PHYSICAL BASIS OF BALLISTOCARDIOGRAPHY. II.

The Quantities That Can Be Measured With Different Types of Ballistocardiographs and Their Mutual Relations

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SINCE it appears from the literature that there might be some confusion in using the terms displacement, velocity, and acceleration (or force) and their relations in ballistocardiographic investigation, a summary of the different possibilities will follow below. It will appear that, if an optimal representation of one of the above-mentioned quantities is desired, then the natural frequency and the damping of the loaded ballistocardiograph (BCG) determine which of these quantities is recorded.

Since we want to know which of the quantities is recorded, we assume for simplicity's sake the binding between body and BCG to be infinitely strong. The subject himself is considered to be a rigid body.

The differential equation of a BCG, loaded with a patient[1] is:

$$M\ddot{x}_c - \beta\dot{x} - Dx = M\ddot{x} \tag{1}$$

in which x is the displacement, $\dot{x}$ the velocity, and $\ddot{x}$ the acceleration of subject and BCG together with respect to the surroundings (only the longitudinal axis is attended). These quantities are positive in headward direction (to the right in Fig. 1). The quantities x_c, $\dot{x}_c$, and $\ddot{x}_c$, respectively, have the same meaning for the common center of gravity of subject and BCG with respect to the skeleton (positive to the left in Fig. 1). M is the total mass of subject and BCG. $M\ddot{x}_c$ is the force exerted on subject and BCG by the circulation. The frictional force $\beta\dot{x}$ is working in the opposite direction of the velocity $\dot{x}$. Dx is the directive force that drives subject and BCG to the zero position. This force is working in the opposite direction of the displacement x. So, the left side of equation (1) gives the forces working on subject and BCG. They equal the product of the total mass of the moving system M and its acceleration $\ddot{x}$. Equation (1) can also be written in the following form:

$$M\ddot{x} + \beta\dot{x} + Dx = M\ddot{x}_c \tag{1a}$$

The center of gravity of subject and BCG is assumed to move sinusoidally (term of a Fourier series) with a displacement x_c and an amplitude $|x_c|$. Then x_c can be written in the usual exponential form:

$$x_c = |x_c|\ e^{j\omega t} \tag{2}$$

$\omega = 2\pi\nu$, ν is the frequency, t the time and j the imaginary unit ($j^2 = -1$).

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Received for publication June 1, 1955.

The solution of the differential equation (1a) (valid after a time long enough) is:

$$x = |x| e^{j(\omega t + \varphi)} \tag{3}$$

in which $|x|$ represents the amplitude of this displacement x of subject and BCG. φ is the phase shift (time-lag) between the mass-movement within the subject and the movement of subject and BCG.

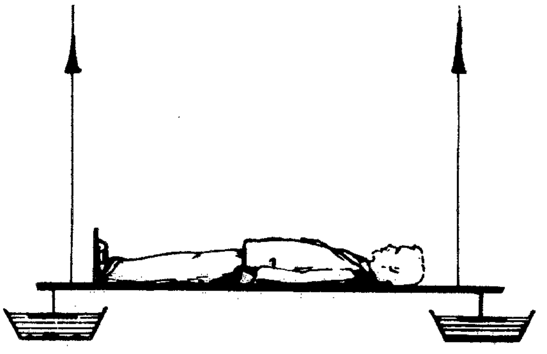


Fig. 1.—Ballistocardiograph loaded with a subject.

The ratio between the amplitude of the displacement of subject and BCG $|x|$ and the amplitude of the displacement of the common center of gravity of subject and BCG $|x_c|$ follows from the differential equation (1a) by substituting the formulas (2) and (3). It can be easily proved that:

$$\frac{|x|}{|x_c|} = \left[\left(1 - \frac{\nu_0^2}{\nu^2}\right)^2 + 4\delta^2 \frac{\nu_0^2}{\nu^2}\right]^{-\frac{1}{2}} \tag{4}$$

or

$$\frac{|x|}{|x_c|} = N. \tag{4a}$$

with

$$N = \left[\left(1 - \frac{\nu_0^2}{\nu^2}\right)^2 + 4\delta^2 \frac{\nu_0^2}{\nu^2}\right]^{-\frac{1}{2}}. \tag{5}$$

If subject and BCG are displaced from their zero position and are released without initial velocity, they would oscillate with the natural frequency ν_0 if there were no damping ($\beta = 0$). The frequency with which the common center of gravity moves is represented by ν. The ratio between the damping β and the critical damping β_c is represented by δ.
So

$$\delta = \beta/\beta_c.$$

By differentiating equation (2) with respect to time we find for the velocity of the common center of gravity $\dot{x}_c$:

$$\dot{x}_c = j\omega\, |x_c|\, e^{j\omega t}. \tag{6}$$

From this equation the amplitude of the velocity $|\dot{x}_c|$ is found to be:

$$|\dot{x}_c| = \omega|x_c|. \tag{6a}$$

The acceleration $\ddot{x}_c$ of the common center of gravity is found by differentiating $\dot{x}_c$ (equation (6)):

$$\ddot{x}_c = -\omega^2|x_c|e^{j\omega t}. \tag{7}$$

The amplitude of the acceleration is:

$$|\ddot{x}_c| = \omega^2|x_c|. \tag{7a}$$

The internal force that causes the center of gravity to move equals the product of the acceleration of the common center of gravity of subject and BCG and their total mass.

In the same way the velocity $\dot{x}$ of subject and BCG is to be found:

$$\dot{x} = j\omega|x|e^{j\omega t}, \tag{8}$$

with the amplitude for this velocity $|\dot{x}|$:

$$|\dot{x}| = \omega|x|. \tag{8a}$$

For the acceleration is found

$$\ddot{x} = -\omega^2|x|e^{j\omega t}. \tag{9}$$

The amplitude of the acceleration follows from equation (9):

$$|\ddot{x}| = \omega^2|x|. \tag{9a}$$

Equation (4a) gives the ratio $|x|/|x_c|$. With the aid of the equations (6a), (7a), (8a), and (9a) the following ratios can be calculated:

$$\frac{|\dot{x}|}{|x_c|}, \frac{|\ddot{x}|}{|x_c|}, \frac{|x|}{|\dot{x}_c|}, \frac{|\dot{x}|}{|\dot{x}_c|}, \frac{|\ddot{x}|}{|\dot{x}_c|}, \frac{|x|}{|\ddot{x}_c|}, \frac{|\dot{x}|}{|\ddot{x}_c|} \quad \text{and} \quad \frac{|\ddot{x}|}{|\ddot{x}_c|}.$$

We find:

$$|x_c| : |\dot{x}_c| : |\ddot{x}_c| : |x| : |\dot{x}| : |\ddot{x}| = 1 : \omega : \omega^2 : N : N\omega : N\omega^2 \tag{10}$$

in which N is dependent on the frequency (formula (5)).

There will be a time-lag between the phenomena occurring within the body and the movement of body and BCG. This time-lag can be indicated by an angle φ, the phase shift. The phase shift φ between the displacement of the common center of gravity of subject and BCG (x_c) and the displacement of subject and BCG (x) is to be calculated by substituting the formulas (2) and (3) in the differential equation (1a). It follows:

$$\mathrm{tg}\ \varphi = \frac{2\delta\nu_0\nu}{\nu^2 - \nu_0^2} \tag{11}$$

The phase shift φ^* in all other cases can be read from Fig. 2. The figure is rotating in the direction of the arrow. φ^* is positive in the direction of the arrow and is counted from the quantity with index c to the quantity without index. For instance: the phase shift φ^* between the acceleration of the common center of gravity ($\ddot{x}_c$) and the displacement of subject and BCG (x) equals:

$$\varphi^* = \varphi + 180°$$

Three main types of ballistocardiographs are used:

A. The *low-frequency*, critically or less than critically damped BCG.[1] The natural frequency ν_0 of the loaded BCG is low with respect to the frequency of the heart ($\nu_0 \ll 1$c/s, $\delta \leqslant 1$).

B. The *middle-frequency* critically damped BCG according to Nickerson. The natural frequency of the loaded BCG is neither high nor low with respect to the frequency of the heart ($\nu_0 = 1$–2 c/s, $\delta = 1$).

C. The *high-frequency* BCG. The natural frequency of the loaded BCG is high with respect to the frequency of the heart. According to Starr ν_0 is about 15 c/s. Dock has chosen a rigid underlayer as BCG. So, in the extreme case $\nu_0 = \infty$; the BCG does not move. Therefore Dock must record the movement of the subject. If the binding between body and BCG were infinitely strong, as has been assumed earlier in this paper, then the subject would not move when a BCG according to Dock is used. So, this BCG does not fit in the scheme of this paper. It will be discussed in a following one.[2]

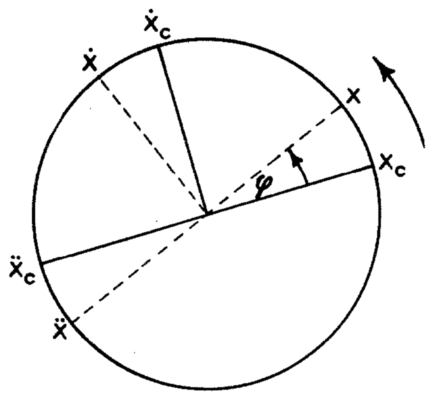


Fig. 2.—If the displacement (x), the velocity ($\dot{x}$), or the acceleration ($\ddot{x}$) of subject and BCG is recorded, the phase shift between the recorded quantity and each of the quantities displacement (x_c), velocity ($\dot{x}_c$), and acceleration ($\ddot{x}_c$) of the common center of gravity of subject and BCG equals the angle φ*. This angle can be found by measuring the angle between the quantity of the common center of gravity one is interested in, and the quantity one records. The angle must be measured in the direction of the arrow.

In this paper we will consider of the high-frequency BCG's only the less than critically damped, high-frequency BCG according to Starr.

We will calculate how each of the three types of BCG's represents:

(a) the displacement of the common center of gravity (x_c);
(b) the velocity of the common center of gravity ($\dot{x}_c$);
(c) the acceleration of the common center of gravity ($\ddot{x}_c$), if either the displacement (x) of subject and BCG or its velocity ($\dot{x}$) or its acceleration ($\ddot{x}$) is recorded. Mostly one of the three last-mentioned quantities are consciously or unconsiously recorded.

So there are the following nine amplitude characteristics for each type of BCG (the rows are indicated with a,b,c, the columns with α, β, γ):

	α	β	γ
a	$\dfrac{\lvert x \rvert}{\lvert x_c \rvert}$	$\dfrac{\lvert \dot{x} \rvert}{\lvert x_c \rvert}$	$\dfrac{\lvert \ddot{x} \rvert}{\lvert x_c \rvert}$
b	$\dfrac{\lvert x \rvert}{\lvert \dot{x}_c \rvert}$	$\dfrac{\lvert \dot{x} \rvert}{\lvert \dot{x}_c \rvert}$	$\dfrac{\lvert \ddot{x} \rvert}{\lvert \dot{x}_c \rvert}$
c	$\dfrac{\lvert x \rvert}{\lvert \ddot{x}_c \rvert}$	$\dfrac{\lvert \dot{x} \rvert}{\lvert \ddot{x}_c \rvert}$	$\dfrac{\lvert \ddot{x} \rvert}{\lvert \ddot{x}_c \rvert}$

Moreover, there are nine corresponding phase characteristics for each type of BCG. The amplitude and phase characteristics aα, bβ, and cγ are equal. Likewise aβ and bγ. Also bα and cβ. The equal ones are joined by broken lines. These equalities follow from formula (10).

The remaining five different amplitude and phase characteristics are represented:

1. In Figs. 3, 4, 5, 6, and 7 for a *low-frequency* BCG with a natural frequency of 0.3 c/s. The damping is critical ($\delta = 1.0$). In Fig. 3 also the characteristics for a less than critical damping ($\delta = 0.4$) are represented. (The ordinates of the amplitude characteristics are calculated in the c.g.s. system, so the amplitude in Fig. 5, e.g., is given in sec^2).

2. In Figs. 8, 9, 10, 11, and 12 for a *middle-frequency* BCG. The natural frequency of the loaded BCG is 1.5 c/s. Two values are chosen for the damping: (1) critical damping ($\delta = 1$) according to Nickerson; (2) damping far more than critical ($\delta = 5$). For the reason of this choice see below.

3. In Figs. 13, 14, 15, 16, and 17 for a *high-frequency* less than critically damped BCG ($\delta = 14.10^{-3}$). The natural frequency of the loaded BCG is 15 c/s. (The properties represented in these figures can also be represented by other methods.)

An amplitude characteristic has to meet the requirement that it must be flat, for instance within 20 per cent, in the frequency range one is interested in (from about 1 c/s to about 20 c/s). The phase shift must be smaller than, e.g., 20 degrees in the same frequency range. (If the phase shift is not diverging more than 20 degrees from 180 degrees, the curve is correct, for a phase shift of 180 degrees means that the curve is inverted.)

It will now be ascertained systematically which frequency characteristics meet these requirements.

A. The *low-frequency* BCG. From the characteristics in Figs. 3, 4, 5, 6, and 7 it follows that only the characteristics of Fig. 3 can be used. Moreover, it follows from Fig. 3 that the characteristics become better if we choose the damping somewhat smaller than critical. If $\delta = 0.4$ is chosen, then the characteristics are quite satisfactory.

B. The *middle-frequency* BCG. If the value of δ is chosen so that the damping is critical, as is done by various investigators, none of the frequency characteristics meet the conditions (Figs. 8, 9, 10, 11, and 12). Only if a very heavy damping is used ($\delta \geqslant 5$) the frequency characteristics in Fig. 9 are useful, after proper calibration of the apparatus.

Fig. 3.

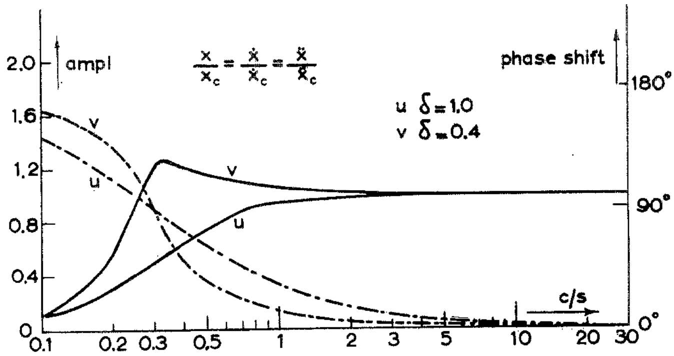


Fig. 4.

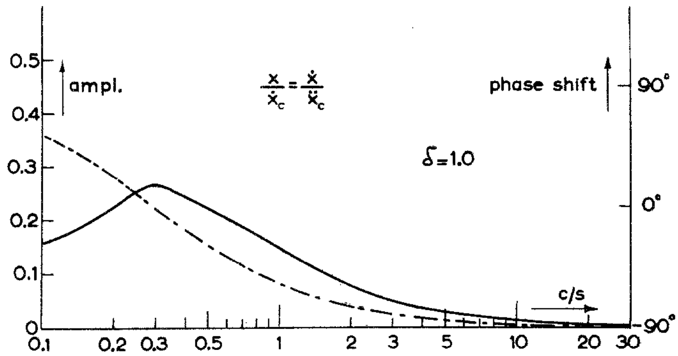


Fig. 5.

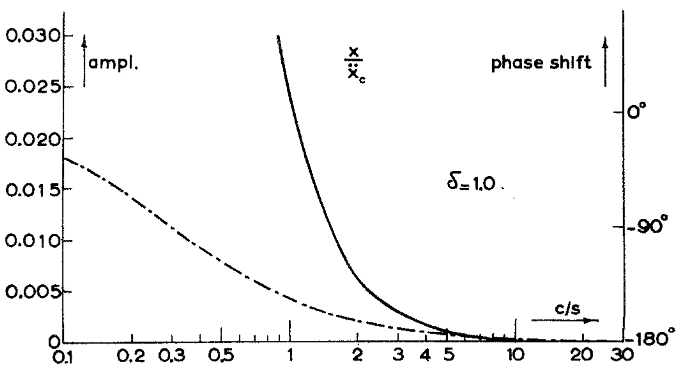


Fig. 3.—Amplitude characteristics (solid lines) and phase characteristics (broken lines) of a *low-frequency* BCG (ν_o = 0.3 c/s; δ = 1.0 and 0.4), indicating how the displacement of the center of gravity, its velocity, and its acceleration are represented by the displacement, the velocity, and the acceleration of the BCG, respectively.

Fig. 4.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *low-frequency* BCG (ν_o = 0.3 c/s; δ = 1.0), indicating how the velocity of the center of gravity and its acceleration are represented by the displacement and the velocity of the BCG, respectively.

Fig. 5.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *low-frequency* BCG (ν_o = 0.3 c/s; δ = 1.0), indicating how the acceleration of the center of gravity is represented by the displacement of the BCG.

C. The *high-frequency* BCG. From the characteristics in Figs. 13, 14, 15, 16, and 17 it follows that only those of Fig. 15 are suitable if the natural frequency of the loaded BCG (ν_o) is high enough and the apparatus is calibrated properly. So, if a reliable representation of the phenomena is wished for frequencies up to 20 c/s the natural frequency must be at least about 25 c/s. In the case of Fig. 15 ($\nu_o = 15$ c/s) the representation is reliable to about 10 c/s. Therefore, if a correct measurement of the quantities is required, the following possibilities remain:

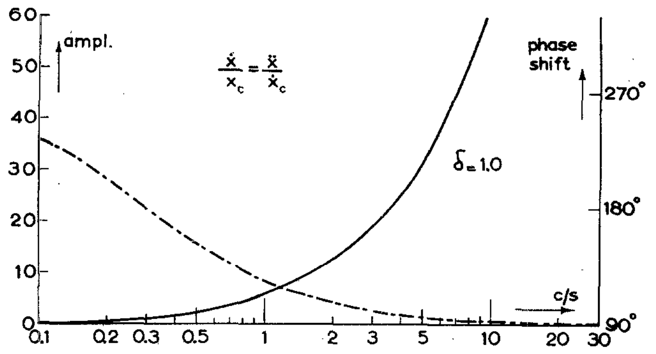


Fig. 6.

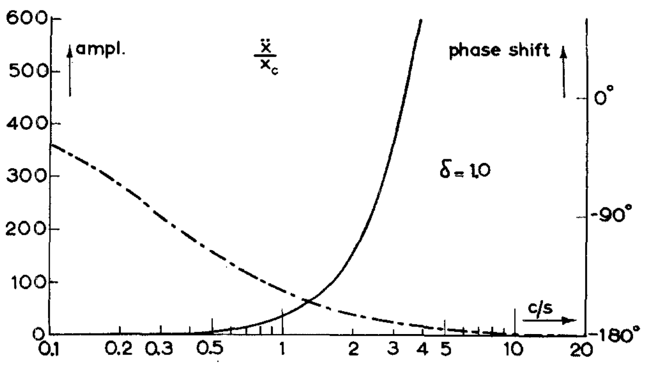


Fig. 7.

Fig. 6.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *low-frequency* BCG ($\nu_o = 0.3$ c/s; $\delta = 1.0$), indicating how the displacement of the center of gravity and its velocity are represented by the velocity and the acceleration of the BCG, respectively.

Fig. 7.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *low-frequency* BCG ($\nu_o = 0.3$ c/s; $\delta = 1.0$), indicating how the displacement of the center of gravity is represented by the acceleration of the BCG.

A. The *low-frequency* BCG (natural frequency about 0.3 c/s, δ about 0.4). (See Fig. 3.)

a. If the displacement of subject or BCG (x) is recorded, the obtained curve represents the displacement of the common center of gravity of subject and BCG (x_c), caused by the heart action.

b. If the velocity of subject or BCG ($\dot{x}$) is recorded, the obtained curve represents the velocity of the common center of gravity ($\dot{x}_c$).

Fig. 8.

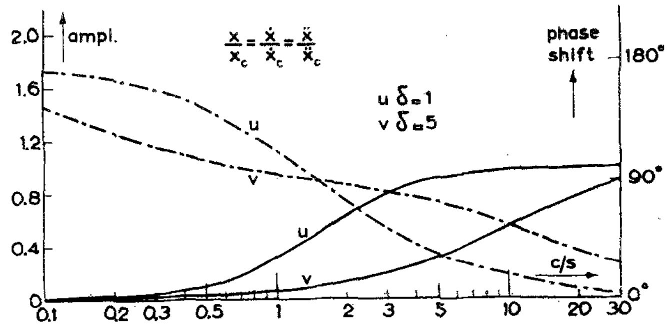


Fig. 9.

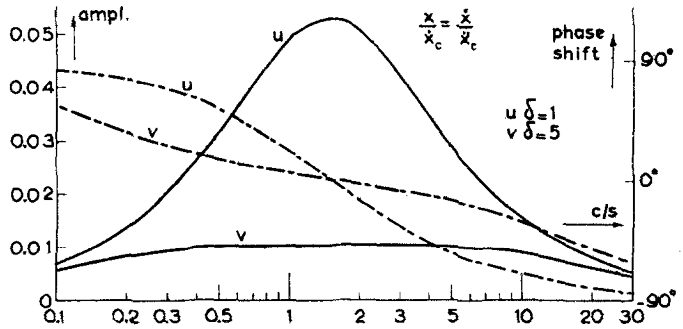


Fig. 10.

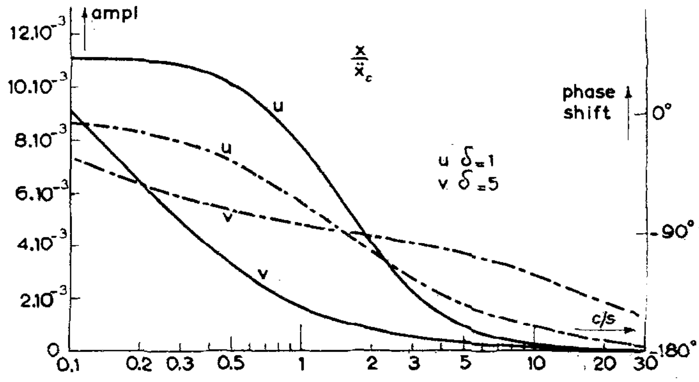


Fig. 8.—Amplitude characteristics (solid lines) and phase characteristics (broken lines) of a *middle-frequency* BCG (ν_0 = 1.5 c/s; δ = 1 and 5), indicating how the displacement of the center of gravity, its velocity, and its acceleration are represented by the displacement, the velocity, and the acceleration of the BCG, respectively.

Fig. 9.—Amplitude characteristics (solid lines) and phase characteristics (broken lines) of a *midde-frequency* BCG (ν_0 = 1.5 c/s; δ = 1 and 5), indicating how the velocity of the center of gravity and its acceleration are represented by the displacement and the velocity of the BCG, respectively.

Fig. 10.—Amplitude characteristics (solid lines) and phase characteristics (broken lines) of a *middle-frequency* BCG (ν_0 = 1.5 c/s; δ = 1 and 5), indicating how the acceleration of the center of gravity is represented by the displacement of the BCG.

c. If the acceleration of subject or BCG ($\ddot{x}$) is recorded, the obtained curve represents the acceleration of the common center of gravity ($\ddot{x}_c$).

(In principal, it is of no importance in which way x, $\dot{x}$, or $\ddot{x}$ is recorded. It is only a matter of experimental methods, and will not be discussed here.)

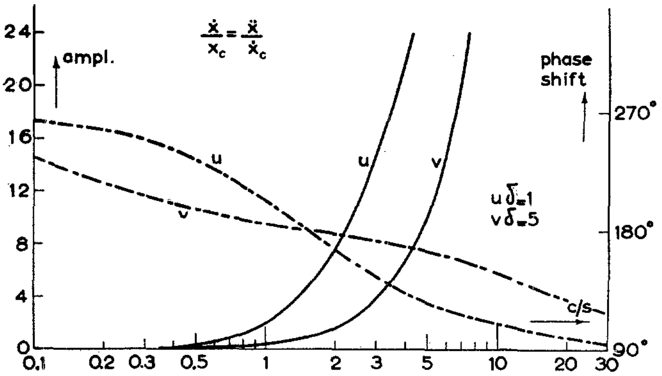


Fig. 11.

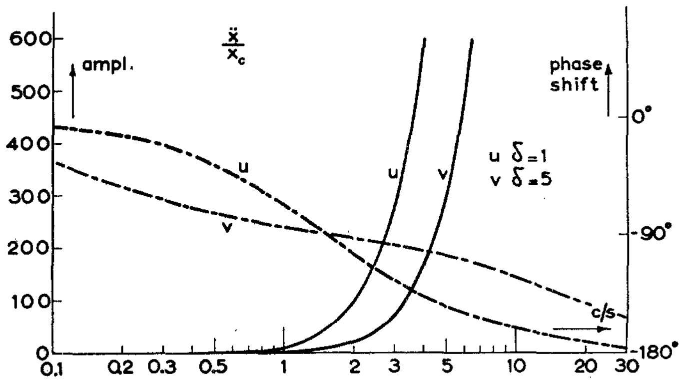


Fig. 12.

Fig. 11.—Amplitude characteristics (solid lines) and phase characteristics (broken lines) of a *middle-frequency* BCG (ν_0 = 1.5 c/s; δ = 1 and 5), indicating how the displacement of the center of gravity and its velocity are represented by the velocity and the acceleration of the BCG, respectively.

Fig. 12.—Amplitude characteristics (solid lines) and phase characteristics (broken lines) of a *middle-frequency* BCG (ν_0 = 1.5 c/s; δ = 1 and 5), indicating how the displacement of the center of gravity is represented by the acceleration of the BCG.

The curves mentioned under a, b, and c are mathematically related: The curve under b can be obtained by differentiating once the curve under a. The curve under c can be obtained by differentiating once the curve under b or, of course, by differentiating twice the curve under a.

The reverse, the curve under b and a can be obtained from the curve under c by integrating once and twice, respectively. This is schematically represented here.

Fig. 13.

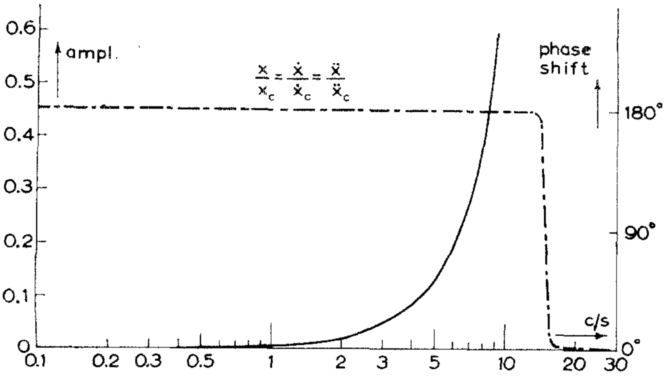


Fig. 14.

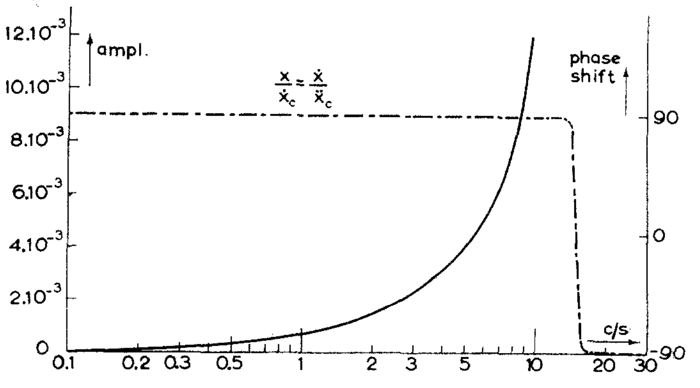


Fig. 15.

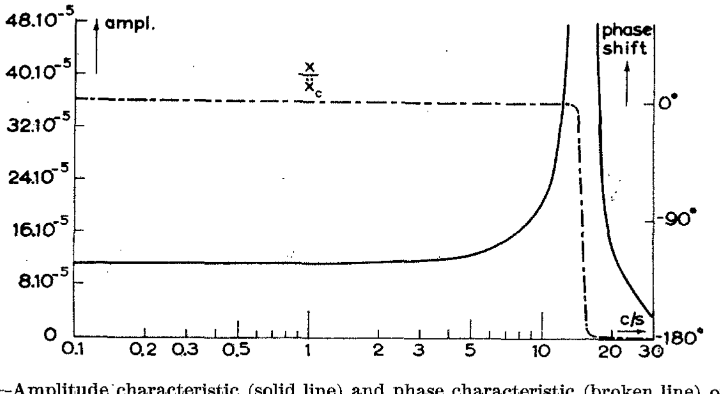
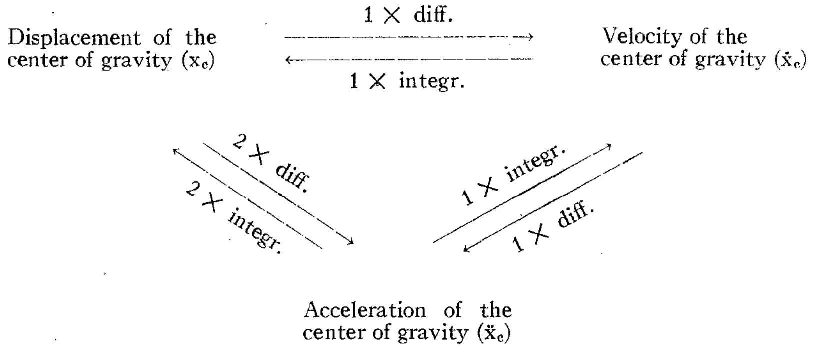


Fig. 13.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *high-frequency* BCG (ν_0 = 15 c/s; $\delta = 14.10^{-3}$), indicating how the displacement of the center of gravity, its velocity, and its acceleration are represented by the displacement, the velocity, and the acceleration of the BCG, respectively.

Fig. 14.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *high-frequency* BCG (ν_0 = 15 c/s; $\delta = 14.10^{-3}$), indicating how the velocity of the center of gravity and its acceleration are represented by the displacement and the velocity of the BCG, respectively.

Fig. 15.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *high-frequency* BCG (ν_0 = 15 c/s; $\delta = 14.10^{-3}$), indicating how the acceleration of the center of gravity is represented by the displacement of the BCG.



B. The *middle-frequency* BCG (natural frequency about 1.5 c/s, δ greater than 5). (See Fig. 9.)

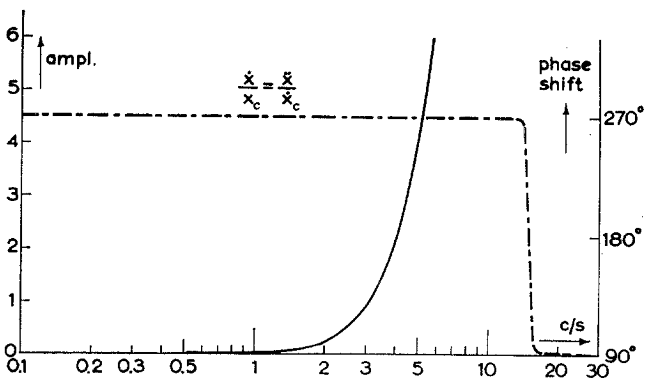


Fig. 16.

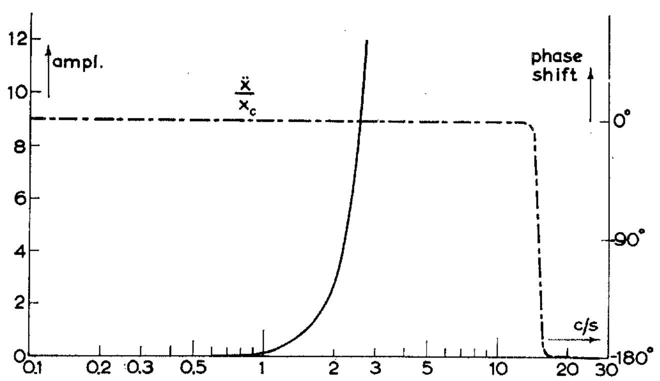


Fig. 17.

Fig. 16.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *high-frequency* BCG (ν_0 = 15 c/s; $\delta = 14.10^{-3}$), indicating how the displacement of the center of gravity and its velocity are represented by the velocity and the acceleration of the BCG, respectively.

Fig. 17.—Amplitude characteristic (solid line) and phase characteristic (broken line) of a *high-frequency* BCG (ν_0 = 15 c/s; $\delta = 14.10^{-3}$), indicating how the displacement of the center of gravity is represented by the acceleration of the BCG.

b. If the displacement of subject or BCG is recorded, the obtained curve represents the velocity of the common center of gravity of subject and BCG, caused by the heart action.

c. If the velocity of subject or BCG is recorded, the obtained curve represents the acceleration of the common center of gravity.

The curves under b and c are mathematically related in the same way as described above. Moreover, the displacement of the center of gravity can be found by integrating once the curve under b or twice the curve under c.

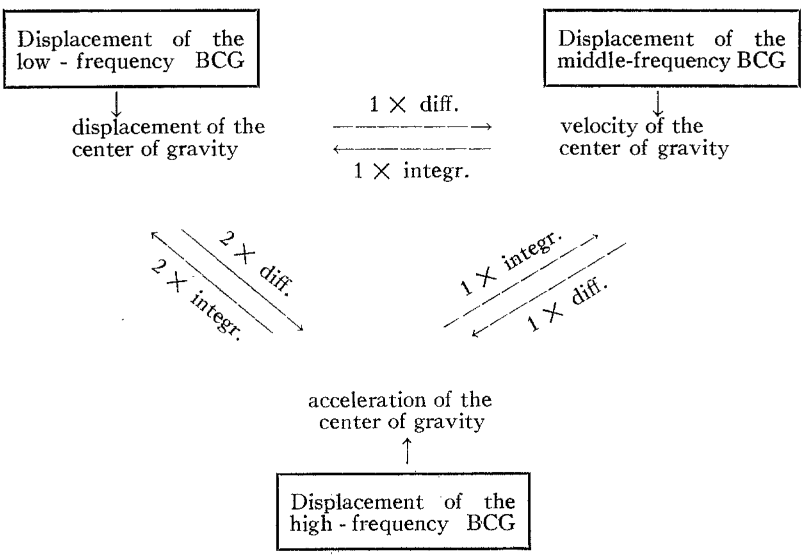
C. The *high-frequency* BCG (natural frequency higher than 20 c/s, δ very small with respect to 1.) (See Fig. 15.)

c. If the displacement of subject or BCG is recorded, the obtained curve represents the acceleration of the common center of gravity of subject and BCG.

The velocity and the displacement of the center of gravity can be obtained from this curve by integrating it once and twice respectively.

From the above it follows that the displacement, the velocity, and the acceleration of the common center of gravity of subject and BCG can be found by means of a low-frequency, a middle-frequency, and a high-frequency BCG, if the natural frequency and the degree of damping meet the requirements mentioned previously.

The results obtained with the different types of ballistocardiographs are mutually related provided that subject and BCG move exactly in the same way, and the different parts of the body do not move with respect to each other (as has been assumed earlier in this paper). The relation when the *displacement* of each type is recorded is shown here.



Analogous schemes can be made if other quantities are recorded (27 schemes in all).

In fact, it is not realizable that the movement of the subject and of the BCG are exactly the same. The stronger the binding of the BCG to the surroundings (the higher the natural frequency and the heavier the damping), the greater the difference in movement (the "relative movement"). As a result of this relative movement the amplitude characteristic of the high-frequency BCG is not half as excellent as in Fig. 15. Because of this phenomenon it is preferable to use a low-frequency BCG if a correct representation of the occurrences is desired. This point will be worked out in a following paper.[2]

SUMMARY

It is investigated which quantity concerning the common center of gravity of a subject is recorded by a low-frequency, a middle-frequency, and a high-frequency ballistocardiograph. It appeared that the displacement of these types of ballistocardiographs represents the displacement, the velocity, and the acceleration of the center of gravity, respectively, if the external circumstances meet the given requirements. Corresponding amplitude and phase characteristics are calculated.

A relation between the quantities of displacement, velocity, and acceleration of the center of gravity is deduced, from which follows the method by which these three quantities can be found from each of the three types of ballistocardiographs.

In this paper the binding between body and ballistocardiograph is assumed to be infinitely strong.

SUMMARIO IN INTERLINGUA

Esseva investigate qual quantitate relative al commun centro de gravitate del subjecto es registrate per le ballistocardiographo a basse, a median, e a alte frequentia. Il pareva que le displaciamento de iste tres typos de cardiographo representa, respectivemente, le displaciamento, le velocitate, e le acceleration del centro de gravitate, providite que le conditiones externe satisface le correspondente requirimentos. Le correspondente amplitude e characteristicas phasic esseva calculate.

Esseva deducite un relation inter displaciamento, velocitate, e acceleration del centro de gravitate. Per medio de iste relation le tres mentionate quantitates es determinabile con omne le tres typos de ballistocardiographo.

In le deductiones del presente reporto il es assumite que le fortia ligante inter corporee balli stocardiographo es infinite.

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