HOMOTOPY THEORY OF PRODUCTS ON SPHERES. II.

BY

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8. PRODUCTS ON S^1, S^3 AND S^7

In this section points of S^n are represented by unit complex numbers when $n=1$, by unit quaternions when $n=3$, and by unit Cayley numbers when $n=7$.

On S^1, S^3, S^7 we have in this way a standard product of type $(1, 1)$, induced by the ordinary complex, quaternionic and Cayley multiplication.

Dots and powers always relate to the standard product, and the base-point e will be the identity.

Consider the maps

$$M_{p,q}, Q_r: S^n \times S^n \to S^n \quad (n=1, 3, 7)$$

defined by

(8.1) $$M_{p,q}(x, y) = x^p \cdot y^q; \ Q_r(x, y) = (x^r \cdot y \cdot x^{-r}) \cdot y^{-1}.$$

Then $M_{p,q}$ is a map of type (p, q) and Q_r is of type $(0, 0)$.

In view of (7.8) we have:

(8.2) **Lemma**

Let $f: S^1 \times S^1 \to S^1$ be a map of type (p, q).
Then f is homotopic to $M_{p,q}$.

In the sequel of this section we assume that $n=3$ *or* 7.
Let $u, v: S^n \times S^n \to S^n$ be the maps given by

$$u(x, y) = x \cdot y \cdot x^{-1}, \quad v(x, y) = y,$$

then $\lambda_n = d(u, v)$ generates $\pi_{2n}(S^n)$ (see § 9 of [12]).

Furthermore we need the map $w: S^n \times S^n \to S^n$, given by $w(x, y) = y^{-1}$. Because of theorem (6.2) we have the relation:

$\lambda_n = d(u, v) = d(u \cdot w, v \cdot w) = d(Q_1, e)$, where e is the constant map. Applying now theorem (3.13), we obtain:

(8.3) $d(Q_r, e) = r\lambda_n$. So $d(e, Q_r) = -r\lambda_n$ by property (3.8).

(8.4) **Theorem**

Let $n=3$ or let $n=7$.

Let $f: S^n \times S^n \to S^n$ be a map of type (p, q).

Then there exists an integer r, such that f is homotopic to $M_{p,q} \cdot Q_r$.

PROOF: By the homotopy extension theorem there exists a map $f': S^n \times S^n \to S^n$ such that $f' \sim f$ and $f'|\Sigma = M_{p,q}|\Sigma$.

Since λ_n generates $\pi_{2n}(S^n)$, $d(f', M_{p,q}) = r\lambda_n$ for some suitable integer r. Now (8.3), theorem (6.2) and the fact that e is the identity enable us to write down:

$$d(f', M_{p,q} \cdot Q_r) = d(f' \cdot e, M_{p,q} \cdot Q_r) = d(f', M_{p,q}) + d(e, Q_r) = r\lambda_n - r\lambda_n = 0.$$

So $f' \sim M_{p,q} \cdot Q_r$ and thus $f \sim M_{p,q} \cdot Q_r$. Q.E.D.

Remark: Since $\pi_6(S^3) \equiv Z_{12}$ and $\pi_{14}(S^7) \equiv Z_{120}$ (see Chapter XIV of [14]), it follows easily from lemma (7.7) and from theorem (8.4):

For any pair of integers p, q there are exactly 12 homotopy classes of products of type (p, q) on S^3, and 120 of them on S^7. Representative products of these classes are $M_{p,q} \cdot Q_r$, where $0 \leqslant r \leqslant 11$ when $n=3$, and $0 \leqslant r \leqslant 119$ when $n=7$.

9. THE REFLECTING PRODUCT ON S^n

Euclidian $(n+1)$-space R^{n+1} is embedded in Hilbert space R in the usual way. On R^{n+1} we have the usual inner product, given by $(x; y) = \Sigma x_i y_i$, where $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1})$. Thus S^n is the subspace of R^{n+1} in which $(x; x) = 1$.

The *reflecting product* ν on S^n is now defined to be the map

$$\nu: S^n \times S^n \to S^n$$

given by

(9.1) $$\nu(x, y) = x \cdot y = -y + 2(x; y)x \quad x, y \in S^n.$$

If n is not specified to be 1, 3 or 7, then a dot always relates to this reflecting product.

By the notation $x^{\langle 1 \rangle} \cdot x^{\langle 2 \rangle} \cdot x^{\langle 3 \rangle} \cdot \ldots \cdot x^{\langle m \rangle}$ we shall always mean

$$x^{\langle 1 \rangle} \cdot (x^{\langle 2 \rangle} \cdot (x^{\langle 3 \rangle} \cdot \ldots \cdot x^{\langle m \rangle})).$$

It is straightforward from (9.1) to verify the following properties of the reflecting product:

(9.2) $$x \cdot x = x, \quad x \cdot x \cdot y = y, \quad (x \cdot y) \cdot z = x \cdot y \cdot x \cdot z \text{ for all } x, y, z \in S^n.$$

To avoid much writing we want to make clear some notations. Let $f(x, y, e)$ and $g(z, e)$ be explicit formulas in x, y, z, e.

Then by writing down $\curlyvee_{(x,y)}f(x, y, e)$ and $\curlyvee_z g(z, e)$ we denote the maps $F: S^n \times S^n \to S^n$ and $G: S^n \to S^n$ which are defined by $F(x, y) = f(x, y, e)$ and $G(z) = g(z, e)$ respectively.

Furthermore, instead of $x \cdot y$ we shall often use the notation $S_x y$, where S_x is conceived as an operator.

For example: $[S_x S_e]^3 y = x \cdot e \cdot x \cdot e \cdot x \cdot e \cdot y$ and $\nu = \curlyvee_{(x,y)} x \cdot y$.

(9.3) **Definition**

The set of (x, y, e)-*expressions* will be the smallest set B, satisfying the following conditions:

(i) $e, x, y \in B$,
(ii) for any two elements $a, b \in B$ we have $a \cdot b \in B$.

(9.4) **Definition**

A *compound reflecting product* (c.r.p.) α on S^n is a map $\alpha: S^n \times S^n \to S^n$, such that the explicit form of $\alpha(x, y)$ is a (x, y, e)-expression. $\curlyvee_{(x,y)}(x \cdot e) \cdot (x \cdot y) \cdot x$ and $\curlyvee_{(x,y)} x \cdot y \cdot e \cdot x \cdot e$, for example, are c.r.p.
In the sequel we shall assume that the explicit form of a c.r.p. is written down without brackets, in the sense of the notation above (9.2) This is always possible because of the third property of (9.2).

(9.5) **Definition**

The *length* of a c.r.p. is the minimum number of factors which are necessary to write down the explicit form as a (x, y, e)-expression without brackets.

(9.6) **Definition**

A *simple compound reflecting product* (s.c.r.p.) on S^n is a c.r.p. on S^n such that in the explicit form (without brackets) there occurs no x between any two occurrences of y, and no y between any two occurrences of x.

(9.7) **Theorem**

The reflecting product on S^n is of type $(2, -1)$ when n is odd, and it is of type $(0, 1)$ when n is even.

PROOF: It is not hard to verify that the map $\curlyvee_y e \cdot y$ has degree $(-1)^n$. Now consider the map $\curlyvee_x x \cdot e$. Notice that $x \cdot e = -e$ if $(x; e) = 0$. Next consider the maps $f, g: S^n \to S^n$, defined by:

$$f(x) = x \cdot e \text{ when } (x; e) \geqslant 0, \qquad g(x) = -e \text{ when } (x; e) \geqslant 0,$$

$$f(x) = -e \text{ when } (x; e) \leqslant 0, \qquad g(x) = x \cdot e \text{ when } (x; e) \leqslant 0.$$

It is easily verified that $\{f\} = \iota_n$ in $\pi_n(S^n)$, and that $g(x) = f(-x)$. Since $\curlyvee_x - x$ has degree $(-1)^{n+1}$, it follows that $\{g\} = (-1)^{n+1}\iota_n$.

Furthermore $\{\Upsilon_x x\cdot e\}=\{f\}+\{g\}$ in $\pi_n(S^n)$, so $\Upsilon_x x\cdot e$ has degree $1+(-1)^{n+1}$.
Q.E.D.

The map $\Upsilon_{(x,y)}y$ is of type (0, 1), so we may ask whether $\Upsilon_{(x,y)}x\cdot y$ is homotopic to $\Upsilon_{(x,y)}y$ (of course only in case n is even).

In order to answer this question, we need some results on rotation groups.

Let R_{n+1} be the rotation group of S^n. In § 6 of [15] G. W. Whitehead introduces the *canonical map* $C_n: S^n \to R_{n+1}$. C_n induces a map

$$C'_n: S^n \times S^n \to S^n$$

which is defined by

$$C'_n(x, y) = (C_n(x))(y).$$

From the definition of C_n it is easy to verify that

(9.8) the map C'_n is homotopic to the map $\Upsilon_{(x,y)}x\cdot e\cdot y$.

In § 9 of [16] G. W. Whitehead shows that $J\{C_n\}=\pm\,[\iota_{n+1}, \iota_{n+1}]$, where $J\{C_n\}=\{G(C'_n)\}$ by the definition of J in (5.12) of [16]. For clearness sake we notice that the map f, defined in (9.3) of [16], is just the canonical map.

Applying now (4.3), (4.6) and (9.8), we obtain:

(9.9) $c(\Upsilon_{(x,y)}x\cdot e\cdot y)=c(\Upsilon_{(x,y)}x\cdot(e\cdot y))=(-1)^n c(\Upsilon_{(x,y)}x\cdot y)=\pm\,[\iota_{n+1}, \iota_{n+1}]$.

If n is even, then $[\iota_{n+1}, \iota_{n+1}]=-[\iota_{n+1}, \iota_{n+1}]$ by (3.3) of [19].

If n is odd, it follows from the appendix that the Hopf invariants $Hc(\Upsilon_{(x,y)}x\cdot y)$ and $H[\iota_{n+1}, \iota_{n+1}]$ are equal.

Therefore the following theorem is a consequence of (9.9).

(9.10) **Theorem**

Let ι_{n+1} be the positive generator of $\pi_{n+1}(S^{n+1})$.
For the reflecting product on S^n applies:

$$c(\Upsilon_{(x,y)}x\cdot y)=[\iota_{n+1}, \iota_{n+1}] \text{ in } \pi_{2n+1}(S^{n+1}).$$

(9.11) **Corollary**

a. Let n be even, but $n\neq 2,6$. (We exclude the case $n=0$). Then the reflecting product $\Upsilon_{(x,y)}x\cdot y$ on S^n is *not* homotopic to the projection to the second factor $\Upsilon_{(x,y)}y$.
b. For $n=2$ and for $n=6$ the maps $\Upsilon_{(x,y)}x\cdot y$ and $\Upsilon_{(x,y)}y$ are homotopic.

Proof: In view of (9.7), (7.6) and (7.7), $\Upsilon_{(x,y)}x\cdot y$ is homotopic to $\Upsilon_{(x,y)}y$ (n even) if, and only if their Hopf suspensions agree. Since $c(\Upsilon_{(x,y)}y)=0$ by (4.7), our object is to determine when $c(\Upsilon_{(x,y)}x\cdot y)$ is zero.

In Theorem 1.1.1 of [1], J. F. Adams proves that $[\iota_{n+1}, \iota_{n+1}]=0$ if and only if $n=-1$, 0, 2 or 6. Because of this fact, the corollary follows from theorem (9.10). Q.E.D.

(9.12) Lemma

a. For all $x, y \in S^n$ applies:
$[S_xS_e]^k S_x\, y = ([S_xS_e]^{\frac{1}{2}(k+1)}\, e)\cdot y$ if k is odd, $k \geqslant 0$,
$= ([S_xS_e]^{\frac{1}{2}k}\, x)\cdot y$ if k is even, $k \geqslant 0$.

b. If n *is odd*, then for maps $S^n \to S^n$ applies:
$\Upsilon_x[S_xS_e]^j e$ has degree $2j$ $(j \geqslant 0)$,
$\Upsilon_x[S_xS_e]^j x$ has degree $2j+1$ $(j \geqslant 0)$.

c. If n *is even*, then the maps under (b) have degree 0, 1 respectively.

PROOF:

a. We only discuss the case k is odd. In case k is even, the proof is fully analogous. The assertion is proved by induction on k. If $k=1$, we have to prove that $S_xS_eS_x\, y = (S_xS_e\, e)\cdot y$. This is immediate from (9.2), for

$$(S_xS_e\, e)\cdot y = (S_x\, e)\cdot y = (x\cdot e)\cdot y = x\cdot e\cdot x\cdot y = S_xS_eS_x\, y.$$

Assume now that the assertion is valid for some odd k, then we have by (9.2):

$$[S_xS_e]^{k+2} S_x y = S_xS_e[S_xS_e]^k S_xS_eS_x y = x\cdot e\cdot [S_xS_e]^k S_x(e\cdot x\cdot y) =$$
$$= x\cdot e\cdot ([S_xS_e]^{\frac{1}{2}(k+1)}\, e)\cdot e\cdot x\cdot y = x\cdot (e\cdot [S_xS_e]^{\frac{1}{2}(k+1)}\, e)\cdot x\cdot y =$$
$$= (x\cdot e\cdot [S_xS_e]^{\frac{1}{2}(k+1)}\, e)\cdot y = ([S_xS_e]^{\frac{1}{2}(k+3)}\, e)\cdot y,$$

which proves the induction.

b.

c. We only prove the first assertion of (b), for the rest is proved in a similar way. We use induction on j.
If $j=0$, then there is nothing to prove.
Suppose now that $\Upsilon_x[S_xS_e]^j e$ has degree $2j$ for some j, so its homotopy class $\{\Upsilon_x[S_xS_e]^j e\}$ equals $(2j)\iota_n$ in $\pi_n(S^n)$. Then we obtain by (6.1), (9.2) and (9.7):

$$\{\Upsilon_x[S_xS_e]^{j+1}\, e\} = \{\Upsilon_x S_xS_e[S_xS_e]^j\, e\} = \{\Upsilon_x\, x\cdot e\cdot [S_xS_e]^j\, e\} =$$
$$= \{(\Upsilon_x\, x)\cdot (\Upsilon_x\, e\cdot [S_xS_e]^j\, e)\} = \{\Upsilon_x\, x\cdot e\} + \{\Upsilon_x\, e\cdot e\cdot [S_xS_e]^j\, e\} =$$
$$= \{\Upsilon_x\, x\cdot e\} + \{\Upsilon_x[S_xS_e]^j\, e\} = 2\iota_n + (2j)\iota_n = 2(j+1)\iota_n \quad (n \text{ odd}),$$

which proves the induction. Q.E.D.

Let n be odd, and let $v_n = [\eta_n, \iota_n]$ be the Whitehead product between generators η_n, ι_n of $\pi_{n+1}(S^n)$, $\pi_n(S^n)$ respectively.

We remark that $2v_n = 0$ if n is odd, for $\pi_2(S^1) = 0$ and $\pi_{n+1}(S^n) \cong Z_2$ when $n \geqslant 3$.

(9.13) Lemma

Consider the reflecting product on S^n, *where n is odd.*

Then in $\pi_{2n}(S^n)$ (n is odd) the following relations are valid:

(1) $d(\curlyvee_{(x,y)} x\cdot e\cdot y\cdot e, \curlyvee_{(x,y)} y\cdot e\cdot x\cdot e)=v_n$.

(2) $d(\curlyvee_{(x,y)} x\cdot y\cdot e, \curlyvee_{(x,y)} e\cdot y\cdot x\cdot e)=v_n$.

(3) $d(\curlyvee_{(x,y)} U^k V^l e, \curlyvee_{(x,y)} V^l U^k e)=klv_n$,
where $U=S_xS_e$, $V=S_yS_e$, and k, l are integers ($k, l \geqslant 0$).

(4) $d(\curlyvee_{(x,y)} U^k S_e V^l e, \curlyvee_{(x,y)} S_e V^l S_e U^k e)=klv_n$,
where the notations are the same as above.

Proof: Item (1) is proved by James in § 6 of [11].

(2) follows from (1) by replacing y by $(e\cdot y)$, for because of (3.12), (3.13) and because of the fact that the map $\curlyvee_y e\cdot y$ has degree -1 (n is odd), we obtain from (1):

$$(*) \qquad d(\curlyvee_{(x,y)} x\cdot e\cdot (e\cdot y)\cdot e, \curlyvee_{(x,y)} (e\cdot y)\cdot e\cdot x\cdot e)=-v_n.$$

But $-v_n=v_n$ (n is odd), $x\cdot e\cdot (e\cdot y)\cdot e=x\cdot y\cdot e$ and $(e\cdot y)\cdot e\cdot x\cdot e=e\cdot y\cdot x\cdot e$, hence (2) is immediate from (*).

Concerning the items (3) and (4), we prove only (3) in case k is odd and l is even. In the latter case there exist integers i, j such that $k=2i+1$ and $l=2j$.

In item (1) we replace x by $U^i x$ and y by $V^j e$.

Then we obtain from (1):

$$(**) \qquad d(\curlyvee_{(x,y)} (U^i x)\cdot e\cdot (V^j e)\cdot e, \curlyvee_{(x,y)} (V^j e)\cdot e\cdot (U^i x)\cdot e)=2j(2i+1)v_n.$$

But by (9.12) (a) we have

$$(U^i x)\cdot e\cdot (V^j e)\cdot e=U^{2i}S_xS_eV^{2j-1}S_ye=U^{2i+1}V^{2j}e,$$

and in the same manner $(V^j e)\cdot e\cdot (U^i x)\cdot e=V^{2j}U^{2i+1}e$.

Thus it follows from (**) that item (3) is valid when k is odd and l is even. In an analogous way one proves (3) for all other possible choices for k and l. In exactly the same manner one proves that (4) follows from (2).

Q.E.D.

Remark: The type of a c.r.p. on S^n (see 9.4)) is easily determined with the help of lemma (9.12) (b) (c) and the consideration that if $f: S^n \to S^n$ has degree p, then $e\cdot f: S^n \to S^n$ has degree p when n is even, and it has degree $-p$ when n is odd.

(9.14) **Theorem**

Let n be odd.

Let α be a c.r.p. on S^n of type (p, q), where p, q are *both even.* Then there exists a s.c.r.p. β on S^n, such that $d(\alpha, \beta)=v_n$ or 0.

PROOF: We prove the theorem by induction on the length $l(\alpha)$ of α. Notice that p, q are both even, if and only if in the explicit form (without brackets) of α the last factor is e. This follows easily from (9.12) (b) (c).

If $l(\alpha) \leqslant 3$, then there is nothing to prove.

Suppose now that the theorem is true for any c.r.p. of length 1, 2, ... up to and including L. Let α be a c.r.p. of length $L+1$. So there exists an explicit form (without brackets) for $\alpha(x, y)$ with $L+1$ factors. Then there are 3 possibilities:

$$\text{(i)}: \ \alpha = e \cdot \alpha', \qquad \text{(ii)}: \ \alpha = (\curlyvee_{(x,y)} x) \cdot \alpha', \qquad \text{(iii)}: \ \alpha = (\curlyvee_{(x,y)} y) \cdot \alpha',$$

wherein α' is a c.r.p. of type (p', q'), p' and q' both even, and of length $l(\alpha') \leqslant L$.

So there is a s.c.r.p. β' such that $d(\alpha', \beta') = v_n$ or 0.

For the explicit form of $\beta'(x, y)$ there are 8 possibilities:

(1) $\beta'(x, y) = [S_x S_e]^i \, [S_y S_e]^j \, e$

(2) $\beta'(x, y) = S_e [S_x S_e]^i \, [S_y S_e]^j \, e$

(3) $\beta'(x, y) = [S_x S_e]^i \, S_e [S_y S_e]^j \, e$

(4) $\beta'(x, y) = S_e [S_x S_e]^i \, S_e [S_y S_e]^j \, e$

(5, 6, 7, 8) The above formulas with x and y interchanged.

As an example, we discuss only the combination (iii), (2).

So $\alpha(x, y) = y \cdot \alpha'(x, y)$, $\beta'(x, y) = S_e [S_x S_e]^i [S_y S_e]^j \, e$ and $d(\alpha', \beta') = v_n$ or 0.

Define the map $\beta'' : S^n \times S^n \to S^n$ by $\beta''(x, y) = S_e [S_y S_e]^j [S_x S_e]^i \, e$. By (6.3) and (9.13) (3) we have:

$$\begin{aligned} d(\beta', \beta'') &= d(\curlyvee_{(x,y)} S_e U^i V^j e, \curlyvee_{(x,y)} S_e V^j U^i e) = \\ &= d(\curlyvee_{(x,y)} e \cdot U^i V^j e, \curlyvee_{(x,y)} e \cdot V^j U^i e) = \\ &= -d(\curlyvee_{(x,y)} U^i V^j e, \curlyvee_{(x,y)} V^j U^i e) = v_n \text{ or } 0, \text{ for } -v_n = v_n \ (n \text{ is odd}). \end{aligned}$$

Furthermore $d(\alpha', \beta') = v_n$ or 0, so $d(\alpha', \beta'') = v_n$ or 0 by (3.8).

Now define the map $\beta : S^n \times S^n \to S^n$ by $\beta(x, y) = y \cdot \beta''(x, y)$; it is clear that β is a s.c.r.p.

Moreover, using (6.3) again, we obtain:

$$d(\alpha, \beta) = d((\curlyvee_{(x,y)} y) \cdot \alpha', (\curlyvee_{(x,y)} y) \cdot \beta'') = -d(\alpha', \beta'') = v_n \text{ or } 0. \qquad \text{Q.E.D.}$$

From (4.4), (4.11) of [6] it follows immediately that the following conclusions are valid:

(9.15) $$v_1 = 0, \quad v_{4k-1} = 0, \quad v_{4k+1} \neq 0 \quad (k \geqslant 1).$$

Therefore, in view of (3.7), theorem (9.14) may be specialized to

(9.16) **Corollary**

Let α be a c.r.p. on S^n of type (p, q) where p, q are *both even.* If $n=1$ or if $n=4k-1$, then there exists a s.c.r.p. β on S^n, such that the maps $\alpha, \beta: S^n \times S^n \to S^n$ are homotopic.

Remark: Corollary (9.16) can not be extended to the case $n=4k+1$ $(k \geqslant 1)$. We construct a counter-example.

Consider the c.r.p. $\alpha: S^{4k+1} \times S^{4k+1} \to S^{4k+1}$, given by $\alpha(x, y)=y \cdot x \cdot y \cdot e$. α is of type $(-2, 4)$, so the only s.c.r.p. that may be homotopic to α are $\beta=\Upsilon_{(x,y)} y \cdot e \cdot y \cdot x \cdot e$ and $\gamma=\Upsilon_{(x,y)} e \cdot x \cdot y \cdot e \cdot y \cdot e$. However $\beta(x, y)=(y \cdot e) \cdot x \cdot e$ and $\gamma(x, y)=e \cdot x \cdot (y \cdot e) \cdot e$. Therefore it follows from (9.13) (2) that $d(\beta, \gamma)=2v_n=0$, so $\beta \sim \gamma$. On the other hand we have by (9.13) (2):

$$d(\alpha, \beta)=d((\Upsilon_{(x,y)} y) \cdot (\Upsilon_{(x,y)} x \cdot y \cdot e), (\Upsilon_{(x,y)} y) \cdot (\Upsilon_{(x,y)} e \cdot y \cdot x \cdot e))=$$
$$=-d(\Upsilon_{(x,y)} x \cdot y \cdot e, \Upsilon_{(x,y)} e \cdot y \cdot x \cdot e)=-v_n=v_n \neq 0.$$

Hence, by (7.5), α and β are not homotopic.

Remark: Let n be even. Then a c.r.p. on S^n of type (p, q), where p, q are both even, is necessarily of type $(0, 0)$. This follows from (9.12) (c).

(9.17) **Theorem**

Let $\alpha: S^n \times S^n \to S^n$ be any c.r.p. of type $(0, 0)$. α is homotopic to the constant map $e: S^n \times S^n \to S^n$ if $n=1$, $n=4k-1$ and if n is even.

PROOF:

a. $n=1$ or $n=4k-1$.
According to corollary (9.16), there is a s.c.r.p. β on S^n such that $\alpha \sim \beta$, hence β is also of type $(0, 0)$. However a s.c.r.p. of type $(0, 0)$ equals the constant map e when n is odd, as is easily verified from (9.12) (b). Thus $\beta=e$, and hence $\alpha \sim e$.

b. n is even.
Suppose we have an explicit form without brackets for α, consisting of m factors, the last factor being e because of (9.12) (c). Let $m>1$. As $\Upsilon_{(x,y)} x \cdot e$, $\Upsilon_{(x,y)} y \cdot e$ and $\Upsilon_{(x,y)} e \cdot e$ are homotopic to $\Upsilon_{(x,y)} e$ (n is even), α is homotopic to a c.r.p. β which consists of $m-1$ factors, the last one being e. Iterating this process, we deduce that α is homotopic to the constant map. Q.E.D.

Remark: Also theorem (9.17) can not be extended to the case $n=4k+1$ $(k \geqslant 1)$. A counter-example is $\Upsilon_{(x,y)} y \cdot x \cdot e \cdot y \cdot x \cdot e$. In view of (9.13) (2) and (6.3) we have:

$$d(\Upsilon_{(x,y)} y \cdot x \cdot e \cdot y \cdot x \cdot e, \Upsilon_{(x,y)} y \cdot x \cdot x \cdot y \cdot e)=v_{4k+1}.$$

Hence $d(\Upsilon_{(x,y)}y\cdot x\cdot e\cdot y\cdot x\cdot e,\ \Upsilon_{(x,y)}e)=v_{4k+1}$, since $y\cdot x\cdot x\cdot y\cdot e=e$. Because of the non-triviality of v_{4k+1}, it follows from (7.5) that $\Upsilon_{(x,y)}y\cdot x\cdot e\cdot y\cdot x\cdot e$ is not homotopic to the constant map.

(9.18) **Lemma**

Let n be *even*. Then for maps $S^n\times S^n\to S^n$ we state the following properties:

(1) Let $f\colon S^n\times S^n\to S^n$ be an abitrary map.
Then the map $e\cdot f$ is homotopic to f.

(2) $\Upsilon_{(x,y)}y\cdot x\cdot y$ is homotopic to $\Upsilon_{(x,y)}x\cdot y$.
$\Upsilon_{(x,y)}x\cdot y\cdot x$ is homotopic to $\Upsilon_{(x,y)}y\cdot x$.

Proof:

(1) If n is even, then the reflecting product ν on S^n is of type (0, 1). Hence the map $\nu''\colon S^n\to S^n$, given by $\nu''(z)=e\cdot z$, has degree 1. Therefore ν'' is homotopic to the identity map on S^n. Statement (1) is now an easy consequence of the fact that the map $e\cdot f$ is just the composite $S^n\times S^n\xrightarrow{f}S^n\xrightarrow{\nu''}S^n$.

(2) $\alpha=\Upsilon_{(x,y)}y\cdot x\cdot y$ and $\beta=\Upsilon_{(x,y)}x\cdot y$ are both of type (0, 1), so, by (7.6) and (7.7), they are homotopic if and only if their Hopf suspensions are equal.
Consider the map $pr_2=\Upsilon_{(x,y)}y$. There exists a map $\gamma\colon S^n\times S^n\to S^n$ such that $\gamma\sim pr_2$ and $\gamma|\Sigma=\beta|\Sigma$. Applying (6.2) ($p=0$, $q=1$), we obtain: $d(\beta,\gamma)=d(pr_2\cdot\alpha,\ pr_2\cdot pr_2\cdot\gamma)=d(\alpha,\ pr_2\cdot\gamma)$.
Because of (4.4) the relation above leads to the equation $c(\gamma)-c(\beta)=$ $=c(pr_2\cdot\gamma)-c(\alpha)$. However $c(\gamma)=c(pr_2\cdot\gamma)=0$, since $\gamma\sim pr_2$ (see (4.7)). Hence $c(\alpha)=c(\beta)$, which proves the first part of statement (2). The second part of (2) is now immediate. Q.E.D.

Let α be a c.r.p. on S^n, where n is even. Consider an explicit form for α. Using (9.17) and applying repeatedly (9.2) and (9.18), we obtain:

(9.19) **Corollary** (see also corollary (9.11))

Let α be a c.r.p. on S^n, where n is *even*.
Then α is homotopic to at least one of the following c.r.p.:

$$\Upsilon_{(x,y)}e,\ \Upsilon_{(x,y)}y,\ \Upsilon_{(x,y)}x\cdot y,\ \Upsilon_{(x,y)}x,\ \Upsilon_{(x,y)}y\cdot x.$$

Let Ξ be a subset of $\pi_{2n}(S^n)$ such that $\pi_{2n}(S^n)$ is generated by the elements of Ξ.

Because of (3.9) for any $\xi\in\Xi$ there exists a map $f_\xi\colon S^n\times S^n\to S^n$ such that $d(e,f_\xi)=\xi$.

(9.20) **Theorem**

Let n be *odd*, but suppose that $n \neq 1, 3, 7$.

Let $g: S^n \times S^n \to S^n$ be a map of type (p, q).

Then the map $g \cdot e$ is homotopic to a map $N \cdot T$, where N is a s.c.r.p. on S^n of type (p, q), and $T: S^n \times S^n \to S^n$ is a composition of reflecting products wherein the elementary factors are only e and f_ξ $(\xi \in \Xi)$.

Instead of proving this theorem, we shall demonstrate an example. In the first place we remark that the product pq must be even; this follows from the theory of the Hopf invariant.

Suppose now that g is of type $(2, 3)$.

Let N be defined by $N(x, y) = x \cdot e \cdot y \cdot e \cdot y$ $(x, y \in S^n)$.

Next we choose a map $g': S^n \times S^n \to S^n$ such that $g' \sim g$ and $g'|\Sigma = N|\Sigma$.

Suppose that $d(g', N) = \xi_1 - 3\xi_2 + 2\xi_3$ $(\xi_1, \xi_2, \xi_3 \in \Xi)$.

Put $T = f_{\xi_1} \cdot e \cdot (e \cdot f_{\xi_2}) \cdot e \cdot (e \cdot f_{\xi_2}) \cdot e \cdot (e \cdot f_{\xi_2}) \cdot e \cdot f_{\xi_3} \cdot e \cdot f_{\xi_3} \cdot e$. Applying (6.3) repeatedly, we obtain: $d(e, T) = 2\xi_1 - 6\xi_2 + 4\xi_3$.

By (6.3) we have now: $d(g' \cdot e, N \cdot T) = 2d(g', N) - d(e, T) = 0$. Then it follows from (3.7) that $g' \cdot e \sim N \cdot T$, so $g \cdot e \sim N \cdot T$.

In the following theorem we gather some results.

(9.21) **Theorem**

(i) *Let n be odd and let p, q be both even.*
Then for every c.r.p. α on S^n of type (p, q) there exists a s.c.r.p. β on S^n such that $d(\alpha, \beta) = v_n$ or 0 (see (9.14)).

(ii) *Assume that $n = 1$, $n = 4k - 1$ or n is even. Furthermore let p, q be both even.*
Then all c.r.p. on S^n of the same type (p, q) are homotopic.

(iii) *Let n be even, but $n \neq 2$ and $n \neq 6$.*
Then $\curlyvee_{(x,y)} x \cdot y$ is not homotopic to $\curlyvee_{(x,y)} y$ (see (9.11)).

(iv) *Let $n = 2$ or $n = 6$.*
Then $\curlyvee_{(x,y)} x \cdot y$ is homotopic to $\curlyvee_{(x,y)} y$ (see (9.11)).

(v) *Let n be even.*
Then a c.r.p. on S^n of type $(0, 1)$ is homotopic to $\curlyvee_{(x,y)} x \cdot y$ or to $\curlyvee_{(x,y)} y$; a c.r.p. on S^n of type $(1, 0)$ is homotopic to $\curlyvee_{(x,y)} y \cdot x$ or to $\curlyvee_{(x,y)} x$ (see (9.19)).

Proof: We have to prove only item (ii).

Suppose that the conditions in item (ii) are fulfilled.

Let α and β be c.r.p. on S^n of the same type (p, q).

Let $\alpha(x, y) = z_1 \cdot z_2 \ldots z_m \cdot e$ be an explicit form without brackets, wherein each z_i stands for x, e or y.

Then the maps $e = \curlyvee_{(x,y)} z_m \cdot z_{m-1} \ldots z_2 \cdot z_1 \cdot \alpha(x, y)$ and

$$\gamma = \curlyvee_{(x,y)} z_m \cdot z_{m-1} \ldots z_2 \cdot z_1 \cdot \beta(x, y)$$

are both c.r.p. of type $(0, 0)$. Therefore $\gamma \sim e$ by (9.17). Hence the maps $\alpha = \curlyvee_{(x,y)} z_1 \cdot z_2 \ldots z_m \cdot e$ and $\beta = \curlyvee_{(x,y)} z_1 \cdot z_2 \ldots z_m \cdot \gamma(x, y)$ are homotopic. Q.E.D.

10. Some remarks on maps $S^n \times S^n \times S^n \to S^n$.

Given a map $f: S^n \times S^n \times S^n \to S^n$, we may ask whether there exist maps $g, h: S^n \times S^n \to S^n$ such that f is homotopic to $g \square h$, where the map

$$g \square h: S^n \times S^n \times S^n \to S^n$$

is defined by

$$(g \square h)(x, y, z) = g(x, h(y, z)). \tag{10.1}$$

In this section we only give a summary of necessary conditions in order that f is homotopic to $g \square h$.

I don't know if these conditions are sufficient.

For a map $f: S^n \times S^n \times S^n \to S^n$ we consider the sections $f_1, f_2, f_3: S^n \to S^n$ and $f_{12}, f_{13}, f_{23}: S^n \times S^n \to S^n$ which are defined by:

$$\begin{array}{lll} f_1(x) = f(x, e, e) & f_{12}(x, y) = f(x, y, e) & \\ f_2(x) = f(e, x, e) & f_{13}(x, y) = f(x, e, y) & \\ f_3(x) = f(e, e, x) & f_{23}(x, y) = f(e, x, y) & x, y \in S^n. \end{array} \tag{10.2}$$

The type of the map f will be the triple (p_1, p_2, p_3), where p_i is the degree of f_i $(i = 1, 2, 3)$.

(10.3) **Lemma**

Let $f: S^n \times S^n \times S^n \to S^n$ be a map of type (p_1, p_2, p_3).
Let $g, h: S^n \times S^n \to S^n$ be maps of type (q_1, q_2), (r_1, r_2) respectively.
If $f \sim (g \square h)$, then $p_1 = q_1$, $p_2 = q_2 r_1$, and $p_3 = q_2 r_2$.

Remark: Suppose that $n \neq 1, 3, 7$ and let f be of type $(1, 2, 2)$. Then it follows from (10.3) that there exists no pair of maps g, h such that $f \sim (g \square h)$. For let $f \sim (g \square h)$, then either g is of type $(1, \pm 1)$ and h is of type $(\pm 2, \pm 2)$, or g is of type $(1, \pm 2)$ and h is of type $(\pm 1, \pm 1)$.

However, by the theory of the Hopf invariant, there exists no map $S^n \times S^n \to S^n$ of type $(\pm 1, \pm 1)$ unless $n = 1$, 3 or 7.

If $f \sim (g \square h)$, then the following conditions must be fulfilled:

$$f_{12} \sim \curlyvee_{(x,y)} g(x, h(y, e)), \quad f_{13} \sim \curlyvee_{(x,y)} g(x, h(e, y))$$

and

$$f_{23} \sim \curlyvee_{(x,y)} g(e, h(x, y)).$$

Applying now (4.5), (4.6) and (4.8), we obtain:

(10.4) **Lemma**

Let $f: S^n \times S^n \times S^n \to S^n$ be a given map.
Let $g, h: S^n \times S^n \to S^n$ be of type (q_1, q_2), (r_1, r_2) respectively.
If $f \sim (g \square h)$, then the following equations are valid in $\pi_{2n+1}(S^{n+1})$: $c(f_{12}) = r_1 c(g)$, $c(f_{13}) = r_2 c(g)$, $c(f_{23}) = (q_2 \iota_{n+1}) \circ c(h)$, where ι_{n+1} is the positive generator of $\pi_{n+1}(S^{n+1})$.

For a map $f: S^n \times S^n \times S^n \to S^n$ we consider a special kind of Hopf suspension, namely the map

$$G_{23}(f): S^n \times \dot{I}^{2n+2} \to S^{n+1}$$

which is defined by

$$(10.5) \qquad G_{23}(f)(x, (y, z, t)) = d_n(f(x, y, z), t),$$

where $x, y, z \in S^n$ and $(y, z, t) \in \dot{I}^{2n+2}$ as in (4.1), (4.2).

(10.6) **Lemma**

For maps $g, h: S^n \times S^n \to S^n$ we have: $G_{23}(g \square h) \sim (E'g) \bigcirc (Gh)$, where $E'g: S^n \times S^{n+1} \to S^{n+1}$ is a right suspension of g, and where the map $(E'g) \bigcirc (Gh): S^n \times \dot{I}^{2n+2} \to S^{n+1}$ is defined by $[(E'g) \bigcirc (Gh)](x, u) = E'g(x, Gh(u))$ $(x \in S^n,\ u \in \dot{I}^{2n+2})$.

Proof: According to (10.1) and (10.5) we have:

$$[G_{23}(g \square h)](x, (y, z, t)) = d_n((g \square h)(x, y, z), t) = d_n(g(x, h(y, z)), t),$$

while

$$[(E'g) \bigcirc (Gh)](x, (y, z, t)) = E'g(x, d_n(h(y, z), t)).$$

Notice that these two maps are the same when $t = 0$, and that both of them map the subspaces of $S^n \times \dot{I}^{2n+2}$ in which $t \geqslant 0$, $t \leqslant 0$ into E_+^{n+1}, E_-^{n+1} respectively (see § 1 and § 5).

Therefore $G_{23}(g \square h)$ and $(E'g) \bigcirc (Gh)$ are homotopic. Q.E.D.

Since the map $(E'g) \bigcirc (Gh)$ is just the composite

$$S^n \times \dot{I}^{2n+2} \xrightarrow{i \times Gh} S^n \times S^{n+1} \xrightarrow{E'g} S^{n+1},$$

where $i: S^n \to S^n$ is the identity map, we obtain from (4.3), (4.6) and (5.3):

$$(10.7) \qquad c((E'g) \bigcirc (Gh)) = c(E'g) \circ ((-1)^n E^{n+1}\{Gh\}) =$$
$$= Ec(g) \circ (-E^{n+1}c(h)) = -E(c(g) \circ E^n c(h)).$$

From (10.6) and (10.7) we deduce the following lemma:

(10.8) **Lemma**

Let $f: S^n \times S^n \times S^n \to S^n$ and $g, h: S^n \times S^n \to S^n$ be maps. If $f \sim (g \square h)$, then we have in $\pi_{3n+2}(S^{n+2})$:

$$c(G_{23}f) = -E(c(g) \circ E^n c(h)).$$

11. Appendix

On the Hopf invariant

Let P be an oriented simplicial complex.

Let $\mathfrak{d}$ be the duality operator with respect to the orientation of P, and let $\mathfrak{D}$ denote the operator $\mathfrak{o}\mathfrak{d}$, such as defined in [4].

Let Q be another simplicial complex, and let

$$f: P \to Q$$

be a simplicial map. For each integral i-cochain u^i of Q and for each integral i-chain c_i of P we have the important relation:

$$(u^i f)c_i = u^i(fc_i),$$

where $u^i f$ denotes the image of u^i under the cochain map defined by f.

Assume that there are given in $P = S^r$ two cycles c_i and d_j, such that $i + j = r - 1$, $i \neq 0$, $j \neq 0$.

Then there is defined a *linking number* $v(c_i, d_j)$ in the following way:

Let $c_i = \partial C_{i+1}$ for some chain C_{i+1} and let $d_j = k^{i+1}\mathfrak{d}$ for some cochain k^{i+1}. We put:

$$v(c_i, d_j) = k^{i+1} C_{i+1}.$$

Notice that $v(c_i, d_j)$ is just the *intersection number* $\emptyset(C_{i+1}, d_j)$ as defined in chapter XI of AH.

Instead of S^{2n-1} we consider $\dot{I}^{2n} = \dot{I}^n \times I^n + (-1)^n I^n \times \dot{I}^n$.

In $\dot{I}^{2n}$ we compute the linking numbers

$$v(\dot{I}^n \times a, b \times \dot{I}^n) \text{ and } v(b \times \dot{I}^n, \dot{I}^n \times a),$$

where a, b are fixed interior points of I^n.

For a point $x \in I^n$, let k_{ex} denote a 1-chain of I^n, such that $\partial\, k_{ex} = [x] - [e]$.

It is not hard to see that the following results are valid:

$$(11.1) \quad v(\dot{I}^n \times a, b \times \dot{I}^n) = \emptyset((-1)^{n-1}\, \dot{I}^n \times k_{ea} + I^n \times e, b \times \dot{I}^n) = \\ = \emptyset(I^n \times e, b \times \dot{I}^n) = (-1)^n.$$

$$(11.2) \quad v(b \times \dot{I}^n, \dot{I}^n \times a) = \emptyset(k_{eb} \times \dot{I}^n + e \times I^n, \dot{I}^n \times a) = \emptyset(e \times I^n, \dot{I}^n \times a) = +1.$$

Let $g: \dot{I}^{2n} \to S^n$ be a map which is simplicial on subdivisions of $\dot{I}^{2n}$ and S^n.

Let t_n and s_n be two *disjoint* simplices of S^n and suppose that their orientations are coherent with the orientation of S^n. Furthermore, let t^n and s^n be the integral n-cocycles of S^n which are defined by the relations

$$t^n t_n = 1, \quad t^n t_n' = 0 \text{ if } t_n' \neq \pm t_n,$$

$$s^n s_n = 1, \quad s^n s_n' = 0 \text{ if } s_n' \neq \pm s_n.$$

Then the *Hopf invariant* $H(g)$ of the map g can be defined by

$$H(g) = v((t^n g)\mathfrak{D}, (s^n g)\mathfrak{d}) \qquad (n \geqslant 2).$$

We want to compute the Hopf invariant $H(g)$ only for the following special maps g:

(i) $g = Gh$, the Hopf suspension of a map $h: S^{n-1} \times S^{n-1} \to S^{n-1}$. Assume that h is of type (p, q), that is to say the restricted maps $S^{n-1} \times e \to S^{n-1}$ and $e \times S^{n-1} \to S^{n-1}$ have degree p and q respectively.

(ii) g is a representative map for the Whitehead product $[\iota_n, \iota_n]$, where ι_n is the positive generator of $\pi_n(S^n)$.

Ad (i). *$g = Gh$, $h: S^{n-1} \times S^{n-1} \to S^{n-1}$ of type (p, q), $n \geqslant 2$.*

Take the simplex t_n in E^n_+ and s_n in E^n_-. Then $(t^n g)\mathfrak{d}$ is a $(n-1)$-cycle of $I^n \times \dot{I}^n$. So there exists an integer p' such that $(t^n g)\mathfrak{d}$ is homologous to $p'(b \times \dot{I}^n)$. To determine p', we form

$$v(\dot{I}^n \times a, (t^n g)\mathfrak{d}) = v(\dot{I}^n \times a, p'(b \times \dot{I}^n)) = (-1)^n p'.$$

On the other hand we have by definition

$$v(\dot{I}^n \times a, (t^n g)\mathfrak{d}) = (t^n g)((-1)^{n-1} \dot{I}^n \times k_{ea} + I^n \times e).$$

$(t^n g)((-1)^{n-1} \dot{I}^n \times k_{ea}) = 0$, since $g(\dot{I}^n \times I^n) \subset E^n_-$.

Therefore we obtain: $v(\dot{I}^n \times a, (t^n g)\mathfrak{d}) = (t^n g)(I^n \times e)$.

The restricted map $g: \dot{I}^n \times e \to S^{n-1}$ has degree p, so the restricted map $g: (I^n \times e, \dot{I}^n \times e) \to (E^n_+, S^{n-1})$ has degree $(-1)^n p$.

Because of this fact we obtain: $(t^n g)(I^n \times e) = (-1)^n p$. Hence $p' = p$.

Thus $(t^n g)\mathfrak{d}$ is homologous to $p(b \times \dot{I}^n)$.

By analogous considerations, we obtain that $(s^n g)\mathfrak{d}$ is a $(n-1)$-cycle of $\dot{I}^n \times I^n$ which is homologous to $(-1)^{n+1}\, q(\dot{I}^n \times a)$.

By the foregoing results we can write down:

$$H(g) = v((t^n g)\mathfrak{D}, (s^n g)\mathfrak{d}) = v(p(b \times \dot{I}^n), (-1)^{n+1} q(\dot{I}^n \times a)) = (-1)^{n+1} pq.$$

(11.3) **Lemma**

Let $h: S^{n-1} \times S^{n-1} \to S^{n-1}$ be a map of type (p, q), and let $Gh: \dot{I}^{2n} \to S^n$ be the Hopf suspension of h.

Then $H(Gh) = (-1)^{n+1} pq$. $(n \geqslant 2)$

Ad (ii). *g is a representative map for $[\iota_n, \iota_n]$, $n \geqslant 2$.*

We may choose the map $g: (\dot{I}^n \times I^n + (-1)^n\ I^n \times \dot{I}^n) \to S^n$ such that

$$g(x, y) = \Phi_n(y) \text{ if } x \in \dot{I}^n,\ y \in I^n,$$

$$g(x, y) = \Phi_n(x) \text{ if } x \in I^n,\ y \in \dot{I}^n.$$

Let t_n and s_n be chosen as before. Then it is clear that $(t^n g)\mathfrak{d} = c_{n-1} + d_{n-1}$, where c_{n-1}, d_{n-1} are $(n-1)$-cycles of $I^n \times \dot{I}^n$, $\dot{I}^n \times I^n$ respectively.

So there exist integers m, m' such that c_{n-1} is homologous to $m(b \times \dot{I}^n)$ and d_{n-1} is homologous to $m'(\dot{I}^n \times a)$. To determine m, we form

$$v(\dot{I}^n \times a', (t^n g)\mathfrak{d}) = v(\dot{I}^n \times a',\ m(b \times \dot{I}^n) + m'(\dot{I}^n \times a)) =$$
$$= mv(\dot{I}^n \times a', b \times \dot{I}^n) = (-1)^n m,$$

where we have chosen a suitable $a' \in I^n$.

On the other hand, we have:

$$v(\dot{I}^n \times a', (t^n g)\mathfrak{d}) = v((-1)^{n-1}\dot{I}^n \times k_{ea'} + I^n \times e,\ (t^n g)\mathfrak{d}) = (t^n g)(I^n \times e) = +1,$$

since $g(\dot{I}^n \times k_{ea'})$ does not contain t_n and since the restricted map

$$g\colon (I^n \times e, \dot{I}^n \times e) \to (S^n, e)$$

has degree $+1$. Therefore $m = (-1)^n$, hence c_{n-1} is homologous to $(-1)^n(b \times \dot{I}^n)$. By similar calculations we obtain that $m' = +1$, so d_{n-1} is homologous to $\dot{I}^n \times a$.

In the same way we deduce that $(s^n g)\mathfrak{d} = c'_{n-1} + d'_{n-1}$, where c'_{n-1}, d'_{n-1} are $(n-1)$-cycles of $I^n \times \dot{I}^n$, $\dot{I}^n \times I^n$ respectively.

It turns out that c'_{n-1} is homologous to $(-1)^n(b' \times \dot{I}^n)$, and that d'_{n-1} is homologous to $\dot{I}^n \times a'$, where $a', b' \in I^n$.

By suitable choices for a' and b', we obtain:

$$v((t^n g)\mathfrak{D}, (s^n g)\mathfrak{d}) = v((-1)^n(b \times \dot{I}^n) + \dot{I}^n \times a,\ (-1)^n(b' \times \dot{I}^n) + \dot{I}^n \times a') =$$
$$= (-1)^n v(b \times \dot{I}^n, \dot{I}^n \times a') + (-1)^n v(\dot{I}^n \times a, b' \times \dot{I}^n) = (-1)^n + 1.$$

(11.4) **Lemma**

$$H([\iota_n, \iota_n]) = (-1)^n + 1. \quad (n \geqslant 2)$$

Notice that our results agree with Theorem (5.1) and (5.38) of [16], if the sign in Theorem (5.1) of [16] is corrected as pointed out by J. H. C. Whitehead in [20].

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