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EXACTLY RENORMALIZABLE MODEL IN QUANTUM FIELD THEORY

II. THE PHYSICAL-PARTICLE REPRESENTATION

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Synopsis

For the simplified model of quantum field theory discussed in a previous paper it is shown how the physical particles can be properly described by means of the so-called asymptotically stationary (a.s.) states. It is possible by formulating the theory in terms of these a.s. states to express it entirely in terms of renormalized quantities and convergent integrals.

§ 1. *Introduction.* At least some of the divergences occurring in the quantum theory of fields are due to the fact that the interaction between the free fields is treated as a perturbation. This perturbation method is taken over from the theory of systems with a finite number of degrees of freedom, especially the theory of scattering problems. Thus for the scattering of a particle by a static potential, the hamiltonian is separated into two parts

$$H = H^0 + V, \tag{1.1}$$

H^0 being the kinetic energy and V the potential, which vanishes outside a finite region. This separation is physically well justified, inasmuch as H^0 describes the particle exactly when it is far away from the region where the potential does not vanish. In quantum field theory, however, the usual separation of the total hamiltonian into a term H^0 for the free fields and an interaction between the lattter does not have such a simple physical foundation. In this case the splitting (1.1) of the hamiltonian is actually unphysical since, in the scattering of two particles for instance, the behaviour of these particles when far apart, is not properly described by H^0. Even when the particles are widely separated, they essentially differ from the unperturbed particles in that they are surrounded by a cloud of virtual particles permanently attached to them in virtue of the field interaction. It is the interaction of these *dressed* or *physical particles* which is physically interesting. However, when applying the usual perturbation theory one automatically represents all states in terms of unperturbed or *bare particles* and one unavoidably gets a clumsy representation of what physical particles are and how they interact. The most familiar consequence of this situation is that the observed particle masses and coupling constants differ from the parameters introduced for them in the original hamiltonian. The former are called the renormalized masses and coupling constants, the latter are the unrenormalized ones.

For a system of particles which are all far away from each other a complete description in terms of dressed particles is possible. It is given formally by the *asymptotically stationary* (a.s.) states, defined and used extensively by Van Hove [1]) and by Hugenholtz [2]). This naturally suggests to reformulate the whole problem of interacting fields by representing all states in terms of a.s. states, *i.e.* in terms of dressed particles, in the hope to avoid occurrence of unphysical quantities like the unrenormalized mass and coupling constant. This proposal is very difficult to carry out for a field theory of actual interest, as quantum electrodynamics, in view of its intrinsic complication. There exist, however, simplified models of interacting fields on which the problem could be investigated with more chance of success. The most elaborate of these models has been described and investigated in a previous paper [3]) to be referred to hereafter as I. Our first aim in the present paper is to study this model in the a.s. state or dressed particle representation. Our main result will be that the Schrödinger equation can be replaced in the a.s. state representation by an equation containing only the renormalized mass and coupling constants (section 3). This truly renormalized equation, when further analyzed by perturbation methods, leads directly to physically observable results (section 4).

§ 2. *The asymptotically stationary states.* In I, eq. (I. 2.6) we obtained an exact expression for the physical, *i.e.* dressed V_q-particle state. It can be written in the form

$$|\phi_q(\mathbf{p})\rangle = O_q^*(\mathbf{p})\ |0\rangle, \tag{2.1}$$

where $O_q^*(\mathbf{p})$ is the following operator

$$O_q^*(\mathbf{p}) = N^{1/2}(q)[\psi_q^*(\mathbf{p}) + \textstyle\sum_{n=1}^{\infty} \sum \prod_{j=q}^{q+n-1} g_j^0 \cdot$$
$$\cdot \psi^*_{q+n}(\mathbf{p}-\textstyle\sum_k l_k \mathbf{k}) \prod_k (l_k!)^{-1} [X(k) a^*(\mathbf{k})]^{l_k}]. \tag{2.2}$$

The second summation runs over all sets of integers $\{l_k\}_n$ with $\sum_k l_k = n$. This will always be understood when in subsequent formulae a $\sum$-sign without summation index occurs. Note that the expression of $|\phi_q(\mathbf{p})\rangle$ in terms of unperturbed states becomes meaningless in the limit of an infinite cut-off because then, as we have seen in (I. 2.9), the normalization factor $N^{1/2}(q)$ becomes zero. The physical states of one θ-particle do not differ from the unperturbed states and can thus be written as $a^*(\mathbf{k})\ |0\rangle$. Instead of remaining in the representation of the unperturbed stationary states, whose expression is

$$|\alpha_n(q)\rangle = \textstyle\prod_k (l_k!)^{-1/2} (a^*(\mathbf{k}))^{l_k} \psi_q^*(\mathbf{p})\ |0\rangle,$$

for the state describing a set $\{l_k\}_n$ of θ-particles in addition to an unperturbed V_q-particle, we consider the states

$$|\,\alpha_n(q)\rangle_{as} = \Pi_k\,(l_k!)^{-1/2}(a^*(\mathbf{k}))^{l_k}\,O_q^*(\mathbf{p})|0\rangle, \tag{2.3}$$

where the V_q particle is taken dressed. The latter states, and similar ones with more than one heavy particle present, are the asymptotic stationary states in the sense of Van Hove. In the following we restrict ourselves to the states with one V particle. The states with two or more V particles, which are not connected to them by the hamiltonian, could be discussed similarly, but a complete extension of our results to them has not yet been obtained.

We first remark that, except for the state (2.1) without θ-particle, the a.s. states are not eigenstates of the hamiltonian (I. 2.4 and 2.5). For the a.s. state $a^*(\mathbf{k})\,O_q^*(\mathbf{p})\,|0\rangle$, for instance, we can easily show that the following equation holds

$$Ha^*(\mathbf{k})\,O_q^*(\mathbf{p})\,|0\rangle = \\ = (m + \omega(k))\,a^*(\mathbf{k})O_q^*(\mathbf{p})\,|0\rangle - \omega(k)X(k)\,T(\mathbf{k})\,O_q^*(\mathbf{p})|0\rangle. \tag{2.4}$$

Nevertheless, we can construct with the states (2.3) time-dependent state vectors which obey the time dependent Schrödinger equation asymptotically for large values of the time t; this fact actually explains the characterization of the states as asymptotic stationary. Consider e.g. the state

$$|\phi(t)\rangle \equiv \\ \equiv v^{-1}\sum_{pkq} c(\mathbf{p},\mathbf{k},q)\exp[-i\{(\mathbf{p}.\mathbf{R}_V+\mathbf{k}.\mathbf{R}_\theta)+(m+\omega(k))t\}]a^*(\mathbf{k})O_q^*(\mathbf{p})|0\rangle. \tag{2.5}$$

It describes a heavy particle and a θ-particle which at the time $t = 0$ are located in the neighbourhood of the points $\mathbf{R}_V$ and $\mathbf{R}_\theta$ respectively. $c(\mathbf{p}, \mathbf{k}, q)$ is assumed to be a smooth function of $\mathbf{p}$ and $\mathbf{k}$. From (2.4) it follows that $|\phi(t)\rangle$ satisfies the following equation

$$(H-i\,\partial/\partial t)|\phi(t)\rangle = -v^{-1}\sum_{pkq} c(\mathbf{p},\mathbf{k},q)\,. \\ .\exp[-i\{(\mathbf{p}.\mathbf{R}_V+\mathbf{k}.\mathbf{R}_\theta) + (m+\omega(k))t\}]\,\omega(k)\,X(k)\,T(\mathbf{k})\,O_q^*(\mathbf{p})|0\rangle. \tag{2.6}$$

Because $T(\mathbf{k})\,O_q^*(\mathbf{p})\,|0\rangle$ depends only on the sum $\mathbf{p} + \mathbf{k}$, the summation in (2.6) can be split in a summation over $\mathbf{p}' = \mathbf{p} + \mathbf{k}$ and one over $\mathbf{k}$. The latter takes the form

$$\sum_k c(\mathbf{p}'-\mathbf{k},\mathbf{k},q)\,\omega(k)\,X(k)\,\exp[-i\{\mathbf{k}.(\mathbf{R}_\theta-\mathbf{R}_V) + (m+\omega(k))t\}]$$

and reveals that the right hand side of (2.6) is only different from zero for those values of t for which the V- and θ-particle overlap, *i.e.* for a finite time interval. This proves that $|\phi(t)\rangle$ gives a proper description of the situations where a θ-particle and a V-particle move independently from each other, their separation being large enough to avoid interaction. Of course (2.5) does not describe the scattering of a θ-particle by a heavy particle.

In general the a.s. states are not mutually orthogonal. With the aid of the commutator

$$[a(\mathbf{k}), O_q^*(\mathbf{p})] = g_q X(k) O^*_{q+1}(\mathbf{p}-\mathbf{k}), \tag{2.7}$$

the calculation of which is an immediate consequence of the definition (2.2) of $O_q^*(\mathbf{p})$ and of the usual commutation relation $[a(\mathbf{k}), a^*(\mathbf{k}')] = \delta_{k,k'}$, we can calculate all scalar products of a.s. states (2.3)

$$\langle 0|O_{q'}(\mathbf{p}')\prod_{k'}(l'_{k'}!)^{-1/2}(a(\mathbf{k}'))^{l'_{k'}}O_q^*(\mathbf{p})\prod_k(l_k!)^{-1/2}(a^*(\mathbf{k}))^{l_k}|0\rangle =$$
$$= \delta_{q'-m,q-n}\,\delta_{\mathbf{p}'+\Sigma_{k'}l'_{k'}\mathbf{k}',\,\mathbf{p}+\Sigma_k l_k \mathbf{k}}\,\varphi_{q-n}(\{l'_{k'}\}_m;\{l_k\}_n). \tag{2.8}$$

We find, for instance,

$$\varphi_q(0;0) = 1;\ \varphi_q(\mathbf{k}';0) = g_q X(k');\ \varphi_{q-1}(\mathbf{k}';\mathbf{k}) = \delta_{k',k} + g_q^2 X(k')X(k). \tag{2.9}$$

A closed expression for the φ-functions can be given:

$$\varphi_q(\{l'_{k'}\}_m;\{l_k\}_n) = \Sigma_{r=0}^{s}\,\Sigma\prod_k\binom{l'_k}{i_k}^{1/2}\binom{l_k}{i_k}^{1/2}\cdot\varphi_{q+n}(\{l'_k-i_k\};0)\varphi_{q+m}(0;\{l_k-i_k\}),$$

with $s = \Sigma_k \min(l_k', l_k)$ and $\binom{p}{q} = p!/q!(p-q)!$. The second summation runs over all sets $\{i_k\}_r$ of r θ-particles contained in the intersection of the sets $\{l'_{k'}\}_m$ and $\{l_k\}_n$. $\varphi_q(\{l_k\}_n;0)$ is given by

$$\varphi_q(\{l_k\}_n;0) = \prod_{j=q}^{q+n-1} g_j\cdot\prod_k(l_k!)^{-1/2}(X(k))^{l_k}.$$

For reference we also give two recurrence relations between the φ-functions,

$$\varphi_{q-n}(\{l'_{k'}\}_m;\{l_k\}_n) = (l_{k'}/l'_{k'})^{1/2}\varphi_{q-n+1}(\{l'_{k'}\}_m-\mathbf{k}';\{l_k\}_n-\mathbf{k}') +$$
$$+\ g_q X(k')(l'_{k'})^{-1/2}\varphi_{q-n+1}(\{l'_{k'}\}_m-\mathbf{k}';\{l_k\}_n) \tag{2.10}$$

$$\varphi_{q-n}(\{l'_{k'}\}_m;\{l_k\}_n) = (l_k'/l_k)^{1/2}\,\varphi_{q-n+1}(\{l'_{k'}\}_m-\mathbf{k};\{l_k\}_n-\mathbf{k}) +$$
$$+\ g_{q+m-n}X(k)(l_k)^{-1/2}\varphi_{q-n+1}(\{l'_{k'}\}_m;\{l_k\}_n-\mathbf{k}) \tag{2.11}$$

With $\{l_k\}_n - \mathbf{k}'$ we denote the set of $(n-1)$ θ-particles, containing for each $\mathbf{k}$, $l_k - \delta_{kk'}$ particles of momentum $\mathbf{k}$. We require $l'_{k'} \geqslant 1$, $l_{k'} \geqslant 1$ in (2.10) and $l'_k \geqslant 1$, $l_k \geqslant 1$ in (2.11). An important feature is the exclusive occurrence of renormalized coupling constants in the functions φ_{q-n}. This will enable us to reformulate in the next section the Schrödinger equation in terms of the renormalized mass and coupling constants only.

In general the functions $\varphi_{q-n}(\{l'_{k'}\}_m;\{l_k\}_n)$, as defined by (2.8) differ from the Kronecker δ-function, which is unity when the two sets $\{l'_{k'}\}_m$ and $\{l_k\}_n$ are equal and zero otherwise, by a sum of positive terms each containing l factors of the form $g_{q'}X(k')$, where l has the values

$$l = m+n,\ m+n-2,\ \ldots,\ |m-n|.$$

This shows that the a.s. states (2.3) are not mutually orthogonal. They exhibit, however, a sort of asymptotic orthonomality for large times. To see this consider e.g. the state (2.5) and the state

$$|\phi'(t)\rangle = v^{-1} \sum_{p'k'q'} c'(\mathbf{p}'\mathbf{k}'q') \exp[-\, i\{\mathbf{p}'.\mathbf{R}_V' + \mathbf{k}'.\mathbf{R}_\theta'\} + \\ + (m + \omega(k'))t]a^*(\mathbf{k}')\, O_{q'}^*(\mathbf{p}')|0\rangle \qquad (2.12)$$

and form the scalar product $\langle\phi(t) \mid \phi'(t)\rangle$. Using the value of $\varphi_{q-1}(\mathbf{k};\mathbf{k}')$ as given in (2.9) we find

$$\langle\phi(t) \mid \phi'(t)\rangle = v^{-2} \sum_{pkq} c^*(\mathbf{pk}q)c'(\mathbf{pk}q) \exp i[\mathbf{p}.(\mathbf{R}_V-\mathbf{R}_V') + \mathbf{k}.(\mathbf{R}_\theta-\mathbf{R}_\theta')] + \\ + v^{-2}g_q{}^2\sum_{pkk'q}c^*(\mathbf{pk}q)c'(\mathbf{p}+\mathbf{k}-\mathbf{k}', \mathbf{k}', q)X(k)X(k') \exp i[\mathbf{p}.(\mathbf{R}_V-\mathbf{R}_V') - \\ - \mathbf{k}.(\mathbf{R}_V'-\mathbf{R}_\theta) + \omega(k)t + \mathbf{k}'.(\mathbf{R}_V'-\mathbf{R}_\theta') - \omega(k')t]. \qquad (2.13)$$

The second term in the right-hand side of (2.13) can be appreciably different from zero only for a finite time interval. So for $t \to \pm\infty$ the scalar product is given by the first term, as would be the case if the a.s. states were considered as orthonormal. More generally we can prove that for large times the scalar product of the states

$$|\phi(t)\rangle = \sum_n v^{-(n+1)/2} \sum_{qpk_1\ldots k_n} c_n(q\mathbf{pk}_1\ldots\mathbf{k}_n)\, e^{-iE(k_1\cdots k_n)t}|\alpha_n(q)\rangle_{as}$$

and

$$|\phi'(t)\rangle = \sum_n v^{-(n+1)/2} \sum_{qpk_1\ldots k_n} c'_n(q\mathbf{pk}_1\ldots\mathbf{k}_n)\, e^{-iE(k_1\cdots k_n)t}\, |\alpha_n(q)\rangle_{as},$$

where $|\alpha_n(q)\rangle_{as}$ is of the form (2.3) and $E(k_1\ldots k_n) = m + \omega(k_1) + + \ldots + \omega(k_n)$, is given by

$$\lim_{t\to\pm\infty} \langle\phi(t) \mid \phi'(t)\rangle = \sum_n v^{-(n+1)} \sum_{qpk_1\ldots k_n} c_n^*(q\mathbf{pk}_1\ldots\mathbf{k}_n)c_n'(q\mathbf{pk}_1\ldots\mathbf{k}_n).$$

§ 3. *The Schrödinger equation.* We can now derive another form of the time-independent Schrödinger equation involving only the renormalized mass and coupling constants, by writing it in terms of a.s. states. To this effect we need the formulae

$$[\Pi_k(a(\mathbf{k}))^{l_k}, H] = \\ = \sum_k l_k\omega(k)\,.\,\Pi_{k'}(a(\mathbf{k}'))^{l_{k'}} - \sum_k l_k\omega(k)X(k)T^*(\mathbf{k})(a(\mathbf{k}))^{l_k-1}.\,\Pi_{k'\neq k}(a(\mathbf{k}'))^{l_{k'}} \quad (3.1)$$

and

$$T^*(\mathbf{k})\, O_q^*(\mathbf{p})|0\rangle = g_q O^*{}_{q+1}(\mathbf{p}-\mathbf{k})|0\rangle. \qquad (3.2)$$

(3.2) is equivalent to the definiton (I. 3.2) of the renormalized coupling constant g_q. With these formulae it is easy to calculate the matrixelements of the total hamiltonian H(I. 2.4 and 2.5) between two a.s. states of the form (2.3). We find

$$\langle 0|O_{q'}(\mathbf{p}')\ \Pi_{k'}(l'_{k'}!)^{-1/2}(a(\mathbf{k}'))^{l'_{k'}}\ HO_q^*(\mathbf{p})\ \Pi_k(l_k!)^{-1/2}(a^*(\mathbf{k}))^{l_k}|0\rangle = \\ = \delta_{q'-m,\, q-n}\, \delta_{\mathbf{p}'+\sum_k l_k'\mathbf{k},\mathbf{p}+\sum_k l_k\mathbf{k}}\cdot[(m+\sum_k l_k'\omega(k))\varphi_{q-n}(\{l'_{k'}\}_m;\ \{l_k\}_n) - \\ - \sum_k(l_k')^{1/2}\omega(k)g_qX(k)\varphi_{q-n+1}(\{l'_{k'}\}_m-\mathbf{k};\ \{l_k\}_n)] = \\ = \delta_{q'-m,q-n}\delta_{\mathbf{p}'+\sum_k l_k'\mathbf{k},\mathbf{p}+\sum_k l_k\mathbf{k}}[m\varphi_{q-n}(\{l'_{k'}\}_m;\ \{l_k\}_n) + \\ + \sum_k(l_k'l_k)^{1/2}\ \omega(k)\varphi_{q-n+1}(\{l'_{k'}\}_m-\mathbf{k};\ \{l_k\}_n-\mathbf{k})]. \qquad (3.3)$$

In the last step the relation (2.10) has been used. We now assume the a.s. states to form a complete set and expand the stationary states of total momentum $\mathbf{p}$ with one V-particle into this new set of basic states:

$$|\psi_q(\mathbf{p})\rangle =$$
$$= [d_0 O_q^*(\mathbf{p}) + \Sigma_{n=1}^{\infty} \Sigma\, d(\{l_k\}_n) O^*_{q+n}(\mathbf{p} - \Sigma_k l_k \mathbf{k})\ \Pi_k (l_k!)^{-1/2} (a^*(\mathbf{k}))^{l_k}]|0\rangle. \quad (3.4)$$

$d_0 = 1$ and $d(\{l_k\}_n) = 0$ for $n \geqslant 1$ gives the physical V_q-particle state, which has the energy m. Other states of the form (3.4) describe the scattering, creation and annihilation of θ-particles by a V-particle. For a general stationary state the Schrödinger equation reads

$$H \mid \psi_q(\mathbf{p})\rangle = (m + \omega_0)\ |\psi_q(\mathbf{p})\rangle. \quad (3.5)$$

By multiplying on the left with the state vector

$$\langle 0|\ O_{q+m}(\mathbf{p} - \Sigma_{k'} l_{k'}'\mathbf{k}')\ \Pi_{k'} (l'_{k'}!)^{-1/2} (a(\mathbf{k}'))^{l'_{k'}},$$

where $\{l_{k'}'\}_m$ is an arbitrary set of m θ-particles, we obtain from (3.5), with the aid of (3.3), the following infinite set of equations for the coefficients d_0 and $d(\{l_k\}_n)$.

$$\omega_0[d_0 \varphi_q(\{l_{k'}'\}_m; 0) + \Sigma_{n=1}^{\infty} \Sigma\, d(\{l_k\}_n) \varphi_q(\{l_{k'}'\}_m; \{l_k\}_n)] =$$
$$= \Sigma_{n=1}^{\infty} \Sigma\, d(\{l_k\}_n)\ \Sigma_{k'}\ (l'_{k'} l_{k'})^{1/2} \omega(k') \varphi_{q+1}(\{l'_{k'}\}_m - \mathbf{k}'; \{l_k\}_n - \mathbf{k}'). \quad (3.6)$$

We have hereby succeeded in replacing the Schrödinger equation by an equivalent set of equations which contains only the renormalized coupling constants g_q (they appear in the φ-functions). This set of equations, to be sure, is far more complicated in algebraic structure than the original Schrödinger equation. It is, however, much more satisfactory from the physical standpoint. It is not expected that, besides the physical V_q-state, other exact solutions of (3.6) can be found.

§ 4. *The perturbation approach to equation* (3.6). We now consider our renormalized Schrödinger equation somewhat closer. Firstly we see from (3.6) that, if the a.s. states formed an orthogonal set, *i.e.* if $\varphi_q(\{l'_{k'}\}_m; \{l_k\}_n)$ were zero when $\{l_{k'}'\}_m$ and $\{l_k\}_n$ are different, these states would also be stationary. Indeed, the equation (3.6) would then be satisfied for all d's except one equal to zero. In this case scattering of one or more θ-particles on a V-particle would have a vanishing cross section (for arbitrary cut off). The non-orthogonality of the a.s. states is therefore essential for the occurrence of scattering processes.

Secondly we can now prove rigorously that for infinite cut off there will be no $V_{q+1} - \theta$ scattering, a result established in I to 6th order only (see formula I.7.4). All renormalized coupling constants g are equal for infinite cut off so that the φ_q-functions become independent of q. The solution of (3.6) for the scattering of a θ-particle of momentum $\mathbf{k}_0$ on a V_{q+1}-particle is then simply

$$\omega_0 = \omega(k_0)\,;\; d_0 = -gX(k_0)\,;\; d(\mathbf{k}) = \delta_{kk_0}\,;\; \mathrm{d}(\{l_k\}_n) = 0 \text{ for } n \geqslant 2. \qquad (4.1)$$

Substitution of (4.1) into (3.6) gives

$$\omega(k_0)[-g\,X(k_0)\varphi_q(\{l'_{k'}\}_m;\,0)+\varphi_q(\{l'_{k'}\}_m;\,\mathbf{k}_0)]=(l'_{k_0})^{1/2}\omega(k_0)\varphi_{q+1}(\{l'_{k'}\}_m-\mathbf{k}_0;\,0),$$

which, according to (2.11), indeed is an identity. From (4.1) the motion of a wave packet is given by a state vector of the form

$$|\phi(t)\rangle = \Sigma_{qpk}\,c(q\mathbf{p}\mathbf{k})\,e^{-i(m+\omega(k))t}[-gX(k)O_q{}^*(\mathbf{p}+\mathbf{k})+a^*(\mathbf{k})O^*{}_{q+1}(\mathbf{p})]|0\rangle. \quad (4.2)$$

For large $|t|$ the second term in the bracket is the only one to contribute (see the consideration after (2.6) where a similar point is established) and it describes the incident wave. This shows that no scattering at all takes place. There is only a deformation of the wave packet, described by the first term in the brackets, when the θ-particle is in the neighbourhood of the V-particle.

Let us now consider $\theta - V_{q+1}$ scattering for arbitrary cut off, and let us first take the one-meson approximation, *i.e.* neglect $d(\{l_k\}_n)$ for $n > 1$. The equations (3.6) then reduce to

$$d_0 + \Sigma_k g_q X(k)\, d(\mathbf{k}) = 0 \qquad (4.3)$$

$$\omega_0[d_0 g_q\, X(k_1) + \Sigma_k\, d(\mathbf{k})\, \{\delta_{kk_1} + g^2{}_{q+1}\, X(k)\, X(k_1)\}] = \omega(k_1)\, \mathrm{d}(\mathbf{k}_1).$$

For an incoming θ-particle of momentum $\mathbf{k}_0$ the exact solution of (4.3) is

$$d_0 = -\, g_q\omega(k_0)\, X(k_0)h^{-1}(\omega(k_0) + i0)\,; \qquad (4.4)$$

$$d(\mathbf{k})=\delta_{kk_0}-(g_q{}^2-g^2{}_{q+1})(\omega(k)-\omega(k_0)-i0)^{-1}.\,\omega^2(k_0)\,X(k_0)\,X(k)h^{-1}(\omega(k_0)+i0).$$

$h(z)$ is the following function of the complex variable z:

$$h(z) = z[1 + (g_q{}^2 - g^2{}_{q+1})\, z\Sigma_k\, X^2(k)/(\omega(k) - z)]. \qquad (4.5)$$

The $N-\theta$ scattering state of the Lee model is included in (4.4) as a special case ($g_{q+1} = 0$). We remark from (4.4) that for $k \gg \omega(k_0)$

$$d(\mathbf{k}) \sim k^{-5/2}, \qquad (4.6)$$

which assures the convergence of the summations in (4.3). This is an important improvement compared with an expansion in unperturbed states where divergences still occur because the coefficients are proportional to $k^{-3/2}$ for large k (cf. Källén and Pauli [4]), eq. 38). We see that the one meson approximation of the renormalized equations (3.6), which is an improved kind of Tamm-Dancoff approximation, contains no divergences, unlike what happens in the usual Tamm-Dancoff approximation.

More generally it is not to be expected that the exact solutions of (3.6) involve divergences. This can be proved very simply to the first non-vanishing order in the coupling constants. Consider again the equations (3.6) for the scattering of a θ-particle of momentum $\mathbf{k}_0$ on a V_{q+1}-particle. We write the coefficients as

$$d_0 = d_0^{(1)} + \ldots ; d(\mathbf{k}) =$$
$$= \delta_{kk_0} + d^{(2)}(\mathbf{k}) + \ldots ; d(\{l_k\}_n) = d^{(n+1)}(\{l_k\}_n) + \ldots , n \geqslant 2.$$

where $d^{(n+1)}(\{l_k\}_n)$ denotes the first non-vanishing approximation of $d(\{l_k\}_n)$ in the coupling constants. It is found to be

$$d_0^{(1)} = -g_q X(k_0) ; d^{(n+1)}(\{l_k\}_n) =$$
$$= h(n)\omega(k_0)X(k_0) \prod_k (l_k!)^{-1/2}(X(k))^{l_k} [\Sigma_k l_k \omega(k) - \omega(k_0) - i0]^{-1}, \quad (4.7)$$

with

$$h(1) = g^2_{q+1} - g_q^2 ; h(n) = \prod_{j=q+1}^{q+n-1} g_j \, \Sigma_{p=0}^n (-1)^{n+p} \binom{n}{p} g^2_{q+p} \text{ for } n > 1.$$

(4.7) is easily checked, using (2.11) and the relation

$$\Sigma \prod_k \binom{l_k'}{l_k} = \binom{m}{n},$$

where the summation runs over all sets $\{l_k\}_n$ of n θ-particles contained in the set $\{l_{k'}'\}_m$.

There are two important remarks concerning the solution (4.7):

i) $d^{(n+1)}(\{l_k\}_n)$ is of the order $(n + 1)$ in the coupling constants and not of order $(n - 1)$ as would be expected at first sight. This remarkable feature originates from the fact that the stationary states are expanded in the a.s. states, in which the effects of order $(n-1)$ are already incorporated;

ii) for every $\mathbf{k}$ for which $l_k \neq 0$ in $\{l_k\}_n$ and $k \gg \omega(k_0)$ we have again

$$d(\{l_k\}_n) \sim k^{-5/2}$$

and the summations in (3.6) are consequently convergent to the lowest order.

For $d(\mathbf{k})$ we have calculated the next order and for $k \gg \omega(k_0)$ we found

$$d(\mathbf{k}) \sim k^{-5/2} \log (k/\mu). \quad (4.8)$$

This still does not make the summations in (3.6) diverge and it may be expected that for all $d(\{l_k\}_n)$ and to any order the equations (3.6) contain only convergent sums. In other words we expect that the renormalized Schrödinger equation (3.6) can be solved by perturbation theory to any order without occurrence of divergences. The overall convergence of the power series in the coupling constants which would thus be obtained is much more difficult to study. The occurrence of the logarithm in the 2^d approximation (4.8), while no such logarithm occurs in the first approximation, suggests that the overall convergence might be only asymptotic.

The fact that for the model here considered it is possible to write down a renormalized Schrödinger equation which can be solved by perturbation theory to give finite answers, suggests the possibility of a similar reformulation of quantum electrodynamics using only the observable mass and charge and the a.s. states. In particular an equation corresponding to our eq. (3.6) should be derived. It will certainly be very complicated, but the main suggestion from our considerations is that it exists and that its treatment by perturbation theory would give convergent expressions to any order.

A final remark refers to the occurrence of abnormal states. In order to obtain a non-vanishing cross section for $V_{q+1} - \theta$ scattering the renormalized coupling constants g_q and g_{q+1} have to be different and for an infinite cut — off this implies that imaginary unrenormalized coupling constants should be taken (see I, section 3). In the Lee model this is known to give rise to an additional stationary state (*ghost state*) with negative energy. It is not excluded that similar abnormal states would occur in our model (see 3), 5) and I 18)). In the one-meson approximation it certainly is present. The function $h(z)$, defined in (4.5), is for $g_{q+1} = 0$ identical with the function $h(z)$ defined by Källén and Pauli [4]). From their investigations we immediately conclude that for $g_q^2 > g^2{}_{q+1}$ our function $h(z)$ has an additional negative zero point and this implies the presence of a ghost state. A conclusive proof, however, for the actual occurrence of ghost states in our complete theory has not yet been given.

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