Journal of Pure and Applied Algebra 2 (1972) 231–247. North-Holland Publishing Company

FAMILIES OF REPRESENTATIONS OF LIE ALGEBRAS IN CHARACTERISTIC p

F.D. VELDKAMP
Mathematisch Instituut, Budapestlaan 6, Utrecht, Netherlands

Communicated by F. Oort
Received 11 August 1971

Introduction

In this paper we consider a finite dimensional Lie algebra $\mathfrak{g}$ over an algebraically closed (sometimes perfect only) field k of characteristic $p > 0$. Zassenhaus [12] has shown the existence of a number m such that every irreducible representation of g has dimension $\leqslant p^m$, equality holding in "most" cases; for the precise meaning of "most", see §4, (iv). The irreducible representations correspond in a 1-1 way to the points of a certain algebraic variety over k. Thus one is lead to considering (algebraic) families of representations, that is, representations of $\mathfrak{g}$ in an algebraic vector bundle over a certain scheme such that in the fibers one gets ordinary linear representations (see §4, (ii)). These families of representations have been introduced by Rudakov and Šafarevič [8]. In this paper we generalize to arbitrary Lie algebras some of their results for the simple Lie algebra of type A_1 in characteristic $p > 2$. These results are the following. In the category of families of irreducible p^m-dimensional representations there is no universal element (Proposition (4.1)). However, if one adds a rigidity to the families of representations (see §4, (v)), one finds that there does exist a universal element in the category of irreducible p^m-dimensional rigid representations (Theorem (4.2)). As a preparation to these propositions we derive in §§2 and 3 information on the structure of the universal enveloping algebra U of g over its center Z and on maximal subfields of the quotient field of U. To conclude the paper, we determine in §5 the irreducible representations of the two different classical Lie algebras of type A_1 over an algebraically closed field of characteristic 2.

§ 1. Recollections

In this paper $\mathfrak{g}$ denotes a finite dimensional Lie algebra over a field k of characteristic p. k will be assumed to be perfect (and later on even algebraically closed). $\mathcal{U}$ is the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{Z}$ the center of $\mathcal{U}$. Following Zassenhaus [12] we consider the subspace $\mathcal{L}$ of $\mathcal{U}$ generated by the p^ith powers of elements of $\mathfrak{g}$, $p = 0, 1, 2, \ldots$:

$$\mathcal{L} = \mathfrak{g} + k\mathfrak{g}^p + k\mathfrak{g}^{p^2} + \ldots + k\mathfrak{g}^{p^i} + \ldots,$$

and

$$\mathcal{M} = \mathcal{L} \cap \mathcal{Z}.$$

Let $\mathcal{O}$ be the subalgebra of $\mathcal{Z}$ generated by 1 and $\mathcal{M}$. Zassenhaus has shown that $\mathcal{U}$ is a free $\mathcal{O}$-module of rank p^s for some finite s.

$\mathcal{U}$ has a left quotient division ring (see [5] or [12]), say, D. Let $K \subseteq D$ be the quotient field of $\mathcal{Z}$ and $Q \subseteq K$ that of $\mathcal{O}$. D is central simple over K of dimension p^{2m}, for some m. By Zassenhaus ([12], Theorem 6) we have

(1.1). *The dimension of any absolutely irreducible representation of $\mathfrak{g}$ is at most p^m.*

§ 2. $\mathcal{U}$ as algebra over its center $\mathcal{Z}$

(i). Let T be the reduced trace of D over K, and

$$T(x, y) = T(xy)$$

the inner product on D defined by T. $T(\,.\,)$ and $T(\,.\,,\,.\,)$ have values in K, and their restrictions to $\mathcal{U}$ have values in $\mathcal{Z}$. For a basis B of D over K, δ_B denotes the discriminant of B, i.e., if $B = \{e_1, \ldots, e_{p^{2m}}\}$, then

$$\delta_B = \det((T(e_i, e_j))_{1 \leq i, j \leq p^{2m}}).$$

For $B \subset \mathcal{U}$, $\delta_B \in \mathcal{Z}$ (see, e.g., [5] for these definitions and results). We define the discriminant ideal Δ of $\mathcal{U}$ over $\mathcal{Z}$ by

$$\Delta = \mathcal{Z}\text{- ideal generated by } \{\delta_B \mid B \text{ a basis of } D \text{ over } K, B \subset \mathcal{U}\}.$$

(ii). Now we assume k is algebraically closed.

Let M be a maximal ideal of $\mathcal{Z}$, so $\mathcal{Z}/M = k$. $\mathcal{U}/M\mathcal{U}$ is a separable algebra over k if and only if its discriminant is not 0. This happens if and only if K has a basis B over K, $B \subset \mathcal{U}$ such that $\delta_B \notin M$, in other words, if $M \not\supseteq \Delta$.

Set $Z = \mathrm{Spec}(\mathcal{Z})$, and let Z_Δ be the Zariski open subset of Z consisting of all prime ideals of $\mathcal{Z}$ not containing Δ. Z_Δ is an open subscheme of the irreducible scheme Z. For M a maximal ideal in $\mathcal{Z}$, $\mathcal{U}/M\mathcal{U}$ is separable if and only if $M \in Z_\Delta$. From general results on separable algebras over rings (see [2] or [3]), it follows that for any prime ideal P in $\mathcal{Z}$, $\mathcal{U}_P$ is separable over $\mathcal{Z}_P$ if and only if $P \in Z_\Delta$; here $\mathcal{Z}_P$ denotes, as usual, the localization of $\mathcal{Z}$ with respect to P, and $\mathcal{U}_P = \mathcal{Z}_P\mathcal{U}$.

Let for $a \in \mathcal{Z}$, $a \neq 0$, $\mathcal{Z}_a$ denote the localization of $\mathcal{Z}$ with respect to the multiplicatively closed system $\{a^i \mid i = 1, 2, \ldots\}$, $\mathcal{U}_a = \mathcal{Z}_a\mathcal{U}$, and Z_a the Zariski open subset of Z of the $P \in Z$ with $a \notin P$. We define a sheaf of algebras $\mathcal{F}$ on Z by

$$\mathcal{F}(Z_a) = \mathcal{U}_a \quad \text{for } a \in \mathcal{Z}, a \neq 0,$$

with the obvious restriction homomorphisms. The results of the preceding paragraph tell us that the restriction of $\mathcal{F}$ to Z_Δ is a locally separable sheaf of algebras on Z_Δ (cf. [1]), and that $\mathcal{F}$ is not separable in the points of Z outsied Z_Δ. It follows from results of Zassenhaus [12] that, for a maximal ideal M in $\mathcal{Z}$, $\mathcal{U}/M\mathcal{U}$ is central simple of dimension p^{2m} over $\mathcal{Z}/M = k$ whenever $M \in Z_\Delta$.

§ 3. Maximal subfields of D

(i). Let the field k be perfect. Zassenhaus has proved that $\mathcal{U}$ is a maximal order in D ([12], Lemma 5, pp. 16–17). He actually showed: if $\mathcal{A}$ is a subring of D having the property:

$$\text{there exists } n \in \mathcal{O} \text{ such that } \mathcal{U} \subseteq \mathcal{A} \subseteq n^{-1}\mathcal{U},$$

then $\mathcal{A} = \mathcal{U}$. Since $\mathcal{O}$ is noetherian, this is easily seen to imply the following somewhat stronger result.

Lemma 3.1. *$\mathcal{U}$ consists of all elements of D which are integral over $\mathcal{O}$.*

Consider a maximal subfield L of D; then $[L:K] = p^m$. Define $\mathcal{V} = L \cap \mathcal{U}$. $\mathcal{V} \supseteq \mathcal{Z}$ and $D = \mathcal{U} \otimes_{\mathcal{Z}} K$, hence L is the quotient field of $\mathcal{V}$. By Lemma 3.1, $\mathcal{V}$ is integrally closed.

(ii). In what follows, k is assumed to be algebraically closed.

Let M be a maximal ideal in Z. Out of a set of generators of V over Z we choose a minimal set of generators of V_M over Z_M, say $a_1, \ldots, a_s$. They generate L over K, hence $s \geqslant p^m$. Similarly, we choose a set of generators $b_1, \ldots, b_t$ of U_M over V_M contained in U which is minimal with these properties; then $t \geqslant p^m$. A basis for V_M/MV_M over Z_M/MZ_M can be chosen from the elements $a_1 \bmod MV_M, \ldots, a_s \bmod MV_M$; say we get as a basis $a_1 \bmod MV_M, \ldots, a_{s'} \bmod MV_M$. Similarly we choose a basis for U_M/MU_M over V_M/MV_M, say $b_1 \bmod MV_M, \ldots, b_{t'} \bmod MV_M$. Then

$$V_M = \langle a_1, \ldots, a_{s'} \rangle + MV_M$$

as a Z_M-module, where $\langle a_1, \ldots, a_{s'} \rangle$ denotes the Z_M-module generated by $a_1, \ldots, a_{s'}$. Since MZ_M is the radical of Z_M, we get from Nakayama's Lemma that V_M is generated by $a_1, \ldots, a_{s'}$. From the choice of $a_1, \ldots, a_s$ it follows that $s' = s$. In a similar way one shows that $t' = t$.

The elements $a_i b_j \bmod MU_M$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$, form a basis of U_M/MU_M over $Z_M/MZ_M = k$. Now assume $M \in Z_\Delta$. Then U_M/MU_M is central simple over k of dimension p^{2m}, hence $st = p^{2m}$. Since $s, t \geqslant p^m$, we get $s = t = p^m$. Let δ be the discriminant of the basis $\{a_i b_j \mid 1 \leqslant i, j \leqslant p^m\}$. Since U_M/MU_M is separable over k, $\delta \notin M$, i.e., $M \in Z_\delta$.

Let N be any maximal ideal in Z, $N \in Z_\delta$. We claim that $\{a_i b_j \mid 1 \leqslant i, j \leqslant p^m\}$ is a basis of U_N over Z_N. The $a_i b_j$ are contained in U and linearly independent over K, hence certainly over Z_N. Let $x \in U_N$; then $x = \Sigma_{i,j} x_{i,j} a_i b_j$ with $x_{i,j} \in K$. Consider the equations

$$T(x, a_k b_l) = \sum_{i,j} x_{i,j} T(a_i b_j, a_k b_l), \qquad 1 \leqslant k, l \leqslant p^m.$$

The coefficients of these equations are in Z_N, and the determinant of the matrix $(T(a_i b_j, a_k b_l))$ is δ. Since $\delta \notin N$, it is invertible in Z_N, so all $x_{i,j} \in Z_N$, and therefore the $a_i b_j$ generate U_N over Z_N. From this it is immediate that $a_1, \ldots, a_{p^m}$ is a basis of V_N over Z_N, and $b_1, \ldots, b_{p^m}$ one of U_N over V_N. Here we have considered $U(U_N)$ as a left module over $V(V_N)$, but the same arguments of course hold for right modules. We summarize our results in the following proposition.

Proposition 3.2. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic p, L a maximal subfield of D and $V = L \cap U$. Then*

(i). *L is the quotient field of V and V is integrally closed.*

(ii). *The sheaf of Z-modules defined on $Z = \operatorname{Spec}(Z)$ by V is locally free of rank p^m on Z_Δ.*

(iii). *The sheaf of (left or right) $\mathcal{V}$-modules defined by $\mathcal{U}$ on $V = \mathrm{Spec}(\mathcal{V})$ is locally free of rank p^m on V_Δ, the open subset of V lying over Z_Δ.*

(iv). *The sheaf $\mathcal{F}$ defined by $\mathcal{U}$ on Z is locally free of rank p^{2m} on Z_Δ.*

(iii). Now suppose, moreover, that the maximal subfield L of D is separable over K. We claim that, for $M \in Z_\Delta$, $\mathcal{V}_M/M\mathcal{V}_M$ is separable over $\mathcal{Z}_M/M\mathcal{Z}_M = k$. In the proof of the above proposition we have seen that elements $a_1, \ldots, a_{p^m}$ exist in $\mathcal{V}$ which form a basis of $\mathcal{V}_M$ over $\mathcal{Z}_M$ and whose cosets modulo $M\mathcal{V}_M$ are a basis of $\mathcal{V}_M/M\mathcal{V}_M$ over $\mathcal{Z}_M/M\mathcal{Z}_M = k$. Since L is separable over K, the elements $a_1^p, \ldots, a_{p^m}^p$ are linearly independent over $\mathcal{Z}_M$. Let $\mathcal{W}$ be the $\mathcal{Z}$-module generated by $a_1^p, \ldots, a_{p^m}^p$. Assume $\mathcal{V}_M/M\mathcal{V}_M$ to be inseparable over $\mathcal{Z}_M/M\mathcal{Z}_M = k$. Then the cosets $a_1^p \bmod M\mathcal{V}_M, \ldots, a_{p^m}^p \bmod M\mathcal{V}_M$ would be linearly dependent over k. Hence we might assume, e.g., that $a_1^p \bmod M\mathcal{V}_M$ is a linear combination of the others, and consequently that $a_1^p \bmod M\mathcal{W}_M$ is a k-linear combination of $a_2^p \bmod M\mathcal{W}_M, \ldots, a_{p^m}^p \bmod M\mathcal{W}_M$ in $\mathcal{W}_M/M\mathcal{W}_M$. Hence

$$\mathcal{W}_M = \langle a_2^p, \ldots, a_{p^m}^p \rangle + M\mathcal{W}_M.$$

By Nakayama's Lemma this implies that

$$\mathcal{W}_M = \langle a_2^p, \ldots, a_{p^m}^p \rangle \text{ as a } \mathcal{Z}_M\text{-module},$$

which contradicts the linear independence of $a_1^p, \ldots, a_{p^m}^p$ over $\mathcal{Z}_M$. Thus we have shown

Proposition 3.3. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic $p > 0$, L a maximal subfield of D which is separable over K, and $\mathcal{V} = L \cap \mathcal{U}$. Then for the maximal ideals M of $\mathcal{Z}$, $M \in Z_\Delta$, the algebra $\mathcal{V}_M/M\mathcal{V}_M$ is separable over $\mathcal{Z}_M/M\mathcal{Z}_M = k$.*

§4. Families of representations

(i). Let X be a scheme. We recall that a *vector bundle of rank r* on X consists of

(1) a scheme E and a morphism $\pi : E \to X$,

(2) an open covering $\{U_i\}$ of X, called atlas,

(3) isomorphisms $\phi_i : \pi^{-1}(U_i) \to \mathbb{A}^r \times U_i$ of schemes over U_i such that for all i, j:

$$\psi_{i,j} = \phi_j \circ \phi_i^{-1} : \mathbb{A}^r \times U_i \cap U_j \to \mathbb{A}^r \times U_i \cap U_j$$

is an isomorphism over $U_i \cap U_j$ such that its cohomomorphism $\psi^*_{i,j}$ takes the coordinates $X_1, \ldots, X_r$ on $\mathbb{A}^r$ into linear forms in the X_i:

$$\psi^*_{i,j}(X_k) = \sum_{l=1}^{r} a^{(i,j)}_{k,l} X_l,$$

where all $a^{(i,j)}_{k,l} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$, $\mathcal{O}_X$ denoting the structure sheaf on X (see [7, pp. 294–5]). Notation: (E, π, X).

To avoid distinguishing one particular atlas, one assumes the atlas to be maximal. The notion of a vector bundle of rank r is equivalent to that of a locally free sheaf of modules of rank r on X, the sheaf $\mathcal{E}$ corresponding to E being the sheaf of sections of E.

A *morphism* of vector bundles over X: $(E, \pi, X) \to (F, \rho, X)$ is a morphism $\phi : E \to F$ over X locally given by morphisms

$$\pi^{-1}(U_i) = \mathbb{A}^r \times U_i \to \mathbb{A}^s \times U_i = \rho^{-1}(U_i)$$

given by linear transformations of $\mathbb{A}^r$ into $\mathbb{A}^s$ with coefficients in the structure sheaf of U_i (as in (3) of the definition of vector bundle). A morphism of a vector bundle into itself is called an endomorphism, and the ring of endomorphisms of (E, π, X) is denoted by $\mathrm{End}(E, \pi, X)$, or $\mathrm{End}\, E$ for short.

(ii). Consider a Lie algebra $\mathfrak{g}$ over an algebraically closed field k (of arbitrary characteristic) and a scheme X over k. Rudakov and Šafarevič [8] have defined a *family of representations of* $\mathfrak{g}$ *with basis* X as a vector bundle (E, π, X) together with a k-linear representation ϕ of $\mathfrak{g}$ in $\mathrm{End}\, E$. Notation: $\mathbf{E} = (E, \pi, X, \phi)$. For each closed point $x \in X$ this defines a k-linear representation

$$\phi_x : \mathfrak{g} \to \mathrm{End}\, E_x,$$

where the linear space E_x over k is the fibre $\pi^{-1}(x)$. The rank of (E, π, X) is called the *dimension* of the family. $\mathbf{E}$ is called *irreducible* if every ϕ_x is irreducible.

Let $\mathbf{E} = (E, \pi, X, \phi)$ and $\mathbf{F} = (F, \rho, X, \psi)$ be families of representations of $\mathfrak{g}$. A *morphism* of $\mathbf{E}$ into $\mathbf{F}$ is a morphism $f : (E, \pi, X) \to (F, \rho, X)$ of vector bundles over X satisfying

$$\psi(Y) \circ f = f \circ \phi(Y) \text{ for } Y \in \mathfrak{g}.$$

The set of isomorphism classes of families of irreducible n-dimensional representations of $\mathfrak{g}$ with basis X is denoted by $R_n(X)$.

Let $\mathbf{E} = (E, \pi, X, \phi)$ be a family of irreducible n-dimensional representations of $\mathfrak{g}$, and $f: Y \to X$ a morphism of schemes over k. Consider the inverse image $(f^*E, f^*\pi, Y)$ of the vector bundle (E, π, X) with respect to f, and let $\tilde{f}$ be the corresponding homomorphism of End E into End f^*E. Then

$$\tilde{f} \circ \phi: \mathfrak{g} \to \mathrm{End}\, f^*E$$

defines a family of irreducible n-dimensional representations of $\mathfrak{g}$ with basis Y: $(f^*E, f^*\pi, Y, \tilde{f} \circ \phi) = f^*\mathbf{E}$. Thus R_n is a contravariant functor from the category of schemes over k to the category of sets. If this functor R_n can be represented by a family $\mathbf{E} = (E, \pi, X, \phi)$, we call $\mathbf{E}$ a *universal* family of irreducible n-dimensional representations of $\mathfrak{g}$. That is, $\mathbf{E}$ is a universal family if

$$\alpha_Y : \mathrm{Hom}_k(Y, X) \to R_n(Y)$$

defined by

$$\alpha_y(f) = f^*\mathbf{E}$$

is a bijection for every scheme Y over k. $R_n(\mathrm{Spec}(k))$ consists of (the equivalence classes of) all the irreducible n-dimensional representations of $\mathfrak{g}$, so in case $\mathbf{E}$ is a universal family, each irreducible representation of $\mathfrak{g}$ is realized in precisely one of the fibers $\pi^{-1}(x)$, x a closed point of X.

(iii). Here is an important example of a family of representations. Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic p, L a maximal subfield of D, $V = L \cap U$ (see §3). Set $V = \mathrm{Spec}(V)$ and $p: V \to Z = \mathrm{Spec}(Z)$ the projection defined by the inclusion of Z in V, $V_\Delta = p^{-1}(Z_\Delta)$, G the restriction to V_Δ of the sheaf of right $\mathcal{O}_V$-modules on V defined by U as a right V-module. By Proposition 3.2, G is locally free of rank p^m, so it defines a vector bundle (U, π, V_Δ) of rank p^m. The action of $\mathfrak{g}$ by left multiplication on U and hence on G, defines a k-linear representation λ of $\mathfrak{g}$ in End U. This will be shown to yield an irreducible representation in each fiber. Consider a maximal ideal $N \in V_\Delta$; $N \cap Z = M$ is a maximal ideal $\in Z_\Delta$. The action of $\mathfrak{g}$ on U can be extended to an action of U on U, by left multiplication. This induces an action of the central simple algebra U/MU on U/MU as a right V/MV-module. In §3, (ii), we have seen that U/MU has dimension p^{2m} over $Z/M = k$, and V/MV has dimension p^m. Since N is a maximal ideal in V, $V/N = k$ and U/UN has dimension p^m over k. This implies that UN/MU is a maximal left ideal of U/MU, hence U/UN is irreducible under the action of $\mathfrak{g}$.

(iv). Since the irreducible representations of degree p^m of a Lie algebra $\mathfrak{g}$ are in 1-1 correspondence with the closed points of the algebraic variety Z_Δ (cf. [12, p. 26 ff.]), one might expect there exists a universal family of irreducible p^m-dimensional representations. Unfortunately, this is false. This was proved for $\mathfrak{g}$ of type A_1, $p > 2$, by Rudakov and Šafarevič ([8, p. 445–446]). Their proof works mutatis mutandis for arbitrary Lie algebras over an algebraically closed field of characteristic $\neq 0$. For convenience of the reader, we shall write up here this generalized version.

Proposition 4.1. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic $p \neq 0$. Then there does not exist a universal family of irreducible p^m-dimensional representations of $\mathfrak{g}$.*

Proof. Consider a family $\mathbf{E} = (E, \pi, X, \phi)$ of irreducible p^m-dimensional representations of $\mathfrak{g}$. For a closed point $x \in X$, ϕ_x is an irreducible representation of $\mathfrak{g}$ in the fiber E_x, which can be extended to a representation of U. Hence for $z \in Z$,

$$\phi_x(z) = \mu(z)(x) 1_{E_X},$$

with a homomorphism

$$\mu : Z \to \Gamma(X, \mathcal{O}_X),$$

the latter symbol denoting, as usual, the k-algebra of global sections of the structure sheaf $\mathcal{O}_X$ on X. μ defines a morphism over k

$$f_{\mathbf{E}} : X \to Z = \operatorname{Spec}(Z).$$

To the point $f_{\mathbf{E}}(x) \in Z$ corresponds at least one irreducible representation ρ of $\mathfrak{g}$, say, in a space V. Considering $z \in Z$ as a function on Z, we see

$$\rho(z) = z(f_{\mathbf{E}}(x)).1_V,$$

but since μ is the cohomomorphism of $f_{\mathbf{E}}$,

$$z(f_{\mathbf{E}}(x)) = \mu(z)(x).$$

So ρ must be ϕ_x. ϕ_x being irreducible of dimension p^m, $f_{\mathbf{E}}(x) \in Z_\Delta$, so we may consider $f_{\mathbf{E}}$ as a morphism of X in Z_Δ.

Now assume $\mathbf{E}$ is universal. In that case each irreducible representation of

dimension p^m is realized precisely once as a ϕ_x, so

$$f_{\mathbf{E}} : X \to Z_\Delta$$

is bijective. This implies that X is irreducible and that its function field $k(X)$ is a purely inseparable extension of the function field of Z, which is K.

If $\mathbf{F} = (F, \rho, Y, \psi)$ is any family of irreducible p^m-dimensional representations, there exists a morphism

$$g_{\mathbf{F}} : Y \to X$$

such that $\mathbf{F} = g_{\mathbf{F}}^* \mathbf{E}$, because $\mathbf{E}$ is universal. From this it is immediate that $f_{\mathbf{F}} = f_{\mathbf{E}} \circ g_{\mathbf{F}}$. Now take $\mathbf{F} = \mathbf{U} = (U, \pi, V_\Delta, \lambda)$ as in (iii) above. Since, for closed $v \in V_\Delta$, ϕ_v is the representation corresponding to $f_{V_\Delta}(v) \in Z_\Delta$, f_{V_Δ} must be the projection p of V_Δ on Z_Δ and μ the injection of Z in V. From $f_{\mathbf{U}} = f_{\mathbf{E}} \circ g_{\mathbf{U}}$ it follows that we have the inclusions

$$L = k(V_\Delta) \supseteq k(X) \supseteq K.$$

L can be chosen so as to be separable over K, and $k(X)$ is purely inseparable over K, hence $k(X) = K$.

Let x be a generic point of X. There exists a nontrivial homomorphism

$$\phi_x : U \to \operatorname{End} E_x.$$

Since $E_x = K^{p^m}$, we get a nontrivial homomorphism

$$D = U \otimes_Z K \to (K)_{p^m},$$

the algebra of $p^m \times p^m$-matrices over K. Since D is a division ring, such a homomorphism cannot exist. Thus we are led to a contradiction.

(v). An analysis of the proof of Proposition 4.1 shows two points to be taken into consideration in an effort to ensure the existence of a universal family of dimension p^m by a modification of the definition of a family of representations. First, K has to be replaced by L in order to split D; this amounts to replacing Z by V. Second, to get a morphism $f_{\mathbf{E}} : X \to V$ (instead of $X \to Z$), we need a homomorphism

$$\mu : V \to \Gamma(X, \mathcal{O}_X)$$

which defines in every stalk E_x eigenvalues of the elements of V. Thus one is led to

the following generalizations of definitions given by Rudakov and Šafarevič [8], who followed an example of Grothendieck (see [4]). In the rest of this section, L is a maximal *separable* subfield of D, fixed once for all, and $V = L \cap U$.

A *rigid representation* of $\mathfrak{g}$ is a representation of $\mathfrak{g}$, and hence of U, together with a common eigenvector for the elements of V. Morphisms of rigid representations are defined in the obvious way. Notice that multiplication in the representation space by a nonzero element of k is an isomorphism of rigid representations.

Let ρ be a p^m-dimensional irreducible representation of $\mathfrak{g}$ in a linear space W, corresponding to a maximal ideal M in Z. Since $M \in Z_\Delta$, V/MV is a p^m-dimensional commutative semisimple algebra over the algebraically closed field k, hence V has p^m linearly independent eigenvectors in W with distinct weight functions. Thus ρ gives rise to precisely p^m nonisomorphic rigid representations.

A *family of rigid representations* $\mathbf{E} = (E, \pi, X, \phi, s)$ of the Lie algebra $\mathfrak{g}$ consists of a family of representations (E, π, X, ϕ) together with a global section s in E which is a weight vector of V, i.e., for every closed point $x \in X$, $s(x) \neq 0$ and $s(x)$ is a common eigenvector for the elements of $\phi_x(V)$. s is called the *rigidity* of $\mathbf{E}$.

Morphisms of families of rigid representations and universal families of rigid representations are defined as in the nonrigid case. A universal family is unique up to isomorphism, if it exists. Let $\mathbf{E} = (E, \pi, X, \phi, s)$ be a family of rigid representations, $\mathcal{E}$ the locally free module of sections of E. For $\lambda \in k$, $\lambda \neq 0$, multiplication by λ in $\mathcal{E}$ defines an isomorphism of families of rigid representations. Hence replacing s by λs means replacing of $\mathbf{E}$ by an isomorphic family.

(vi). Let $(U, \pi, V_\Delta, \lambda)$ be the family of irreducible p^m-dimensional representations of $\mathfrak{g}$ defined in (iii). Remember that now L is assumed to be separable over K. $1 \in U$ defines a global section over V_Δ, which is a weight vector of V, so it defines a rigidity in the family. Thus we get a family of irreducible p^m-dimensional rigid representations $\mathbf{U}_1 = (U, \pi, V_\Delta, \lambda, 1)$.

(vii). In this, the final part of §4, we shall prove that the family $\mathbf{U}_1$ is universal. For $\mathfrak{g}$ of type A_1, $p \neq 2$, this result is again due to Rudakov and Šafarevič [8]. Their proof can be easily adapted to the general case; for convenience of the reader we shall give it here.

Theorem 4.2. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic $p \neq 0$. Then $\mathbf{U}_1 = (U, \pi, V_\Delta, \lambda, 1)$ is a universal family of irreducible p^m-dimensional rigid representations of $\mathfrak{g}$.*

Proof. Let $\mathrm{RR}_n(Y)$ denote the set of isomorphism classes of families of irreducible n-dimensional rigid representations. For every scheme Y over k, there is a mapping

$$\alpha : \mathrm{Hom}_k(Y, V_\Delta) \to \mathrm{RR}_{pm}(Y)$$

which maps a k-morphism

$$f : Y \to V_\Delta$$

on the family $f^*\mathbf{U}_1 = (f^*U, f^*\pi, \tilde{f} \circ \lambda, f^*1)$. We have to show that α is bijective. To this end we shall construct an inverse mapping

$$\beta : \mathrm{RR}_{pm}(Y) \to \mathrm{Hom}_k(Y, V_\Delta).$$

Let $\mathbf{F} = (F, \rho, Y, \psi, s)$ be a family in a class of RR_{pm}. s being a weight vector of V, there exists a homomorphism

$$\mu : V \to \Gamma(Y, \mathcal{O}_Y)$$

such that

$$\psi(v)s = s\mu(v) \text{ for } v \in V.$$

Note that we identify F with the corresponding sheaf of sections $\mathcal{F}$, considered as a right $\mathcal{O}_Y$-module. μ defines a k-morphism

$$f : Y \to V.$$

As in the proof of Proposition 4.1 one sees that f maps the closed points of Y into V_Δ, so we may consider f as an element of $\mathrm{Hom}_k(Y, V_\Delta)$. We set

$$\beta(\mathbf{F}) = f.$$

$\beta\alpha = 1$ is immediate from the definitions of $\mathbf{U}_1$, α and β. To show that $\alpha\beta = 1$, we have to construct an isomorphism between $f^*\mathbf{U}$ and $\mathbf{F}$, if $f = \beta(\mathbf{F})$. Let X be an affine open set in Y; $X = \mathrm{Spec}(A)$ for a k-algebra A. The restriction of the sheaf $\mathcal{F}$ of sections of F to X can be writen as $\mathcal{F} \mid X = \tilde{M}$, the sheaf of modules defined by the right A-module M. Let $s \mid X = \tilde{m}$, $m \in M$. We may assume X to be chosen in such a way that f maps X into an affine open set $V_\delta = p^{-1}Z_\delta$ of V for some $\delta \in \Delta$ (p = projection of V on Z).

$$f \mid X : X \to V_\delta$$

has a cohomomorphism

$$\tilde{f}: V_\delta \to A.$$

Notice that for $v \in V$,

$$\tilde{f}(v) = \mu(v) \mid X,$$

if we identify $\Gamma(X, \mathcal{O}_Y)$ with A. Consider the right A-module

$$M' = U \otimes_V A.$$

A considered as a left V-module via $\tilde{f}$. M' defines a sheaf of modules $\tilde{M}'$ on X, and clearly

$$\tilde{M}' = f^* G \mid X,$$

G being the sheaf of sections of the vector bundle (U, π, X), hence the sheaf defined by U on V_δ. Define

$$h : U \to M$$

by

$$\widetilde{h(u)} = \psi(u)\,\tilde{m} \text{ for } u \in U.$$

For $u \in U$, $v \in V$, we get

$$\begin{aligned}\widetilde{h(uv)} &= \psi(uv)\,\tilde{m}\\ &= \psi(u)\,\psi(v)\,\tilde{m}\\ &= \psi(u)\,\tilde{m}\,\mu(v)\\ &= \widetilde{h(u)}\,\tilde{f}(v),\end{aligned}$$

hence h is a homomorphism of right V-modules. So we can extend h to a homomorphism of right A-modules

$$h' : M' = M \otimes_V A \to M,$$

which induces a homomorphism of sheaves of $\mathcal{O}_Y \mid X$-modules

$$h^* : f^*G \mid X = \widetilde{M}' \to \widetilde{M} = F \mid X.$$

It is straightforward to verify that these homomorphisms for the various affine open sets X which cover Y can be pasted together so as to give a homomorphism from f^*G to F, which we shall also denote by h^*. h^* defines a morphism of vector bundles

$$\bar{h} : (f^*U, f^*\pi, Y) \to (F, \rho, Y).$$

It is immediate that

$$h^*(f^*(1)) = s.$$

From the definitions of h and $\bar{h}$ one easily derives that $\bar{h}$ induces a k-linear homomorphism of left $\mathfrak{g}$-modules

$$\bar{h}_y : (f^*U)_{f(y)} \to F_y$$

for closed points $y \in Y$. The representations of $\mathfrak{g}$ in F_y and $(f^*U)_{f(y)}$ are irreducible, $\bar{h}_y$ is nontrivial, $h^*(f^*(1)) = s$ and $s_y \neq 0$, so we must conclude that $\bar{h}_y$ is an isomorphism of rigid $\mathfrak{g}$-representations, for every closed point $y \in Y$. Hence we have an isomorphism of families of rigid representations

$$\bar{h} : (f^*U, f^*\pi, Y, \widetilde{f} \circ \lambda, f^*(1)) \simeq (F, \rho, Y, \psi, s).$$

This completes the proof of the theorem.

§5. Representations of Lie algebras of type A_1 in characteristic 2.

Let $\mathfrak{g}$ be the Lie algebra of an almost simple algebraic group of type A_1 over an algebraically closed field k. In case $p > 2$ the representations of $\mathfrak{g}$ have been determined in [8]. Rather than trying to obtain these results again in the framework of the general theory presented in the preceding sections, which would require a good deal of computations, we shall consider here the case $p = 2$. Then there are two forms of Lie algebras of type A_1, viz., the Lie algebra of $\mathrm{SL}_2(k)$ and that of $\mathrm{PGL}_2(k)$.

(i). $\mathfrak{g}$ is the Lie algebra of $\mathrm{SL}_2(k)$. From a Chevalley basis in characteristic 0 we get by reduction mod 2 the basis H, X, Y satisfying

$$[X, Y] = H, \quad [H, X] = [H, Y] = 0.$$

So the Cartan subalgebra $\mathfrak{h}$ is the center of $\mathfrak{g}$. By Proposition 1.2, [11], $\mathcal{O} = k[H, X^2, Y^2]$ and H, X^2 and Y^2 are algebraically independent over k. $\mathcal{O}$, being a polynomial ring, is integrally closed. From the existence of the Steinberg representation of dimension 2 ([9], [10]) we infer that $[D : K] \geqslant 4$. But $Q = k(H, X^2, Y^2)$, hence $[D : Q] = 4$ and $K = Q$. $\mathcal{U}$ is a finitely generated $\mathcal{O}$-module, hence $\mathcal{Z}$ is integral over $\mathcal{O}$. Therefore $\mathcal{Z} = \mathcal{O}$.

Since $\mathcal{U}$ is a free module over $\mathcal{Z} = \mathcal{O}$ with basis $1, X, Y, XY$, the discriminant ideal Δ is generated by the discriminant δ of this basis. The trace bilinear form of D over K is easily computed for $\mathrm{char}(k) = 2$:

$$T(\lambda + \mu X + \nu Y + \rho XY, \lambda' + \mu' X + \nu' Y + \rho' XY) = (\lambda\rho' + \rho\lambda' + \mu\nu' + \nu\mu' + \rho\rho' H)H.$$

For the discriminant of the basis $1, X, Y, XY$ we get from this

$$\delta = H^4,$$

hence

$$\Delta = (H^4).$$

As a maximal separable subfield of D we take

$$L = K(XY).$$

Then

$$V = L \cap \mathcal{U} = \mathcal{Z}[XY] = k[H, X^2, Y^2, XY]$$
$$\cong k[T_1, T_2, T_3, T_4]/(T_4^2 + T_1 T_4 + T_2 T_3).$$

V_Δ is the open set given by $T_1 \neq 0$. The only singular point of V is $(0, 0, 0, 0)$, which is not in V_Δ. Every point of Z_Δ gives precisely one irreducible 2-dimensional representation of $\mathfrak{g}$, every point of V_Δ precisely one irreducible 2-dimensional rigid representation. Here a rigid representation is a representation together with an eigenvector of XY.

To see what happens in the points of Z outside Z_Δ, we consider a maximal ideal M of $\mathcal{Z} = k[T_1, T_2, T_3]$ (here $T_1 = H$, $T_2 = X^2$, $T_3 = Y^2$) with $T_1 \in M$, say

$$M = (T_1, T_2 - \alpha_2, T_3 - \alpha_3).$$

Then $\mathcal{U}/M\mathcal{U}$ has a k-basis $1, x, y, xy$ with

$$x^2 = \alpha_2,\ y^2 = \alpha_3,\ xy = yx.$$

This algebra is commutative and its only maximal ideal is

$$(x + \sqrt{\alpha_2}, y + \sqrt{\alpha_3}),$$

hence it yields precisely one irreducible representation, of dimension 1 if $(\alpha_2, \alpha_3) \neq (0,0)$, of dimension 0 if $(\alpha_2, \alpha_3) = (0,0)$.

(ii). $\mathfrak{g}$ is the Lie algebra of $PGL_2(k)$. From an integral basis in characteristic 0 one gets by reduction mod 2 a basis H, X, Y with

$$[X, Y] = 0, \quad [H, X] = X, \quad [H, Y] = Y.$$

The Casimir operator in characteristic 0 is $H^2 + H - XY$, so in characteristic 2 we get as an invariant in $\mathcal{U}$ under the adjoint operation of $\mathfrak{g}$, i.e., as an element of the center

$$T_1 = H^2 + H + XY.$$

In this case

$$\mathcal{O} = k[H^2 + H, X^2, Y^2]$$

and

$$\begin{aligned} \mathcal{Z} \supseteq \mathcal{O}[T_1] &= k[H^2 + H, X^2, Y^2, XY] \\ &\cong k[X_1, X_2, X_3, X_4]/(X_4^2 + X_2 X_3). \end{aligned}$$

We see that $\mathrm{Spec}(\mathcal{O}[T_1])$ is a 3-dimensional hypersurface whose singularities are given by $X_2 = X_3 = X_4 = 0$, i.e., the set of singularities has codimension 2. Then it is well known that $\mathrm{Spec}(\mathcal{O}[T_1])$ is a normal variety [7, p. 391, Prop. 2], hence $\mathcal{O}[T_1]$ is integrally closed.

Since $[D : K]$ is a square, $[D : K] \geqslant 4$. Obviously, D is spanned over $Q(T_1) = k(H^2 + H, X^2, Y^2, XY)$ by $1, H, X, HX$, since $Y = (X^2)^{-1}(XY)X$. Hence $[D : Q(T_1)] \leqslant 4$. It follows that $K = Q(T_1)$ and that $[D : K] = 4$. Again, $\mathcal{Z}$ is integral over $\mathcal{O}[T_1]$ and hence

$$\mathcal{Z} = \mathcal{O}[T_1] = k[H^2 + H, X^2, Y^2, XY].$$

The trace bilinear form of D over K is easily computed:

$$T(\lambda + \mu H + \nu X + \rho HX, \lambda' + \mu' H + \nu' X + \rho' HX)$$
$$= \lambda\mu' + \mu\lambda' + \mu\mu' + X^2(\nu\rho' + \rho\nu'),$$

and thus one gets the discriminant of the basis $1, H, X, HX$: $\delta_1 = X^4$. Similarly, the discriminant of the basis $1, H, Y, HY$ is $\delta_2 = Y^4$. Hence

$$Z_\Delta \subseteq Z_{X_2} \cup Z_{X_3} \text{ in } Z = \operatorname{Spec}(\mathcal{Z}).$$

Now consider a closed point of Z not in $Z_{X_2} \cup Z_{X_3}$, i.e., a maximal ideal in $\mathcal{Z}$ of the form

$$M = (X_1 - \alpha, X_2, X_3, X_4).$$

$\mathcal{U}/M\mathcal{U}$ is generated over $k = \mathcal{Z}/M$ by elements $1, h, x, y$ with

$$x^2 = y^2 = xy = yx = h^2 + h + \alpha = 0,$$
$$hx + xh = x, \; hy + yh = y.$$

Its radical is $(x, y) \neq 0$, so $\mathcal{U}/M\mathcal{U}$ is not separable, i.e., $M \notin Z_\Delta$. Thus we conclude

$$Z_\Delta = Z_{X_2} \cup Z_{X_3}.$$

We take

$$L = K(H) = k(H, X^2, Y^2, XY).$$
$$V = L \cap \mathcal{U} = \mathcal{Z}[H] = k[H, X^2, Y^2, XY]$$
$$\cong k[X_1, X_2, X_3, X_4]/(X_4^2 + X_2X_3).$$

On $V_\Delta = V_{X_2} \cup V_{X_3}$ one has a rigid representation in each closed point. The singular points ($X_2 = X_3 = X_4 = 0$) form precisely the complement of V_Δ in X.

Consider a closed point of $Z \backslash Z_\Delta$, i.e., a maximal ideal in $\mathcal{Z}$ of the form

$$M = (X_1 - \alpha_1, X_2, X_3, X_4).$$

As seen above, $\mathcal{U}/M\mathcal{U}$ has the radical $R = (x, y)$.

$$(\mathcal{U}/M\mathcal{U})/R = k[h]$$

with

$$h^2 + h + \alpha = 0,$$

hence $(\mathcal{U}/M\mathcal{U})/R \cong k \oplus k$. Thus we get two inequivalent irreducible representations

of dimension 1 in every point where $\alpha \neq 0$, whereas for $\alpha = 0$ we get one nonzero 1-dimensional representation and the null representation. The only [2]-representations (or restricted-) are obtained for

$$M = (X_1, X_2, X_3, X_4),$$

since in that case $H^2 + H = X_1$, $X^2 = X_2$ and $Y^2 = X_3$ are represented by 0, and $X_4^2 = X_2 X_3 = 0$. Thus we get a 1-dimensional irreducible [2]-representation and the null representation.

References

[1] B. Auslander, The Brauer group of a ringed space, J. Algebra 4 (1966) 220–273.

[2] M. Auslander and O. Goldman, The Brauer group of a commutative ring, Trans. Amer. Math. Soc. 97 (1960) 367–409.

[3] F. DeMeyer and E. Ingraham, Separable algebras over commutative rings, Lecture Notes in Mathematics 181 (Springer, Berlin, 1971).

[4] A. Grothendieck, Technique de descente et théorèmes d'existence en géométrie algébrique, V. Sém. Bourbaki 1961/62, exposé 232.

[5] N. Jacobson, The theory of rings (Amer. Math. Soc., New York, 1943).

[6] N. Jacobson, Lie algebras (Interscience, New York, 1962).

[7] D. Mumford, Introduction to algebraic geometry (preliminary version), Dept. Math., Harvard Univ.

[8] A.N. Rudakov and I.R. Šafarevič, The irreducible representations of the simple three-dimensional Lie algebra over a field of nonzero characteristic, Mat. Zametki 2 (1967) 489–454 (in Russian; English translation: Math. Notes 2 (1967) 760–767).

[9] R. Steinberg, Prime power representations of finite linear groups, II, Can. J. Math. 9 (1957) 347–351.

[10] R. Steinberg, Representations of algebraic groups, Nagoya Math. J. 22 (1963) 33–56.

[11] F.D. Veldkamp, The center of the universal enveloping algebra of a Lie algebra in characteristic p, Ann. Sci. Ecole Norm. Sup. (1972), to appear.

[12] H. Zassenhaus, The representations of Lie algebras of prime characteristic, Proc. Glasgow Math. Assoc. 2 (1954) 1–36.