

A Remark on the Orthogonality Relations for Green Functions

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TO J. A. GREEN

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Let G be a connected linear algebraic group defined over the finite field \mathbb{F}_q of characteristic p . We denote by k an algebraic closure of \mathbb{F}_q and by F the Frobenius automorphism ($a \mapsto a^q$) of k over \mathbb{F}_q . Then F operates on G and its group G^F of fixed points is the finite group of \mathbb{F}_q -rational points of G .

Now assume G to be reductive and let \mathcal{U} be the \mathbb{F}_q -variety of unipotent elements of G . The *Green functions* of G^F are class functions $Q_{T,G}$ defined on the set \mathcal{U}^F of unipotent elements of G^F , parametrized by a maximal torus T of G defined over \mathbb{F}_q (and depending only on the G^F -conjugacy class of T). They are of importance for the representation theory of G^F .

In his fundamental paper [7] on the character theory of the finite groups $GL_n(\mathbb{F}_q)$, J. A. Green first introduced such functions, in a combinatorial guise. Subsequently, various definitions of Green functions for general reductive groups were given (in [13], elaborated in [14], in [6] and most recently in [9]).

Conjecturally, all definitions should be equivalent, at least if the characteristic p is good in the sense of [14] (As far as I know this has not yet been established, and the equivalence of the various definitions is known only under more severe restrictions on p .)¹ A basic property of the Green functions are their *orthogonality relations*, which can be written as

$$|G^F|^{-1} \sum_{u \in \mathcal{U}^F} Q_{T,G}(u) Q_{T_1,G}(u) = |T^F|^{-1} |T_1^F|^{-1} |N_G(T, T_1)^F|. \quad (1)$$

Here T and T_1 are two maximal tori of G , defined over \mathbb{F}_q (or F -stable),

$$N_G(T, T_1) = \{x \in G \mid xTx^{-1} = T_1\},$$

¹ *Note added in proof.* The equivalence has now been proved by G. Lusztig. See his preprint "On the character values of finite Chevalley groups at unipotent elements" (II Università degli Studi di Roma, 1985).

and $|A|$ denotes the number of elements of the finite set A . In particular, the left-hand side of (1) is zero when T and T_1 are not G^F -conjugate.

Equation (1) has been proved in the references given above, for each of the definitions of Green functions (in [14] with some restrictions on p and q). In this note we shall discuss a consequence of the orthogonality relations (1) and of the fact that the Green functions of [14] are polynomials in q (in a sense to be specified below).

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We assume from now on that the characteristic p is good for G . We shall also have to assume sometimes that q is sufficiently large, in the sense that \mathbb{F}_q contains a suitable finite field \mathbb{F}_{p^s} . In establishing nontrivial results about Green functions for general reductive groups one needs tools from algebraic topology over finite fields, even if as in [13] the definition of Green functions is “elementary.”

We use the definition of Green functions of [14], and we recall some results, most of them being contained in [14].

We fix an F -stable Borel subgroup B_0 of G and an F -stable maximal torus $T_0 \subset B_0$. Let $W = N_G T_0 / T_0$ be the Weyl group of $(G, T_0)(N_G(\cdot))$ denoting a normalizer in G . Then F operates on W . If G is semisimple and split over \mathbb{F}_q (which we shall assume soon to be the case) then F acts trivially on W . Let $w = n T_0 \in W$. By Lang’s theorem there is $x \in G$ such that $x^{-1}(Fx) = n$. Writing (somewhat incorrectly) $T_w = x T_0 x^{-1}$ we have that T_w is an F -stable maximal torus of G . Each torus is G^F -conjugate to a T_w , and $T_w, T_{w'}$ are G^F -conjugate if and only if there is $w_1 \in W$ with $w' = w_1^{-1} w (Fw_1)$. In particular, if G is semisimple and split over \mathbb{F}_q then the G^F -conjugacy classes of F -stable maximal tori of G are parametrized by the conjugacy classes in W .

If $u \in \mathcal{U}$ put $C(u) = Z_G(u) / Z_G(u)^0$, the quotient of the centralizer of u by its identity component. Then $C(u)$ is a finite group. If $u \in \mathcal{U}^F$ then F operates on $C(u)$.

For simplicity, we assume G to be semisimple and split and q to be sufficiently large. It is known that the classification of unipotent conjugacy classes in G and G^F and the structure of the groups $C(u)$ is independent of the characteristic p (which was assumed to be good), see [4, 5.11]. We can choose a set of representatives S of the unipotent conjugacy classes of G which is contained in G^F . We may assume (taking q sufficiently large) that F operates trivially on all $C(u)$ ($u \in S$). Now the unipotent conjugacy classes of G^F can be described as follows. Fix $u \in S$ and $c \in C(u)$. Let $x \in Z_G(u)$ represent C and take $y \in G$ such that $y^{-1}(Fy) = x$. Then $u_c = yxy^{-1}$ lies in U^F and is G -conjugate to u , but not necessarily G^F -conjugate.

Its conjugacy class in G^F depends only on that of c in $C(u)$. If u and c vary, the u_c provide representatives of the unipotent conjugacy classes of G^F . We have

$$|Z_{G^F}(u_c)| = |Z_{C(u)}(c)| |Z_{G^F}^0(u_c)|. \tag{2}$$

For these results see [14, no. 6] (there similar results are discussed for nilpotent elements in the Lie algebra of G).

We denote by \mathcal{B} the variety of Borel subgroups of G . This is a projective algebraic variety over \mathbb{F}_q on which G operates, which is isomorphic to the quotient variety G/B_0 . For $u \in \mathcal{U}$ denote by \mathcal{B}_u the fixed point set of u in \mathcal{B} . This is a projective variety which is defined over \mathbb{F}_q if $u \in \mathcal{U}^F$. It is known that \mathcal{B}_u is connected and of pure dimension $e(u) = \frac{1}{2}(\dim Z_G(u) - rkG)$ (for this last statement see [12], a proof for large p is given in [4, 5.10]).

We now introduce the l -adic cohomology groups $H^i(\mathcal{B}_u, E)$, with (constant) coefficients in a suitable field E (e.g., $E = \mathbb{Q}_l$ where $l \neq p$). If $u \in \mathcal{U}^F$ then F operates linearly in the vector spaces $H^i(\mathcal{B}_u, E)$. It was shown in [14] that there is a representation $r_u^i \otimes s_u^i$ of $W \times C(u)$ in $H^i(\mathcal{B}_u, E)$ such that, with the previous notations,

$$Q_{T_w, G}(u_c) = \sum_{i \geq 0} (-1)^i \text{Tr}(r_u^i(w^{-1}) s_u^i(c), H^i(\mathcal{B}_u, E)).$$

(The representation r_u^i is not the one of [14], it is the product of the latter one with the sign representation of W .)

The following result has been established in [1].

PROPOSITION. (i) *If p is good then $H^i(\mathcal{B}_u, E) = 0$ if i is odd.*

(ii) *If p is good and q is sufficiently large than F operates in $H^{2i}(\mathcal{B}_u, E)$ as scalar multiplication by q^i .*

The proposition shows that for p good and q large we have

$$Q_{T_w, G}(u_c) = \sum_{i \geq 0} \text{Tr}(r_u^i(w^{-1}) s_u^i(c), H^{2i}(\mathcal{B}_u, E)) q^i. \tag{3}$$

This shows that $Q_{T_w, G}(u_c)$ “is a polynomial in q .” Since $H^i(\mathcal{B}_u, E) = 0$ if $i > 2e(u)$ ($= 2 \dim \mathcal{B}_u$), the degree is at most $2e(u)$. Multiplying both sides of (1) with $|G^F|$, which is also a polynomial in q , we obtain a formula which can be viewed as a polynomial identity in q . A comparison of the leading coefficients in both sides of that identity has, essentially, been carried out in [14]. (For this one does not need the proposition.) This leads to the parametrization of the irreducible characters of W given in [14]. Below we shall consider the next highest coefficients in both sides of the polynomial identity, and discuss the meaning of their equality.

Using the description of the unipotent conjugacy classes in G^F given above one sees that the left-hand side of (1) can be written as

$$\sum_{u \in S} |C(u)|^{-1} \sum_{c \in C(u)} |Z_G^0(u_c)^F|^{-1} Q_{T,G}(u_c) Q_{T_1,G}(u_c). \tag{4}$$

If A is a finite group denote by A^\vee the set of its E -valued irreducible characters. From (3) we see that we can write

$$Q_{T_w,G}(u_c) = \sum_{\varphi \in C(u)^\vee} \varphi(c) (\chi_{u,\varphi}(w) q^{e(u)} + \chi'_{u,\varphi}(w) q^{e(u)-1} + \dots), \tag{5}$$

where $\chi_{u,\varphi}$ and $\chi'_{u,\varphi}$ are characters of representations of W . It has been shown in [14] that the nonzero $\chi_{u,\varphi}$ are precisely the absolutely irreducible characters of W . We write

$$C(u)_0^\vee = \{ \varphi \in C(u)^\vee \mid \chi_{u,\varphi} \neq 0 \}.$$

Next we need some information about the number $|Z_G^0(u_0)^F|$. This is contained in the following (essentially known) lemma. If G is any connected linear algebraic group we denote by $X^*(G)$ its character group, i.e. the group of homomorphisms of algebraic groups $G \rightarrow \mathbb{G}_m$ and we denote by $V(G)$ the vector space $X^*(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. Also, we denote by $R(G)$ the radical of G .

LEMMA. *Let G be a connected linear algebraic group over \mathbb{F}_q and let T_0 be an F -stable maximal torus of G which is contained in an F -stable Borel subgroup of G . There is a polynomial $\Phi \in \mathbb{Z}[X]$ whose coefficients only depend on the action of F on $X^*(T_0)$ such that $|G^F| = \Phi(q)$. We have*

$$\Phi(X) = X^{\dim G} - aX^{\dim G - 1} + \dots,$$

where

$$a = q^{-1} \text{Tr}(F, V(R(G))).$$

The important point for us is the explicit description of a .

Let U be the unipotent radical of G . Using that $|G^F| = |(G/U)^F| |U^F|$ and that $|U^F| = q^{\dim U}$ we see that it suffices to consider the case that G is reductive. Using that isogeneous groups have the same number of rational points one reduces the proof to the two cases that G is either semisimple or a torus. In the first case $a=0$, as follows from well-known formulas [15, 11.16]. For the case of a torus see [4, Chap. 3]. For G reductive the lemma is also immediate from the formula for $|G^F|$ given in [5, p. 230].

We now return to (4). If $u \in S$ denote by λ_u the character of the representation of $C(u)$ on $V(Z_G^0(u))$, induced by the conjugation action of $Z_G(u)$ on $Z_G^0(u)$. It then follows from the lemma that

$$|Z_G^0(u_c)^F| = q^{\dim Z_G(u)} - \lambda_u(c) q^{\dim Z_G(u) - 1} + \dots \tag{6}$$

Take $T = T_u$, $T_1 = T_{W_1}$ in (4). Using (5), (6), and the orthogonality relations of the group characters of the groups $C(u)$, we see that we can write (4) as a power series in q^{-1} , which starts off with

$$\begin{aligned} & q^{-r} \sum_{\substack{u \in S \\ \varphi \in C(u)^\vee}} \chi_{u,\varphi}(w) \chi_{u,\varphi}(w_1) \\ & + q^{-r-1} \left[\sum_{\substack{u \in S \\ \varphi, \varphi_1 \in C(u)_0^\vee}} \lambda_u(c) \varphi(c) \varphi_1(c) \chi_{u,\varphi}(w) \chi_{u,\varphi_1}(w_1) \right. \\ & \left. + \sum_{\substack{u \in S \\ \varphi \in C(u)_1^\vee}} \chi_{u,\varphi}(w) \chi'_{u,\varphi}(w_1) + \chi_{u,\varphi}(w_1) \chi'_{u,\varphi}(w) \right], \end{aligned}$$

where r is the rank of G . Recall that $\chi'_{u,\varphi}$ is defined by (5).

Using the parametrization of the irreducible characters of W alluded to above it is easily seen that the right-hand side of (1), viewed as a power series in q^{-1} , starts off with

$$q^{-r} \sum_{\substack{u \in S \\ \varphi \in C(u)_0^\vee}} \chi_{u,\varphi}(w) \chi_{u,\varphi}(w_1) (1 + q^{-1} \tau(w)) + \dots,$$

where τ is the character of W (acting in the vector space $V(T_0)$). We then can conclude that

$$\begin{aligned} \tau(w) &= \sum_{\substack{u \in S \\ \varphi \in C(u)_0^\vee}} \chi_{u,\varphi}(w) \chi_{u,\varphi}(w_1) \\ &= \sum_{\substack{u \in S \\ \varphi \in C(u)_0^\vee}} (\chi_{u,\varphi}(w) \chi'_{u,\varphi}(w_1) + \chi_{u,\varphi}(w_1) \chi'_{u,\varphi}(w)) \\ &+ \sum_{\substack{u \in S \\ \varphi, \varphi_1 \in C(u)_0^\vee}} \lambda_u(c) \varphi(c) \varphi_1(c) \chi_{u,\varphi}(w) \chi_{u,\varphi_1}(w_1). \end{aligned}$$

Write for $u \in S$, $\varphi \in C(u)^\vee$,

$$\chi'_{u,\varphi} = \sum_{\substack{v \in S \\ \psi \in C(u)_0^\vee}} a_{u,\varphi,v,\psi} \chi_{v,\psi} \tag{7}$$

where the $a_{u,\varphi;v,\psi}$ are nonnegative integers. Inserting this into the preceding formula one obtains the following expression for the product of the character τ and an irreducible character $\chi_{u,\varphi}$ ($u \in S, \varphi \in C(u)_0^\vee$)

$$\tau\chi_{u,\varphi} = \sum_{\varphi \in C(u)_0^\vee} \langle \lambda_u \varphi, \psi \rangle_{C(u)} \chi_{u,\varphi} + \sum_{\substack{v \in S \\ \varphi \in C(u)_0^\vee}} (a_{u,\varphi;v,\psi} + a_{v,\psi;u,\varphi}) \chi_{v,\psi}, \quad (8)$$

where $\langle , \rangle_{C(u)}$ denotes the standard inner product of class functions on $C(u)$. This formula is the consequence of (1) alluded to in the last paragraph of no. 1.

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We make a number of comments about (8).

(a) If $u, v \in S$ write $u \leq v$ if the conjugacy class of u in G is contained in the closure of the conjugacy class of v . It then follows from the results of Borho–MacPherson [3] that if $\chi_{u,\varphi;v,\psi} \neq 0$ we have $u < v$ (i.e., $u \leq v$ and $u \neq v$). It follows that at least one of the two integers $a_{u,\varphi;v,\psi}, a_{v,\psi;u,\varphi}$ is zero.

(b) If W is any finite group and τ a representation of W over an algebraically closed field E of characteristic 0 then McKay has defined a graph $\Gamma_\tau(W)$ as follows [10]. The vertex set is W^\vee , the set of irreducible representations (or characters) of W over E . For $\chi \in W^\vee$ write

$$\tau\chi = \sum_{\psi \in W^\vee} a_{\chi\psi} \psi.$$

The vertices $\chi, \psi \in W^\vee$ are joined by $a_{\chi\psi}$ directed edges.

It is clear that if W is a Weyl group and τ its standard representation, formula (8) gives a description of the integers $a_{\chi\psi}$. Now the vertex set of $\Gamma_\tau(W)$ can be viewed as the set of pairs (u, φ) , with $u \in S, \varphi \in C(u)_0^\vee$. The previous remark shows that if $u \neq v$, the vertices (u, φ) and (v, ψ) can lie on an edge of the McKay graph only if $u < v$ or $v < u$. Moreover, (8) also shows that for fixed u , the full subgraph of $\Gamma_\tau(W)$ whose vertices are the (u, φ) with $\varphi \in C(u)_0$, is a subgraph of the graph $\Gamma_{\lambda_u}(C(u))$.

(c) If W is an irreducible Weyl group of classical type, formulas for the decomposition of the tensor product of the standard representations and an irreducible representation have been given by Tokayama [16] in terms of the combinatorial parametrization of the irreducible representations of W . Below, we shall say a bit more about the case $W = S_n$. For the Weyl groups of exceptional type the Green functions are explicitly known (see [14, 11, 2]). The information contained in these references combined with (8) will lead to an explicit description of the decomposition of the tensor product for the exceptional cases.

(d) In the situation of no. 3, let $G = PGL_n$. Then W is the symmetric group S_n . In that case all groups $C(u)$ are trivial, so the irreducible characters χ_u can be parametrized by the elements $u \in S$. The results of [16] together with the combinatorial description of the order \leq on S [8] now give the following form of (8). If $u, v \in S$ write $u \sim v$ if u and v are neighbors for the order \leq . Then

$$\tau\chi_u = m_u\chi_u + \sum_{v \sim u} \chi_v,$$

where $m_u = \dim V(Z_G(u))$.

We see that if one discards the loops of the McKay graph $R_\tau(S_n)$, one obtains the graph describing the closure relations of the unipotent conjugacy classes of G .

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