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1958

Physica XXIV
363–376

A THEOREM ON THE SINGLE PARTICLE ENERGY IN A FERMI GAS WITH INTERACTION

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Synopsis

This paper investigates single particle properties in a Fermi gas with interaction at the absolute zero of temperature. In such a system a single particle energy has only a meaning for particles of momentum $|k|$ close to the Fermi momentum k_F. These single particle states are metastable with a life-time approaching infinity in the limit $|k| \to k_F$. The limiting value of the energy is called the Fermi energy E_F. As a special case of a more general theorem, it is shown that for a system with zero pressure (i.e. a Fermi liquid at absolute zero) the Fermi energy E_F is equal to the average energy per particle E_0/N of the system. This result should apply both to liquid He_3 and to nuclear matter.

The theorem is used as a test on the internal consistency of the theory of Brueckner [1]) for the structure of nuclear matter. It is seen that the large discrepancy between the values of E_F and E_0/N, as calculated by Brueckner and Gammel [2]), arises from the fact that Brueckner neglects important cluster terms contributing to the single particle energy. This neglection strongly affects the calculation of the optical potential.

1. *Introduction.* In Brueckner's theory [1]) on the structure of nuclear matter the interior of a nucleus is considered as a gas of strongly interacting Fermi particles. To each particle a separate energy E_l is assigned, which depends on the momentum l of the particle. This energy is written as the sum of the kinetic energy $l^2/2M$ and a potential energy V_l. The computation of V_l from a set of implicit equations is the main problem in this theory. Once V_l is known, the energy of the whole system in its ground state is given by the simple formula

$$E_0 = \sum_{|l|<k_F} (l^2/2M + \tfrac{1}{2} V_l). \tag{1}$$

The summation is extended over all occupied states, *i.e.* over all momenta smaller than the Fermi momentum k_F *).

One might ask the question, what is the physical meaning of this single particle energy E_l or the "potential energy" V_l in a system of strongly interacting particles. To answer this question we consider the theory of Brueckner as a special approximation of a general time-independent

*) We put $\hbar = 1$ throughout this paper.

perturbation formalism which was developed earlier by the authors [3]) (to be quoted as I, II and III). As will be shown in section 2, it then turns out that only to particles with momentum l in the neighbourhood of the Fermi momentum k_F an approximate energy E_l can be assigned. Only in the limit that $|l|$ approaches k_F the energy E_l gets a precise meaning. This limiting value of E_l is called the *Fermi energy* E_F.

Section 3 will be devoted to an important theorem concerning this Fermi energy. It will be shown rigorously that for a system of Fermi particles at its ground state the Fermi energy as defined above is equal to the mean energy per particle, provided the system has zero pressure. Nuclear matter is an example of such a system.

This theorem, which is a special case of a more general formula, derived in the first half of section 3, can be used as a test for the validity of the approximation of Brueckner. In recent calculations of Brueckner and Gammel [2]) the ground state energy per particle is found to be -15 MeV, whereas these authors find for the Fermi energy the value -34 MeV *). The cause of this discrepancy is investigated in the last section. Indications are presented that the largest part of the discrepancy comes from the inaccuracy of E_F.

2. *The single particle energy*. The considerations of this and the following sections are mainly based on I and III. We consider a system of a large number N of Fermi particles enclosed in a box of volume Ω. For simplicity we assume the particles to have no spin or charge. We are interested in particular in the case that both N and Ω are very large with a finite density $\rho = N/\Omega$. The hamiltonian H of the complete system is written as a sum of the kinetic energy H_0 and the interaction V, which in the occupation number representation for plane wave states have the form

$$H_0 = \int_l (|l|^2/2M)\, \xi_l{}^* \xi_l,$$
$$V = \tfrac{1}{4} \int_{l_1 l_2 l_3 l_4} v(l_1 l_2 l_3 l_4) \xi_{l_1}{}^* \xi_{l_2}{}^* \xi_{l_3} \xi_{l_4}.$$

For the notation we refer to III. ξ_l and $\xi_l{}^*$ are annihilation and creation operators for a particle with momentum l, obeying the anticommutation relations

$$\{\xi_k, \xi_l{}^*\} = \Omega(2\pi)^{-3}\, \delta_{kl}.$$

In the limit $\Omega \to \infty$ the right-hand side goes over into the Dirac δ-function $\delta(k - l)$.

The ground state $|\varphi_0\rangle$ of the unperturbed system is the state where all states of the Fermi sea, *i.e.* all one particle states with momenta less than the Fermi momentum k_F, are occupied. The Fermi momentum k_F is related to the particle density by $\rho = k_F{}^3/6\pi^2$.

*) As Dr. Brueckner kindly pointed out to us, the numbers quoted here are not quite correct and must be replaced by -14.6 MeV and -27.5 MeV. The discrepancy is therefore 13 MeV. (*Note added in proof*).

All other stationary states of the unperturbed system are characterized by the momenta $k_1, k_2, \ldots$. of the additional particles present and the momenta $m_1, m_2, \ldots$. of the holes present (holes are unoccupied states of the Fermi sea). We respectively use the letters k and m to indicate momenta larger and smaller than the Fermi momentum k_F. Because the annihilation of a particle in the Fermi sea is equivalent to the creation of a hole, it is useful to reinterpret ξ_m and $\xi_m{}^*$ for $|m| \leqslant k_F$ as creation and annihilation operators for holes.

We have thus obtained a hamiltonian which exhibits a close formal resemblance to a field theory with pair creation. There is, however, an important difference, which will be considered in this section. Whereas in field, theory, for not too strong coupling, to each unperturbed state corresponds at least one stationary state of the complete system; this is not the case in our system, which is essentially dissipative. In I and II a simple criterion was given for the existence of a perturbed stationary state corresponding to a state $|\alpha\rangle$ of the unperturbed system. It amounts to the existence of a pole for the expectation value of the resolvent $R(z) = (H - z)^{-1}$ for the state $|\alpha\rangle$. As shown in III the expectation value $D_0(z)$ of $R(z)$ for $|\varphi_0\rangle$ has always a pole. Consequently there exists a stationary state $|\psi_0\rangle$, the ground state of the system of interacting particles, which corresponds to the unperturbed ground state $|\varphi_0\rangle$. The energy of $|\psi_0\rangle$ we call E_0. The explicit expression of $|\psi_0\rangle$ and E_0 was determined in III.

Next we consider an unperturbed state with one additional particle with momentum k ($|k| > k_F$); it will be denoted by $|k;\rangle$. According to I we must study the function $D_k(z) = \bar{D}_k(z) * D_0(z)$ *) of the complex variable z. $D_k(z)$ is the expectation value of the resolvent $R(z)$ for $|k;\rangle$ except for a factor $\delta(o)$: $\langle ;k\,|R(z)|\,k';\rangle = \delta(k - k')\, D_k(z)$. The product $*$ is the convolution product defined and extensively used in III. $\bar{D}_k(z)$ was defined in III (section 10) by a series in increasing powers of the interaction V, all terms of which can be represented by means of connected diagrams with one external particle line at both ends (the diagrams used are defined in III, section 3; particle lines have arrows pointing to the left, lines corresponding to holes the opposite direction). The decisive point is now whether or not $D_k(z)$ has a pole. A pole would mean that the complete system has a stationary state corresponding to the unperturbed state $|k;\rangle$. The absence of a pole would reveal the dissipative nature of the unperturbed state $|k;\rangle$. As shown previously (see a fourth paper [4]) to be quoted as IV) $\bar{D}_k(z)$ has no pole and consequently $D_k(z)$ can have none, so that the state $|k;\rangle$ is a dissipative one †). The only singularity of $\bar{D}_k(z)$ is a cut in the complex plane along the real axis, running from some point E_F, independent of k, up to $+\infty$. Whereas the

*) To avoid the unnecessary appearance of the term ε_0 in our formulae the function $\bar{D}_k(\varepsilon_0 + z)$ defined in III is denoted here simply as $\bar{D}_k(z)$.

†) For a further discussion of dissipative states see [5]).

real part of $\bar{D}_k(z)$ varies continuously if we cross this cut, the imaginary part changes its sign. If we now consider the discontinuity of the imaginary part of $\bar{D}_k(z)$ for all points of the cut, we find, in the case that $|k|$ is very close to the Fermi-momentum k_F, a high narrow peak for some point E_k *). This situation is to be compared with the δ-singularity, which one would find if E_k was a pole of $\bar{D}_k(z)$. In the limit $|k| \to k_F$ the point E_k approaches the branching point E_F, the difference $E_k - E_F$ being proportional to $|k| - k_F$. The width Γ_k of the peak decreases as $(E_k - E_F)^2$, so that for $|k| - k_F$ small enough, the width of the peak is small compared to its distance from E_F.

Such a situation was analysed in III (section 14). In the case that $\Gamma_k \ll E_k - E_F$ a state vector $|\psi_k\rangle$ can be constructed, which corresponds to a metastable state with an approximate energy $E_k + E_0$ and a life-time equal to Γ_k^{-1}. The metastable character of $|\psi_k\rangle$ is exhibited by the equation

$$\langle\psi_{k'}|e^{-iHt}|\psi_k\rangle = \delta^3(k' - k)\exp[-i(E_0 + E_k)t - \Gamma_k|t|],$$

which holds for values of t of the order of Γ_k^{-1} †). The energy E_k can then be interpreted as the energy of a metastable particle with momentum $|k| > k_F$, moving in the Fermi gas with slow dissipation of its momentum and energy into collective types of motion of the gas. The success of the optical model for the scattering of nucleons on heavy nuclei is experimental evidence for the existence of such metastable states in nuclear matter. Conversely we can say that our theory of the Fermi gas with interaction accounts for the low energy behaviour of the optical potential.

In the limit of $|k| \to k_F$ the single particle energy E_k tends to E_F. We call this limit the *Fermi energy*. The life-time Γ_k^{-1} tends then to infinity, and it can even be shown that E_F is the pole (in the somewhat broadened sense defined in III section 9) of the function $\bar{D}_{k_F}(z)$. Hence a state with one additional particle at the surface of the Fermi sea is exactly stationary, with an energy $E_0 + E_F$.

Instead of states with an additional particle one can also consider states with a hole of momentum $|m| < k_F$. This case is very much analogous to the former one. The function $\bar{D}_m(z)$, which is defined in terms of connected diagrams with one external hole line at both ends, has for $|m|$ close to k_F a similar behaviour as $\bar{D}_k(z)$ for $|k|$ close to k_F. This implies for the case that $|m|$ is close to k_F the existence of a metastable state of a hole, with

*) In IV this quantity was denoted by $\bar{E}_k$, whereas the notation E_k was there used for $E_0 + \bar{E}_k$. The notation used here agrees with the usual one in the Brueckner theory.

†) In III, eq. (14.8) and the subsequent equation as well as their derivation are incorrect. The definition of the two states $|\psi_\alpha\rangle^\pm$ as given by eq. (14.2) of III, however, is correct. In the case that $|\alpha\rangle = |k;\rangle$ these two states are identical and are denoted by $|\psi_k\rangle$.

an approximate energy $E_0 - E_m$. Here $-E_m$ is the point on the real axis where $\bar{D}_m(z)$ is strongly peaked *). It can be interpreted as the energy of a hole of momentum $-m$ near the surface of the Fermi sea, and E_m therefore can be regarded as the energy of a particle of momentum m in the Fermi sea. In the limit $|m| = k_F$, $\bar{D}_m(z)$ does have a pole which, as was surmised in IV and will be confirmed in the next section, is equal to $-E_F$, where E_F is the Fermi energy as defined above.

We should like to stress here that all our considerations are based on the assumption of convergence of all series involved. It may very well be that in addition to the ground state and metastable excited states here considered for the Fermi gas with interaction there exist another "abnormal" stationary state and metastable excitations of it, depending in a singular way on the two-body interaction and therefore not directly accessible to our methods. The possibility of such abnormal states for a Fermi gas with attractive forces has been established by Bardeen, Cooper and Schrieffer [6]) in their theory of superconductivity. How the abnormal states can be obtained in the perturbation formalism based on diagrams has been shown by Bogolubov [7]). The possible existence and observability of such abnormal states for nuclear matter and liquid helium 3 are questions of great importance which we shall not discuss here.

3. *Theorem on the Fermi energy* E_F. We start this section with the derivation of a formula for $\bar{D}_k(z)$, which brings to light a close similarity between this function and the ground state expectation value $\langle\varphi_0 |R(z)| \varphi_0\rangle \equiv D_0(z)$. We shall make an extensive use of the methods presented in III. Before doing so we want, however, to stress the following point. As is well known, the general perturbation method as developed in I, II and III is only exact if the particle number N and the volume Ω of the system are so large that terms proportional to Ω^{-1} or N^{-1} can be neglected. Nevertheless several definitions and results of III are also exactly valid for systems with arbitrary finite N and Ω. This is the case in particular with the definitions and calculation rules of diagrams, diagonal diagrams, connectedness and also with the theorem on the convolution of the contributions of two diagrams (section 7, eq. 4). We use this important fact in the following derivation.

We take a finite cubic box with volume Ω, and impose, as usual, periodic boundary conditions. Let the state vector $|\varphi\rangle$, which is normalized to one, describe a state of the unperturbed system where N particles occupy N given single particle plane-wave states. This set of N single-particle states we shall call the "sea". The state $|\varphi\rangle$ may be different from the unperturbed ground state $|\varphi_0\rangle$. All other states of the unperturbed system can be obtained from $|\varphi\rangle$ by the application of suitable operators $\xi_k{}^*$ or ξ_m, thereby creating

*) E_m in this paper corresponds to the quantity $-\bar{E}_m$ in IV. The single particle energy for particles in the Fermi sea is now E_m.

additional particles or holes. Clearly the momenta k of the additional particles must be outside the sea, whereas the momenta m of the holes must belong to it.

In calculating the diagonal matrix element $\langle\varphi|R(z)|\varphi\rangle$ we make use of diagrams. If, just as in III, lines running from right to left (from left to right) represent particles (holes), we obtain diagrams identical with those which were used in III for calculating $D_0(z) \equiv \langle\varphi_0|R(z)|\varphi_0\rangle$. Their contributions are, however, different, because the momenta k and m of the virtual particles and holes have now to be summed over different, discrete sets of values. The diagrams contributing to $\langle\varphi|R(z)|\varphi\rangle$ are either connected or consist of two or more connected parts. If we denote the total contribution to $\langle\varphi|R(z+\varepsilon)|\varphi\rangle$ of all connected diagrams by $B(z)$, with ε the energy of $|\varphi\rangle$, the total contribution to $\langle\varphi|R(z+\varepsilon)|\varphi\rangle$ of all diagrams consisting of two connected parts is equal to

$$\tfrac{1}{2}\,B(z) * B(z).$$

Here we used the convolution in the complex plane introduced in III (section 7). The factor $\frac{1}{2}$ accounts for the fact that this convolution gives each term twice. Proceeding in the same way with diagrams consisting of three and more components, one finds easily

$$\langle\varphi|R(\varepsilon+z)|\varphi\rangle = -z^{-1} + B(z) + \tfrac{1}{2}B(z)*B(z) + \tfrac{1}{6}B(z)*B(z)*B(z) + \ldots . \quad (2)$$

For the special choice where $|\varphi\rangle \equiv |\varphi_0\rangle$ equation (2) leads to

$$D_0(\varepsilon_0 + z) = -z^{-1} + B_0(z) + \tfrac{1}{2}B_0(z) * B_0(z) + \\ + \tfrac{1}{6}B_0(z) * B_0(z) * B_0(z) + \ldots .\ ^{8)}, \quad (3)$$

where $B_0(z)$ is defined as the sum of the contributions of connected ground state diagrams; ε_0 is the energy of the unperturbed ground state $|\varphi_0\rangle$.

We now also apply (2) for another choice of $|\varphi\rangle$. We take for $|\varphi\rangle$ the unperturbed state $|\varphi_k\rangle$, where in addition to the N particles in the Fermi sea of $|\varphi_0\rangle$ there is an extra particle of momentum k ($|k|>k_F$). The total contribution of all connected diagrams (without external lines) to $\langle\varphi_k|R(\varepsilon+z)|\varphi_k\rangle$, where $\varepsilon = \varepsilon_0 + k^2/2M$, we denote by $B_k(z)$. Equation (2) reads for this case

$$\langle\varphi_k|\,R(\varepsilon_0 + k^2/2M + z)\,|\varphi_k\rangle = -z^{-1} + B_k(z) + \tfrac{1}{2}B_k(z) * B_k(z) + \\ + \tfrac{1}{6}\,B_k(z) * B_k(z) * B_k(z) + \ldots .$$

Introducing the notation $B_k(z) - B_0(z) = \bar{B}_k(z)$ we are lead to the equation

$$\langle\varphi_k|\,R(\varepsilon_0 + k^2/2M + z)|\,\varphi_k\rangle = -z^{-1} + (B_0(z) + \bar{B}_k(z)) + \\ + \tfrac{1}{2}(B_0(z) + \bar{B}_k(z)) * (B_0(z) + \bar{B}_k(z)) + \ldots .$$

If we compare this series with the exponential series we see immediately that it can be written as the convolution of two functions one of which, by equation (3), is equal to $D_0(\varepsilon_0 + z)$. Thus

$$\langle\varphi_k\,|R(\varepsilon_0 + k^2/2M + z)|\,\varphi_k\rangle = \\ = D_0(\varepsilon_0 + z) * [-z^{-1} + \bar{B}_k(z) + \tfrac{1}{2}\bar{B}_k(z) * \bar{B}_k(z) + \ldots.]. \quad (4)$$

The state vectors $|\varphi_k\rangle$ and $|k;\rangle \equiv \xi_k|\,\varphi_0\rangle$ describe the same state. Remembering their different normalization we can write

$$|k;\rangle = \Omega^{1/2}(2\pi)^{-3/2}\,|\varphi_k\rangle.$$

Hence

$$D_k(z)\,\delta^3(k-k') \equiv \langle;k'\,|R(z)|\,k;\rangle = \delta_{k',k}\,\langle;k\,|R(z)|\,k;\rangle = \\ = \Omega(2\pi)^{-3}\,\delta_{k',k}\,\langle\varphi_k\,|R(z)|\,\varphi_k\rangle = \langle\varphi_k\,|R(z)|\,\varphi_k\rangle\,\delta^3(k-k'),$$

where we used the relation between Kronecker symbol and δ-function for finite Ω (see III, section 2):

$$\delta^3(k-k') = \Omega(2\pi)^{-3}\,\delta_{k,k'}.$$

We see that

$$\langle\varphi_k\,|R(z)|\varphi_k\rangle = D_k(z). \quad (5)$$

As we know $D_k(z)$ can be expressed very simply in terms of $\bar{D}_k(z)$, which is defined by means of connected one particle diagrams, and $D_0(z)$ by the formula (see III (10.1))

$$D_k(\varepsilon_0 + z) = \bar{D}_k(z) * D_0(\varepsilon_0 + z). \quad (6)$$

Comparing (4) and (6) we get

$$\bar{D}_k(k^2/2M + z) = -\,z^{-1} + \bar{B}_k(z) + \tfrac{1}{2}\bar{B}_k(z) * \bar{B}_k(z) + \\ + \tfrac{1}{6}\bar{B}_k(z) * \bar{B}_k(z) * \bar{B}_k(z) + \ldots.. \quad (7)$$

This equation, which is formally quite similar to equation (3) for $D_0(z)$, is strictly valid for a finite system. We are , however, specially interested in the case that both Ω and N are infinite. We therefore study the function $\bar{B}_k(z)$ in this limit. As follows from its definition the function $B_k(z)$ can be obtained from $B_0(z)$, if in the latter each summation $\int_{k_i}$ corresponding to a particle line is replaced by ($\int_{k_i} - (2\pi)^3\,\Omega^{-1} \times$ term with $k_i = k$) and each summation $\int_{m_j}$ for a hole line is replaced by ($\int_{m_j} + (2\pi)^3\,\Omega^{-1} \times$ term with $m_j = k$). Keeping in mind that $B_0(z)$, which was defined in terms of connected ground state diagrams, is proportional to Ω in the limit of $\Omega \to \infty$, we see that $\bar{B}_k(z) = B_k(z) - B_0(z)$ contains a main term independent of Ω, and other terms which vanish if Ω tends to infinity. The function $\bar{B}_k(z)$ is therefore well defined also for an infinitely large system. Replacing summations by integrations and keeping only those terms which are independent of the volume Ω, $\bar{B}_k(z)$ is calculated in the following way. It is a sum of terms, each of which is obtained from the function $(2\pi)^3 \Omega^{-1} B_0(z)$ by putting the momentum of one of the lines equal to k and performing the integration over all other momenta. If the momentum which is put equal to k belongs to a particle line, the corresponding term gets a minus sign. Both sides of equation (7) have well defined finite limits for $\Omega \to \infty$. We can now return to this limiting case.

Although equation (7) for general k is interesting in itself, giving an alternative way of calculating $\bar{D}_k(z)$, we are here particularly interested in the limit of $|k|$ tending to k_F. In this limit the relation between $\bar{B}_k(z)$ and $B_0(z)$ has the following very simple form

$$\bar{B}_{k_F}(z) = 2\pi^2 k_F^{-2} \frac{\mathrm{d}}{\mathrm{d}k_F} (B_0(z)/\Omega). \tag{8}$$

To prove equation (8) we notice that $B_0(z)/\Omega$ depends on k_F only through the limits of integration of the integrals over particle and hole momenta. Differentiation of $B_0(z)/\Omega$ with respect to k_F gives a sum of terms, in each of which the momentum of one line is put equal to k_F. There is in addition a common factor $4\pi k_F^2$ resulting from integration over the surface of the Fermi sphere. Also here one gets a minus sign if the fixed momentum belongs to a particle because then k_F appears in the lower integration limit. The factors $4\pi k_F^2$ and $2\pi^2/k_F^2$ give together exactly $(2\pi)^3$, thus establishing equation (8). Using the well known relation between k_F and the density $\rho \equiv N/\Omega$:

$$\rho = k_F^3/6\pi^2,$$

equation (8) gets the simpler form

$$\bar{B}_{k_F}(z) = \frac{\mathrm{d}}{\mathrm{d}\rho} (B_0(z)/\Omega). \tag{9}$$

We now make essential use of the great formal similarity of equations (3) and (7). Clearly $D_0(\varepsilon_0 + z)$ changes into $\bar{D}_k(k^2/2M + z)$ if in (3) $B_0(z)$ is replaced by $\bar{B}_k(z)$. It was shown in III (section 9) that $D_0(\varepsilon_0 + z)$ can be expressed very simply in terms of the function $\bar{G}_0(\varepsilon_0 + z) \equiv z^2 B_0(z)$. In particular $D_0(\varepsilon_0 + z)$ was found to have a simple pole at $z = -\bar{G}_0(\varepsilon_0)$ with the residue $\exp(-\bar{G}_0'(\varepsilon_0))$, where the prime means the derivative with respect to z. This was a consequence of the fact that $z^2 B_0(z) = \bar{G}_0(\varepsilon_0 + z)$ had no singularities on the negative real axis of the z-plane. The same property holds for $z^2 \bar{B}_k(z)$ when $|k| = k_F$. By analogy we therefore conclude immediately that $\bar{D}_{k_F}(k_F^2/2M + z)$ has a pole at the point

$$z = -\lim_{z_1 \to 0} [z_1^2 \bar{B}_{k_F}(z_1)] = -\frac{\mathrm{d}}{\mathrm{d}\rho} (\bar{G}_0(\varepsilon_0)/\Omega), \tag{10}$$

with a residue

$$\exp\left[-\frac{\mathrm{d}}{\mathrm{d}\rho} (\bar{G}_0'(\varepsilon_0)/\Omega)\right].$$

As follows from the definition of the Fermi energy E_F, the pole of $\bar{D}_{k_F}(k_F{}^2/2M + z)$ is equal to $\Delta E_F = E_F - k_F{}^2/2M$. We have thus from (10)

$$\Delta E_F = \frac{\mathrm{d}}{\mathrm{d}\rho} (\Delta E_0/\Omega).$$

The same relation holds for the kinetic parts of E_F and E_0, hence

$$E_F = \frac{\mathrm{d}}{\mathrm{d}\rho} (E_0/\Omega). \tag{11}$$

This equation, if written in the equivalent form

$$E_F = \left(\frac{\partial E_0}{\partial N}\right)_\Omega,$$

where the derivative is taken at constant Ω, shows that the Fermi energy E_F, as defined in the previous section in terms of one-particle diagrams, is equal to the change in ground state energy of the system produced by addition or removal of one particle at constant volume.

For the function $\bar{D}_m(z)$ ($|m| < k_F$), which is the counterpart of $\bar{D}_k(z)$ for holes, one can proceed in exactly the same way. Instead of (7) one finds

$$\bar{D}_m(-m^2/2M + z) =$$
$$= -z^{-1} + \bar{B}_m(z) + \tfrac{1}{2}\bar{B}_m(z) * \bar{B}_m(z) + \tfrac{1}{6}\bar{B}_m(z) * \bar{B}_m(z) * \bar{B}_m(z) + \ldots, \tag{12}$$

where $\bar{B}_m(z)$ is defined in exactly the same way as $\bar{B}_k(z)$, except for the momentum k being replaced by m and the roles of particle and hole lines being interchanged. It is easily seen that the limit of $\bar{B}_m(z)$ for $|m| \to k_F$ is equal to $-\bar{B}_{k_F}(z)$. Forming now the convolution of $\bar{D}_k(k^2/2M + z)$ and $\bar{D}_m(-m^2/2M + z)$ for $|k| = |m| = k_F$ one finds, after an obvious shift of z in both functions

$$\bar{D}_k(z) * \bar{D}_m(z) = -z^{-1}, \text{ for } |k| = |m| = k_F.$$

This equation implies, that the poles of $\bar{D}_k(z)$ and $\bar{D}_m(z)$ for $|k| = |m| = k_F$ add up to zero, while the corresponding residues have a product equal to one. Since the sum of the poles is zero, the energy of a hole at the surface of the Fermi sea is equal to $-E_F$. Therefore the energy E_l of a particle of momentum $|l|$ close to k_F, as defined in section 2 for $|l|$ smaller or larger than k_F, is continuous at $|l| = k_F$.

Equation (11) can be expressed in terms of the energy per particle instead of the energy per unit volume:

$$E_F = E_0/N + \rho \frac{\mathrm{d}}{\mathrm{d}\rho} (E_0/N).$$

In terms of the pressure

$$p = -\left(\frac{\partial E_0}{\partial \Omega}\right)_N = \rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho} (E_0/N),$$

this equation reads

$$E_F = E_0/N + p/\rho.$$

In the case that the system is in equilibrium, *i.e.*, at a density such that the pressure vanishes, we obtain the equation

$$E_F = E_0/N. \tag{13}$$

This equality of the Fermi energy and the average energy, which we have proved generally, was derived recently by Weisskopf [9]) on the basis of the independent particle model. Bethe [10]) considered it to be only a rough approximation.

4. *Test on the accuracy of the theory of Brueckner.* In this last section the theorem (13) derived in section 3 will be used as a test on the validity of the Brueckner theory. Recently very accurate calculations on the basis of this theory have been made by Brueckner and Gammel [2]). The following discussion will be based mainly on the results of their work.

Our considerations will be of special interest because the calculations of Brueckner and Gammel show that their results vary strongly with slight changes in the forces between the particles *). Good agreement with the experiments does therefore not guarantee the accuracy of the theory. The test to be discussed here, on the contrary, is independent of the choice of the forces, for equation (13) must hold for all forces.

For the average energy E_0/N and the Fermi energy E_F Brueckner and Gammel find -15 MeV and -34 MeV respectively. There is a discrepancy of about 20 MeV, which shows that at least one of these values is very inaccurate. To investigate the origin of the discrepancy we consider the theory of Brueckner as an approximation of our exact perturbation formalism, as was done in IV †). It was shown there how one can obtain the theory of Brueckner from the exact theory by selecting only those terms which correspond to a certain class of diagrams. The relevant terms for E_0, E_k and E_m ($|k| > k_F$ and $|m| < k_F$) are represented by the diagrams of type *a*, *b* and *c* of fig. 1 **).

Let us consider equation (3) and equation (7) where $\bar{B}_k(z)$ is obtained

*) We are indebted to Dr. J. L. Gammel for communication of this and many other as yet unpublished results.

†) The equation for the scattering matrix G in IV at the bottom of page 537 contains an error. The energy denominator must read $\bar{E}_{k_1} + \bar{E}_{k_2} - |\bar{E}_{l_3}| - |\bar{E}_{l_4}|$.

**) The additional complications originating from the use of shifted energies in the denominators are not relevant for our discussion and are omitted for simplicity.

from $B_0(z)$ in the way prescribed in section 3. If we approximate $B_0(z)$ in these equations by taking the diagrams of fig. 1*a* only, we must still expect that the approximate values one then finds for E_F and E_0/N coincide (the latter value is the Brueckner approximation for the binding energy).

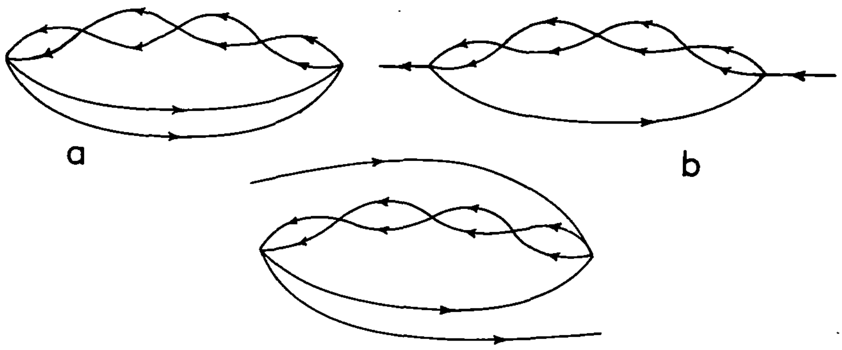

Fig. 1. The Brueckner diagrams. The diagrams *a, b* and *c* correspond to the ground state energy E_0 and the energies E_k and E_m of particles and holes respectively.

The function $\bar{B}_k(z)$ in the approximation now considered is equal to the sum of the contributions of all single particle diagrams, obtained from the ground state diagrams of fig. 1*a* by replacing any internal line by two external particle lines. This leads to two types of diagrams. The first type, where one of the hole lines is replaced by two external particle lines, is shown in fig. 1*b*. The other type, shown in fig. 2*a*, is obtained from fig. 1*a* by replacing one of the many internal particle lines by two external particle

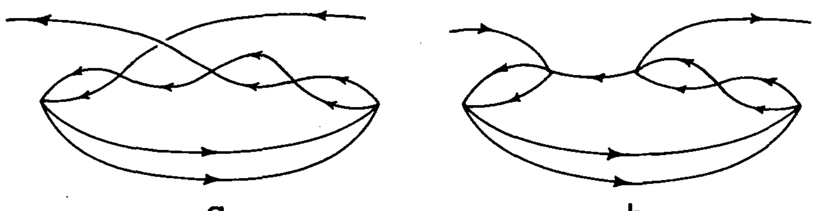

Fig. 2. This figure shows some single particle energy diagrams neglected in the theory of Brueckner; the diagrams *a* and *b* correspond to particles outside and inside the Fermi sea respectively.

lines. It is seen from (7) that in the present approximation $\bar{D}_k(z)$ is a sum of the contributions of these diagrams and of the more complicated ones constructed by linking together two or more of such diagrams. All these single particle diagrams, with the exception of the one in fig. 1*b*, are neglected in the theory of Brueckner. They contain three and more particle clusters. From the numerical discrepancy between E_0/N and E_F found, as mentioned above, by Brueckner and Gammel, we must conclude that for

$|k| = k_F$ the total contribution of the diagrams neglected in the Brueckner theory is considerable. It must account for a difference of about 20 MeV. It seems reasonable to suppose that among the neglected terms the most important ones are those represented by diagrams of the type of fig. 2*a* and the corresponding diagrams for holes in fig. 2*b*. This is also suggested by the following consideration.

The theory of Brueckner can be considered as the first term in the so-called cluster expansion [11]). Using the K-matrix instead of the interaction V all quantities are expressed by means of a very much smaller number of diagrams, namely those diagrams, where no two successive vertices are connected by two particle lines (Goldstone [11]) called them irreducible; we have used this term in III already with another meaning). The diagrams corresponding to the first three terms of the cluster expansion for E_0 are shown in fig. 3. To each dot there corresponds a K-matrix. The first term in the figure gives the Brueckner approximation; it corresponds to diagram *a* of fig. 1. The cluster expansion can be considered as a power series in the

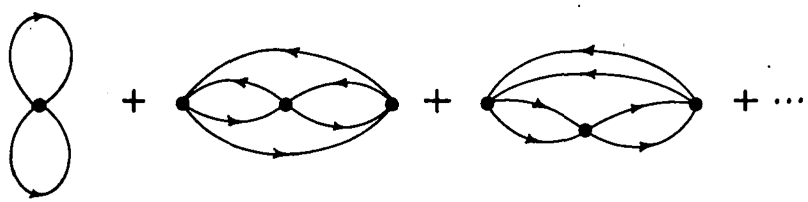

Fig. 3. The first three diagrams of the cluster expansion for E_0.

K-matrix. The Brueckner approximation is based on the assumption that this series converges rapidly. The second term in fig. 3 was calculated by Bethe [10]) for the case of Yukawa forces. It was found to be less than 1 MeV, which is indeed very small compared to the main term. We notice from fig. 3 that the cluster expansion for E_0 contains no term with two K-matrices. This has the consequence that even for a comparatively slow convergence the first term can be a reasonably accurate approximation.

The first two diagrams of the cluster expansion for the single particle energy E_l, are given in fig. 4*a* for $|l| \geqslant k_F$, in fig. 4*b* for $|l| \leqslant k_F$. Also here the first diagrams of *a* and *b* give the Brueckner approximation and correspond to diagrams *b* and *c* of fig. 1. Comparing the first diagrams in fig. 3 and fig. 4*b* we find the well-known relation, characteristic of the Brueckner theory, between the energy shift ΔE_0 of the ground state and the shift $V_l = E_l - l^2/2M$ of the single particle energy:

$$\Delta E_0 = \tfrac{1}{2}\Omega(2\pi)^{-3} \int_0^{k_F} d^3m V_m,$$

which is another form of (1). In the case of particles with spin and isobaric spin $\frac{1}{2}$ a factor 4 must be added at the right-hand side. One sees again that (1) is not an exact equation *).

The cluster expansion for E_l involves a term with two K-matrices which might be quite appreciable in case of a slow convergence of the series. This term corresponds exactly to the type of diagrams shown in fig. 2, so that we must expect the neglection of the diagrams in fig. 2 to be largely responsible for the discrepancy between E_0/N and E_F in the theory of Brueckner. We have made a rough estimate of this term, for spin and charge independent Yukawa forces. Making the same approximation as Bethe did in his calculation of the three-particle cluster term in E_0, we find approximately 12 MeV for the second term in fig. 4*a* or *b*, for a momentum $|l| = k_F$. This shows that even for these unrealistic forces the main single-particle energy term left out by Brueckner is quite large. A calculation of this term and other cluster terms neglected in the Brueckner theory, on the basis of more realistic forces with a repulsive core, would be very interesting. We may conclude already, however, that in the theory of Brueckner the single-particle energy is treated very inaccurately. The influence of this inaccuracy on the calculation of the ground state energy, which manifests itself only through the energy denominators, is probably not very large in the nuclear case. For the calculation of the optical potential the situation is completely different and one clearly must take into account the terms which we discussed in the present section.

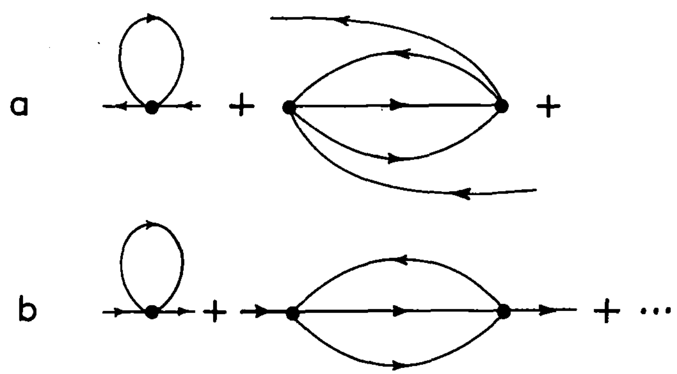

Fig. 4. The first two terms of the cluster expansion for the single particle energy E_l; *a* and *b* correspond to $|l| \geq k_F$ and $|l| \leq k_F$ respectively.

Quite recently, one of the present authors having brought the large internal inconsistency revealed in Brueckner's theory by the theorem here discussed to his attention, Brueckner reconsidered the problem in the framework of his theory and suggested to use the theorem itself for obtaining

*) Differentiation of (1) with respect to the density ρ would lead to (11), provided V_l would not depend on ρ. We know, however, that such is not the case.

a better definition of the single-particle energy *). The new definition amounts to replacing the single-particle energy E_l^B of the original Brueckner approximation (first term in fig. 4*a* of *b*) by a shifted value $E_l^B + \varepsilon$, where the quantity ε, assumed independent of the momentum l, is defined by the condition

$$E_l^B + \varepsilon = E_0/N \text{ for } |l| = k_F.$$

An obvious correction term is then added to the formula expressing E_0 in terms of the single-particle energies. This elementary way of circumventing the inconsistency suffers from two obvious defects. The momentum independence of ε is completely unfounded in a theory where, as in Brueckner's, the potential energy part of E_l^B has an important momentum variation. In the second place, a proper definition of the single-particle energy should be entirely formulated in terms of the propagation of an additional particle (or a hole) of given momentum through the given medium. Such is the case with the definition of E_l in the general theory used here and this is the only reason why our theorem is not trivial. Brueckner's definition of ε, on the contrary, is in fact based on a comparison between two states of the medium with two different densities.

The authors are very grateful to Dr. J. L. Gammel for many stimulating discussions, some of which were at the origin of the present work. They are also indebted to him for much valuable unpublished information on the numerical aspects of Brueckner's theory.

This work is part of the research program of the „Stichting voor Fundamenteel Onderzoek der Materie", which is financially supported by the Netherlands Organization for pure Scientific Research (Z.W.O.).

Received 1-2-58.

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