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1957

Physica XXIII
481–532

PERTURBATION THEORY OF LARGE QUANTUM SYSTEMS

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Synopsis

The time-independent perturbation theory of quantum mechanics is studied for the case of very large systems, *i.e.* systems with large spatial dimensions (large volume Ω), and a large number of degrees of freedom. Examples of such systems are met with in the quantum theory of fields, solid state physics, the theory of imperfect gases and in the theory of nuclear matter. Only systems at or near the ground state (*i.e.*, systems at zero temperature) are treated in this paper. In the application of the conventional perturbation theory to such large quantum systems one encounters difficulties which are connected with the fact that even small perturbations produce large changes of the energy and wave function of the whole system. These difficulties manifest themselves through the occurrence of terms containing arbitrarily high powers of the volume Ω in the perturbation expansion of physical quantities. An extremely bad convergence of the perturbation expansion is the result.

For the analysis of the Ω-dependence of the terms in the expansion a new formulation of the time-independent perturbation theory is used, which was introduced by Van Hove. Making extensive use of diagrams to represent the different contributions to matrix elements it is possible to locate and separate the Ω-dependent terms, and to carry out partial summations in the original expansion. These separations and summations solve the above difficulties completely. Improved perturbation theoretical expressions are obtained for energies and wave functions of stationary states, as well as for the life-times of metastable states. All terms in these expressions are, in the limit of large Ω, either independent of Ω or proportional to Ω, corresponding to intensive or extensive physical quantities. The convergence of the improved perturbation expansions is no longer affected by the large magnitude of Ω.

CHAPTER I. INTRODUCTION

1. *The problem.* This paper is devoted to the perturbation theory of large quantum systems *i.e.*, quantum systems which have large spatial dimensions and a large number of degrees of freedom. The systems met with in the quantum theory of fields are, as is well known, of this type. Also in other branches of physics, such as quantum statistics and the Fermi gas model of heavy nuclei, one has to deal with such large systems. We shall in this paper only be interested in systems at or near the ground state. Our results are, therefore, only applicable to quantum systems at zero temperature.

The separation of the hamiltonian into an unperturbed part and a perturbation is not unique, but in most problems of interest there is a separation which presents itself in a most natural way. In quantum electrodynamics for example, the unperturbed system consists of the electron-positron field and the photon field without interaction. In the theory of an imperfect gas the unperturbed system will be taken as the ideal gas obtained by neglecting interparticle interactions.

In the application of perturbation theory to large quantum systems one encounters problems not met with in the usual perturbation theory of systems with a finite number of degrees of freedom. These problems are related to the following phenomena:

1. Self-energy and cloud effects of individual particles in excited states.
2. The perturbation of the system as a whole.

We shall discuss briefly the first point. The effects mentioned are well known in field theory. A state of one single electron is changed by the perturbation into a superposition of many different unperturbed states, where the one-electron-state is admixed with states containing one or more photons and electron-positron pairs. One usually says that the electron is surrounded by a cloud of photons and pairs. The self-energy of the electron manifests itself by a change of its mass. Also for a scattering state of two or more particles the interaction gives rise to the self-energy and cloud effects just mentioned, in addition to the directly observable scattering effects. While the latter are transient, *i.e.* take place (for general wave packets) within a finite time interval, the former are persistent effects which cause a permanent change of wave function and energy. Effects of this type are not limited to field theory, but occur also in many other systems.

Recently Van Hove [1]) made an extensive study of these phenomena. He developed a time-independent perturbation formalism which is adapted to the treatment of perturbations causing persistent effects. The developments in this paper are largely based on his work.

The effects just discussed concern the motion of one or more particles of the system, which is itself in a quantum state distinct from the ground state (the vacuum state of field theory). The self-energy is a shift caused by the perturbation in the distance between the energy level of the system in the state at hand and the ground state level. It is to be expected that such effects are independent of the volume Ω of the system, in the limit of $\Omega \to \infty$. For instance the self-energy of an electron is not appreciably changed if the fields are enclosed in a box of variable volume, at least for sufficiently large values of the volume.

The problems of the second type mentioned above, which form the subject of the present investigation, are connected with the overall shift of the energy levels, both of the ground state and of excited states. For large systems, as considered in this paper, one must expect that even weak

perturbations give rise to large changes of the wave function and energy of the system. Such effects occur in field theory whenever virtual pair production from the ground state is possible. Disregarding surface effects one must expect on physical grounds that the energy of the ground state, both in the unperturbed and in the perturbed system, is proportional to the volume Ω of the system. This implies that the energy shift ΔE_0 caused by the perturbation is also proportional to Ω. One often uses the words *extensive* and *intensive* for quantities which are respectively proportional to and independent of the volume of the system. Using this terminology one can say that the energy shift ΔE_0 of the ground state is an extensive effect.

In studying states distinct from the ground state we have to deal with both types of phenomena mentioned above. For example, the total energy shift of a one-electron-state is the sum of the vacuum energy shift and the self-energy. Generally, the energy of an excited state can be written as the sum of two terms, one being the energy of the ground state, the other the excitation energy. In the limit $\Omega \to \infty$ the excitation energy is independent of Ω. Both terms are affected by the perturbation; hence the energy shift of an excited state must be the sum of an extensive and an intensive term. This expectation will be confirmed by our results. A separation of the same kind will be shown to exist for the change of the wave function of an excited state. This change is partly a consequence of the change of the ground state wave function, and involves in addition effects due to the excitation. The latter have an intensive character, while the change in the ground state wave function will be found extensive. It is a shortcoming of the conventional perturbation theory that these effects are not separated. It will be seen that this leads to serious difficulties. The expansion of matrix elements in powers of the perturbation contains terms with arbitrarily high powers of the volume Ω of the system, and this gives rise to an extremely bad convergence in the case of large systems.

It is the object of the present investigation to make a clear and complete separation between extensive and intensive effects. The hamiltonian of the system is written in the occupation number representation for the one-particle plane wave states, and we use diagrams to represent the different contributions to matrix elements, as is conventionally done in field theory. This appears to be a valuable tool for the analysis. Although the method and the developments of the following chapters are of a rather general nature and can be applied generally to field theoretical problems and to problems in quantum statistics, especially in solid state physics, the method will be illustrated mainly by considering the example of a Fermi gas with two-body interaction between the particles. This example is described in the next section. In a forthcoming paper our methods will be used for a discussion of the Fermi gas model of nuclear matter. The example adopted here will become there the main object of study. Chapter II contains an

exposition of the perturbation formalism of Van Hove, which is based on a systematic use of the resolvent operator. Our exposition differs from the original presentation only in that extensive use is made of diagrams. In chapter III we investigate how the contributions to the resolvent, represented by different kinds of diagrams, depend on the volume Ω. This analysis shows that a separation is possible between Ω-dependent and Ω-independent quantities. The separation is carried through in chapters IV and V, for the energies and the wave functions of stationary states respectively. Explicit expressions in the form of improved perturbation expansions are derived. In the series expansions for intensive quantities all terms are independent of Ω, whereas all terms are proportional to Ω for extensive quantities.

In the last section of chapter V another phenomenon occurring in large systems is investigated. There are systems where no perturbed stationary states correspond in any simple way to the unperturbed excited states. Such systems are well known from statistical mechanics. Their most striking property is the occurrence of dissipative processes. In the case of small dissipation one can show the existence of metastable states. We shall derive explicit expressions for the life-time, energy and wave function of such states. The life-time will be found independent of Ω, in accordance with physical expectations. A very interesting example of such metastable states is encountered in the optical model description for the scattering of nucleons on heavy nuclei [2]) where a complex potential is introduced to account for the finite mean free path of nucleons in nuclear matter. This will be further analysed in the forthcoming paper already announced, where the present formalism will be applied to a system of interacting nucleons. It will be seen that the theory of Brueckner [3]) for the structure of nuclear matter can be considered as a special approximation to our general formalism. The latter will be helpful for getting new insight into the significance and limits of validity of Brueckner's method.

2. *The Fermi gas.* In this section we shall give a formulation of the N-particle problem, which is adapted to the treatment of a Fermi gas where both the number N and the volume Ω are large. This system will be used as a working example in the rest of the paper. For the interaction between the particles we take central two-body forces, and we shall neglect the spin of the particles. We enclose the whole system in a large cubic box with side L and volume $\Omega = L^3$, and we impose periodic boundary conditions. We have chosen these boundary conditions for mathematical convenience. Because we are particularly interested in large systems, the influence of surface effects is comparatively small.

The wave functions

$$\psi_k(x) = \Omega^{-\frac{1}{2}} \exp(ikx) \tag{2.1}$$

where the three components of k can have the values $2\pi n/L (n=0, \pm 1, \pm 2, \ldots)$, describing the motion of a single particle with momentum k *), form a complete orthonormal set of single particle states. A state of N identical particles moving independently of each other is determined by a series of occupation numbers N_k, each giving the number of particles in the one-particle state (2.1) with momentum k. These states form the basic set of the unperturbed system. In order to obtain a simple expression of the hamiltonian in this representation, we introduce the annihilation and creation operators η_k and η_k^* which obey in the case of Fermi particles the anti-commutation laws

$$\{\eta_k, \eta_l\} = \{\eta_k^*, \eta_l^*\} = 0 \text{ and } \{\eta_k, \eta_l^*\} = \delta_{kl}. \tag{2.2}$$

Using these operators the hamiltonian can be written

$$H = H_0 + V,$$

where

$$H_0 = \textstyle\sum_k \frac{1}{2}(|k|^2/M)\, \eta_k^* \eta_k \tag{2.3}$$

and

$$V = \tfrac{1}{2}\, \Omega^{-1} \textstyle\sum_{klmn} \delta_{Kr}\, (k + l - m - n)\, v(k - n) \eta_k^* \eta_l^* \eta_m \eta_n.$$

Here $v(k)$ is the Fourier-transform of the central two-body potential $v(r)$:

$$v(k) = \int \mathrm{d}^3x\, v(r) \exp(-ikx), \tag{2.4}$$

and depends only on the modulus $|k|$ of k. The Kronecker symbol δ_{Kr} is equal to one if the argument is zero and vanishes otherwise. It expresses the fact that momentum is conserved in the interaction.

The creation operators η_k^* can be used to obtain simple expressions for the states of our basic set of unperturbed states. By $|0\rangle$ we denote the normalized state without any particles. It is determined by the condition that $\eta_k\, |0\rangle = 0$ for all k. A state of N particles with momenta $k_1, k_2, \ldots, k_N$ can be written

$$\eta_{k_1}^* \eta_{k_2}^* \ldots \eta_{k_N}^* \, |0\rangle. \tag{2.5}$$

The commutation rules (2.2) imply that this state vector is normalized to one and is antisymmetric in the N particles.

The formulation given thus far is not very suitable for our case. Of physical interest is the case where Ω and N are very large for a given value of the density $\varrho = N/\Omega$. In the limit $\Omega \to \infty$ one has a continuous spectrum and summations are replaced by integrations. The normalization of states must be changed as can be seen from (2.1) where $\psi_k(x)$ vanishes in the limit

*) We use no special notation to indicate vectors. The letters k, l, m, n are used for momenta, whereas in (2.1) and (2.4) x is a vector in configuration space. We put $\hbar = 1$ throughout this paper.

$\Omega \to \infty$. It seems therefore appropriate for a finite but large Ω to adopt another normalization of the wave vectors. We introduce the following new notations (L is the side of the cubic volume Ω)

$$\xi_k = (L/2\pi)^{3/2}\,\eta_k \text{ and } \delta(k-l) = (L/2\pi)^3\,\delta_{Kr}(k-l). \tag{2.6}$$

The function $\delta(k)$ of the discrete variable k goes over into the 3-dimensional delta-function of Dirac in the limit of $\Omega \to \infty$.

The commutation relations of ξ_k and ξ_k^* read

$$\{\xi_k, \xi_l\} = \{\xi_k^*, \xi_l^*\} = 0 \text{ and } \{\xi_k, \xi_l^*\} = \delta(k-l).$$

Introducing the notation

$$\textstyle\int_k = (2\pi/L)^3 \sum_k$$

the hamiltonian (2.3) reads

$$H = \textstyle\int_k (|k|^2/2M)\,\xi_k^*\xi_k + \frac{1}{2}(2\pi)^{-3}\int_{klmn}\delta(k+l-m-n)\,v(k-n)\,\xi_k^*\xi_l^*\xi_m\xi_n. \tag{2.7}$$

With this notation it is extremely simple to pass over to the limit of $\Omega \to \infty$, the only change being that the summation symbol $\int_k$ is replaced by the integration sign $\int d^3k$. The states of the unperturbed system will now also be expressed by means of the operators ξ_k^*. For a N-particle state one gets

$$\xi_{k_1}^*\,\xi_{k_2}^* \ldots \xi_{k_N}^*\,|0\rangle, \tag{2.8}$$

which only differs from (2.5) by the normalization.

Obviously (2.8) is not very suitable for the case of a very large number of particles. Therefore we shall proceed in a different way. We draw in momentum space a sphere with centre in the origin and radius k_F and consider a normalized unperturbed N-particle state $|\varphi_0\rangle$, such that all one-particle states with momenta within the sphere are occupied, whereas there are no particles with momenta outside the sphere. This state $|\varphi_0\rangle$ is obviously the ground state of the unperturbed system of N particles. The sphere is often called the *Fermi sphere* and the set of particles occupying the states within this sphere we shall call the *Fermi sea*, in analogy with the Dirac sea of Dirac's hole theory for electrons and positrons.

The number N is a discontinuous function of the Fermi momentum k_F which we keep fixed and consider as a substitute for the density parameter. In the limit of $\Omega \to \infty$ we have however asymptotically

$$N = k_F^3\Omega/6\pi^2, \tag{2.9}$$

as follows from the fact that each one-particle state in momentum space occupies a volume $(2\pi)^3/\Omega$, $2\pi/L$ being the distance of the lattice points. The number of states within a sphere with the volume $\frac{4}{3}\pi k_F^3$ is then given by (2.9). Also the total kinetic energy ε_0 of $|\varphi_0\rangle$ is strictly speaking a discontinu-

ous function of k_F. Again neglecting terms that vanish in the limit of $\Omega \to \infty$ one has however

$$\varepsilon_0 = k_F^5 \Omega / 20\pi^2 M. \tag{2.10}$$

We shall now characterize the states of the unperturbed system by comparing them with $|\varphi_0\rangle$. An arbitrary state of the basic system can be obtained from $|\varphi_0\rangle$ by removing a number of particles from the Fermi sea and adding some others with momenta outside the Fermi sphere. In other words one can get any state of the basic system by the application to the state $|\varphi_0\rangle$ of a number of annihilation operators ξ_m, with $|m| < k_F$, and of a number of creation operators ξ_k^* with $|k| > k_F$. We introduce the notation

$$|k_1 k_2 \dots k_p \,;\, m_1 m_2 \dots m_q\rangle = \xi_{k_1}^* \xi_{k_2}^* \dots \xi_{k_p}^* \xi_{m_1} \xi_{m_2} \dots \xi_{m_q} |\varphi_0\rangle, \tag{2.11}$$

where

$$|k_i| > k_F \text{ and } |m_j| < k_F.$$

The conjugate wave function shall be denoted by

$$\langle m_q \dots m_2 m_1 \,;\, k_p \dots k_2 k_1|.$$

The state (2.11) differs from $|\varphi_0\rangle$ by the absence of q particles, with momenta $m_1, m_2, \dots, m_q$, from the Fermi sea while there are p additional particles, with momenta $k_1, k_2, \dots, k_p$, outside the Fermi sphere.

An unoccupied one-particle state will often be called a *hole*. By the energy and momentum of a hole we shall mean the energy and momentum of the missing particle, taken both with the opposite sign. Hence the energy carried by a hole is negative. In this terminology, which is selected in analogy to the hole theory of Dirac, the additional particles with momenta outside the Fermi sphere are briefly called particles. Thus the state (2.11) contains q holes and p particles. In this way we are led to a reinterpretation of the operators ξ and ξ^*.

For $|k| > k_F$, ξ_k annihilates a particle and ξ_k^* creates a particle.

For $|m| < k_F$, ξ_m creates a hole and ξ_m^* annihilates a hole.

Finally we shall study some different types of transitions which can be brought about by the interaction V. It is often convenient to have a more symmetrical expression for V than in (2.7).

One can write

$$V = \tfrac{1}{4} \int_{l_1 l_2 l_3 l_4} v(l_1 l_2 l_3 l_4)\, \xi_{l_1}^* \xi_{l_2}^* \xi_{l_3} \xi_{l_4}, \tag{2.12}$$

where

$$v(l_1 l_2 l_3 l_4) = (2\pi)^{-3} (v(l_1 - l_4) - v(l_1 - l_3))\, \delta(l_1 + l_2 - l_3 - l_4).$$

The function $v(l_1 l_2 l_3 l_4)$ has the following symmetry properties

$$v(l_1 l_2 l_3 l_4) = -\, v(l_2 l_1 l_3 l_4) = v(l_3 l_4 l_1 l_2).$$

In (2.12) the summation is extended over all momenta $l_1 l_2 l_3 l_4$, both inside and outside the Fermi sphere.

We make the following convention. The letters m and k will be used for momenta inside and outside the Fermi sphere respectively.

Hence the expression

$$\tfrac{1}{4}\int_k v(k_1k_2k_3k_4)\ \xi^*_{k_1}\xi^*_{k_2}\xi_{k_3}\xi_{k_4}$$

differs from (2.12) by restricting the summation to momenta k such that $|k| > k_F$. This term describes the absorption of two particles and the emission of two other particles, a process which can be interpreted as the scattering of two particles. In exactly the same way the term

$$\tfrac{1}{4}\int_m v(m_1m_2m_3m_4)\ \xi^*_{m_1}\xi^*_{m_2}\xi_{m_3}\xi_{m_4}$$

gives rise to the scattering of two holes. Let us finally consider

$$\tfrac{1}{4}\int_{k_1k_2k_3m} v(k_1k_2k_3m)\ \xi^*_{k_1}\xi^*_{k_2}\xi_{k_3}\xi_m,$$

where one particle is absorbed and two particles and a hole are emitted. This process can also be described in another way. A particle interacts with a particle in the Fermi sea, thereby removing it to a state outside the Fermi sphere. This leads to a state of two particles and one hole.

Chapter II. Formulation of the time-independent perturbation method by means of diagrams

3. *Diagrams.* We consider a large but finite quantum system with a hamiltonian $H = H_0 + V$. The basic set of unperturbed states $|\alpha\rangle$ are eigenstates of H_0 with the eigenvalue ε_α. If the system is a gas of Fermi particles as was studied in section 2, the states $|\alpha\rangle$ are to be identified with the states $|k_1k_2 \dots k_p\ ;\ m_1m_2 \dots m_q\rangle$ defined in (2.11), with arbitrary p and q. The unperturbed energy ε_α is then given by

$$\varepsilon_\alpha = \varepsilon_0 + \sum |k_j|^2/2M - \sum |m_i|^2/2M. \tag{3.1}$$

If $|\psi\rangle$ is some time-independent wave function then the wave function $|\psi(t)\rangle = U(t)|\ \psi\rangle$ with $U(t) = \exp(-iHt)$ solves the Schrödinger equation. Instead of the operator $U(t)$, we investigate, following Van Hove, a related time-independent operator, the *resolvent* $R(z)$, which depends on the complex number z. $R(z)$ is defined by

$$R(z) = (H - z)^{-1} = (H_0 + V - z)^{-1}, \tag{3.2}$$

and, because H is hermitian, $R(z)$ is a bounded operator for non-real z. The connection between $R(z)$ and $U(t)$ is given by the formula

$$U(t) = -\ (2\pi i)^{-1} \oint \mathrm{d}z\ R(z) \exp(-\ izt). \tag{3.3}$$

The path of integration is a contour around a sufficiently large portion of the real axis of the z-plane. It is to be described counterclockwise. Therefore we are only interested in the behaviour of $R(z)$ in the neighbourhood of the real axis.

From (3.2) follows

$$R(z) = (H_0 - z)^{-1} - (H_0 - z)^{-1}\, VR(z) = (H_0 - z)^{-1} - R(z)\; V(H_0 - z)^{-1}. \quad (3.4)$$

Iterating this formula one finds the series expansion:

$$R(z) = (H_0 - z)^{-1} - (H_0 - z)^{-1}\, V(H_0 - z)^{-1} + \\ + (H_0 - z)^{-1}\, V(H_0 - z)^{-1}\, V(H_0 - z)^{-1} - \ldots \quad (3.5)$$

In this paper the convergence of this series for z away from the real axis will be assumed. Whether this assumption is legitimate has to be investigated in each case.

For the calculation of matrix elements of $R(z)$ we represent the contributions to the various terms in (3.5) by diagrams. To be more specific, we shall turn to the case of the Fermi gas. Let us, as a simple example, consider the matrix element $\langle\beta\,|R(z)|\,\alpha\rangle$ between the initial state $|\alpha\rangle = |k_1\,;\,\rangle$ and the final state $|\beta\rangle = |k_2k_3\,;\,m\rangle$, and see how one calculates the second order term in the expansion (3.5). Using (2.11), and (2.12) we get the expression

$$4^{-2} \int_{l_1l_2l_3l_4} \int_{n_1n_2n_3n_4} v(l_1l_2l_3l_4)\; v(n_1n_2n_3n_4)\,. \\ .\,\langle\varphi_0|\xi^*_m\xi_{k_3}\xi_{k_2}(H_0-z)^{-1}\xi^*_{l_1}\xi^*_{l_2}\xi_{l_3}\xi_{l_4}(H_0-z)^{-1}\xi^*_{n_1}\xi^*_{n_2}\xi_{n_3}\xi_{n_4}(H_0-z)^{-1}\xi^*_{k_1}|\varphi_0\rangle, \quad (3.6)$$

with the summation symbols $\int_{l_1l_2l_3l_4}$ introduced in section 2. The summation is extended over all momenta both inside and outside the Fermi sphere. The ground state to ground state matrix element in the integrand will only have a value different from zero provided (when reading from right to left) each particle or hole created in a virtual transition is reabsorbed in a later transition. In each non-vanishing contribution there must exist a one-to-one correspondence between creation and annihilation operators. Each associated pair consists of a creation operator and an annihilation operator belonging to the same particle or hole. Reading always from right to left the creation operator comes first.

We shall now see how one can represent such a contribution by a *diagram* †). Each interaction operator V is represented by a point (also called *vertex*). The operators ξ and ξ^* are represented by directed lines joining at this point. The direction is indicated by an arrow. If the direction of a line is pointing to the vertex, the line represents a ξ operator in this point, in the other case a ξ^* operator. The distinction between holes and particles is made in the following way. Lines directed to the left correspond to particles, lines directed to the right to holes. This results in the four possibilities shown

†) The use of diagrams is well known in field theory, where they were first introduced by Feynman [4]). Goldstone [5]) introduced them for the many particle problem. His diagrams are slightly different from the diagrams used here.

in the following table (the convention $|k| > k_F$, $|m| < k_F$ should be kept in mind)

Operator	Represented by
ξ_k	$\bullet \leftarrow$
ξ_m^*	$\bullet \rightarrow$
ξ_k^*	$\leftarrow \bullet$
ξ_m	$\rightarrow \bullet$

One sees that creation operators ξ_k^*, ξ_m (annihilation operators ξ_k, ξ_m^*) are represented by lines reaching the vertex from the left (right). Of the four lines joining at one point two and only two are directed towards that point.

Some diagrams representing different contributions to (3.6) are drawn in figure 1. Each diagram contains two points, the order from right to left

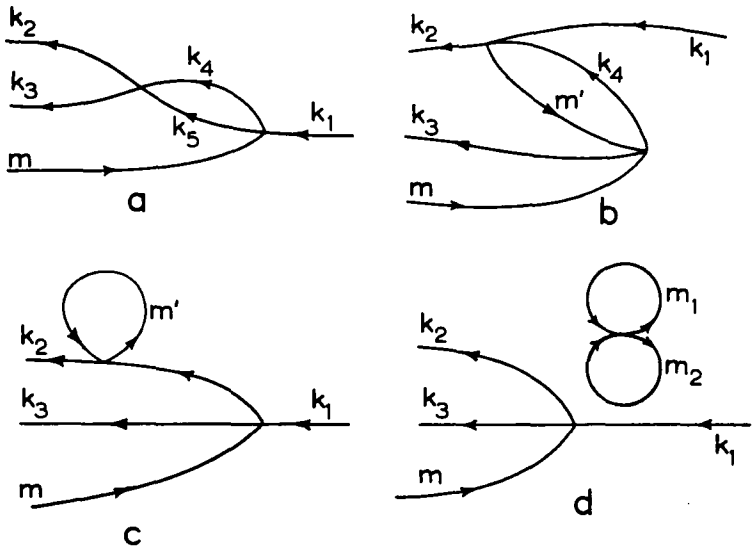

Fig. 1. Some second order diagrams contributing to the matrix element $\langle m;\, k_3k_2\, |R(z)|\, k_1;\rangle$.

corresponding to the order of the V's in (3.6). A line joining two points corresponds to an associated pair of a creation and an annihilation operator. We call this an *internal line*. Lines running from a point towards the right or left edge of the diagram or from one edge of the diagram to the other correspond to associated pairs of which one or both belong to the initial or final state. Such lines are called *external lines*. All diagrams of figure 1 have two internal lines and four external lines.

Diagrams give a schematic picture of the transition process that takes place. In the diagram of fig. 1*a* the particle of the initial state interacts with the Fermi sea giving a hole and two particles. In the second transition these

two particles are scattered on each other. The process represented by fig. 1*b* is different. The first interaction gives rise to the formation of two particles and two holes. In the second transition one particle and one hole are annihilated together with the particle of the initial state. Another particle is created.

Some remarks must be made regarding the diagrams of fig. 1, *c* and *d*. These give contributions to equation (3.6) with associated pairs of creation and annihilation operators belonging to the same interaction V. In V, reading from right to left, the ξ's precede the ξ^*'s. Therefore such associated pairs are possible for holes. In the diagrams they are represented by closed loops through a vertex point, the lines that represent the ξ_m^* and the ξ_m at the same point being the continuation of each other.

Before proceeding we introduce some definitions to be used frequently in the following. Diagrams that can be divided into two or more partial diagrams without cutting any lines are called *disconnected*. All other diagrams are *connected*. The diagrams shown in fig. 1*a*, *b* and *c* are obviously connected, whereas the diagram in *d* is disconnected. The connected parts, a disconnected diagram is composed of, will be called the *components* of the diagram. Diagrams without external lines will often be referred to as *ground state diagrams* and diagram components without external lines as *ground state components*. The diagram of fig. 1*d* has two components of which one is a ground state component.

We shall now show how one calculates the contribution of a given diagram. Let us take the diagram of fig. 1*a*. Putting in (3.6) $l_1 = k_2$, $l_2 = k_3$, $l_3 = n_2 = k_4$, $l_4 = n_1 = k_5$, $n_3 = m$ and $n_4 = k_1$ we find the expression

$$\tfrac{1}{16} \int_{k_4 k_5} v(k_2 k_3 k_4 k_5)\, v(k_5 k_4 m k_1)\,.$$
$$.\ \langle \varphi_0 | \xi_m^* \xi_{k_3} \xi_{k_2} (H_0 - z)^{-1} \xi_{k_2}^* \xi_{k_3}^* \xi_{k_4} \xi_{k_5} (H_0 - z)^{-1} \xi_{k_5}^* \xi_{k_4}^* \xi_m \xi_{k_1} (H_0 - z)^{-1} \xi_{k_1}^* | \varphi_0 \rangle . \quad (3.7)$$

The unperturbed energies of the initial, final and intermediate state can be obtained from (3.1). The last factor in (3.7) (without the energy denominators) is ± 1, the sign depending on the order of the operators ξ and ξ^*. Here the number of permutations necessary to bring the operators of associated pairs next to each other is even, hence we get a plus sign. One obtains identical contributions if one interchanges the role of the two ξ's or of the two ξ^*'s belonging to the same V. Hence a factor 4 must be added for each point, which gives a factor 16 in this example, exactly canceling the factor $1/16$. However each pair of equivalent lines, *i.e.* lines between the same two points and with the same direction, is counted twice, so that a factor $\frac{1}{2}$ must be added for each such pair. In our example the lines k_4 and k_5 are equivalent and the total factor is $\frac{1}{2}$. For the total contribution of the diagram of fig. 1*a* to

$$\langle m\ ;\ k_3 k_2\, |R(z)|\, k_1\ ;\ \rangle$$

one finds

$$\tfrac{1}{2}\int_{k_4k_5} \frac{v(k_2k_3k_4k_5)\, v(k_5k_4mk_1)}{(\varepsilon_0 + k_2^2/2M + k_3^2/2M - m^2/2M - z)\,(\varepsilon_0 + k_4^2/2M + k_5^2/2M - m^2/2M - z)\,(\varepsilon_0 + k_1^2/2M - z)}$$

This example suffices to indicate how one calculates matrix elements of the resolvent (3.5) to any order in the perturbation. One draws all possible diagrams of the given order and one adds their contributions, each of which is calculated in the way shown.

A remark must still be made concerning the Pauli principle for intermediate states. The various particles must have different momenta and the same must hold for holes. Thus, for the example treated above, the term with $k_4 = k_5$ should be excluded in the summation. However $v(k_2k_3k_4k_5)$ and $v(k_5k_4k_1m)$ are antisymmetric in k_4 and k_5 so that the term with $k_4 = k_5$ would automatically give no contribution and we are justified in dropping the restriction on the summation. It has been remarked by Wick [6]) that this holds quite generally for Fermi particles and that for Bose particles one is similarly allowed to forget the modifications in the production and absorption matrix elements which occur when more than one boson is in a given state. Quite generally, the errors made if one does not take into account the influence of the Fermi or Bose statistics on intermediate states with particles of equal momenta, cancel each other exactly.

We shall now introduce the important concept of *diagonal diagrams*. In fig. 2 different diagrams are drawn describing the interaction of two particles. Momentum is conserved in each elementary interaction. One will therefore have the relation $k_1 + k_2 = k_3 + k_4$ in all diagrams. In the diagrams c and d however one has $k_1 = k_3$ and $k_2 = k_4$. These diagrams are called diagonal because their contributions contain the factors $\delta^3(k_1 - k_3)\,\delta^3(k_2 - k_4)$. We shall in general call a diagram of the matrix element $\langle\beta\,|R(z)|\,\alpha\rangle$ diagonal if the states $|\alpha\rangle$ and $|\beta\rangle$ contain the same numbers of particles and holes, and if the contribution of the diagram to $\langle\beta\,|R(z)|\,\alpha\rangle$ contains the factor $\delta(\alpha-\beta)$, where $\delta(\alpha - \beta)$ is the product of the 3-dimensional δ-functions for the momenta of all particles and holes, as defined by (2.6).

Diagonal diagrams play a very important part in the theory of large systems. This is shown by the following consideration. Diagonal diagrams give contributions only to diagonal matrix elements of the resolvent $R(z)$, whereas non-diagonal diagrams contribute both to its diagonal and non-diagonal matrix elements. Comparing now the contributions of a diagonal and a non-diagonal diagram to some diagonal element $\langle\alpha\,|R(z)|\,\alpha\rangle$, one finds that the contribution of the first diagram is larger than that of the second by at least one factor Ω. This is an immediate consequence of the fact that the contribution of a diagonal diagram contains more δ-factors than the contribution of a non-diagonal one. According to (2.6) each δ-factor

gives rise to a factor $\Omega/8\pi^3$. The origin of this extra factor Ω must be sought in the much larger number of intermediate states occurring in the contributions of diagonal diagrams.

There exist essentially three types of diagonal diagrams. In the first place diagrams without external lines, the so-called ground state diagrams, are diagonal. They contribute to the diagonal element $\langle\varphi_0 \,|R(z)|\, \varphi_0\rangle$. Secondly all diagrams contributing to matrix elements $\langle\, ;\, k_2 \,|R(z)|\, k_1\, ;\, \rangle$ between one-particle states are diagonal. Each contribution contains the factor $\delta^3(k_1 - k_2)$ which results from conservation of momentum. The diagrams of this type have one external particle line at each end. Also the diagrams with one external hole line at each end are diagonal. These three types of diagrams correspond in field theory with vacuum diagrams and self-energy diagrams. We have learned from the examples of fig. 2 that disconnected diagrams, the components of which belong to the categories just mentioned, are also diagonal. It is easily established that no other diagonal diagrams exist.

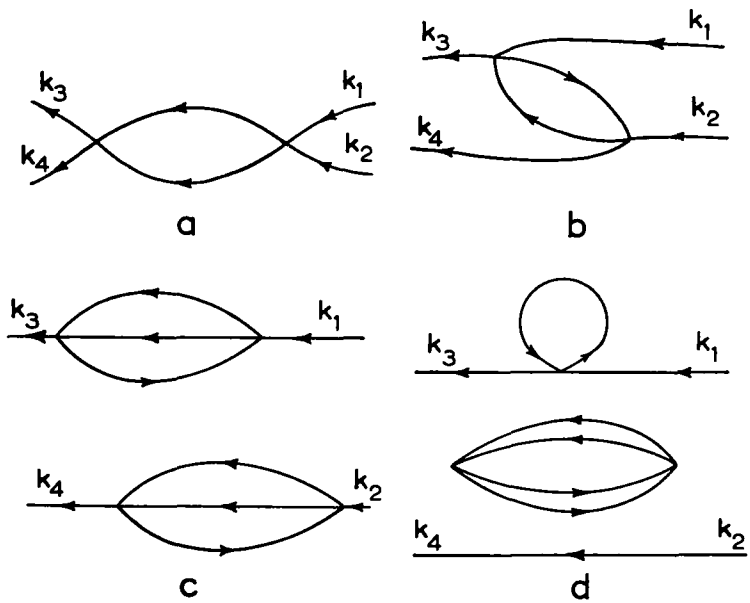

Fig. 2. Some diagonal and non-diagonal diagrams of the matrix element $\langle\, ;\, k_4k_3 \,|R(z)|\, k_1k_2;\rangle$

Each matrix element $\langle\beta \,|R(z)|\, \alpha\rangle$ can unambiguously be written as

$$\langle\beta \,|R(z)|\, \alpha\rangle = D_\alpha(z)\, \delta(\beta - \alpha) + F_{\beta\alpha}(z), \tag{3.8}$$

where the first term is the sum of the contributions of all diagonal diagrams. It is of course only present when the states $|\alpha\rangle$, $|\beta\rangle$ involve the same numbers of particles and holes. The operator $D(z)$, with matrix elements $\langle\beta \,|D(z)\, |\alpha\rangle = D_\alpha(z)\, .\, \delta(\beta - \alpha)$, is called the *diagonal part* of $R(z)$.

4. *The reduction of diagrams.* Let us consider as an example the non-diagonal diagram shown in fig. 3. It is of the 6th order, so that there are five intermediate states, indicated in the figure. These intermediate states cannot all be varied independently. According to the arguments which were used in section 3 in connection with the diagonal diagrams, it is seen that the intermediate state $|\gamma_4\rangle$ is related to $|\beta\rangle$ by the factor $\delta(\beta - \gamma_4)$. Also the states $|\gamma_1\rangle$ and $|\gamma_3\rangle$ are connected by a factor $\delta(\gamma_1 - \gamma_3)$. When cut at the intermediate states $|\gamma_1\rangle$, $|\gamma_3\rangle$, and $|\gamma_4\rangle$, the diagram falls apart into four parts, two of which are diagonal. Removing the diagonal parts, to be called *diagonal subdiagrams*, and joining the remaining pieces, one gets exactly the diagram *a* of fig. 1. The process of elimination of diagonal subdiagrams is called the *reduction* of a diagram. If a diagram cannot be reduced, as for example diagram *a* of fig. 1, we call it *irreducible.*

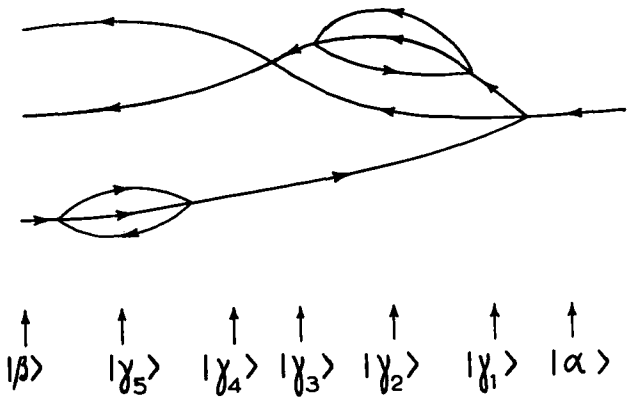

Fig. 3. A non-diagonal diagram which can be reduced to the diagram *a* of fig. 1.

Conversely, one can obtain any non-diagonal diagram in a unique way by inserting suitable diagonal subdiagrams in an irreducible non-diagonal diagram. In other words, in order to calculate the contribution of a reducible diagram, one should take the expression for the corresponding reduced diagram, and substitute for the factors $(\varepsilon_\alpha - z)^{-1}$, $(\varepsilon_\gamma - z)^{-1}$, $(\varepsilon_\beta - z)^{-1}$ belonging to initial, intermediate and final states the contributions of the appropriate diagonal diagrams, without the δ-factor. The sum of the contributions of all non-diagonal diagrams of $\langle\beta\,|R(z)|\,\alpha\rangle$ is obtained by taking the sum of the contributions of all irreducible non-diagonal diagrams and substituting for the factors $(\varepsilon_\alpha - z)^{-1}$, $(\varepsilon_\gamma - z)^{-1}$ and $(\varepsilon_\beta - z)^{-1}$ the functions $D_\alpha(z)$, $D_\gamma(z)$, and $D_\beta(z)$ defined by (3.8). We can express this simply by the formula

$$\{R(z)\}_{nd} = [-D(z)\,V\,D(z) + D(z)V\,D(z)\,V\,D(z) - \ldots\ldots]_{ind}\,.$$

The subscript *nd* indicates the non-diagonal part of $R(z)$, whereas *ind* means restriction to the contributions of all irreducible non-diagonal diagrams.

For $R(z)$ itself we find

$$R(z) = D(z) + [- D(z)\, V\, D(z) + D(z)\, V\, D(z)\, V\, D(z) - \ldots\ldots]_{ind}. \quad (4.1)$$

Having discussed the reduction of non-diagonal diagrams, we now go over to an analysis of the diagonal ones. We want to derive a formula which expresses $D(z)$ in terms of irreducible diagrams. To that end we give a unique prescription how to reduce a diagonal diagram. We make the following convention. The reduced diagonal diagram is obtained by removing all diagonal subdiagrams not containing the first vertex (from the right) of the original diagram. The contribution of any diagonal diagram of $\langle\beta\,|R(z)|\,\alpha\rangle$ can now again be derived unambiguously from the reduced diagram by replacing the factors $(\varepsilon_\beta - z)^{-1}$, $(\varepsilon_\gamma - z)^{-1}$ by the contributions of the appropriate diagonal diagrams, the factor $(\varepsilon_\alpha - z)^{-1}$ being left unaltered. This leads to the following formula for $D(z)$:

$$D(z) = (H_0 - z)^{-1} + D(z)\,[-V + V\,D(z)\,V - \\ -V\,D(z)\,V\,D(z)\,V + \ldots]_{id}\,(H_0 - z)^{-1}. \quad (4.2)$$

The subscript *id* means that only contributions of irreducible diagonal diagrams should be taken.

The second term in the right-hand side of (4.2) contains only diagonal factors. The order of factors is therefore immaterial and the result is independent of the way we defined the reduction of diagonal diagrams.

Let us define a diagonal operator $G(z)$ by

$$G(z) = [- V + V\,D(z)\,V - V\,D(z)\,V\,D(z)\,V + \ldots]_{id}. \quad (4.3)$$

Substituting (4.3) in (4.2) one gets

$$D(z) = (H_0 - z)^{-1} + D(z)\,G(z)\,(H_0 - z)^{-1}$$

or

$$D(z) = (H_0 - z - G(z))^{-1}. \quad (4.4)$$

This is one of the basic equations of Van Hove (compare formulae (3.12), (3.15) of H I). It was derived here by an alternative method. For the sake of comparison a remark must be made on the fact that in H I (section 2) an assumption had to be made on the occurrence of $\delta(\gamma_i - \gamma_k)$ factors for the intermediate states. It was assumed that, whenever one has two pairs of intermediate states $|\gamma_j\rangle$, $|\gamma_k\rangle$ and $|\gamma_l\rangle$, $|\gamma_m\rangle$ related by the factors $\delta(\gamma_j - \gamma_k)$ and $\delta(\gamma_l - \gamma_m)$, the order, in which these four intermediate states occur in the matrix element, is never such that the states of one pair are separated by only one state of the other pair. In the present work, where diagonal parts of operators are described by means of diagrams, the assumption is automatically satisfied. It is an immediate consequence of the structure of the diagrams.

5. *Energies and wave functions of stationary states.* From (4.4), the operators $G(z)$ and $D(z)$ being diagonal in the $|\alpha\rangle$ representation, their eigenvalues $G_\alpha(z)$ and $D_\alpha(z)$ for some state $|\alpha\rangle$ are related by the equation

$$D_\alpha(z) = (\varepsilon_\alpha - z - G_\alpha(z))^{-1}. \tag{5.1}$$

It has been shown in H I that the numerical functions $G_\alpha(z)$, $D_\alpha(z)$ are both holomorphic for non-real z and satisfy the inequalities

$$Im\, D_\alpha(z) > 0 \text{ and } Im\, G_\alpha(z) \geqslant 0 \text{ for } Im\, z > 0. \tag{5.2}$$

The fact that the hamiltonian H is a hermitian operator implies the relations

$$R(z^*) = R^*(z),\ D_\alpha(z^*) = D_\alpha^*(z),\ G_\alpha(z^*) = G_\alpha^*(z). \tag{5.3}$$

$G_\alpha(z)$ and $D_\alpha(z)$ have singularities only on the real axis, where for a finite system they have a large number of poles. The analytical behaviour in the neighbourhood of the real axis becomes however very simple in the asymptotic limit of large Ω, *i.e.* if one neglects all effects which tend to zero for $\Omega \to \infty$.

We shall study the functions $G_\alpha(z)$ and $D_\alpha(z)$ in this approximation. From the definition (4.3) of $G(z)$, where in our approximation all sums over intermediate states are replaced by integrals, one can conclude that $G_\alpha(z)$ has no poles but has finite discontinuities for z crossing the real axis in all points of certain intervals which usually depend on α. In most cases these points of discontinuity cover a portion of the real axis from a finite number on up to $+\infty$.

Let x be a point on the real axis, and $\eta > 0$, then the real functions $K_\alpha(x)$ and $J_\alpha(x)$ are defined by

$$\lim_{\eta\to 0} G_\alpha(x + i\eta) = K_\alpha(x) + iJ_\alpha(x). \tag{5.4}$$

According to (5.3) a similar equation holds with i replaced by $-i$. Equation (5.2) implies

$$J_\alpha(x) \geqslant 0. \tag{5.5}$$

It is clear that $J_\alpha(x) = 0$ in those points of the real axis where $G_\alpha(z)$ is regular. The points x where $J_\alpha(x) > 0$ are the points where $G_\alpha(z)$ has a finite discontinuity for z crossing the real axis. In these points $D_\alpha(z)$ has also a finite discontinuity for z crossing the real axis.

In contrast to $G_\alpha(z)$, the function $D_\alpha(z)$ may have poles even in the limit of $\Omega \to \infty$. This will be the case if the equation $\varepsilon_\alpha - z - G_\alpha(z) = 0$ has a solution. This solution must necessarily be real and hence we consider the equation

$$\varepsilon_\alpha - x - K_\alpha(x) = 0. \tag{5.6}$$

Equation (5.6) has at least one root, and in most applications only one. We suppose such to be the case and call this root E_α. A necessary and sufficient

condition for E_α to be a pole of $D_\alpha(z)$ is that $J_\alpha(x) = 0$ for x in the neighbourhood of E_α. In section 14 we shall investigate the case $J_\alpha(E_\alpha) \neq 0$, but for the time being we shall suppose that $J_\alpha(x) = 0$ for x near E_α.

An important difference between $D_\alpha(z)$ and the corresponding function in the unperturbed system, $(\varepsilon_\alpha - z)^{-1}$, is the shift of the pole from ε_α to E_α, *i.e.* by an amount $G_\alpha(E_\alpha) = K_\alpha(E_\alpha)$. It was shown in H I and we shall derive again that E_α is the energy of a stationary state which arises from $|\alpha\rangle$ through the influence of the perturbation. As we have seen the pole E_α is not the only singularity of $D_\alpha(z)$; $D_\alpha(z)$ has the same interval (or intervals) of discontinuity as $G_\alpha(z)$, formed of all points x where $J_\alpha(x) > 0$. It is evident that E_α is not such a point.

It might be of interest to compare the situation just described with what would be found if one took into account all corrections which vanish for $\Omega \to \infty$. In an exact treatment of a large but finite system one would find a very dense but discrete energy spectrum. $D_\alpha(z)$ would have a large number of poles and no other singularities. All these poles except one, which becomes E_α in the limiting case of large Ω, would be very densely distributed on the real axis, with a separation less than $k_F M^{-1}\Omega^{-\frac{1}{2}}$. The behaviour of $D_\alpha(z)$ at a distance from the real axis large compared to the separation of the poles would be approximately the same as in the limit of $\Omega \to \infty$, when the poles merge together into a line of discontinuity. One can say that the function $D_\alpha(z)$ in the limit of $\Omega \to \infty$ gives a good description of the corresponding quantity in the finite case if one is interested in a kind of average behaviour over energy intervals large compared to $k_F M^{-1}\Omega^{-\frac{1}{2}}$, or in the motion of the system over time intervals short compared to $M\Omega^{\frac{1}{2}}/k_F$.

Returning to the limiting case $\Omega \to \infty$ we shall now derive a formula for the wave function of a stationary state, on the basis of our assumption that $D_\alpha(z)$ has a pole at $z = E_\alpha$. Calculating the matrix element $\langle\beta\,|R(z)|\,\alpha\rangle$ by means of (4.1) one finds

$$\langle\beta\,|R(z)|\,\alpha\rangle = \langle\beta|[1 + D(z)\,\{-V + V\,D(z)V - \ldots\}_{ind}]\,|\alpha\rangle\,D_\alpha(z). \quad (5.7)$$

Of the two factors on the right-hand side the second one has a pole in E_α, while in general the first factor has a finite discontinuity if z crosses the real axis at E_α. Hence one can define two residues of $\langle\beta\,|R(z)|\,\alpha\rangle$ in E_α, one for the upper half plane and one for the lower half plane, by

$$\mathfrak{R}^{\pm}_{E_\alpha}\,[\langle\beta\,|R(z)|\,\alpha\rangle] = \lim_{z\to E_\alpha}(z - E_\alpha)\,\langle\beta\,|R(z)|\,\alpha\rangle,$$

where the plus sign must be chosen if z approaches E_α from above, and the minus sign if z approaches E_α from below. Taking the residue of both sides of equation (5.7) one finds

$$\mathfrak{R}^{\pm}_{E_\alpha}\,[\langle\beta\,|R(z)|\,\alpha\rangle] =$$
$$= -N_\alpha\,\langle\beta\,|[1 + D(E_\alpha \pm i0)\{-V + V\,D(E_\alpha \pm i0)\,V - \ldots\}_{ind}]|\,\alpha\rangle,$$

where $N_\alpha = (1 + G'_\alpha(E_\alpha))^{-1}$. The quantity $-N_\alpha$ is the residue of $D_\alpha(z)$ in E_α.

We shall now prove that the states defined by

$$\int d\beta\, |\beta\rangle \,.\, \mathfrak{R}^{\pm}_{E_\alpha}\,[\langle\beta\,|R(z)|\,\alpha\rangle] = \mathfrak{R}^{\pm}_{E_\alpha}[R(z)\,|\alpha\rangle]$$

are stationary states with the energy E_α.From the definition (3.2) of $R(z)$ one derives easily

$$R(z) - R(z') = (z - z')\, R(z)\,.\, R(z'). \tag{5.8}$$

By application of this operator relation to the state $|\alpha\rangle$, an equation is obtained where both sides, as functions of z', have a pole of the type just considered. Equating the residues $\mathfrak{R}^{+}_{E_\alpha}$ or $\mathfrak{R}^{-}_{E_\alpha}$ of both sides and dividing by $z - E_\alpha$ one finds

$$R(z)\, \mathfrak{R}^{\pm}_{E_\alpha}\,[R(z')\,|\alpha\rangle] = \frac{1}{E_\alpha - z}\, \mathfrak{R}^{\pm}_{E_\alpha}\,[R(z')\,|\alpha\rangle]\,.$$

Substituting in (3.3) one concludes immediately

$$U(t)\, \mathfrak{R}^{\pm}_{E_\alpha}[R(z)\,|\alpha\rangle] = \exp\,(-\,iE_\alpha t)\,.\, \mathfrak{R}^{\pm}_{E_\alpha}\,[R(z)\,|\alpha\rangle].$$

The states $\mathfrak{R}^{\pm}_{E_\alpha}\,[R(z)\,|\alpha\rangle]$ are not yet properly normalized. In H II the normalization constant is shown to be $N_\alpha^{-\frac{1}{2}}$. The stationary states are therefore given by

$$\begin{aligned} |\psi_\alpha\rangle^{\pm} &= -\,N_\alpha^{-\frac{1}{2}}\, \mathfrak{R}^{\pm}_{E_\alpha}\,[R(z)\,|\alpha\rangle] = \\ &= N_\alpha^{\frac{1}{2}}\,[1 + D(E_\alpha \pm i0)\{-\,V + V\,D(E_\alpha \pm i0)\,V - \ldots\}_{ind}]\,|\alpha\rangle. \end{aligned} \tag{5.9}$$

In the case that the states $|\psi_\alpha\rangle^{+}$ and $|\psi_\alpha\rangle^{-}$ are different, they describe scattering with outgoing and ingoing waves respectively. More details are given in H II, where it is moreover proved that, provided $J_\alpha(E_\alpha) = 0$ for all states $|\alpha\rangle$, the set of states $|\psi_\alpha\rangle^{+}$ form a complete orthonormal set, and the states $|\psi_\alpha\rangle^{-}$ as well. This is not necessarily the case if there exist states $|\alpha\rangle$ for which $J_\alpha(E_\alpha) \neq 0$, *i.e.* for which the only singularities of $D_\alpha(z)$ are finite discontinuities.

Chapter III. Separation method for the Ω-dependent parts of the diagrams

6. *Ω-dependence of the diagram contributions.* As explained in the introduction, one is often interested in the way the different physical quantities vary with the volume Ω of the system, at least asymptotically for large Ω. In the last chapter a method was studied to calculate energies and wave functions for the case of large Ω. However the formulae derived there are not very suitable for analysing the dependence on Ω of observable quantities. Although for example we expect on physical grounds that the energy differ-

ence $E_\alpha - E_{\alpha'}$ between two low-lying states must become independent of Ω for large Ω (intensive quantity), it will not be easy to derive such a conclusion from the fact that the perturbed energy is the root of (5.6). The origin of this difficulty must be found in the rather complicated Ω-dependence of $D_\alpha(z)$ and $G_\alpha(z)$, which we shall investigate presently. We restrict ourselves throughout to states $|\alpha\rangle$ differing from the unperturbed ground state by the presence of a finite number of excited particles and holes.

We go back to the series expansion (3.5) for $R(z)$ and see how the contributions of the different diagrams of $\langle\beta\,|R(z)|\,\alpha\rangle$ depend on Ω. All energy denominators contain a term proportional to Ω, for the unperturbed energy ε_α can be written as the sum of two terms $\varepsilon_\alpha = \varepsilon_0 + \varepsilon'_\alpha$, where the ground state energy ε_0 is proportional to Ω according to (2.10) and ε'_α, the sum of the unperturbed energies of the additional particles and holes, is independent of Ω. Henceforth we shall consider $\langle\beta\,|R(\varepsilon_0 + z)|\,\alpha\rangle$ instead of $\langle\beta\,|R(z)|\,\alpha\rangle$; the denominators of this new expression do not depend on Ω.

We consider now the contributions to $\langle\beta\,|R(\varepsilon_0 + z)|\,\alpha\rangle$ of different types of diagrams and study their Ω-dependence. Let us take first a connected diagram with external lines. As remarked before, the conservation of momentum is expressed by a factor $\delta(K_\alpha - K_\beta)$ in the contribution of the diagram, where K_α and K_β are the total momenta of the states $|\alpha\rangle$ and $|\beta\rangle$. If we now replace the summation by an integration we get an expression independent of Ω (see section 3). The terms one should add to correct for the replacement of the summation by an integration tend to zero for $\Omega \to \infty$. Such terms will be neglected as before.

Next we take a connected ground state diagram, *i.e.* a connected diagram without external lines. As always there is a δ-function for each point, expressing the conservation of momentum in each elementary transition. In the present case, however, through the absence of external lines, one of these δ-functions is dependent on the others. This gives rise to a factor $\delta(0)$, which by (2.6) leads to a factor $\Omega/8\pi^3$. If, in the remaining expression, we replace the sums by integrals we get again an expression independent of Ω, except for correction terms which vanish for $\Omega \to \infty$. However we have the factor Ω multiplying not only the integral but also the correction terms, which cannot be all neglected in this case. Hence we conclude that a connected ground state diagram gives a contribution containing a main term proportional to Ω and possibly other terms which, although not all negligible, are small compared to the main term for large Ω.

These considerations are easily extended to more complex diagrams containing one or more ground state components: they exhibit a Ω-dependence such that the highest power of Ω is equal to the number of ground state components. Consequently it is clear that all matrix elements of $R(\varepsilon_0 + z)$ contain terms with arbitrarily high powers of Ω.

7. *Decomposition of diagrams.* After having located the Ω-dependence in the contributions of the diagrams our next task is to make a separation between the Ω-dependent and the Ω-independent parts of a diagram. We must clearly base such a separation on the distinction between the different components of the diagram. We shall derive a formula which makes it possible to express the contribution of each disconnected diagram in terms of the contributions of its components.

Consider two diagrams A and B with or without external lines and denote their contributions to $R(z)$ by $\langle\alpha'|A(z)|\alpha\rangle$ and $\langle\beta'|B(z)|\beta\rangle$. The states $|\alpha\rangle$, $|\alpha'\rangle$, $|\beta\rangle$, and $|\beta'\rangle$ contain certain numbers (possibly zero) of particles and holes. They can be obtained from the state $|\varphi_0\rangle$, which describes the

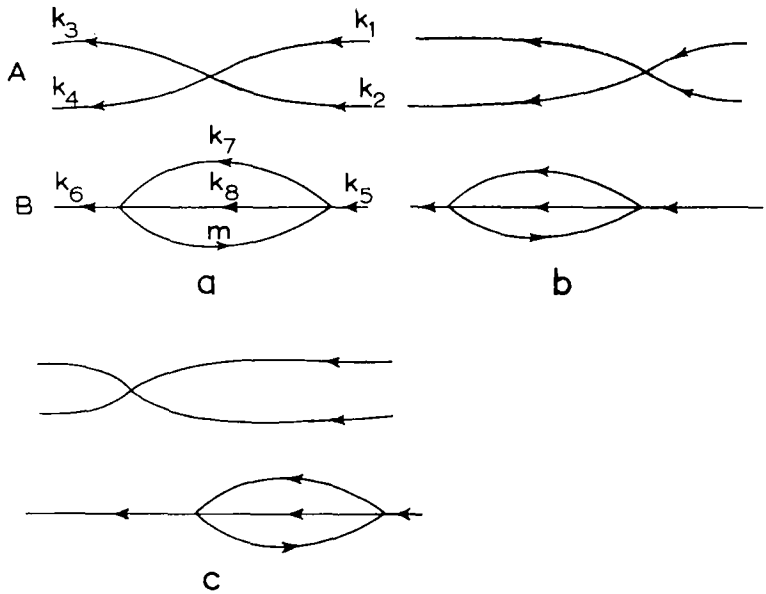

Fig. 4. This figure shows the three ways in which two diagrams of the first and second order in V can be combined to form a diagram of order three.

unperturbed Fermi sea, by applying to $|\varphi_0\rangle$ products of creation operators ξ_k^* and ξ_m for particles and holes respectively. We suppose that $|\alpha\rangle$ and $|\beta\rangle$ do not contain particles or holes with identical momenta. It is then possible to define a state $|\alpha\beta\rangle$ containing all the particles and holes of $|\alpha\rangle$ and $|\beta\rangle$ together. We define $|\alpha\beta\rangle$ as the state obtained, when one applies to $|\beta\rangle$ all the creation operators which must be applied to $|\varphi_0\rangle$ in order to give $|\alpha\rangle$, and in the same order. The notation $\langle\beta\alpha|$ is used for the conjugate of $|\alpha\beta\rangle$. For the states $|\alpha'\rangle$ and $|\beta'\rangle$ we make the same assumption as for $|\alpha\rangle$, $|\beta\rangle$, and we define $|\alpha'\beta'\rangle$ in exactly the same way.

The two given diagrams can be combined in various ways to form a composite diagram. If n and m are the numbers of vertices of A and B, the

composite diagrams have $m + n$ vertices. The various composite diagrams differ by the order in which the points of the two original diagrams A and B occur from right to left, the number of possibilities being $(n + m)!/n!m!$. Figure 4 shows the three possible ways in which two diagrams A and B of order one and two in V respectively can be taken together to form a composite diagram. The sum of the contributions to $\langle\beta'\alpha'|R(z)|\alpha\beta\rangle$ of all composite diagrams found in this way we shall denote by $\langle\beta'\alpha'|C(z)|\alpha\beta\rangle$.

The important point is now that a simple relation exists between $\langle\beta'\alpha'|C(z)|\alpha\beta\rangle$ and the contributions $\langle\alpha'|A(z)|\alpha\rangle$ and $\langle\beta'|B(z)|\beta\rangle$ of the original diagrams A and B to the resolvent. It is expressed by the equation

$$\langle\beta'\alpha'|C(z)|\alpha\beta\rangle = -(2\pi i)^{-1}\oint d\zeta\,\langle\alpha'|A(z-\zeta)|\alpha\rangle\,\langle\beta'|B(\varepsilon_0+\zeta)|\beta\rangle. \quad (7.1)$$

The path of integration is a contour encircling all singular points of the integrand on the real axis, but not encircling the singular points situated on the straight line through z parallel to the real axis. It is to be described counterclockwise.

Before deriving (7.1), we note that the integral in the right-hand side is a type of convolution integral, taken in the complex plane. We shall encounter such convolutions very often and it is therefore convenient to introduce a special notation for them. Let $f(z)$ and $g(z)$ be two functions, holomorphic for non-real z, for which $zf(z)$ and $zg(z)$ are bounded for $|z| \to \infty$. The symbol $f(z) * g(z)$ indicates another function of z, defined by

$$f(z) * g(z) = -(2\pi i)^{-1}\oint d\zeta\, f(z-\zeta)\, g(\zeta), \quad (7.2)$$

with a path of integration as defined above. Using the property that $zf(z)$ and $zg(z)$ are bounded for large $|z|$, one can deform the path of integration into a contour encircling the straight line through z parallel to the real axis. This leads to the equation

$$f(z) * g(z) = g(z) * f(z). \quad (7.3)$$

With our new notation (7.1) reads

$$\langle\beta'\alpha'|C(\varepsilon_0+z)|\alpha\beta\rangle = \langle\alpha'|A(\varepsilon_0+z)|\alpha\rangle * \langle\beta'|B(\varepsilon_0+z)|\beta\rangle. \quad (7.4)$$

We start now the proof of the relation (7.1) which constitutes the basic tool for all the derivations in the following sections.

We first establish the validity of (7.1) for the simple diagrams of fig. 4. The states $|\alpha\rangle$, $|\alpha'\rangle$, $|\beta\rangle$, and $|\beta'\rangle$ are denoted by $|k_1k_2\,;\,\rangle$, $|k_3k_4\,;\,\rangle$, $|k_5\,;\,\rangle$, and $|k_6\,;\,\rangle$ respectively.

According to the rules of section 3 one finds

$$\langle\,;\,k_4k_3|A(z+\varepsilon_0)|k_1k_2\,;\,\rangle = -v(k_4k_3k_1k_2)(\varepsilon'_{\alpha'}-z)^{-1}(\varepsilon'_{\alpha}-z)^{-1}, \quad (7.5)$$

$$\langle\,;\,k_6|B(z+\varepsilon_0)|k_5\,;\,\rangle = \tfrac{1}{2}\int_{k_7k_8m} v(k_6mk_7k_8)\,v(k_7k_8k_5m)\,.$$
$$.\,(\varepsilon'_{\beta'}-z)^{-1}(\varepsilon'_{\beta_1}-z)^{-1}(\varepsilon'_{\beta}-z)^{-1}. \quad (7.6)$$

The energies in the denominators are the excitation energies of the states as compared with the energy ε_0 of $|\varphi_0\rangle$. Diagram a of fig. 4 gives the contribution

$$-\tfrac{1}{2}\int_{k_7k_8m} v(k_6mk_7k_8)\, v(k_4k_3k_1k_2)\, v(k_7k_8k_5m)\ .$$
$$(\varepsilon'_{\alpha'} + \varepsilon'_{\beta'} - z)^{-1}\,(\varepsilon'_{\alpha'} + \varepsilon'_{\beta_1} - z)^{-1}\,(\varepsilon'_{\alpha} + \varepsilon'_{\beta_1} - z)^{-1}\,(\varepsilon'_{\alpha} + \varepsilon'_{\beta} - z)^{-1}.$$

In writing down this equation we made essential use of the remark in section 3 concerning intermediate states with two or more particles in the same plane wave state. The diagrams b and c give the same expression except for the energy denominators. The sum of the products of energy factors in the expressions for a, b and c is easily calculated. After some simple algebraic manipulations one gets

$$(\varepsilon'_{\alpha} - \varepsilon'_{\alpha'})^{-1}\,(\varepsilon'_{\alpha'} + \varepsilon'_{\beta'} - z)^{-1}\,(\varepsilon'_{\alpha'} + \varepsilon'_{\beta_1} - z)^{-1}\,(\varepsilon'_{\alpha'} + \varepsilon'_{\beta} - z)^{-1} +$$
$$+ (\varepsilon'_{\alpha'} - \varepsilon'_{\alpha})^{-1}\,(\varepsilon'_{\alpha} + \varepsilon'_{\beta'} - z)^{-1}\,(\varepsilon'_{\beta} + \varepsilon'_{\beta_1} - z)^{-1}\,(\varepsilon'_{\alpha} + \varepsilon'_{\beta} - z)^{-1},$$

which again is equal to

$$(\varepsilon'_{\beta'} - z)^{-1}\,(\varepsilon'_{\beta_1} - z)^{-1}\,(\varepsilon'_{\beta} - z)^{-1} * (\varepsilon'_{\alpha} - z)^{-1}\,(\varepsilon'_{\alpha'} - z)^{-1}$$

as follows immediately from our definition (7.2) of convolution.
For the sum of the contributions of the diagrams a, b, and c of fig. 4 we find using (7.5) and (7.6)

$$\langle\ ;\ k_6k_4k_3\,|C(\varepsilon_0 + z)|\ k_1k_2k_5\ ;\ \rangle = -\tfrac{1}{2}\int_{k_7k_8m} v(k_6mk_7k_8)\ .$$
$$.\,v(k_4k_3k_1k_2)\ v(k_7k_8k_5m)\ [(\varepsilon'_{\alpha'}-z)^{-1}\,(\varepsilon'_{\alpha}-z)^{-1} * (\varepsilon'_{\beta'}-z)^{-1}\,(\varepsilon'_{\beta_1}-z)^{-1}\,(\varepsilon'_{\beta}-z)^{-1}] =$$
$$= \langle\ ;\ k_4k_3\,|A(z + \varepsilon_0)|\ k_1k_2\ ;\ \rangle * \langle\ ;\ k_6\,|B(z + \varepsilon_0)|\ k_5\ ;\ \rangle,$$

which proves equation (7.1) for the special case considered.

This simple example shows that to prove equation (7.1) it is sufficient to establish the corresponding equation for the products of energy factors only. In the case of our example this equation could be proved by a direct calculation, which however cannot easily be extended to the general case. It is more convenient to proceed by induction. The products of energy factors corresponding to the two diagrams A and B can be written in the form

$$\prod_{k=0}^{n} (a_k - z)^{-1} \text{ and } \prod_{l=0}^{m} (b_l - z)^{-1}$$

where

$$a_0 = \varepsilon'_{\alpha},\ a_n = \varepsilon'_{\alpha'},\ b_0 = \varepsilon'_{\beta},\ b_m = \varepsilon'_{\beta'},$$

while $a_k (k = 1, 2, \ldots, n - 1)$ and $b_l (l = 1, 2, \ldots, m - 1)$ are the excitation energies of the intermediate states of A, and B relative to the ground state energy ε_0. We introduce ordered products of $n + m + 1$ factors

$$(a_k + b_l - z)^{-1}\ (k = 0, \ldots, n\ ;\ l = 0, \ldots, m)$$

with the following property: if $(a_k + b_l - z)^{-1}$ and $(a_{k'} + b_{l'} - z)^{-1}$ are consecutive factors, the first being to the left, one has either $k = k'$, $l = l' + 1$ or $k = k' + 1$, $l = l'$. The number of products which can be constructed according to this rule is $(n + m)!/n!m!$ and we shall denote their sum by $\Phi(a_n \dots a_1 a_0; b_m \dots b_1 b_0)$. The first factor from the left of each ordered product will always be $(a_n + b_m - z)^{-1}$, the last one to the right $(a_0 + b_0 - z)^{-1}$. The second factor will be either $(a_n + b_{m-1} - z)^{-1}$ or $(a_{n-1} + b_m - z)^{-1}$. In the first case the product can be written as $(a_n + b_m - z)^{-1}$ multiplied by an ordered product of factors $(a_k + b_l - z)^{-1}$ $(k = 0, \dots, n\ ;\ l = 0, \dots, m - 1)$. For the sum of all products of this kind, we find therefore $(a_n + b_m - z)^{-1} \cdot \Phi(a_n \dots a_1 a_0\ ;\ b_{m-1} \dots b_1 b_0)$. In exactly the same way we get for the sum of all products with $(a_{n-1} + b_m - z)^{-1}$ as the second factor the expression $(a_n + b_m - z)^{-1} \cdot \Phi(a_{n-1} \dots a_1 a_0\ ;\ b_m \dots b_1 b_0)$. These results taken together give the reduction formula

$$\Phi(a_n \dots a_0\ ;\ b_m \dots b_0) = (a_n + b_m - z)^{-1} [\Phi(a_{n-1} \dots a_0\ ;\ b_m \dots b_0) + \\ + \Phi(a_n \dots a_0\ ;\ b_{m-1} \dots b_0)]. \tag{7.7}$$

Let us define the quantity $\Psi(a_n \dots a_0\ ;\ b_m \dots b_0)$ by the formula

$$\Psi(a_n \dots a_0\ ;\ b_m \dots b_0) = \prod_{k=0}^{n} (a_k - z)^{-1} * \prod_{l=0}^{m} (b_l - z)^{-1}.$$

The equation we want to prove is then simply

$$\Phi = \Psi. \tag{7.8}$$

We show first that (7.7) is satisfied by Ψ as well as by Φ. Indeed

$$(a_n + b_m - z)^{-1} [\Psi(a_{n-1} \dots a_0\ ;\ b_m \dots b_0) + \Psi(a_n \dots a_0\ ;\ b_{m-1} \dots b_0)] = \\ = -(2\pi i)^{-1} \oint d\zeta \frac{1}{a_n + b_m - z} \left(\frac{1}{b_m - \zeta} + \frac{1}{a_n - z + \zeta} \right) \frac{1}{a_{n-1} - z + \zeta} \dots \\ \dots \frac{1}{a_0 - z + \zeta} \cdot \frac{1}{b_{m-1} - \zeta} \dots\dots \frac{1}{b_0 - \zeta} = \Psi(a_n \dots a_0\ ;\ b_m \dots b_0).$$

On the other hand the definitions of Φ and Ψ imply immediately

$$\Phi(a_0\ ;\ b_l \dots b_0) = \Psi(a_0\ ;\ b_l \dots b_0),$$

$$\Phi(a_k \dots a_0\ ;\ b_0) = \Psi(a_k \dots a_0\ ;\ b_0).$$

By induction we can now conclude to the validity of (7.8), thus completing the proof of equation (7.1).

In the appendix A1 another derivation of (7.1) will be discussed. It is based on a comparison with a system consisting of two completely uncoupled subsystems, a situation for which equation (7.1) is a direct consequence of the additivity of the energies.

Chapter IV. The separation of ground state diagrams. Calculation of energies

8. *The integral equation for* $D_0(z)$. We consider a matrix element $\langle\alpha'|R(z)|\alpha\rangle$ of $R(z)$ and suppose that $|\alpha\rangle$ and $|\alpha'\rangle$ are not both identical with $|\varphi_0\rangle$. This means that the contributing diagrams contain external lines. Each diagram consists of one or more components (*i.e.* connected parts) with external lines and possibly some components without external lines. The sum of the contributions to $\langle\alpha'|R(z)|\alpha\rangle$ of all diagrams $\bar{A}$ which do not contain ground state components, will be denoted by $\langle\alpha'|\bar{R}(z)|\alpha\rangle$. Each diagram of $\langle\alpha'|R(z)|\alpha\rangle$ can be obtained in an unambiguous way from a diagram $\bar{A}$ of $\langle\alpha'|\bar{R}(z)|\alpha\rangle$ by the addition of a ground state diagram A_0. The contribution to $\langle\alpha'|R(\varepsilon_0+z)|\alpha\rangle$ of all diagrams one can form from $\bar{A}$ and A_0 is, according to (7.4), expressed by

$$\langle\alpha'|\bar{A}(\varepsilon_0+z)|\alpha\rangle * A_0(\varepsilon_0+z),$$

where $\langle\alpha'|\bar{A}(z)|\alpha\rangle$ is the contribution of $\bar{A}$ to $\langle\alpha'|\bar{R}(z)|\alpha\rangle$ and $A_0(z)$ is the contribution of A_0 to $\langle\varphi_0|R(z)|\varphi_0\rangle \equiv D_0(z)$.

One clearly gets the total value of $\langle\alpha'|R(\varepsilon_0+z)|\alpha\rangle$ by summing over all possible diagrams $\bar{A}$ and A_0. This gives the formula

$$\langle\alpha'|R(\varepsilon_0+z)|\alpha\rangle = \langle\alpha'|\bar{R}(\varepsilon_0+z)|\alpha\rangle * D_0(\varepsilon_0+z). \tag{8.1}$$

We have seen in section 6 that diagrams not containing ground state components give contributions which, for large volume Ω, are independent of Ω, whereas ground state diagrams are Ω-dependent. The importance of (8.1) is due to the fact that it gives in the resolvent a complete separation between Ω-dependent and Ω-independent quantities.

We shall study now the dependence on Ω of the ground state expectation value $D_0(z)$ of the resolvent. $D_0(z)$ is, by definition, the sum of the contributions of all ground state diagrams. As discussed in section 6 these contributions contain arbitrarily high powers of Ω. In order to investigate the explicit Ω-dependence of $D_0(z)$, we derive for this function a simple integral equation which will enable us to calculate it using only connected ground state diagrams.

In the second of the operator equations (3.4) we take on both sides the diagonal element for the state $|\varphi_0\rangle$ and replace z by ε_0+z. We find

$$D_0(\varepsilon_0+z) = -1/z + \langle\varphi_0|R(\varepsilon_0+z)\,V|\varphi_0\rangle . 1/z,$$

or, summing over intermediate states,

$$D_0(\varepsilon_0+z) = -1/z + D_0(\varepsilon_0+z)\,\langle\varphi_0|V|\varphi_0\rangle/z + \\ + 1/z\int'\langle\varphi_0|R(\varepsilon_0+z)|\alpha\rangle\,\langle\alpha|V|\varphi_0\rangle.$$

The summation is extended over all states $|\alpha\rangle \neq |\varphi_0\rangle$. This equation can be

transformed by applying (8.1) to the matrix element $\langle\varphi_0|R(\varepsilon_0+z)|\alpha\rangle$. One finds after elementary manipulations

$$\begin{aligned} D_0(\varepsilon_0+z) &= -z^{-1}[1+D_0(\varepsilon_0+z) * (z^{-1}\langle\varphi_0|V|\varphi_0\rangle - \\ &\qquad - \int' \langle\varphi_0|\bar{R}(\varepsilon_0+z)|\alpha\rangle\langle\alpha|V|\varphi_0\rangle)] = \\ &= -z^{-1}[1-D_0(\varepsilon_0+z) * z^{-1}\bar{G}_0(\varepsilon_0+z)]. \end{aligned} \tag{8.2}$$

The function $\bar{G}_0(z)$ is defined by

$$\bar{G}_0(z) = \langle\varphi_0|[-V+V(H_0-z)^{-1}V-\ldots\ldots]_C|\varphi_0\rangle, \tag{8.3}$$

where the subscript C means that only connected ground state diagrams contribute to $\bar{G}_0(z)$. Equation (8.2) is an integral equation for $D_0(z)$ which, by (7.2), can be written in the more explicit form

$$zD_0(\varepsilon_0+z) = -1-(2\pi i)^{-1}\oint d\zeta\,\zeta^{-1}\bar{G}_0(\varepsilon_0+\zeta)\,D_0(\varepsilon_0+z-\zeta). \tag{8.4}$$

The path of integration is, according to (7.2), a contour around that portion of the real axis, which contains all the singularities of the first factor in the integrand. It is described counterclockwise and it is chosen in such a way that it does not cross the line through the point z parallel to the real axis.

As we shall see below, (8.4) can be solved explicitly. It expresses $D_0(\varepsilon_0+z)$ in terms of the function $\bar{G}_0(\varepsilon_0+z)$, which is much simpler since it involves connected diagrams only. In particular the definition (8.3) implies that $\bar{G}_0(\varepsilon_0+z)$ contains a main term proportional to the volume Ω of the system, and other terms which, for large Ω, are small compared to the first.

The connected ground state diagrams contributing to (8.3) contain in general diagonal subdiagrams. With the methods of section 4 we derive an expression for $\bar{G}_0(z)$ in terms of irreducible diagrams only. Starting from irreducible ground state diagrams one can construct all diagrams occurring in (8.3) in an unambiguous way by inserting suitable diagonal subdiagrams between any pair of successive points. These diagonal subdiagrams should not contain any ground state components. This suggests considering for each state $|\alpha\rangle \neq |\varphi_0\rangle$ the sum $\bar{D}_\alpha(z)$ of the contributions to $D_\alpha(z)$ of all diagonal diagrams which do not contain ground state components. Using this definition we obtain the following expression for $\bar{G}_0(z)$:

$$\bar{G}_0(z) = \langle\varphi_0|[-V+V\bar{D}(z)V-V\bar{D}(z)V\,\bar{D}(z)V+\ldots]_{idC}|\varphi_0\rangle. \tag{8.5}$$

The subscript idC means that, in calculating the right-hand side, one should limit oneself to irreducible connected ground state diagrams.

9. *Solution of the integral equation.* It is our aim to solve explicitly the integral equation (8.4) for $D_0(\varepsilon_0+z)$ in the limiting case of large systems. To that end we shall first discuss briefly some properties of the function $\bar{G}_0(\varepsilon_0+z)$ for this limiting case.

All terms in $\bar{G}_0(\varepsilon_0+z)$ except the term proportional to Ω will be neglected.

This means in particular that we replace in (8.5) all sums over intermediate states by integrals. In this approximation, from (8.5), the function $\bar{G}_0(\varepsilon_0+z)$ has no poles. Its only singularities are the points of a cut along the real axis from a point B up to $+\infty$. In each point of this cut the function has a finite discontinuity for z crossing the real axis. The function can, in most practical cases, be continued analytically across the cut, from above and from below. B plays then the role of a branch point.

We shall assume, as is consistent with the fact that $|\varphi_0\rangle$ is the ground state and has an energy ε_0, that B lies on the positive real axis or at the origin. In the latter case a further assumption will be made, concerning the real function

$$\bar{J}_0(x) = -\tfrac{1}{2}i \lim_{\eta\to 0} [\bar{G}_0(x+i\eta) - \bar{G}_0(x-i\eta)],\ \eta > 0, \tag{9.1}$$

namely

$$|\bar{J}_0(\varepsilon_0 + x)| < A \,.\, x^{\alpha+1}, \text{ for } x > 0, \tag{9.2}$$

where A and α are positive constants.

If we write

$$\bar{G}_0(\varepsilon_0 + z) = -\langle\varphi_0|V|\varphi_0\rangle + g(z),$$

the function $g(z)$ has, according to (8.5), the property that $zg(z)$ is bounded for large $|z|$. This implies that the path of integration C in Cauchy's formula

$$g(z) = (2\pi i)^{-1}\int_C d\zeta\, g(\zeta)(\zeta - z)^{-1} \text{ for non-real } z,$$

can be deformed into a contour around the singular points of $g(z)$ on the positive real axis. One finds in this way using (9.1)

$$\bar{G}_0(\varepsilon_0 + z) = \pi^{-1}\int_0^\infty d\xi\, \bar{J}_0(\varepsilon_0 + \xi)(\xi - z)^{-1} - \langle\varphi_0|V|\varphi_0\rangle, \tag{9.3}$$

and, taking the derivative at both sides,

$$\bar{G}_0'(\varepsilon_0 + z) = \pi^{-1}\int_0^\infty d\xi\, \bar{J}_0(\varepsilon_0 + \xi)(\xi - z)^{-2}. \tag{9.4}$$

As an immediate consequence of our assumptions we see that both $\bar{G}_0(\varepsilon_0 + z)$ and its first derivative exist at the origin.

The validity of the assumptions we made to reach these conclusions must of course be established in each special case. One can however easily see that the assumption that the branch point B is situated on the negative real axis would lead to unphysical results. In the case that $\bar{G}_0(\varepsilon_0 + z)$ has negative singularities, equation (8.4) can only be solved by a function $D_0(\varepsilon_0 + z)$ having singularities extending to $-\infty$. This would correspond to a perturbed energy spectrum without lower bound, a situation not realized in actual physical systems. In quantum electrodynamics the branch point B is at the point $z = 2m$ (m is the observable mass of the electron), corresponding to the lowest energy necessary for the creation of an electron-positron pair and

a photon. For a Fermi gas with interaction, the case considered as main example in the present paper, B is at the origin with $\alpha > 0$, as will be shown in the forthcoming paper already announced.

To reduce (8.4) to a simpler form we introduce the function

$$h(z) = (\bar{G}_0(\varepsilon_0 + z) - \bar{G}_0(\varepsilon_0))z^{-1} \tag{9.5}$$

which, under our assumptions, has no singularities except for a cut on the positive real axis, from B to $+\infty$. At the origin it has the well-defined value $h(0) = \bar{G}_0'(\varepsilon_0)$. Substituting (9.5) in (8.4) we find

$$z \,.\, D_0(\varepsilon_0 + z) = -1 - \bar{G}_0(\varepsilon_0)\, D_0(\varepsilon_0 + z) - (2\pi i)^{-1} \oint \mathrm{d}\zeta\, h(\zeta)\, D_0(\varepsilon_0 + z - \zeta)$$

which, by the substitution $z \to z - \bar{G}_0(\varepsilon_0)$ can be written

$$zD_0(\varepsilon_0 - \bar{G}_0(\varepsilon_0) + z) = -1 - (2\pi i)^{-1} \oint \mathrm{d}\zeta\, h(\zeta)\, \mathrm{D}_0(\varepsilon_0 - \bar{G}_0(\varepsilon_0) + z - \zeta).$$

If we introduce the notation

$$f(z) = D_0(\varepsilon_0 - \bar{G}_0(\varepsilon_0) + z) \tag{9.6}$$

our integral equation is reduced to the simple form

$$zf(z) = -1 - (2\pi i)^{-1} \oint \mathrm{d}\zeta\, h(\zeta)\, f(z - \zeta). \tag{9.7}$$

Both $h(z)$ and the desired solution $f(z)$ have the property that $zh(z)$ and $zf(z)$ remain bounded for $|z| \to \infty$, which means that according to (7.3) the equation can be written in the equivalent form

$$zf(z) = -1 - (2\pi i)^{-1} \oint \mathrm{d}\zeta\, h(z - \zeta)\, f(\zeta). \tag{9.8}$$

The second term on the right-hand side tends to zero for $|z| \to \infty$. This gives the relation

$$-(2\pi i)^{-1} \oint \mathrm{d}z\, f(z) = 1, \tag{9.9}$$

which also follows from (3.3) by taking the expectation value for $|\varphi_0\rangle$ and putting $t = 0$.

To solve equation (9.7) we proceed in the following way. It can be shown that (9.7) has at most one solution $f(z)$ belonging to the class of functions which are holomorphic outside the real axis and bounded for large $|z|$. A proof will be given in appendix A2. If we now can find a function $f(z)$ of the class mentioned which solves (9.7) we are certain that it is the only one. From a physical point of view we are not interested in other solutions if they exist.

Equation (9.7) suggests a solution of the form

$$f(z) = -N_0/z + \psi(z), \tag{9.10}$$

where the function $\psi(z)$, just as $h(z)$, has no other singularities than a cut

on the positive real axis running from B to $+\infty$ †). We shall show that a solution of the form (9.10) exists and we shall determine N_0 and $\psi(z)$. The relations (5.3) imply $h(z^*) = h(z)^*$ and $\psi(z^*) = \psi(z)^*$. Consequently the functions $p(x)$ and $\varphi(x)$, defined by

$$2\pi i\, p(x) = \lim_{\eta\to 0} [h(x + i\eta) - h(x - i\eta)],\ \eta > 0 \tag{9.11}$$

$$2\pi i\, \varphi(x) = \lim_{\eta\to 0} [\psi(x + i\eta) - \psi(x - i\eta)],\ \eta > 0 \tag{9.12}$$

are real. They vanish for $x < B$. Furthermore, from (5.2), $\varphi(x) \geqslant 0$, while (9.1) and (9.5), imply

$$p(x) = \bar{J}_0\,(\varepsilon_0 + x)/x. \tag{9.13}$$

It will be sufficient for us to determine $\varphi(x)$. The function $\psi(z)$ will then follow by application of Cauchy's theorem. Substituting (9.10) in (9.8) one finds, using (9.12),

$$-N_0 + z\psi(z) = -1 + N_0\, h(z) + \int_0^\infty \mathrm{d}\xi\, h(z - \xi)\, \varphi(\xi)$$

which by (9.11) and (9.12) can be reduced to the following integral equation for $\varphi(x)$:

$$x\varphi(x) = N_0\, p(x) + \int_0^x \mathrm{d}\xi\, p(x - \xi)\, \varphi(\xi). \tag{9.14}$$

This equation is solved by means of a Laplace transformation. If one defines

$$\hat{\varphi}(s) = \int_0^\infty \varphi(x)\, e^{-sx}\mathrm{d}x,\ \hat{p}(s) = \int_0^\infty p(x)\, e^{-sx}\, \mathrm{d}x,$$

(9.14) can be transformed into

$$-\frac{d}{\mathrm{d}s}\,\hat{\varphi}(s) = N_0\,\hat{p}(s) + \hat{p}(s)\ .\ \hat{\varphi}(s). \tag{9.15}$$

The general solution of this linear first order differential equation is

$$\hat{\varphi}(s) = -N_0 + C \exp\left(-\int \hat{p}(s)\, \mathrm{d}s\right),$$

where C is an arbitrary constant. The indefinite integral in the exponent can be taken equal to $-\int_0^\infty p(x)\, x^{-1} \exp(-sx)\, \mathrm{d}x$. For $\hat{\varphi}(s)$ to be the Laplace-transform of a function $\varphi(x)$ the following condition must be satisfied: $\lim_{s\to\infty}\hat{\varphi}(s) = 0$. It implies $C = N_0$ and gives us the following expression for $\hat{\varphi}(s)$:

$$\hat{\varphi}(s) = N_0\left[\exp\left(\int_0^\infty \frac{p(x)}{x}\, e^{-sx}\, \mathrm{d}x\right) - 1\right]. \tag{9.16}$$

Whether this function $\hat{\varphi}(s)$ is the Laplace-transform of a solution $\varphi(x)$ of

†) In the case $B = 0$, the origin is not a proper pole of $f(z)$. We shall however continue to call such a point a pole, provided the function $\varphi(x)$, defined in (9.12) has near the origin the behaviour $\varphi(x) = O(x^{\alpha-1})$ with $\alpha > 0$. As seen from (9.19) this condition is fulfilled.

(9.14) depends on the behaviour of $p(x)$ near the origin. It will be shown in appendix A3 that a sufficient condition for this to hold is

$$|p(x)| = O(x^{\alpha}),\ \alpha > 0. \tag{9.17}$$

Equation (9.17) however follows immediately from (9.13) and (9.2) and is thus always satisfied under our assumptions. For $\varphi(x)$ one has the expression

$$\varphi(x) = (2\pi i)^{-1}\, x^{-1} \int_{-i\infty+\delta}^{+i\infty+\delta} \hat{\varphi}'(s)\, e^{sx}\, \mathrm{d}s,\ \delta > 0 \tag{9.18}$$

which is independent of δ. It is also shown in A3 that the behaviour of $\varphi(x)$ near the origin is given by

$$\varphi(x) = O(x^{\alpha-1}),\ \alpha > 0 \tag{9.19}$$

(see footnote on page 508).

The constant N_0 must still be determined. Substituting (9.10) in (9.9) and using (9.12) one finds

$$\int_0^{\infty} \varphi(x)\, \mathrm{d}x = 1 - N_0. \tag{9.20}$$

Remembering that $\int_0^{\infty}\varphi(x)\, \mathrm{d}x = \hat{\varphi}(0)$ one obtains from (9.16) and (9.20)

$$N_0 = \exp\left(-\int_0^{\infty} p(x)\, x^{-1}\, \mathrm{d}x\right)$$

or, from (9.13) and (9.4),

$$N_0 = \exp\left(-\bar{G}_0'(\varepsilon_0)\right). \tag{9.21}$$

Finally, Cauchy's theorem gives rise to the following expression of $\psi(z)$ in terms of $\varphi(x)$

$$\psi(z) = \int_0^{\infty} \mathrm{d}\xi\, \varphi(\xi)(\xi - z)^{-1}. \tag{9.22}$$

By direct substitution it is now easily shown that (9.10) actually represents a solution of (9.7) if for N_0 and $\psi(z)$ we adopt the expressions just found. The solution of the integral equation (8.4) is now completed. The result reads, by (9.6),

$$D_0(z) = \frac{\exp\left(-\bar{G}_0'(\varepsilon_0)\right)}{\varepsilon_0 - \bar{G}_0(\varepsilon_0) - z} + \int_0^{\infty} \mathrm{d}\xi\, \frac{\varphi(\xi)}{\xi - z + \varepsilon_0 - \bar{G}_0(\varepsilon_0)}, \tag{9.23}$$

where $\varphi(x)$, by (9.18) and (9.16), is explicitly expressed in terms of $p(x) = \bar{J}_0(\varepsilon_0 + x)/x$. It also contains the factor $N_0 = \exp\left(-\bar{G}_0'(\varepsilon_0)\right)$.

Equation (9.23) gives an explicit expression for $D_0(z)$ in terms of $\bar{G}_0(\varepsilon_0)$, $\bar{G}_0'(\varepsilon_0)$ and $\bar{J}_0(\varepsilon_0 + x)$, quantities which, for large Ω, are all proportional to Ω. Let us analyse our results in more detail, in particular for the pole of $D_0(z)$.

As shown in section 5 the pole of $D_0(z)$ is the energy E_0 of the perturbed ground state $|\psi_0\rangle$. From (9.23) we obtain its expression in very simple terms

$$E_0 = \varepsilon_0 - \bar{G}_0\,(\varepsilon_0). \tag{9.24}$$

As was to be expected the energy shift $\Delta E_0 = -\bar{G}_0(\varepsilon_0)$ and consequently the perturbed energy E_0 are proportional to Ω for large Ω. This important result is established to general order in the perturbation. It will be commented upon later. On the basis of (8.3) the energy shift can be written

$$\Delta E_0 = \\ = -\langle\varphi_0|[-V+V(H_0-\varepsilon_0)^{-1}V-V(H_0-\varepsilon_0)^{-1}V(H_0-\varepsilon_0)^{-1}V+\ldots]_C|\varphi_0\rangle. \quad (9.25)$$

This formula was recently derived by Goldstone [5]), who used an adiabatic switching-on of the interaction and a time-dependent perturbation method. A slightly different formula is obtained by summing the contributions of all diagrams that can be reduced to the same irreducible form, thus replacing (8.3) by (8.5). It reads

$$\Delta E_0 = -\langle\varphi_0|[-V+V\,\bar{D}(\varepsilon_0)V-V\bar{D}(\varepsilon_0)V\bar{D}(\varepsilon_0)V+\ldots]_{\mathrm{idc}}|\varphi_0\rangle. \quad (9.26)$$

One should note that the ground state energy shift is the only quantity for which the Ω-dependence was studied before to general order in the perturbation. The argument used to this end by Goldstone does not extend to the investigation of other quantities. As we shall see in the following, our method is of much greater generality.

The residue of the function $D_0(z)$ at its pole is $-N_0$. The Ω-dependence of this quantity is not linear but, as follows from (9.21), exponential. This is in accordance with the remark in section 8 that terms of arbitrarily high powers in Ω occur in the expansion of $D_0(z)$. As seen from equation (5.9) the factor $N_0^{\frac{1}{2}}$ is a normalization constant for the wave function $|\psi_0\rangle$. In fact, N_0 is the probability that one finds the actual ground state $|\psi_0\rangle$ in the state $|\varphi_0\rangle$. That this factor should decrease exponentially with the size of the system can be understood on the basis of the same general arguments sometimes used to explain why the total energy of a large system is proportional to its volume. One subdivides the system in a large number of identical cells, themselves large enough for the interactions across cell boundaries to be negligible. Just as the total energy is then approximately the sum of the energies of the individual cells and consequently proportional to the size of the system, the total wave function takes the form of a product and must therefore depend exponentially on the size.

10. *Energies of excited states.* We shall now turn to the determination of $D_\alpha(z)$ for a state different from the ground state. Taking in (8.1) the diagonal element for a state $|\alpha\rangle \neq |\varphi_0\rangle$ one finds

$$D_\alpha(\varepsilon_0 + z) = \bar{D}_\alpha(\varepsilon_0 + z) * D_0(\varepsilon_0 + z), \quad (10.1)$$

where, as defined in section 8, $\bar{D}_\alpha(z)$ is the sum of the contributions to $D_\alpha(z)$ of all diagrams which do not contain ground state components. This

equation determines $D_\alpha(\varepsilon_0 + z)$ in terms of $D_0(\varepsilon_0 + z)$ and $\bar{D}_\alpha(\varepsilon_0 + z)$, thus leading us to a study of the function $\bar{D}_\alpha(\varepsilon_0 + z)$.

Applying the methods of section 4 and taking into account that only diagrams without ground state components are involved, we derive, in exactly the same way as (4.2), the formula

$$\bar{D}_\alpha(\varepsilon_0 + z) = (\varepsilon'_\alpha - z)^{-1} + (\varepsilon'_\alpha - z)^{-1}[-V + V\bar{D}(\varepsilon_0 + z)V - \ldots]_{idL}(\alpha)\bar{D}_\alpha(\varepsilon_0 + z),$$

where $\varepsilon'_\alpha = \varepsilon_\alpha - \varepsilon_0$.

The subscript idL means that only irreducible diagonal diagrams without ground state components contribute. The L stands for "linked clusters", an expression used by Brueckner, e.a., to indicate contributions from diagrams without ground state components. Defining

$$\bar{G}_\alpha(\varepsilon_0 + z) = [-V + V\bar{D}(\varepsilon_0 + z)V - \ldots]_{idL}(\alpha) \tag{10.2}$$

one obtains for $\bar{D}_\alpha(\varepsilon_0 + z)$ the expression

$$\bar{D}_\alpha(\varepsilon_0 + z) = (\varepsilon'_\alpha - z - \bar{G}_\alpha(\varepsilon_0 + z))^{-1}, \tag{10.3}$$

which has just the same form as (5.1).

The considerations of section 6 tell us that $\bar{G}_\alpha(\varepsilon_0 + z)$ and $\bar{D}_\alpha(\varepsilon_0 + z)$, being defined by means of diagrams without ground state components, have a finite limit for $\Omega \to \infty$. We see here clearly the important advance made with respect to chapter II. While in the developments of chapter II we were forced to keep Ω finite in order to avoid the occurrence of infinite quantities, we can here in the expressions of $\bar{D}_\alpha(\varepsilon_0 + z)$ and $\bar{G}_\alpha(\varepsilon_0 + z)$ carry out completely the limit $\Omega \to \infty$. In this limit $\bar{G}_\alpha(\varepsilon_0 + z)$ has no poles as can be concluded from (10.2). There will be one or more cuts in the complex plane along the real axis. Exactly as for $D_\alpha(z)$ and $G_\alpha(z)$ we have

$$\bar{G}_\alpha(z^*) = \bar{G}_\alpha(z)^* \text{ and } \bar{D}_\alpha(z^*) = \bar{D}_\alpha(z)^*.$$

If we define the real functions $\bar{K}_\alpha(x)$ and $\bar{J}_\alpha(x)$ of the real variable x by the equation

$$\lim_{\eta \to 0} \bar{G}_\alpha(x + i\eta) = \bar{K}_\alpha(x) + i\bar{J}_\alpha(x) \; ; \; \eta > 0, \tag{10.4}$$

the above relations imply

$$\lim_{\eta \to 0} \bar{G}_\alpha(x - i\eta) = \bar{K}_\alpha(x) - i\bar{J}_\alpha(x) \; ; \; \eta > 0, \tag{10.5}$$

The singular points of $\bar{G}_\alpha(\varepsilon_0 + z)$ on the real axis are the points where $\bar{J}_\alpha(\varepsilon_0 + x) \neq 0$. Equation (10.3) shows that these points will also be singularities of $\bar{D}_\alpha(\varepsilon_0 + z)$. In addition $\bar{D}_\alpha(\varepsilon_0 + z)$ can have a pole, when the equation

$$\varepsilon'_\alpha - x - \bar{K}_\alpha(\varepsilon_0 + x) = 0 \tag{10.6}$$

has a root in the neighbourhood of which $\bar{J}_\alpha(\varepsilon_0 + x) = 0$. Equation (10.6) has at least one and in most cases it has only one root $\bar{E}_\alpha$; for simplicity we

assume the latter to be the case. The present section, as well as sections 12 and 13 hereafter, deal with the case that this point $\bar{E}_\alpha$ is a pole of $\bar{D}_\alpha(\varepsilon_0 + z)$. For $\bar{E}_\alpha$ to be a true pole of $\bar{D}_\alpha(\varepsilon + z)$ one must have $\bar{J}_\alpha(\varepsilon_0 + x) = 0$ in a neighbourhood of $\bar{E}_\alpha$. This condition is, however, too strong for our purpose. It will be enough to assume that $\bar{J}_\alpha$ satisfies for small x the relation

$$\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha + x) = O(|x|^{1+\alpha}) \text{ with } \alpha > 0. \tag{10.7}$$

Under this circumstance we still call $\bar{E}_\alpha$ a pole of the function $\bar{D}_\alpha(\varepsilon_0 + z)$. The condition (10.7) is sufficient for the finiteness of $\bar{G}_\alpha(\varepsilon_0 + z)$ and its derivative $\bar{G}'_\alpha(\varepsilon_0 + z)$ at $\bar{E}_\alpha$. From (10.6) and (10.7) we also notice

$$\bar{E}_\alpha = \varepsilon'_\alpha - \bar{G}_\alpha(\varepsilon_0 + \bar{E}_\alpha). \tag{10.8}$$

The residue of $\bar{D}_\alpha(\varepsilon_0 + z)$ at $\bar{E}_\alpha$ we denote by $-\bar{N}_\alpha$. It follows easily from (10.3) that

$$\bar{N}_\alpha^{-1} = 1 + \bar{G}'_\alpha(\varepsilon_0 + \bar{E}_\alpha). \tag{10.9}$$

Equation (10.1) enables us to express the pole $E_\alpha - \varepsilon_0$ and the residue $-N_\alpha$ of $D_\alpha(\varepsilon_0+z)$ in the corresponding quantities of $\bar{D}_\alpha(\varepsilon_0+z)$ and $D_0(\varepsilon_0+z)$. We must simply add up the poles of $\bar{D}_\alpha(\varepsilon_0 + z)$ and $D_0(\varepsilon_0 + z)$ to find the pole of $D_\alpha(\varepsilon_0 + z)$, whereas the residue of $D_\alpha(\varepsilon_0 + z)$ is minus the product of the residues of $\bar{D}_\alpha(\varepsilon_0 + z)$, and $D_0(\varepsilon_0 + z)$. This leads to the equations

$$N_\alpha = N_0 \,.\, \bar{N}_\alpha, \tag{10.10}$$

and

$$E_\alpha = E_0 + \bar{E}_\alpha. \tag{10.11}$$

It can furthermore be shown that the relation (10.7) implies that the discontinuity of $\bar{D}_\alpha(\varepsilon_0 + z)$ at a point x of the real axis in the neighbourhood of $\bar{E}_\alpha$ behaves like $O(|x - \bar{E}_\alpha|^{\alpha-1})$ and that by (10.1) the same holds for the discontinuity of $D_\alpha(z)$ in the neighbourhood of E_α. We shall forgo the proof. These facts imply that we are justified in calling $\bar{E}_\alpha$ and E_α poles (see the footnote on page 508).

The inequality (5.2) for $D_\alpha(z)$ and $D_0(z)$ implies that both N_α and N_0 are positive quantities. Hence (10.10) shows that also $\bar{N}_\alpha > 0$.

We shall now discuss the physical significance of the result (10.11). In section 5 we saw that the pole E_α of $D_\alpha(z)$ represents the energy of the perturbed states $|\psi_\alpha\rangle^\pm$ which, through the influence of the perturbation, originate from the unperturbed state $|\alpha\rangle$ with energy ε_α. Equation (10.11) shows that $\bar{E}_\alpha$ represents the energy of the system in the state $|\psi_\alpha\rangle^\pm$ as compared with the energy E_0 of the perturbed ground state $|\psi_0\rangle$, *i.e.* $\bar{E}_\alpha$ is the excitation energy of $|\psi_\alpha\rangle^\pm$. From the Ω-independence of $\bar{D}_\alpha(\varepsilon_0 + z)$, the energy $\bar{E}_\alpha$ is independent of the size of the system, in agreement with physical expectation. We thus see that $\bar{E}_\alpha$ is not only a convenient auxiliary quantity, but that it also has a simple and direct physical meaning. In field

theory in particular, where the true (perturbed) energy E_0 of the vacuum is not measurable, $\bar{E}_\alpha$ is the only energy of physical interest.

An important remark must be added concerning the case where the state $|\alpha\rangle$ involves several particles or holes. The diagonal diagrams contributing to $\bar{D}_\alpha(\varepsilon_0 + z)$ for such states are not connected. An example is shown in figure 2. The diagram c represents a contribution to $\bar{D}_\alpha(\varepsilon_0 + z)$ where $|\alpha\rangle$ is a state of two particles. As seen in section 3 the diagonal diagrams of states containing several particles and holes are always composed of diagonal diagrams of one-particle-states and one-hole-states. Let us, to be more explicit, take the state $|\alpha\rangle = |k_1 k_2\ ;\ \rangle$, and consider $\bar{D}_\alpha(\varepsilon_0 + z)$. Each diagram is composed of a diagonal diagram of the state $|k_1\ ;\ \rangle$, and a diagonal diagram of $|k_2\ ;\ \rangle$. Applying (7.4), and summing over all possible diagrams one gets

$$\bar{D}_\alpha(\varepsilon_0 + z) = \bar{D}_{k_1}(\varepsilon_0 + z) * \bar{D}_{k_2}(\varepsilon_0 + z), \tag{10.12}$$

where $\bar{D}_k(\varepsilon_0 + z)$ denotes the function $\bar{D}_\gamma(\varepsilon_0 + z)$ for $|\gamma\rangle = |k\ ;\ \rangle$. Repeating the arguments which led to (10.10) and (10.11) one finds from (10.12)

$$\bar{N}_\alpha = \bar{N}_{k_1} \bar{N}_{k_2},\ \bar{E}_\alpha = \bar{E}_{k_1} + \bar{E}_{k_2}. \tag{10.13}$$

The last equation expresses the additivity of the perturbed excitation energies of the two particles. It is clear that our argument extends to states with an arbitrary number of particles and holes.

Chapter V. The perturbed wave functions

11. *The wave function of the ground state.* In this and the following sections we are concerned with the application of the considerations of chapter III to the calculation of wave functions. We have learned in section 5 that to each unperturbed state $|\alpha\rangle$, such that $D_\alpha(z)$ has a pole E_α, can be associated two perturbed stationary states $|\psi_\alpha\rangle^\pm$ respectively characterized by the outgoing and incoming nature of the scattered waves. For such a state $|\alpha\rangle$, according to (5.7), the matrix element $\langle\beta\,|R(z)|\,\alpha\rangle$ can be written as a product of two factors. The second factor is $D_\alpha(z)$ and has a pole E_α, whereas the first factor has a cut on the real axis running from a point B to $+\infty$. For most states $|\alpha\rangle$, $B < E_\alpha$ and E_α is not a proper pole of the function $\langle\beta\,|R(z)|\,\alpha\rangle$. Instead of one, there are two residues in such a point E_α defined by

$$\mathfrak{R}^\pm_{E_\alpha}\,[\langle\beta\,|R(z)|\,\alpha\rangle] = \lim_{z\to E_\alpha} (z - \mathrm{E}_\alpha)\,\langle\beta\,|R(z)|\,\alpha\rangle,$$

where the plus (minus) sign must be chosen if z approaches E_α from the upper (lower) half of the complex plane. It is clear that both residues coincide if $E_\alpha < B$. Using this, slightly more general, definition of the residue we found in section 5 that the wave functions

$$|\psi_\alpha\rangle^\pm = -\,N_\alpha^{-\frac{1}{2}}\,\mathfrak{R}^\pm_{E_\alpha}\,[R(z)\,|\alpha\rangle] \tag{11.1}$$

are normalized eigenfunctions of $H = H_0 + V$.

We now write this equation in an alternative form. We define the operator

$$A_\alpha(z) = \int_\beta \langle\beta\,|R(z)|\,\alpha\rangle \,.\, A_\beta, \tag{11.2}$$

where A_β is for each $|\beta\rangle$ the ordered product of creation operators such that $A_\beta\,|\varphi_0\rangle = |\beta\rangle$. Applying (11.2) to $|\varphi_0\rangle$ we get the equality

$$A_\alpha(z)\,|\varphi_0\rangle = R(z)\,|\alpha\rangle. \tag{11.3}$$

$A_\alpha(z)$ contains the factor $D_\alpha(z)$ which has a pole at $z = E_\alpha$. We define the residues $\mathfrak{R}^{\pm}_{E_\alpha}[A_\alpha(z)]$ of $A_\alpha(z)$ (in the extended sense defined above) by taking the residues of each matrix element in the expansion (11.2). We then define the operators O^+_α and O^-_α

$$O^{\pm}_\alpha = -\,N_\alpha^{-\frac{1}{2}}\,\mathfrak{R}^{\pm}_{E_\alpha}\,[A_\alpha(z)], \tag{11.4}$$

where $-N_\alpha$ is as usual the residue of $D_\alpha(z)$ in E_α.

From (11.4) and (11.3) the formula (11.1) can be written

$$|\psi_\alpha\rangle^{\pm} = O^{\pm}_\alpha\,|\varphi_0\rangle. \tag{11.5}$$

The purpose of this section is to achieve a far-reaching simplification of (11.4) by means of the results of chapter III.

We study first the ground state wave function $|\psi_0\rangle$. Equation (5.7) for $|\alpha\rangle = |\varphi_0\rangle$ reads

$$\langle\beta\,|R(z)|\,\varphi_0\rangle = \langle\beta|[1 + \{-D(z)\,V + D(z)\,V\,D(z)\,V - \ldots\}_{ind}]\,|\varphi_0\rangle\,D_0(z). \tag{11.6}$$

The intermediate states $|\gamma\rangle$ occurring in (11.6) are also intermediate states in the expression for $G_0(z)$, which one obtains from (4.3) by taking the diagonal matrix element for $|\varphi_0\rangle$. Furthermore, according to section 5, the fact that E_0 is a pole of $D_0(z)$ implies that $G_0(z)$ is single-valued at the point E_0. This requires, as was shown in H I, that for none of the intermediate states $|\gamma\rangle$ under consideration the cut of $D_\gamma(z)$ would extend through E_0. Consequently the cut of the first factor on the right-hand side of (11.6) does not extend through E_0. In other words $E_0 \leqslant B$. Accordingly the matrix element $\langle\beta\,|R(z)|\,\varphi_0\rangle$ and the operator

$$A_0(z) = \int_\beta \langle\beta\,|R(z)|\,\varphi_0\rangle\,A_\beta$$

have only one residue, whether calculated from the upper half or the lower half of the z-plane. Hence

$$|\psi_0\rangle = O_0\,|\varphi_0\rangle, \tag{11.7}$$

where

$$O_0 = -\,N_0^{-\frac{1}{2}}\,\mathfrak{R}_{E_0}\,[A_0(z)]. \tag{11.8}$$

$A_0(z)$ can be written

$$A_0(z) = D_0(z) + \int' \langle\beta\,|R(z)|\,\varphi_0\rangle\,A_\beta, \tag{11.9}$$

where the sum extends over all states $|\beta\rangle$ except $|\varphi_0\rangle$. Replacing z by $\varepsilon_0 + z$ and applying (8.1) to the matrix element $\langle\beta\,|R(\varepsilon_0 + z)|\,\varphi_0\rangle$ we obtain the formula

$$A_0(\varepsilon_0 + z) = \bar{A}_0(\varepsilon_0 + z) * D_0(\varepsilon_0 + z), \tag{11.10}$$

where $\bar{A}_0(\varepsilon_0 + z)$ is defined by

$$\bar{A}_0(\varepsilon_0 + z) = -\,1/z + \int' \langle\beta\,|\bar{R}(\varepsilon_0 + z)|\,\varphi_0\rangle\, A_\beta. \tag{11.11}$$

The matrix element $\langle\beta\,|\bar{R}(\varepsilon_0 + z)|\,\varphi_0\rangle$ for $|\beta\rangle \neq |\varphi_0\rangle$ was defined in section 8 as the sum of the contributions to $\langle\beta\,|R(\varepsilon_0 + z)|\,\varphi_0\rangle$ of all diagrams, which do not contain ground state components.

Applying the methods of section 4 it is easy to express $\langle\beta\,|\bar{R}(\varepsilon_0 + z)|\,\varphi_0\rangle$ in terms of irreducible diagrams only. One finds, remembering that $|\beta\rangle \neq |\varphi_0\rangle$,

$$\langle\beta\,|\bar{R}(\varepsilon_0 + z)|\,\varphi_0\rangle =$$
$$= \langle\beta|\,[-\bar{D}(\varepsilon_0 + z)\,V + \bar{D}(\varepsilon_0 + z)\,V\bar{D}(\varepsilon_0 + z)\,V - \ldots]_{iL}\,|\varphi_0\rangle\,(-z)^{-1}, \tag{11.12}$$

where the subscript iL means that one should take only irreducible diagrams without ground state components. Instead of the factor D_0 at the extreme right of (11.6), one finds in (11.12) the factor $(-z)^{-1}$. This shows that $\langle\beta\,|\bar{R}(\varepsilon_0 + z)\,|\varphi_0\rangle$ has a pole in the origin, and the same holds true for $\bar{A}_0(\varepsilon_0 + z)$, as seen from (11.11). From (11.10) $A_0(\varepsilon_0 + z)$ is obtained by convolution of $\bar{A}_0(\varepsilon_0 + z)$ and $D_0(\varepsilon_0 + z)$. The latter functions have poles at $z = 0$ and $z = E_0 - \varepsilon_0$, and these poles determine the pole of $A_0(\varepsilon_0 + z)$ at $E_0 - \varepsilon_0$. Consequently the residue of $A_0(\varepsilon_0 + z)$ at its pole $E_0 - \varepsilon_0$ is simply minus the product of the residues of $\bar{A}_0(\varepsilon_0 + z)$, and $D_0(\varepsilon_0 + z)$. This gives the formula

$$\mathfrak{R}_{E_0}\,[A_0(z)] = N_0\,\mathfrak{R}_0\,[\bar{A}_0(\varepsilon_0 + z)]. \tag{11.13}$$

Defining the operator $\bar{O}_0$ by

$$\bar{O}_0 = -\,\mathfrak{R}_0\,[\bar{A}_0(\varepsilon_0 + z)], \tag{11.14}$$

we get from (11.8) and (11.13)

$$O_0 = N_0^{\frac{1}{2}} \cdot \bar{O}_0. \tag{11.15}$$

This formula already presents an important simplification with respect to (11.8), inasmuch as the definition of $\bar{O}_0$ only involves diagrams without ground state components.

Equation (11.15) is equivalent to a result derived recently by Goldstone [5]). To show the equivalence we write (11.11) in a more explicit form using (3.5).

$$\bar{A}_0(\varepsilon_0 + z)\,|\varphi_0\rangle =$$
$$=[1-(H_0-\varepsilon_0-z)^{-1}\,V+(H_0-\varepsilon_0-z)^{-1}\,V(H_0-\varepsilon_0-z)^{-1}\,V-\ldots]_L\,|\varphi_0\rangle\,(-z)^{-1}$$

which, together with (11.14) and (11.15), gives

$$|\psi_0\rangle = N_0^{\frac{1}{2}}\,[1 - (H_0-\varepsilon_0)^{-1}V+(H_0-\varepsilon_0)^{-1}V(H_0-\varepsilon_0)^{-1}V-\ldots]_L|\varphi_0\rangle. \quad (11.16)$$

The subscript L excludes diagrams with ground state components. This equation is, except for the normalization factor $N_0^{\frac{1}{2}}$, identical with eq. (3.2) of Goldstone's paper.

We now proceed to derive a still simpler expression for O_0. We notice that the diagrams occurring in the definition of $\bar{A}_0(\varepsilon_0+z)$ are, in general, not connected. They can be composed of an arbitrary number of components, each of which has external lines at the left end. We define the operator

$$\tilde{A}_0(\varepsilon_0+z) = \int_\beta' \langle\beta|\,\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle\, A_\beta, \quad (11.17)$$

where $\langle\beta\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle$ is the sum of all connected diagrams contributing to the matrix element $\langle\beta\,|R(\varepsilon_0+z)|\,\varphi_0\rangle$, for $|\beta\rangle \neq |\varphi_0\rangle$. In exactly the same way as (11.12) we derive for $|\beta\rangle \neq |\varphi_0\rangle$ the formula

$$\langle\beta\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle = \\ = \bar{D}_\beta(\varepsilon_0+z)\,\langle\beta\,|[-V+V\bar{D}(\varepsilon_0+z)V-\ldots]_{iC}|\,\varphi_0\rangle\,(-z)^{-1}, \quad (11.18)$$

where the subscript iC means that one sums over the contributions of irreducible connected diagrams only. As we see $\langle\beta\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle$ has a pole in the origin, and by (11.17) the same holds for $\tilde{A}_0(\varepsilon_0+z)$. We now define the operator $\tilde{O}_0$ by

$$\tilde{O}_0 = -\,\mathfrak{R}_0\,[\tilde{A}_0(\varepsilon_0+z)]. \quad (11.19)$$

Just as before for O_0 and $\bar{O}_0$, the residue is unique. Using (11.17) and (11.18) we can write (11.19) in a more explicit form

$$\tilde{O}_0 = \int_\beta' \langle\beta\,|[-\bar{D}(\varepsilon_0)\,V+\bar{D}(\varepsilon_0)\,V\,\bar{D}(\varepsilon_0)\,V-\ldots]_{iC}\,|\varphi_0\rangle\,.\,A_\beta, \quad (11.20)$$

where the sum is extended over all states $|\beta\rangle \neq |\varphi_0\rangle$.

Only connected diagrams contribute to $\tilde{O}_0$. This class of diagrams is much smaller than the class of all diagrams without ground state components, which we had to use in the expression of $\bar{O}_0$. Still, as we shall see now, $\bar{O}_0$ can be expressed very simply in terms of $\tilde{O}_0$. We write

$$\bar{A}_0(\varepsilon_0+z) = \textstyle\sum_{\nu=0}^{\infty} A_\nu, \quad (11.21)$$

where A_ν is the sum of the contributions of all diagrams of $\bar{A}_0(\varepsilon_0+z)$ containing exactly ν components. From (11.11) and (11.17) follows immediately that $A_0 = -\,z^{-1}$ and $A_1 = \tilde{A}_0\,(\varepsilon_0+z)$. Let us calculate A_2. Consider all diagrams which are composed of a diagram of $\langle\beta'\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle$ and a diagram of $\langle\beta''\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle$. These diagrams have two components and their contribution to A_2 is given by

$$\langle\beta'\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle * \langle\beta''\,|\tilde{R}(\varepsilon_0+z)|\,\varphi_0\rangle\, A_{\beta'}A_{\beta''},$$

as follows immediately from the fundamental formula (7.4). Summing this expression over all states $|\beta'\rangle$ and $|\beta''\rangle$ distinct from $|\varphi_0\rangle$ we get

$$A_2 = \tfrac{1}{2}\tilde{A}_0(\varepsilon_0 + z) * \tilde{A}_0(\varepsilon_0 + z).$$

The factor $\frac{1}{2}$ arises from the fact that in the summation over $|\beta'\rangle$ and $|\beta''\rangle$ each diagram of A_2 is counted twice. The diagrams of A_3 can be obtained by the combination of a diagram of A_2 with a diagram of A_1. Another factor $\frac{1}{3}$ must be added to compensate for redundant counting and one gets

$$A_3 = \tfrac{1}{6}\tilde{A}_0(\varepsilon_0 + z) * \tilde{A}_0(\varepsilon_0 + z) * \tilde{A}_0(\varepsilon_0 + z).$$

Continuing in the same way we get generally

$$A_\nu = (\nu!)^{-1}\, \tilde{A}_0(\varepsilon_0 + z) * \tilde{A}_0(\varepsilon_0 + z) * \ldots\ldots * \tilde{A}_0(\varepsilon_0 + z),$$

with ν factors $\tilde{A}_0(\varepsilon_0 + z)$ in the convolution. If we now take the residue in $z = 0$ of each term in the expansion (11.21) we find

$$\bar{O}_0 = 1 + \tilde{O}_0 + \tilde{O}_0^2/2! + \tilde{O}_0^3/3! + \ldots\ldots,$$

a result which reduces to the compact form

$$\bar{O}_0 = \exp(\tilde{O}_0). \tag{11.22}$$

Inserting this in (11.15) we obtain the important result

$$O_0 = N_0^{\frac{1}{2}} \exp(\tilde{O}_0) = \exp(-\tfrac{1}{2}\bar{G}_0'(\varepsilon_0) + \tilde{O}_0), \tag{11.23}$$

where $\tilde{O}_0$ is given by (11.20) and where the value (9.21) of N_0 has been used.

The derivations of (11.15) and (11.23) are actually valid for a finite but very large volume Ω. The operators $\bar{O}_0$ and $\tilde{O}_0$ are defined by means of diagrams which do not contain ground state components.

In section 6 we have shown that the contributions of such diagrams have a finite limit for $\Omega \to \infty$. This means that in the expansion of $\bar{O}_0\,|\varphi_0\rangle$ or of $\tilde{O}_0^\nu\,|\varphi_0\rangle$, $\nu = 1, 2, \ldots$, in unperturbed states all coefficients have a finite limit. Owing however to the large number of terms in the expansion the norm of $\tilde{O}_0^\nu\,|\varphi_0\rangle$ is large as $\Omega^{\nu/2}$, while the norm of $\bar{O}_0\,|\varphi_0\rangle$ is exponentially large for $\Omega \to \infty$. As seen from (11.15) this behaviour is compensated in (11.23) by the normalization factor $N_0^{\frac{1}{2}}$, which approaches zero exponentially in the limit $\Omega \to \infty$. In fact we saw before that $\bar{G}_0'(\varepsilon_0)$ is proportional to Ω for large Ω.

12. *Wave functions of excited states.* Having determined O_0 in terms of contributions of connected diagrams we turn to an analysis of $O_\alpha^\pm$ where $|\alpha\rangle \neq |\varphi_0\rangle$ is a state such that $D_\alpha(z)$ has a pole E_α. $O_\alpha^\pm$ is defined by (11.4). Let us consider an arbitrary diagram of $\langle\beta\,|R(\varepsilon_0 + z)|\,\alpha\rangle$. It contains one or more components with external lines at the right end, and other components without such lines. Diagrams composed only of components of the latter type give contributions to $\langle\beta'\,|R(\varepsilon_0 + z)|\,\varphi_0\rangle$. On the other hand we

can take together all diagrams of $\langle\beta''|R(\varepsilon_0+z)|\alpha\rangle$, all components of which have one or more external lines at their right end. The sum of the contributions of these diagrams we denote by $\langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle$.

Consider all diagrams which are composed of a diagram of $\langle\beta'|R(\varepsilon_0+z)|\varphi_0\rangle$ and a diagram of $\langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle$. These diagrams contribute to the matrix element $\langle\beta'\beta''|R(\varepsilon_0+z)|\alpha\rangle$ where $|\beta'\beta''\rangle = A_{\beta'}\,.\,A_{\beta''}|\varphi_0\rangle$.
Their contribution is, as follows immediately from (7.4),

$$\langle\beta'|R(\varepsilon_0+z)|\varphi_0\rangle * \langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle. \tag{12.1}$$

The operator $A_\alpha(\varepsilon_0+z)$, defined by (11.2), can now be written

$$\begin{aligned} A_\alpha(\varepsilon+z) &= \int_{\beta'\beta''}\langle\beta'\beta''|R(\varepsilon_0+z)|\alpha\rangle A_{\beta'}A_{\beta''} = \\ &= \int_{\beta'\beta''}\langle\beta'|R(\varepsilon_0+z)|\varphi_0\rangle * \langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle A_{\beta'}A_{\beta''}. \end{aligned} \tag{12.2}$$

Defining the operator

$$\hat{A}_\alpha(\varepsilon_0+z) = \int_\beta \langle\beta|\hat{R}(\varepsilon_0+z)|\alpha\rangle A_\beta \tag{12.3}$$

we obtain for equation (12.2) the simple form

$$A_\alpha(\varepsilon_0+z) = \hat{A}_\alpha(\varepsilon_0+z) * A_0(\varepsilon_0+z). \tag{12.4}$$

We must now study the matrix element $\langle\beta|\hat{R}(\varepsilon_0+z)|\alpha\rangle$ in somewhat more detail. All diagrams with ground state components are excluded from its definition. By the methods of section 4 we can easily express $\langle\beta|\hat{R}(\varepsilon_0+z)|\alpha\rangle$ in terms of contributions of irreducible diagrams. The formula we obtain reads

$$\begin{aligned} &\langle\beta|\hat{R}(\varepsilon_0+z)|\alpha\rangle = \\ &\langle\beta|[1+\{-\bar{D}(\varepsilon_0+z)\,V+\bar{D}(\varepsilon_0+z)\,V\,\bar{D}(\varepsilon_0+z)\,V-\ldots\}_{indR}]|\alpha\rangle\,\bar{D}_\alpha(\varepsilon_0+z), \end{aligned} \tag{12.5}$$

where $|\alpha\rangle \neq |\varphi_0\rangle$. The subscript *indR* indicates restriction to irreducible non-diagonal diagrams, all components of which have at least one external line at the right-hand side. We see, from (12.5), that $\langle\beta|\hat{R}(\varepsilon_0+z)|\alpha\rangle$ contains the factor $\bar{D}_\alpha(\varepsilon_0+z)$, which has a pole $\bar{E}_\alpha$. The other factor on the right-hand side of (12.5) has a cut along the real axis; in most cases this factor is double-valued at the point $\bar{E}_\alpha$, giving exactly the same situation as met before with $\langle\beta|R(\varepsilon_0+z)|\alpha\rangle$. In (12.2) we have the convolution

$$\langle\beta'|R(\varepsilon_0+z)|\varphi_0\rangle * \langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle.$$

The residue of this expression at its pole $E_\alpha - \varepsilon_0$ is

$$\begin{aligned} &\mathfrak{R}^{\pm}_{\bar{E}_\alpha-\varepsilon_0}[\langle\beta'|R(\varepsilon_0+z)|\varphi_0\rangle * \langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle] = \\ &\quad = -\,\mathfrak{R}^{\pm}_{\bar{E}_\alpha}[\langle\beta''|\hat{R}(\varepsilon_0+z)|\alpha\rangle]\,\mathfrak{R}_{E_0-\varepsilon_0}[\langle\beta'|R(\varepsilon_0+z)|\varphi_0\rangle], \end{aligned}$$

where we have used $\bar{E}_\alpha = E_\alpha - E_0$. Inserting this in (12.2) we find with the definitions (11.2) and (12.3)

$$\mathfrak{R}^{\pm}_{\bar{E}_\alpha}[A_\alpha(z)] = \mathfrak{R}^{\pm}_{\bar{E}_\alpha}[\hat{A}_\alpha(\varepsilon_0+z)]\,.\,\mathfrak{R}_{E_0}[A_0(z)]. \tag{12.6}$$

If we now define the operators $\hat{O}_\alpha^\pm$ by

$$\hat{O}_\alpha^\pm = -\bar{N}_\alpha^{-\frac{1}{2}} \mathfrak{R}_{\bar{E}_\alpha}^\pm[\hat{A}_\alpha(\varepsilon_0 + z)], \tag{12.7}$$

where $-\bar{N}_\alpha$ is the residue of $\bar{D}_\alpha(\varepsilon_0 + z)$ in $\bar{E}_\alpha$, we obtain, using (10.10), the important result

$$O_\alpha^\pm = \hat{O}_\alpha^\pm \,.\, O_0. \tag{12.8}$$

$O_\alpha^\pm$ and O_0 were defined by (11.4) and (11.8). Applying the operator equation (12.8) to $|\varphi_0\rangle$ we get

$$|\psi_\alpha\rangle^\pm = \hat{O}_\alpha^\pm \, |\psi_0\rangle. \tag{12.9}$$

This equation shows clearly the physical meaning of $\hat{O}_\alpha^\pm$. It "creates" the state $|\psi_\alpha\rangle^\pm$ from the actual (perturbed) ground state $|\psi_0\rangle$. It is the analogue, for the perturbed system, of the operator A_α which creates $|\alpha\rangle$ from $|\varphi_0\rangle$ in absence of the interaction. The importance of the operator $\hat{O}_\alpha^\pm$ is further stressed by considering the case of large systems. From the definition (12.7) and from the fact that in the calculation of $\bar{D}_\alpha(\varepsilon_0 + z)$ and $\hat{A}_\alpha(\varepsilon_0 + z)$ only diagrams without ground state components are involved, we conclude that $\hat{O}_\alpha^\pm$ has a finite limit for $\Omega \to \infty$. Although for an infinitely large system a proper expansion of $|\psi_0\rangle$ in unperturbed states strictly speaking no longer exists (remember the vanishing of N_0 for $\Omega \to \infty$ in (11.15)), the operator $\hat{O}_\alpha^\pm$, which describes the change of $|\psi_0\rangle$ introduced by the presence of particles outside the Fermi sphere and holes inside it, keeps a simple and meaningful form.

Throughout this paper we have often used a terminology inspired by the special problem of a gas of Fermi particles with interaction. As mentioned before, however, all results are of a quite general nature and are applicable to a broad range of problems. Up till now we investigated states $|\alpha\rangle$ for which the function $D_\alpha(z)$ has a pole. As will be shown elsewhere, in the theory of the Fermi gas with interaction strictly speaking no such state except $|\varphi_0\rangle$ exists. Low-lying states satisfy however this requirement with a very good approximation and the results of the present section will provide us with an excellent starting point for their study. In the quantum theory of fields, on the other hand, all states $|\alpha\rangle$ satisfy the requirement that $D_\alpha(z)$ has a pole and the whole discussion of this section is immediately applicable.

We end this section with a remark concerning one-particle-states. Let $|\alpha\rangle$ be a state differing from the unperturbed ground state $|\varphi_0\rangle$ by the presence of one single particle. Taking e.g. the case of interacting meson and nucleon fields, $|\varphi_0\rangle$ is the free vacuum and $|\alpha\rangle$ would be for example a one-nucleon-state. We denote this state by $|k\rangle$, k referring to the momentum of the particle. From (12.5) we see that the diagrams contributing to $\langle\beta\,|\hat{R}(\varepsilon_0 + z)|\,k\rangle$ are connected and have one external line at the right end. As is easily seen the intermediate states in (12.5) are the same as those oc-

curring in the expression (10.2) for $\bar{G}_\alpha(\varepsilon_0 + z)$ if we take $|\alpha\rangle = |k\rangle$. By our assumption $\bar{D}_k(\varepsilon_0 + z)$ has a pole $\bar{E}_k$, a fact which requires that $\bar{G}_k(\varepsilon_0 + z)$ be single-valued at the point $\bar{E}_k$, *i.e.*, that the cut of $\bar{G}_k$ does not go through $\bar{E}_k$. As shown in H I, this has the consequence that the same property holds for all functions $\bar{D}_\gamma(\varepsilon_0 + z)$ belonging to states $|\gamma\rangle$ which occur as intermediate states in (10.2). We conclude that also $\langle\beta\,|\hat{R}(\varepsilon_0 + z)|\,k\rangle$, and by (12.3) $\hat{A}_k(\varepsilon_0 + z)$ are single-valued in $\bar{E}_k$. Thus we have shown that to each single particle state $|k\rangle$, such that $D_k(z)$ has a pole, the operators $\hat{O}_k^+$ and $\hat{O}_k^-$ are identical. The expression (12.9) reduces to a single stationary state

$$|\psi_k\rangle = \hat{O}_k\,|\psi_0\rangle. \tag{12.10}$$

If $|\alpha\rangle$ is a state of more than one free particle or hole the states $|\psi_\alpha\rangle^+$ and $|\psi_\alpha\rangle^-$ will in general be different. They correspond to scattering states with outgoing and incoming scattered waves respectively. This was shown in H II by an investigation of the asymptotic behaviour of wave packets for large times.

13. *Asymptotically stationary states.* In this section we shall briefly consider in the light of the diagram analysis, the theory of asymptotically stationary states as developed by Van Hove in H I and H II. It will appear that these states can be expressed very simply by means of the operators $\hat{O}_k$ introduced in the last section.

Let $|\alpha\rangle = \xi^*_{k_1}\xi^*_{k_2}\ldots\ldots\xi^*_{k_p}\,|\varphi_0\rangle$ be an unperturbed p-particle state and assume that the function $D_\alpha(z)$ has a pole E_α. Consider the state $|\psi_\alpha\rangle^{as}$ defined by

$$|\psi_\alpha\rangle^{as} = \hat{O}_{k_1}\hat{O}_{k_2}\ldots\hat{O}_{k_p}\,|\psi_0\rangle. \tag{13.1}$$

As will be shown hereafter, this state is identical with the asymptotically stationary state $|\alpha\rangle_{as}$ defined by Van Hove in H I (equation 5.12). The physical meaning of $|\psi_\alpha\rangle^{as}$ is particularly clear in field theory. It is a state of p "dressed" particles without mutual interaction. In meson-theory for example it would represent a physical situation where one has e.g., p nucleons with their surrounding cloud of mesons and nucleon-antinucleon pairs, moving independently of each other in plane wave states. Such a state is clearly not stationary, and a wave function $|\psi(t)\rangle^{as}$, defined as a linear combination

$$|\psi(t)\rangle^{as} = \int_\alpha c_\alpha \exp\,(-iE_\alpha t)\,|\psi_\alpha\rangle^{as}, \tag{13.2}$$

is not a solution of the Schrödinger equation. It does however approach such a solution for large $|t|$. Indeed, considering the two wave functions

$$|\psi(t)\rangle^\pm = \int_\alpha c_\alpha \exp\,(-iE_\alpha t)\,|\psi_\alpha\rangle^\pm \tag{13.3}$$

which obviously verify the Schrödinger equation, one has the following relations

$$\lim_{t\to-\infty} \mid |\psi(t)\rangle^{as} - |\psi(t)\rangle^{+}| = 0,$$
$$\lim_{t\to+\infty} \mid |\psi(t)\rangle^{as} - |\psi(t)\rangle^{-}| = 0.$$

The bars refer to the norm of the wave functions enclosed. These formulae are identical with (3.2) in H II.

All we want to do here is to establish the identity of the states (13.1) with the states $|\alpha\rangle_{as}$ in H I. From the definition (13.1) of $|\psi_\alpha\rangle^{as}$ we derive, using (12.7), (12.3) and (11.8),

$$|\psi_\alpha\rangle^{as} = -N_\alpha^{-\frac{1}{2}}\,\mathfrak{R}_{E_\alpha-\varepsilon_0}\,[\int_{\beta_0\beta_1\ldots\beta_p} |\beta_0\beta_1\ldots\beta_p\rangle\,.$$
$$.\ \langle\beta_0\,|R(\varepsilon_0+z)|\,\varphi_0\rangle * \langle\beta_1\,|\hat{R}(\varepsilon_0+z)|\,k_1\rangle * \ldots\ldots * \langle\beta_p\,|\hat{R}(\varepsilon_0+z)|\,k_p\rangle].$$

This again can be written

$$|\psi_\alpha\rangle^{as} = -N_\alpha^{-\frac{1}{2}}\,\mathfrak{R}_{E_\alpha}\,[R'(z)\,|\alpha\rangle], \tag{13.4}$$

where $\langle\beta\,|R'(z)|\,\alpha\rangle$ for arbitrary $|\beta\rangle$ is the sum of the contributions to $\langle\beta\,|R(z)|\,\alpha\rangle$ of all diagrams, each component of which has at most one external line at its right end. We shall call such diagrams *completely disconnected.* Substituting (4.1) in (13.4) and taking the residue in E_α one finds

$$|\psi_\alpha\rangle^{as} = N_\alpha^{\frac{1}{2}}\,[1 - D(E_\alpha)\,V + D(E_\alpha)\,V\,D(E_\alpha)\,V - \ldots]_{iD}\,|\alpha\rangle. \tag{13.5}$$

The subscript iD indicates that in calculating $|\psi_\alpha\rangle^{as}$ one should limit oneself to irreducible non-diagonal diagrams which are completely disconnected. Comparing equation (13.5) with equation (5.12) of H I one sees that they are equivalent. The projection operators Y_α in (5.12) of H I were intended to limit the intermediate states to those states which contribute to $D_\alpha(z)$. In the language of diagrams this means that only completely disconnected diagrams should be taken. The identity of the states $|\psi_\alpha\rangle^{as}$ defined in (13.1) with the states $|\alpha\rangle_{as}$ in H I is thereby established.

14. *Metastable states.* Up till now we have considered states $|\alpha\rangle$ such that $D_\alpha(z)$ has a pole in E_α. This value E_α is a root of the equation

$$\varepsilon_\alpha - x - K_\alpha(x) = 0, \tag{14.1}$$

where ε_α is the unperturbed energy and $K_\alpha(x)$ is defined by (5.4). If $J_\alpha(x)$, also defined by (5.4), vanishes in a neighbourhood of E_α (or if this quantity approaches zero sufficiently fast for $x \to E_\alpha$) the states $|\psi_\alpha\rangle^{\pm}$ defined by

$$|\psi_\alpha\rangle^{\pm} = N_\alpha^{\frac{1}{2}}\,[1 - D(E_\alpha \pm io)\,V + D(E_\alpha \pm io)\,V\,D(E_\alpha \pm io)\,V - \ldots]_i\,|\alpha\rangle, \tag{14.2}$$

where $N_\alpha^{-1} = 1 + K_\alpha'(E_\alpha)$, are stationary states.

We already mentioned that in field theory the above condition is always fulfilled. There are however many quantum mechanical systems where for

some or all of the states $|\alpha\rangle \neq |\varphi_0\rangle$, one has $J_\alpha(E_\alpha) \neq 0$. By (5.5) $J_\alpha(E_\alpha)$ is then positive. Under such conditions it is still possible to define states $|\psi_\alpha\rangle^\pm$ by the relation (14.2), but these states have in general no simple physical meaning. However in the special case that $J_\alpha(E_\alpha)$ is very small, we shall see that $|\psi_\alpha\rangle^+$ still approximately behaves as a stationary state. States of this type will be called *metastable*. They are frequently encountered in the many-particle systems of statistical mechanics and play also an important role in the theory of nuclear matter. They are investigated in the present section.

According to (5.1) we can write

$$D_\alpha(z)^{-1} = \varepsilon_\alpha - z - G_\alpha(z).$$

At the point $z = E_\alpha + io$ we have $D_\alpha^{-1}(E_\alpha + io) = -iJ_\alpha(E_\alpha)$. We expand $D_\alpha^{-1}(z)$ in a power series of $(z - E_\alpha)$ for z in the neighbourhood of $E_\alpha + io$ and obtain in this way an analytic continuation of $D_\alpha^{-1}(z)$ from the upper to the lower half of the complex plane. Thus, to first order,

$$D_\alpha^{-1}(z) = -iJ_\alpha(E_\alpha) + (E_\alpha - z)(1 + G'_\alpha(E_\alpha + io)).$$

$J_\alpha(x)$ is assumed to be small for x near E_α. We put accordingly

$$G'_\alpha(E_\alpha + io) = K'_\alpha(E_\alpha)$$

and find

$$D_\alpha^{-1}(z) = N_\alpha^{-1}\,[E_\alpha - iN_\alpha J_\alpha(E_\alpha) - z], \tag{14.3}$$

with

$$N_\alpha^{-1} = 1 + K'_\alpha(E_\alpha).$$

Equation (14.3) shows that $D_\alpha(z)$, if continued analytically from above to below the real axis, has a pole

$$F_\alpha = E_\alpha - iN_\alpha J_\alpha(E_\alpha). \tag{14.4}$$

The time-dependent wave function corresponding to $|\psi_\alpha\rangle^+$ is, according to (3.3), given by

$$U(t)\,|\psi_\alpha\rangle^+ = (2\pi i)^{-1} \int_{-\infty+io}^{+\infty+io} \mathrm{d}\zeta\, R(\zeta)\, e^{-i\zeta t}\, |\psi_\alpha\rangle^+, \tag{14.5}$$

where we assume $t > 0$. This leads us to a study of the matrix element $\langle\beta\,|R(\zeta)|\,\psi_\alpha\rangle^+$ for arbitrary $|\beta\rangle$ and $\operatorname{Im}\zeta > 0$. If we apply both sides of (5.8) to the state $|\alpha\rangle$, putting $z = \zeta$, $z' = E_\alpha + io$, we get after some simple manipulations using (14.2),

$$\langle\beta\,|R(\zeta)|\,\psi_\alpha\rangle^+ =$$
$$= N_\alpha^{\frac{1}{2}}(\zeta - E_\alpha)^{-1}\,\langle\beta\,|R(\zeta)|\,\alpha\rangle\, D_\alpha^{-1}(E_\alpha + io) - (\zeta - E_\alpha)^{-1}\,\langle\beta|\psi_\alpha\rangle^+. \tag{14.6}$$

We have seen that $D_\alpha^{-1}(E_\alpha + io) = -iJ_\alpha(E_\alpha)$, a quantity assumed to

be small. Nonetheless, owing to the singularity at $\zeta = E_\alpha$, the first term on the right-hand side cannot be neglected when inserting (14.6) in (14.5). Equation (14.6) can be rewritten as

$$\begin{aligned}\langle\beta\,|R(\zeta)|\,\psi_\alpha\rangle^+ = &-iJ_\alpha(E_\alpha)\,.\,N_\alpha^{\frac{1}{2}}\,.\,(\zeta - E_\alpha)^{-1}\,[\langle\beta\,|R(\zeta)\,D^{-1}(\zeta)|\,\alpha\rangle - \\ &- \langle\beta\,|R(E_\alpha + io)\,D_\alpha^{-1}(E_\alpha + io)|\,\alpha\rangle]\,D_\alpha(\zeta) + \\ &+ D_\alpha(\zeta)(\zeta - E_\alpha)^{-1}[D^{-1}(E_\alpha + io) - D_\alpha^{-1}(\zeta)]\,.\,\langle\beta|\psi_\alpha\rangle^+. \quad (14.7)\end{aligned}$$

It is now legitimate to drop the first term in the right-hand side. The second term no longer has a singularity at $\zeta = E_\alpha$, but its analytical continuation in the lower half of the ζ-plane has a pole at $\zeta = F_\alpha = E_\alpha - iN_\alpha J_\alpha(E_\alpha)$. Neglecting terms of the order of $J_\alpha(E_\alpha)$ we are left with

$$\langle\beta\,|R(\zeta)|\,\psi_\alpha\rangle^+ = (\zeta - E_\alpha + iN_\alpha J_\alpha(E_\alpha))^{-1}\,\langle\beta|\psi_\alpha\rangle^+, \text{ for } \operatorname{Im}\zeta > 0.$$

If we substitute this approximate result in (14.5) we find, for positive t large of order $J_\alpha(E_\alpha)^{-1}$ and arbitrary $|\beta\rangle$,

$$\langle\beta\,|U(t)|\,\psi_\alpha\rangle^+ = \exp\,(-iE_\alpha t - N_\alpha J_\alpha(E_\alpha)t)\,\langle\beta|\psi_\alpha\rangle^+\,{}^*). \quad (14.8)$$

This equation shows that $|\psi_\alpha\rangle^+$ is a metastable state with an energy E_α and a mean life-time $T_\alpha = 1/N_\alpha J_\alpha(E_\alpha) = 1/\Gamma_\alpha$. The quantity Γ_α plays for our case of continuous spectra a role analogous to the level width of discrete spectra. Just as we derived (14.8) we could establish for $t < 0$

$$\langle\beta|\,U(t)\,|\psi_\alpha\rangle^- = \exp\,(-iE_\alpha t + N_\alpha J_\alpha(E_\alpha)t)\,\langle\beta\,|\psi_\alpha\rangle^-,$$

a formula which however has little physical interest.

Before commenting upon the significance of (14.8) we shall derive, along the lines of sections 10 and 12, simpler expressions for Γ_α and $|\psi_\alpha\rangle^+$. In section 10 we introduced the function $\bar{D}_\alpha(\varepsilon_0 + z)$ which was defined by means of diagrams without ground state components. It was established that the validity of the equation $\bar{J}_\alpha(\varepsilon_0 + x) = 0$ for x in a neighbourhood of $\bar{E}_\alpha$ ($\bar{E}_\alpha$ was the root of equation (10.6)) implies $J_\alpha(x) = 0$ in a neighbourhood of E_α. In the case considered here we have clearly $\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha) \neq 0$, and, as we shall see below, $\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha)$ is positive but small. Using (9.23) and (10.1) we can write

$$D_\alpha(\varepsilon_0 + z) = N_0(E_0 - \varepsilon_0 - z)^{-1} * \bar{D}_\alpha(\varepsilon_0 + z) + \psi(\varepsilon_0 - E_0 + z) * \bar{D}_\alpha(\varepsilon_0 + z).$$

To study the singular behaviour of $D_\alpha(z)$ obtained by analytical continuation in the neighbourhood of E_α the second term on the right-hand side can be neglected and we have approximately

$$D_\alpha(z) = N_0\,\bar{D}_\alpha(\varepsilon_0 - E_0 + z) \text{ for } |z - E_\alpha| \text{ small.} \quad (14.9)$$

*) It is essential for the validity of this equation that $|\beta\rangle$ is a state with only a finite number of particles and holes.

Taking the value at $z = E_\alpha = E_0 + \bar{E}_\alpha$ we get

$$- iJ_\alpha(E_\alpha) = - iN_0\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha).$$

This gives the important formula

$$\Gamma_\alpha \equiv N_\alpha J_\alpha(E_\alpha) = \bar{N}_\alpha\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha), \tag{14.10}$$

which expresses the life-time Γ_α^{-1} in terms of diagrams without ground state components. In particular we conclude that Γ_α is independent of the volume Ω of the system, since both $\bar{N}_\alpha$ and $\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha)$ have this property.

The wave function $|\psi_\alpha\rangle^+$, defined by (14.2), can be written

$$|\psi_\alpha\rangle^+ = O_\alpha^+ \, |\varphi_0\rangle,$$

where

$$O_\alpha^+ = N_\alpha^{\frac{1}{2}} A_\alpha(E_\alpha + io) \, D_\alpha^{-1}(E_\alpha + io). \tag{14.11}$$

The operator $A_\alpha(z)$ was defined in section 11 by (11.2). According to (12.4) we have

$$A_\alpha(\varepsilon_0 + z) = \hat{A}_\alpha(\varepsilon_0 + z) * A_0(\varepsilon_0 + z).$$

Following exactly the same arguments as in the derivation of (14.9) we find

$$A_\alpha(z) = - \mathfrak{R}_{E_0}[A_0(z)] \, . \, \hat{A}_\alpha(\varepsilon_0 - E_0 + z), \text{ for } |z - E_\alpha| \text{ small.}$$

Substituting this expression in (14.11), one obtains, using (11.8) and (14.9),

$$O_\alpha^+ = \hat{O}_\alpha^+ \, . \, O_0, \; |\psi_\alpha\rangle^+ = \hat{O}_\alpha^+ \, |\psi_0\rangle \tag{14.12}$$

with

$$\hat{O}_\alpha^+ = - i\bar{J}_\alpha(\varepsilon_0 + \bar{E}_\alpha) \, \bar{N}_\alpha^{\frac{1}{2}} \, \hat{A}_\alpha(\varepsilon_0 + \bar{E}_\alpha + io).$$

Notice that this formula, which only has an approximate validity, is exactly of the same form as (12.8).

Examples of metastable states in systems with continuous spectra, in the sense defined here, are actually well known. We mention only one, the so-called cloudy-crystal-ball model of heavy nuclei, which is meant to describe the scattering of nucleons at low energies [2]). The imaginary part of the potential, introduced in the model to describe the "absorption" of the incident nucleon in nuclear matter, (leading to compound nucleus formation) corresponds to our quantity Γ_α. In a forthcoming paper, which will deal with the application of the Fermi gas model to the problem of nuclear structure, we shall have opportunity to come back to this point in greater detail. Let us only mention here that in the Fermi gas model, states containing in addition to the Fermi sea one particle with momentum k are metastable in our sense (small Γ_α) when k is near the Fermi momentum k_F. Application of (10.12) shows immediately that states containing more additional particles are then also metastable. Furthermore it is easy to

verify the approximate validity of (10.13) for the case at hand. This formula expresses now the excitation energy of a state containing several additional particles as the sum of the excitation energies of the single particles, whereas the inverse life-time of the total state becomes the sum of the individual inverse life-times. Also the considerations concerning asymptotically stationary states are approximately valid for metastable cases. The asymptotically stationary states are essential for the description of scattering processes. In H I and H II they are used to establish a formula for the S-matrix. If one is interested in collision processes in dissipative systems (e.g., collisions between two additional nucleons in nuclear matter) one can apply this S-matrix formalism provided the life-time of the metastable states is long compared to the time in which the collision takes place.

Chapter VI. Conclusion

15. *Summary of results.* We have now come to the end of our analysis and we shall briefly summarize what has been achieved. Our starting point was the resolvent operator $R(z)$ from which one can derive most of the desired information, such as energies and wave functions of stationary states and life-times of metastable states. The resolvent was expanded in powers of the perturbation, as shown in (3.5). The different contributions to each term in (3.5) were analysed by means of diagrams. In section 6 we investigated the dependence of these contributions upon the volume Ω (or the total number of particles) of the system under consideration. We found that diagrams containing a certain number n of ground state components give, in the limit of $\Omega \to \infty$, a contribution proportional to Ω^n, whereas in the same limit all other diagrams give finite contributions. Because diagrams containing any number of ground state components contribute to $R(z)$, the straight perturbation expansion (3.5) has terms with arbitrarily high powers of Ω. Clearly such an expansion is extremely inadequate for the application to large systems.

The analysis of chapters III and IV showed how this important difficulty can be overcome. On the basis of a general theorem (expressed by (7.4)) we derived the basic formula (8.1) which expresses an arbitrary matrix element $\langle\beta\,|R(z)|\,\alpha\rangle$ by means of a convolution integral involving the unperturbed ground state matrix element $\langle\varphi_0\,|R(z)|\,\varphi_0\rangle = D_0(z)$ and the matrix element $\langle\beta\,|\bar{R}(z)|\,\alpha\rangle$. The latter differs from $\langle\beta\,|R(z)|\,\alpha\rangle$ in this respect that the only diagrams contributing to $\langle\beta\,|\bar{R}(z)|\,\alpha\rangle$ are those without ground state components. Consequently $\langle\beta\,|\bar{R}(z)|\,\alpha\rangle$ is, in the limit of $\Omega \to \infty$, finite and independent of Ω. The whole Ω-dependence is thereby isolated in $D_0(z)$.

To investigate $D_0(z)$ we derived by a new application of (8.1) the integral equation (8.4). This equation can be solved explicitly, the solution being

given by (9.23). It expresses $D_0(z)$ in terms of the function $\bar{G}_0(z)$. According to its definition (8.3), only connected ground state diagrams contribute to $\bar{G}_0(z)$, so that it is, for large Ω, simply proportional to Ω.

Two quantities of physical interest can be derived from $D_0(z)$. They are the pole E_0 of this function, giving the perturbed energy of the ground state, and the residue $-N_0$ in this pole. The factor $N_0^{\frac{1}{2}}$ plays the role of a normalization factor in the expansion of the exact ground state wave function $|\psi_0\rangle$ in unperturbed states. The values of E_0 and N_0 are given by (9.24) and (9.21). We see that these expressions involve only the values of $\bar{G}_0(z)$ and of its derivative at $z = \varepsilon_0$, both proportional to Ω (ε_0 is the unperturbed ground state energy). Two of our intermediate formulae, (9.25) and (11.16), which we used for the shift $\Delta E_0 = E_0 - \varepsilon_0$ of the ground state energy and for the wave function $|\psi_0\rangle$were found recently by Goldstone [5]) who derived them from a time-dependent perturbation method originally introduced by Gell-Mann and Low [7]). These expressions differ considerably from our final formulae (9.26) and (11.23). The simplicity of (11.23) as compared to (11.16) lies in the fact that the former involves only connected diagrams. In addition the reduction of the diagrams to their irreducible form (using the method introduced by Van Hove in H I) makes our formulae much more suitable for the application to infinitely large systems. The perturbed ground state wave function is obtained by application to its unperturbed analogue of the operator (11.23) involving the exponential of a very simple operator $\tilde{O}_0$.

Going over to the consideration of excited states (*i.e.*, for the example of a Fermi gas, of states differing from the ground state by the presence of some additional particles and some holes), the main results of this paper are expressed by (10.11) and (12.9). The importance of these formulae can be expressed by saying that the excitation energy $\bar{E}_\alpha$ and the operator $\hat{O}_\alpha^\pm$ have a finite and simple limit for $\Omega \to \infty$. $\bar{E}_\alpha$ is the perturbed energy difference between excited and ground state, while $\hat{O}_\alpha^\pm$ is the operator which transforms the perturbed ground state wave function into the perturbed wave function of the excited state. As a further result we might mention formula (13.1) which gives a very concise and transparent expression for the asymptotically stationary states as defined by Van Hove in H I and H II. These states play an important role in the theory of collisions.

A striking property of many systems with a large number of degrees of freedom is the existence of dissipative effects. For systems with an excitation energy of the order of the total number of particles, these effects are responsible for the trend towards thermal equilibrium; they were studied extensively by Van Hove [8]). Also for smaller excitation energies (a situation corresponding to zero-temperature) such dissipative effects can play an important role. One aspect of them has been investigated here: the case of metastable states, *i.e.* of states which would be stationary were it not that

they show, as a result of the perturbation, an exponential decay with a long life-time. Such states have been studied in section 14. An important example is provided by a slow nucleon penetrating into nuclear matter and traveling a considerable distance before the compound nucleus is formed.

16. *Final remarks.* At the start of our investigation we assumed the convergence of the expansion (3.5) of $R(z)$ in powers of the perturbation, at least for z non-real. Our final results are still expressed as series expansions but the latter differ from the original expansion by the fact that a number of partial summations have been performed explicitly. This circumstance manifests itself clearly in our results, inasmuch as the class of diagrams contributing to the final expressions is very much smaller than the class contributing to the original ones. We have therefore every reason to believe that the convergence of our resulting expressions is much better. We know in particular that this convergence is no longer affected in any way by the large size of the system and its large number of degrees of freedom. The question under what condition on the strength and form of the two-body potential our final series converge is however unsolved. Let us devote a few comments to this difficult point.

Let us take a large vessel, with volume Ω, filled with a gas of interacting Fermi particles. Considering the ground state of the system we distinguish the following cases.

1. The particles are distributed homogeneously throughout the vessel, exerting a pressure on the walls. This situation certainly occurs whenever the forces are repulsive, but also partly attractive forces can obviously give rise to it. For not too singular forces it is to be expected that our expansions converge.

2. The particles are bound together by their mutual interaction, thus occupying only a part Ω' of the volume Ω. The volume Ω' and the energy are proportional to the number N of particles, as will be the case for saturating forces. A large nucleus of volume Ω' enclosed in a vessel of volume $\Omega > \Omega'$ is an example of the case considered here. For such a system the perturbation theory, even in case of convergence, is not strictly valid: the state $|\psi_0\rangle$, obtained by perturbation of the unperturbed ground state, would then not represent the state of lowest energy. If, however, we reduce the vessel to a volume $\leqslant \Omega'$, thus increasing the particle density, we are back to the first case.

3. In contrast with case 2, the forces may be such that the ground state corresponds to a particle density and an energy density increasing with the total number of particles. This corresponds to non-saturating forces. For such forces the perturbation method will break down completely.

Another remark concerns the normalization factor $N_0^{\frac{1}{2}}$ in (11.23). This factor has the simple form $\exp(-\bar{G}_0'(\varepsilon_0))$. The exponent is proportional to Ω and

consequently N_0 approaches zero in the limit of $\Omega \to \infty$. Consequently, if this limit is actually carried out, all expansion coefficients are zero and a proper expansion of $|\psi_0\rangle$ in unperturbed states is no longer possible. Another way of stating this remarkable fact is to say that the ground state $|\psi_0\rangle$ as well as any other eigenstate $|\psi_\alpha\rangle^\pm$ of the total hamiltonian, become orthogonal to all unperturbed states $|\alpha\rangle$ in the limit $\Omega \to \infty$. This is connected with the fact that in this limit the system has infinitely many degrees of freedom (*e.g.*, infinitely many particles), so that the set of basic vectors spanning the Hilbert space of its state vectors is no longer countable. In this non-separable Hilbert space many separable subspaces can be formed. On the one hand, the unperturbed ground state $|\varphi_0\rangle$, and all unperturbed states $|\alpha\rangle$ differing from $|\varphi_0\rangle$ by excitation of a finite number of particles (and holes) span a separable Hilbert space. On the other hand a separable Hilbert space is formed by the perturbed states $|\psi_0\rangle$ and $|\psi_\alpha\rangle^\pm$. The vanishing of N_0 for $\Omega = \infty$ implies that these separable Hilbert spaces are orthogonal to each other. As was remarked by Van Hove [9]), a similar situation occurs also in field theories where the vacuum is not affected by the perturbation (no pair creation). It is then caused by ultra-violet divergencies. In such theories it is irrelevant whether the volume in the configuration space is finite or not, the essential fact is the occurrence of divergencies in momentum integrations, *i.e.*, the occurrence of an infinite "effective" volume in momentum space.

Finally some words must be said on the relation of the present investigation to the formalism of current field theory. Applying our formalism to field theory, one would be tempted to identify the function $\bar{D}_k(\varepsilon_0 + z)$, being the diagonal element of $R(\varepsilon_0 + z)$ for a state of one particle with momentum k, calculated with omission of all disconnected diagrams, with the Fourier transform of the one-particle propagation function $\Delta'_F(x, t)$ as introduced by Dyson [10]). This, however, is not generally true. One can show that the identity exists only in those theories where the free vacuum is not affected by the perturbation. It can nevertheless be established that the singularities of both quantities, which determine the mass renormalization of the particle, are the same [11]).

Acknowledgements. The author is glad to acknowledge his indebtedness to Professor L. Van Hove who not only suggested many of the ideas underlying this investigation but also assisted the author with advice and valuable criticism during all stages of this work. The author also profited from discussions with other members of the Institute for Theoretical Physics in Utrecht, in particular Dr. N. G. van Kampen and Mr. Th. W. Ruijgrok.

This work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie", which is financially supported by the Netherlands Organization for pure scientific Research (Z.W.O.).

APPENDICES

Appendix 1. An alternative proof of equation (7.1) can be obtained by comparison with the case of a system composed of two completely independent subsystems. The hamiltonian H of the total system can be written

$$H = H_1 + H_2,$$

where H_1 and H_2 are the hamiltonians of the two independent subsystems. We shall denote the resolvent operators by $R(z)$, $R_1(z)$ and $R_2(z)$. They are commuting operators.

If one multiplies the identity

$$H - z = (H_1 - \zeta) + (H_2 - z + \zeta)$$

by the product $R(z)\, R_1(\zeta)\, R_2(z - \zeta)$ one gets

$$R_1(\zeta)\, R_2(z - \zeta) = R(z)\, (R_1(\zeta) + R_2(z - \zeta)).$$

Taking $Im\ z \neq 0$ and integrating on both sides over the variable ζ along the path defined in section 7 eq. (7.2) one obtains the formula

$$-(2\pi i)^{-1} \oint d\zeta\, R_1(\zeta)\, R_2(z-\zeta) = R(z)[-(2\pi i)^{-1} \oint d\zeta\, R_1(\zeta) - (2\pi i)^{-1} \oint d\zeta R_2(z-\zeta)].$$

The second term on the right-hand side is zero, owing to the fact that $R(z - \zeta)$ has no singularities on the real ζ-axis. The first term within the brackets is equal to one, as one sees from (3.3) by putting $t = 0$. We are left with

$$R(z) = R_1(z) * R_2(z), \tag{A1.1}$$

where we used the notation introduced in section 7.

Suppose now that the two subsystems are identical though independent systems of the type studied in this paper. As an example one could think of two vessels of equal volume filled with the same number of identical Fermi particles. The hamiltonians are $H_1 = H_1^0 + V_1$ and $H_2 = H_2^0 + V_2$. The total resolvent $R(z)$ can be expanded in powers of $V = V_1 + V_2$

$$R(z) = \frac{1}{H_1^0 + H_2^0 - z} - \frac{1}{H_1^0 + H_2^0 - z}(V_1 + V_2)\frac{1}{H_1^0 + H_2^0 - z} + \ldots\ldots$$

and the contributions to the different terms can again be represented by diagrams. Consider such an arbitrary diagram. It contains two different kinds of vertices corresponding to V_1 and V_2. The diagram falls apart into two subdiagrams A' and B' which contain all vertices of systems 1 and 2 respectively. A' and B' are not connected with each other, there being no lines joining a vertex of system 1 with a vertex of system 2. Together with this diagram we consider all diagrams which can be obtained from this one by changing the positions of the vertices of A' with respect to B'. The con-

tributions of these diagrams differ only by the energy denominators. Let us denote the sum by $\langle\alpha'\beta'\,|C'(z)\,|\alpha\beta\rangle$ where $|\alpha\rangle$ and $|\alpha'\rangle$ are the initial and final state of system 1, $|\beta\rangle$ and $|\beta'\rangle$ of system 2. If we denote the contribution of A' to $R_1(z)$ by $\langle\alpha'\,|A'(z)|\,\alpha\rangle$ and the contribution of B' to $R_2(z)$ by $\langle\beta'\,|B'(z)|\,\beta\rangle$, application of (A1.1) gives immediately the formula

$$\langle\alpha'\beta'\,|C'(z)|\,\alpha\beta\rangle = \langle\alpha'\,|A'(z)|\,\alpha\rangle * \langle\beta'\,|B'(z)|\,\beta\rangle. \tag{A1.2}$$

It is valid for two uncoupled systems and must not be confused with (7.4). We can however use (A1.2) to establish the validity of (7.4) by means of the following argument. Let us take the diagrams A' and B' identical with A and B of section 7. The quantities $\langle\alpha'\,|A'(z)|\,\alpha\rangle$ and $\langle\beta'\,|B'(z)|\,\beta\rangle$ are then formally identical with $\langle\alpha'\,|A(z)|\,\alpha\rangle$ and $\langle\beta'\,|B(z)|\,\beta\rangle$ of section 7. Let us now compare $\langle\alpha'\beta'\,|C'(z)|\,\alpha\beta\rangle$ with $\langle\beta'\alpha'\,|C(z)|\,\alpha\beta\rangle$. Although these quantities clearly have different meanings, the only formal difference is the fact that in the latter the energy ε_0 of the unperturbed ground state is counted only once and not twice in the energy denominators. This difference can be compensated for by substituting $z - \varepsilon_0$ for z in $\langle\beta'\alpha'\,|C(z)|\,\alpha\beta\rangle$. This leads to the formula

$$\langle\beta'\alpha'\,|C(z-\varepsilon_0)|\,\alpha\beta\rangle = \langle\alpha'\,|A(z)|\,\alpha\rangle * \langle\beta'\,|B(z)|\,\beta\rangle,$$

an alternative form of (7.4) or (7.1).

Appendix 2. It will be shown in this appendix that the integral equation (9.7) has at most one solution $f(z)$ which is holomorphic outside the real axis and bounded for large $|z|$. It is sufficient to prove that the homogeneous equation

$$zf(z) = -\,(2\pi i)^{-1} \oint \mathrm{d}\zeta\, h(\zeta)\, f(z-\zeta)$$

has no such solution $f(z)$.

By (9.11) this equation can be written

$$zf(z) = \int_0^\infty \mathrm{d}\xi\, p(\xi)\, f(z-\xi). \tag{A2.1}$$

We shall make use of the fact that the integrals

$$L = \int_0^\infty |p(x)|\,\mathrm{d}x \quad\text{and}\quad M = \int_0^\infty \frac{|p(x)|}{x}\,\mathrm{d}x \tag{A2.2}$$

are convergent. These properties of $p(x)$ are an immediate consequence of (9.13) and (9.2). We choose an arbitrary point z, not on the real axis, for which $Re\, z < 0$. The function $f(z+a)$ of the real variable a is bounded; consequently there exists a positive number $N(z)$ such that

$$|f(z+a)| < N(z) \text{ for all } a.$$

Iterating equation (A2.1) n times, one obtains

$$f(z) = \frac{1}{z}\int_0^\infty \mathrm{d}\xi_1 \ldots\ldots \int_0^\infty \mathrm{d}\xi_n \int_0^\infty \mathrm{d}\xi_{n+1}\, \frac{p(\xi_1)}{z-\xi_1} \ldots\ldots \frac{p(\xi_n)}{z-\xi_1-\xi_2-\ldots-\xi_n}\,\cdot$$
$$\cdot\, p(\xi_{n+1})\, f(z-\xi_1-\ldots-\xi_{n+1}).$$

This equality can be changed into an inequality: take the absolute value of both sides and replace the integrand by its absolute value. The right-hand side is increased further if one replaces the denominators $|z - \xi_1 - \ldots - \xi_k|$ by $\xi_1 + \ldots + \xi_k$, and $|f(z - \xi_1 - \ldots - \xi_{n+1})|$ by $N(z)$. One finds, using (A2.2),

$$|f(z)| \leqslant \frac{1}{|z|} \int_0^\infty \mathrm{d}\xi_1 \ldots\ldots \int_0^\infty \mathrm{d}\xi_n \frac{|p(\xi_1)|}{\xi_1} \ldots\ldots \frac{|p(\xi_n)|}{\xi_1 + \xi_2 + \ldots + \xi_n} L \,.\, N(z).$$

The integral at the right-hand side is not changed by a permutation of the variables $\xi_1, \xi_2, \ldots, \xi_n$ in the integrand. Using the simple algebraic equation

$$\Sigma_P\, \xi_1^{-1}(\xi_1 + \xi_2)^{-1} \ldots\ldots (\xi_1 + \xi_2 + \ldots + \xi_n)^{-1} = \xi_1^{-1}\, \xi_2^{-1} \ldots \xi_n^{-1},$$

where the sum is extended over the $n!$ permutations of the n variables, one is led to the inequality

$$|f(z)| \leqslant \frac{1}{|z|} \frac{1}{n!} \int_0^\infty \mathrm{d}\xi_1 \ldots\ldots \int_0^\infty \mathrm{d}\xi_n \frac{|p(\xi_1)|}{\xi_1} \ldots\ldots \frac{|p(\xi_n)|}{\xi_n} L.N(z),$$

or by (A2.2)

$$|f(z)| \leqslant LN(z)\, |z|^{-1} \,.\, M^n/n!.$$

This inequality holds for all n. Hence, noticing that $\lim_{n\to\infty} M^n/n! = 0$, we find $f(z) = 0$ for all non-real z, for which $Re\, z < 0$. This is enough to conclude that $f(z) \equiv 0$ and that the homogeneous integral equation has no non-zero solution which is holomorphic outside the real axis and bounded for $|z| \to \infty$. This proves our statement.

Appendix 3. In section 9 the integral equation (9.14) was solved by means of a Laplace-transformation. We shall prove here that the function $\hat{\varphi}(s)$, given by (9.16), is the Laplace-transform of a function $\varphi(x)$ given by (9.18). The known properties of $p(x)$ imply the absolute convergence of $\int_0^\infty p(x) \exp(-sx)\, \mathrm{d}x$ and $\int_0^\infty x^{-1} p(x) \exp(-sx)\, \mathrm{d}x$ for $s = 0$ as can be seen from (A2.2). From this we conclude that both $\hat{p}(s) = \int_0^\infty p(x) \exp(-sx)\, \mathrm{d}x$ and $\int_0^\infty x^{-1} p(x) \exp(-sx)\, \mathrm{d}x$ are analytical functions of s for $Re\, s > 0$ (see e.g. G. Doetsch, Handbuch der Laplace-Transformation I, Satz 1[3.2]). Consequently the same holds true for the functions $\hat{\varphi}(s)$ and $\hat{\varphi}'(s)$ given by (9.16) and (9.15). From the behaviour of $p(x)$ near the origine (9.17) we can determine the asymptotic behaviour of $\hat{p}(s)$ and $\int_0^\infty x^{-1} p(x) \exp(-sx)\, \mathrm{d}x$ for large s. Application of one of the Abelian theorems (see e.g. Doetsch, Satz 5 [14.1]) immediately gives

$$\hat{p}(s) = O(|s|^{-1-\alpha}) \text{ and } \int_0^\infty x^{-1} p(x) \exp(-sx)\, \mathrm{d}x = O(|s|^{-\alpha}) \text{ for } |s| \to \infty.$$

Hence, from (9.16),

$$\lim_{|s|\to\infty} \hat{\varphi}(s) = 0$$

and from (9.15)

$$\hat{\varphi}'(s) = O(|s|^{-1-\alpha}) \text{ for } |s| \to \infty.$$

This asymptotic property of $\hat{\varphi}'(s)$ is sufficient (see e.g. Doetsch, Satz 3 [7.2]) to ensure the existence of a function $\chi(x)$ such that

$$\hat{\varphi}'(s) = \int_0^\infty \chi(x) \exp(-sx)\, dx.$$

$\chi(x)$ is given by the complex integral

$$\chi(x) = (2\pi i)^{-1} \int_{-i\infty+\delta}^{+i\infty+\delta} \varphi'(s) \exp(sx)\, ds, \; \delta > 0.$$

Another Abelian theorem (see e.g. Doetsch Satz 1 [15.5]) predicts the behaviour of $\chi(x)$ near the origin from the asymptotic behaviour of $\hat{\varphi}'(s)$ for large $|s|$:

$$\chi(x) = O(x^\alpha).$$

This enables us to define the Laplace-transform

$$F(s) = \int_0^\infty x^{-1} \chi(x) \exp(-sx)\, dx.$$

$F(s)$ is an analytical function for $Re\, s > 0$, with the same derivative as $\hat{\varphi}(s)$. Both $F(s)$ and $\hat{\varphi}(s)$ tend to zero for $s \to \infty$ and are consequently equal. This proves that $\hat{\varphi}(s)$ is the Laplace-transform of a function $\varphi(x) = x^{-1} \chi(x)$.

Received 8-4-57.

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