the eqs. (I.7) it can be resolved in a product of traces

$$\text{Tr}(e^{-\beta'H_0} e^{(i/\hbar)((x\dot{A}+yA)P+(x\ddot{A}+y\dot{A})Q)}) \times \prod_n \text{Tr}(e^{-\beta H_N} e^{(i/\hbar)((x\dot{A}_n+yA_n)p_n+(x\ddot{A}_n+y\dot{A}_n)q_n)}. \quad (41)$$

These traces are again calculated by means of the theorem of Bloch. After integration one obtains

$$G(\beta', \beta'', t) = \frac{1}{1 - e^{-\beta'\hbar\Omega_0}} \frac{1}{1 - e^{-\beta''\hbar\Omega_0}} \times \frac{1}{\sqrt{\Sigma_1 \Sigma_2 - \Sigma_3^2}}, \quad (42)$$

where

$$\Sigma_1 = \frac{1}{2\Omega_0} \coth \frac{\hbar\Omega_0\beta''}{2} + \left(A^2 + \frac{\dot{A}^2}{\Omega_0^2}\right) \frac{\Omega_0}{2} \coth \frac{\hbar\Omega_0\beta'}{2} + \sum_n \left(A_n^2 + \frac{\dot{A}_n^2}{\omega_n^2}\right) \frac{\omega_n}{2} \coth \frac{\hbar\omega_n\beta}{2}, \quad (43)$$

$$\Sigma_2 = \frac{\Omega_0}{2} \coth \frac{\hbar\Omega_0\beta''}{2} + \left(A^2 + \frac{\dot{A}^2}{\Omega_0^2} \right) \frac{\Omega_0}{2} \coth \frac{\hbar\Omega_0\beta'}{2} + \sum_n \left(A_n^2 + \frac{\dot{A}_n^2}{\omega_n^2} \right) \frac{\omega_n}{2} \coth \frac{\hbar\omega_n\beta}{2}, \quad (44)$$

$$\Sigma_3 = \frac{1}{2}\dot{\Sigma}_1. \quad (45)$$

In app. A.g. we show $\Sigma_1\Sigma_2 - \Sigma_3^2 > 0$, which is needed for the validity of eq. (42). Putting

$$z = e^{-\beta'\hbar\Omega_0}, \quad (46)$$

$$\Sigma_1 = \frac{1}{2\Omega_0} \frac{1+z}{1-z} + \varphi, \quad \Sigma_2 = \frac{\Omega_0}{2} \frac{1+z}{1-z} + \chi, \quad (47)$$

we obtain after rearranging the terms

$$G(\beta', z, t) = \frac{1}{1 - e^{-\beta'\hbar\Omega_0}} \frac{1}{\sqrt{\nu^2 - 2\mu z + \lambda^2 z^2}}. \quad (48)$$

Here

$$\lambda^2 = \frac{1}{4} - \frac{\chi + \Omega_0^2\varphi}{2\Omega_0} + \varphi\chi - \frac{\dot{\varphi}^2}{4}, \quad (49)$$

$$\mu = \varphi\chi - \frac{\dot{\varphi}^2}{4} - \frac{1}{4}, \quad (50)$$

$$\nu^2 = \frac{1}{4} + \frac{\chi + \Omega_0^2\varphi}{2\Omega_0} + \varphi\chi - \frac{\dot{\varphi}^2}{4}. \quad (51)$$

In terms of the Legendre polynomials $\mathfrak{P}_L(\zeta)$ $G(\beta', z, t)$ reads

$$G(\beta', z, t) = \frac{1}{1 - e^{-\beta'\hbar\Omega_0}} \frac{1}{\nu} \sum_L \left(\frac{\lambda z}{\nu} \right)^L \mathfrak{P}_L \left(\frac{\mu}{\lambda\nu} \right). \quad (52)$$

One easily verifies that $|\lambda z/\nu| < 1$. If we define (cf. eq. (31)),

$$P_L(\beta', t) = \sum_M e^{-\beta'M\hbar\Omega_0} P_L^M(t), \quad (53)$$

we obtain the result

$$P_L(\beta', t) = \frac{1}{1 - e^{-\beta'\hbar\Omega_0}} \frac{1}{\nu} \left(\frac{\lambda}{\nu} \right)^L \mathfrak{P}_L \left(\frac{\mu}{\lambda\nu} \right). \quad (54)$$

7. *Derivation of the master equation.* Simple expressions for Σ_1 , Σ_2 and Σ_3 , analogous to the classical expressions (9) and (10) are not possible because of the factor $(\omega_n/2) \coth(\hbar\omega_n/2kT)$. Due to the presence of resonance denominators the temperature factor can be considered for $t \gg \hbar/kT$ as a constant, and subsequently put before the sum (cf. app. B). This approxi-

mation is not correct for orders higher than the first in Γ/Ω . This implies that we need to calculate the functions \bar{A} , A , \bar{A} and \bar{A} only to first order in Γ/Ω . Inserting these approximations into the expressions for Σ_1 , Σ_2 and Σ_3 (cf. eq. (A.19)) and putting

$$\rho = \frac{1}{2} \coth \frac{\hbar\Omega_0\beta}{2}, \quad \rho' = \frac{1}{2} \coth \frac{\hbar\Omega_0\beta'}{2}, \quad (55)$$

one gets for λ , μ and ν to first order in Γ/Ω

$$\begin{aligned} \lambda &= \{\rho + (\rho' - \rho) e^{-2\Gamma t}\} - \frac{1}{2}, \\ \mu &= \{\rho + (\rho' - \rho) e^{-2\Gamma t}\}^2 - \frac{1}{4}, \\ \nu &= \{\rho + (\rho' - \rho) e^{-2\Gamma t}\} + \frac{1}{2}. \end{aligned} \quad (56)$$

Hence follows

$$\mu = \lambda\nu \quad (57)$$

so that with eq. (48) for $t \gg \tau_t$ and $t \gg \hbar kT/\hbar$

$$G(\beta', z, t) = \frac{1}{1 - e^{-\beta'\hbar\Omega_0}} \frac{1}{\nu} \frac{1}{1 - (\lambda/\nu)z} = \quad (58)$$

$$= \frac{1}{1 - e^{-\beta'\hbar\Omega_0}} \frac{\{\rho + (\rho' - \rho) e^{-2\Gamma t}\} + \frac{1}{2}}{(1 - z)\{\rho + (\rho' - \rho) e^{-2\Gamma t}\} + \frac{1}{2}(1 + z)}. \quad (59)$$

This satisfies the equation

$$\frac{\partial G}{\partial t} = (1 - z) \Theta \left\{ (1 - e^{-\hbar\Omega_0\beta} z) \frac{\partial G}{\partial z} - e^{-\hbar\Omega_0\beta} G \right\}, \quad (60)$$

where

$$\Theta = \Gamma \left(\coth \frac{\hbar\Omega_0\beta}{2} + 1 \right). \quad (61)$$

From eq. (60) follows for $P_L(t)$ the equation

$$\frac{dP_L}{dt} = \Theta [(L + 1) P_{L+1} - \{e^{-\beta\hbar\Omega_0} (L + 1) + L\} P_L + e^{-\beta\hbar\Omega_0} L P_{L-1}], \quad L \geq 1, \quad (62)$$

$$\frac{dP_0}{dt} = \Theta (P_1 - e^{-\beta\hbar\Omega_0} P_0).$$

By means of eq. (53) one easily verifies that $P_L^M(t)$ also satisfies this equation.

We have now derived an equation for the conditional probability $P_L^M(t)$. The master equation, however, describes the time behaviour of $P_L(t)$, i.e. the probability that the central oscillator at time t is in the state L , starting from an arbitrary superposition of eigenstates at $t = 0$. Let us expand such

an initial state into eigenstates

$$\sum_{M, m_1, \dots, m_N} c_{M, m_1, \dots, m_N} |M, m_1, \dots, m_N\rangle, \quad (63)$$

where we require normalization

$$\sum_{M, m_1, \dots, m_N} c_{M, m_1, \dots, m_N}^* c_{M, m_1, \dots, m_N} = 1. \quad (64)$$

Then for this special initial state one obtains for the probability to find the central oscillator in the state L

$$\begin{aligned} & \sum_{\substack{M, m_1, \dots, m_N \\ M', m'_1, \dots, m'_N}} c_{M, m_1, \dots, m_N}^* c_{M', m'_1, \dots, m'_N} \times \\ & \times \sum_{l_1, \dots, l_N} \langle M, m_1, \dots, m_N | e^{iHt/\hbar} |L, l_1, \dots, l_N\rangle \cdot \\ & \cdot \langle L, l_1, \dots, l_N | e^{-iHt/\hbar} |M', m'_1, \dots, m'_N\rangle. \end{aligned} \quad (65)$$

Now we assume that the initial phases of the expansion coefficients c_{M, m_1, \dots, m_N} are randomly distributed, and take into account that the bath of small oscillators at $t = 0$ is in thermal equilibrium. This means

$$\begin{aligned} \overline{c_{M, m_1, \dots, m_N}^* c_{M', m'_1, \dots, m'_N}} &= \delta_{M, M'} \delta_{m_1, m'_1} \dots \delta_{m_N, m'_N} \times \\ & \times \prod_n (e^{\beta \hbar \omega_n / 2} - e^{-\beta \hbar \omega_n / 2}) \times \hat{p}_M \times \\ & \times \exp[-\beta \{(m_1 + \frac{1}{2}) \hbar \omega_1 + \dots + (m_N + \frac{1}{2}) \hbar \omega_N\}], \end{aligned} \quad (66)$$

where the bar means averaging over the phases of the expansion coefficients and over the canonical ensemble of the bath of small oscillators. Furthermore,

$$\sum_M \hat{p}_M = 1. \quad (67)$$

This averaging yields for $P_L(t)$

$$P_L(t) = \sum_M \hat{p}_M P_L^M(t). \quad (68)$$

Clearly $P_L(t)$, as $P_L^M(t)$, satisfies now eq. (62), because this equation does not contain M . Consequently, the master equation holds.

Remarks. In the limit of the classical case ($\hbar \Omega_0 \ll kT$) eq. (62) becomes an F.-P. equation for the energy of the central oscillator

$$\frac{\partial P(E)}{\partial t} = 2\Gamma \frac{\partial}{\partial E} E \left[kT \frac{\partial P}{\partial E} + P \right]. \quad (69)$$

This equation is extensively discussed by Rubin and Shuler⁷⁾.

If \hat{p}_M from eq. (66) equals the Boltzmann-factor

$$\hat{p}_M = (1 - e^{-\hbar \Omega_0 \beta'}) e^{-\beta' M \hbar \Omega_0}, \quad (70)$$

one obtains for $P_L(t)$

$$P_L(t) = (1 - e^{-\hbar\Omega_0\beta'(t)}) e^{-L\hbar\Omega_0\beta'(t)}, \quad (71)$$

where

$$\beta'(t) = -\frac{1}{\hbar\Omega_0} \log \left\{ 1 - \frac{(1 - e^{-\hbar\Omega_0\beta})(1 - e^{-\hbar\Omega_0\beta'})}{(1 - e^{-\hbar\Omega_0\beta'}) - e^{-2\Gamma t} (e^{-\hbar\Omega_0\beta} - e^{-\hbar\Omega_0\beta'})} \right\}. \quad (72)$$

One verifies easily that $\beta'(0) = \beta'$ and $\beta'(\infty) = \beta$. Furthermore, one concludes that for this special initial distribution the central oscillator reaches equilibrium via a series of states which can be characterized by a time dependent temperature $T'(t) = k^{-1}\beta'^{-1}(t)$. This agrees with a theorem derived by Mathews, Shapiro and Falkoff⁸). For $\beta'(t)$ one derives the equation

$$\frac{d\beta'(t)}{dt} = \frac{2\Gamma}{\hbar\Omega_0} \frac{(1 - e^{-\hbar\Omega_0\beta'(t)})(1 - e^{-\hbar\Omega_0(\beta - \beta'(t))})}{(1 - e^{-\hbar\Omega_0\beta})}. \quad (73)$$

This leads in the classical limit ($\hbar\Omega_0 \ll kT$) to the well known equation

$$\frac{dT'(t)}{dt} = -2\Gamma(T' - T), \quad (74)$$

where

$$T = (k\beta)^{-1}. \quad (75)$$

8. *Conclusions.* In the preceding sections the Fokker-Planck equation and master equation could be derived without making use of perturbation theory. We had to make the following assumptions and restrictions (cf. I sect. 11):

1. at the initial time the bath of small oscillators is in thermal equilibrium,

2. the frequency spectrum of the small oscillators is approximated by a continuous spectrum, so that the Poincaré period becomes infinite,

3. the strength function varies slowly in the region of the resonance frequency Ω_0 , and satisfies the Hölder condition,

4. approximations are considered to first order of Γ/Ω_0 , so that it must be required that

$$\Gamma \ll \Omega_0,$$

5. times are considered which are large compared to the transient time τ_t ,

$$t \gg \tau_t.$$

This transient time (cf. I sect. 5) is due to the interaction between the central oscillator and the bath of small oscillators, and can be compared with the collision time. It is equal to the reciprocal length of variation of the strength function (cf. I sect. 7). Thus it would vanish if the strength

function is constant, but this is not allowed because then there exist self-accelerating solutions (cf. I sect. 3);

6. obviously, the transient time has to be small compared to the relaxation time,

$$\tau_t \ll \Gamma^{-1}.$$

In the derivation of the master equation we had to make the additional assumptions and restrictions:

7. the phases of the expansion coefficients are at random distributed at the initial time,

8. times are larger than the quantumstatistical transient time $\tau_q = \hbar/kT$,

$$t \gg \tau_q.$$

The existence of this τ_q is a pure quantum effect and it is independent of the interaction;

9. like the transient time τ_t , τ_q has to be small compared to the relaxation time,

$$\tau_q \ll \Gamma^{-1},$$

otherwise the master equation loses its sense.

In the classical case the approximation of the resulting expressions to first order of Γ/Ω_0 was not significant but only convenient for the calculation of relaxation and oscillation times. On the contrary in the quantum-mechanical case this approximation was essential because we had to replace $E(\omega, T)$ by the constant $E(\pm\Omega, T)$ (cf. sect. B).

Other authors have also derived the F.-P. equation for an oscillator coupled linearly with a bath of harmonic oscillators. Toda⁹⁾10) applies perturbation theory and assumes that the bath remains exactly in thermal equilibrium to derive the F.-P. equation. Kogure¹¹⁾ derived a F.-P. equation for two coupled harmonic oscillators in interaction with the bath by solving exactly the equations of motion. Both, however, did not discuss the transient effects.

As should be known we only need to average over the ensemble of the bath at the initial time, and in the quantummechanical case, moreover, over the phases of the expansion coefficients at the initial time. Thus, repeated averaging, as in the conventional derivations, is not needed, owing to the fact that the equations of motion can be solved exactly.

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APPENDIX

A. Summation formulae and inequalities

a. The orthonormality conditions for the transformation X , defined in I sect. 3, are

$$\sum_{\nu} X_{n\nu} X_{m\nu} = \delta_{n,m}, \quad (n, m = 0, 1, \dots, N), \quad (\text{A.1})$$

$$X_{0\nu} X_{0\mu} + \sum_n X_{n\nu} X_{n\mu} = \delta_{\nu,\mu}, \quad (\nu, \mu = 0, 1, \dots, N). \quad (\text{A.2})$$

b. From the secular equation (I.15) follows immediately

$$\sum_n \varepsilon_n^2 (z_{\nu} - \omega_n^2)^{-1} = z_{\nu} - \Omega_0^2. \quad (\text{A.3})$$

c. After multiplying eq. (I.13) with $X_{n\mu} \omega_n^{-2}$ and summing over n (writing s_{ν}^2 for z_{ν}), one finds by using the eqs. (I.14), (A.2) and (A.3)

$$\sum_n \frac{X_{n\nu} X_{n\mu}}{\omega_n^2} = \frac{\delta_{\nu,\mu}}{s_{\nu}^2} - \Omega_1^2 \frac{X_{0\nu} X_{0\mu}}{s_{\nu}^2 s_{\mu}^2}, \quad (\text{A.4})$$

where

$$\Omega_1^2 = \Omega_0^2 - \sum_n \frac{\varepsilon_n^2}{\omega_n^2}. \quad (\text{A.5})$$

d. Multiplying eq. (I.13) with $X_{0\nu} s_{\nu}^{-2}$ and summing over ν one finds

$$\frac{\varepsilon_n}{\omega_n^2} \sum_{\nu} \frac{X_{0\nu}^2}{s_{\nu}^2} = - \sum_{\nu} \frac{X_{n\nu} X_{0\nu}}{s_{\nu}^2}. \quad (\text{A.6})$$

On the other hand one finds by multiplying eq. (I.12) with $X_{0\nu} s_{\nu}^{-2}$ and summing over ν

$$\Omega_0^2 \sum_{\nu} \frac{X_{0\nu}^2}{s_{\nu}^2} = 1 - \sum_n \varepsilon_n \sum_{\nu} \frac{X_{n\nu} X_{0\nu}}{s_{\nu}^2}. \quad (\text{A.7})$$

Elimination of $\sum_{\nu} (X_{0\nu} X_{n\nu} / s_{\nu}^2)$ from both formulae yields

$$\sum_{\nu} \frac{X_{0\nu}^2}{s_{\nu}^2} = \frac{1}{\Omega_1^2}. \quad (\text{A.8})$$

e. In the derivation of the Fokker-Planck equation one encounters the expression

$$\sigma_1(t) = \sum_n (A_n^2 + \dot{A}_n^2 \omega_n^{-2}). \quad (\text{A.9})$$

Using the definition formulae (I.9) and applying the relations (A.1), (A.4) and (A.8),

$$\sigma_1(t) = \frac{1}{\Omega_1^2} - A^2(t) - \Omega_1^2 \dot{A}^2(t), \quad (\text{A.10})$$

where \dot{A} is defined by eq. (11). In a similar way one finds

$$\sigma_2(t) = \sum_n (A_n^2 + \dot{A}_n^2 \omega_n^{-2}) = 1 - A^2(t) - \Omega_1^2 A^2(t). \quad (\text{A.11})$$

f. For the existence of the distribution function (8) and the validity of the F.-P. equation one has to show

$$\sigma_1 \sigma_2 - \sigma_3^2 > 0, \quad t > 0. \quad (\text{A.12})$$

Rearranging the terms one finds

$$\begin{aligned} \sigma_1 \sigma_2 - \sigma_3^2 = & \frac{1}{2} \sum_{n,m} \left\{ (A_n A_m - A_m A_n)^2 + \frac{(A_n \dot{A}_m - A_m \dot{A}_n)^2}{\omega_n^2 \omega_m^2} \right\} + \\ & + \sum_{n,m} \frac{1}{\omega_m^2} (A_n A_m - \dot{A}_m A_n)^2. \end{aligned} \quad (\text{A.13})$$

Hence

$$\sigma_1 \sigma_2 - \sigma_3^2 \geq 0. \quad (\text{A.14})$$

It is only zero for $t = 0$, because for $t \neq 0$ A_n differs from A_m if $n \neq m$.

g. By a similar rearrangement one can show that

$$\Sigma_1 \Sigma_2 - \Sigma_3^2 > 0, \quad (\text{A.15})$$

where Σ_1 , Σ_2 and Σ_3 are defined by the relations (43), (44) and (45). One needs eq. (A.15) in the derivation of the master equation.

B. Calculation of Σ_1 , Σ_2 and Σ_3

In the previous section we derived expressions for σ_1 and σ_2 . This is not possible for the summations in Σ_1 and Σ_2 because of the factor

$$\omega_n \coth(\hbar \omega_n / 2kT)$$

unless one expands it in powers of $\hbar \omega / kT$. It is more satisfactory to use an appropriate approximation. Considering the expression (I.A.16) for $\dot{A}_n(t)$ it is clear that (cf. eq. (44))

$$\Sigma_2' = \sum_n (A_n^2 + \dot{A}_n^2 \omega_n^{-2}) (\omega_n / 2) \coth(\hbar \omega_n / 2kT) \quad (\text{A.16})$$

contains denominators of the type $(\Omega_0 - \omega) \pm i\Gamma$. This means that an important contribution to the sum (A.16) comes from the region where $\omega_n \simeq \Omega_0$. We use the property that the frequency spectrum is dense and replace the sum (A.16) by an integral

$$\Sigma_2' = \int_0^\infty d\omega \rho(\omega) \{A^2(\omega) + \dot{A}^2(\omega) \omega^{-2}\} (\omega / 2) \coth(\hbar \omega / 2kT). \quad (\text{A.17})$$

Here $\rho(\omega)$ represents the density of the spectrum. The integral is a sum of

Fourier integrals as

$$\int_{-\infty}^{\infty} d\omega \frac{\gamma(\omega) \exp(\pm i\omega t)}{(\Omega^2 - \omega^2 - \Gamma^2)^2 + 4\Gamma^2\Omega^2} \frac{\omega}{2} \coth \frac{\hbar\omega}{2kT}. \quad (\text{A.18})$$

The presence of the resonance denominator now suggests to replace the factor $\omega \coth(\hbar\omega/2kT)$ by $\Omega_0 \coth(\hbar\Omega_0/2kT)$ so that this factor can be put before the integral and also before the sum in eq. (A.16). Thus in this approximation one gets

$$\Sigma'_{1,2,3} = \frac{\Omega_0}{2} \coth \frac{\hbar\Omega_0}{2kT} \cdot \sigma_{1,2,3}, \quad (\text{A.19})$$

where σ_1 , σ_2 and σ_3 are known from sect. A. The approximation used is reasonable if the variation of $E(\omega, T)$ is not too large over the width 2Γ of the resonance peak. Hence one must have

$$2\Gamma \left[\frac{d}{d\omega} \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2kT} \right]_{\omega=\Omega_0} \ll \frac{\hbar\Omega_0}{2} \coth \frac{\hbar\Omega_0}{2kT}, \quad (\text{A.20})$$

or

$$\frac{2\Gamma}{\Omega_0} \left[1 - \frac{2\hbar\Omega_0}{kT} \frac{\exp(-\hbar\Omega_0/kT)}{1 - \exp(-2\hbar\Omega_0/kT)} \right] \ll 1. \quad (\text{A.21})$$

In the extreme quantummechanical case ($\hbar\Omega_0 \gg kT$) eq. (A.21) reduces to

$$\Gamma/\Omega_0 \ll 1, \quad (\text{A.22})$$

whereas in the classical case ($\hbar\Omega_0 \ll kT$) it becomes

$$\Gamma/\Omega_0 \ll (kT/\hbar\Omega_0)^2. \quad (\text{A.23})$$

Since the oscillation time of the central oscillator is much larger than its relaxation time, i.e. $\Gamma \ll \Omega$ (cf. eq. (I.A.11)), both conditions are fulfilled. Since the factor [...] in eq. (A.21) is a monotonic function of $\hbar\Omega_0/kT$ the condition (A.20) is also satisfied for intermediate values of T , provided $\Gamma \ll \Omega_0$. Thus the approximation is valid for each value of T .

This approximation was based on taking $E(\omega, T) = (\hbar\omega/2) \coth \hbar\omega/2kT$ constant. The error made by neglecting the variation of $E(\omega, T)$ can be estimated by considering the contribution of the poles of the resonance factors and of the poles of $E(\omega, T)$ itself, when we apply contour integration.

The poles of the resonance denominators yield the factors $E(\Omega + i\Gamma, T)$ and $E(-\Omega + i\Gamma, T)$. Expansion gives

$$E(\pm\Omega, T) + i\Gamma \left(\frac{\partial E(\omega, T)}{\partial \omega} \right)_{\omega=\pm\Omega_0} + \dots \quad (\text{A.24})$$

The total expression must be real. This implies that the first order terms

will cancel each other and that the approximation to replace $E(\omega, T)$ by the constant $E(\Omega, t) = E(-\Omega, T)$ is correct to first order in Γ/Ω_0 .

The poles of $E(\omega, T)$ contribute terms involving the factor

$$\exp[-(2\pi kT/\hbar) \cdot lt], \quad (l = 1, 2, \dots). \quad (\text{A.25})$$

Hence these terms can be neglected after times t , much larger than $\tau_q = \hbar/kT$. Consequently only times may be considered which are much larger than τ_q . The approximation is only meaningful if this time is small compared to the relaxation time Γ^{-1} .

In the previous considerations we assumed tacitly that $\gamma(\omega)$ is an analytic function by applying contour integration. If $\gamma(\omega)$ is not analytic and satisfies only the Hölder condition the conclusions remain right. For, by evaluation of the Fourier integral (A.18) the factor $E(\omega, T) = (\hbar\omega/2) \cdot \coth(\hbar\omega/2kT)$ will generate time dependent terms which correspond to the range of variation of $E(\omega, T)$. In the applied approximation the bigger weight of the higher frequencies, $\omega > kT/\hbar$, caused by $E(\omega, T)$ is not taken into account, so that the behaviour of the integral is only well described for times larger than \hbar/kT .

Summarizing we must require for the validity of the approximations in addition to the conditions, used in calculating $A(t)$ and $A_n(t)$ (cf. (I sect. A))

1. the width of the resonance peak must be small compared to the thermal energy, or

$$\hbar\Gamma \ll kT \quad (\text{A.26})$$

and

2. the times for which the expressions are considered, are restricted by a quantumstatistical transient time τ_q (cf. I. sect. 5),

$$t \gg \tau_q = \hbar/kT. \quad (\text{A.27})$$

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