MATHEMATICS

SPHERICAL FUNCTIONS ON THE $\mathfrak{p}$-ADIC GROUP $PGL(2)$. II.

BY

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CHAPTER II: APPLICATION TO $PGL(2, k)$, THE PROJECTIVE LINEAR GROUP OF RANK 2 OVER A $\mathfrak{p}$-ADIC FIELD k.

6. *Preliminaries about $\mathfrak{p}$-adic fields*

By k we denote a (commutative) $\mathfrak{p}$-adic field, i.e. a commutative locally compact, totally disconnected, non-discrete field. We choose a Haar measure dx on the additive group k^+ of k. A natural (non-archimedian) valuation $|\cdot|$ on k is obtained by the formula $d(ax)=|a|\,dx$ $(a\neq 0)$, $|0|=0$. A Haar measure on the multiplicative group k^* of k is given by $d^*x=|x|^{-1}dx$. The open set $\mathcal{O}=\{x\colon |x|\leqslant 1\}$ is a subring of k, called the ring of integers of k. Usually dx is normalized such that $\mathcal{O}$ has measure one. The ring $\mathcal{O}$ is a local ring; the unique maximal ideal is given by $P=\{x\colon |x|<1\}$. The residue class field $\mathcal{O}/P$ is a finite field with (say) q elements. *We always assume that its characteristic is odd.* Since the valuation on k, defined above, is discrete, it follows that P is a principal ideal. Let π be a generator of P. Then $|\pi|=q^{-1}$ and for each $x\in k$ either $|x|=0$ (if and only if $x=0$) or $|x|=q^n$ for some integer n. Put $P^n=\pi^n\mathcal{O}$ for all integers n. Clearly P^n has measure q^{-n} and $P^n=\{x\colon |x|\leqslant q^{-n}\}$. The collection $\{P^n\}_{n=0}^{\infty}$ is a fundamental system of neighbourhoods of the zero-element in k^+, consisting of open compact subgroups of k^+. Let $\mathcal{U}$ be the group of units of $\mathcal{O}$. A unique cyclic subgroup of $\mathcal{U}$ of order $q-1$ exists, whose elements, together with the zero of k, constitute a complete set of representatives for $\mathcal{O}/P$. Put $\mathcal{U}_n=1+P^n$ for $n\geqslant 1$, $\mathcal{U}_0=\mathcal{U}$. Let ε be a generator of the cyclic subgroup just mentioned. Each element x in k^* can be written uniquely in the form $x=\pi^n\varepsilon^k x_1$, where $|x|=q^{-n}$, $0\leqslant k<q-1$, $x_1\in\mathcal{U}_1$. It follows that k^* is isomorphic to the direct product of three groups: $k^*\simeq \mathbf{Z}\times\mathbf{Z}_{q-1}\times\mathcal{U}_1$. Clearly the collection $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is a fundamental system of neighbourhoods for the identity in k^*, consisting of open compact subgroups.

For any integer $m\geqslant 1$, $x\mapsto x^2$ induces on $\mathcal{U}_m$ an automorphism of $\mathcal{U}_m$. It follows that the squares in k^*, denoted $(k^*)^2$, form an open subgroup of k^*. From $k^*\simeq\mathbf{Z}\times\mathbf{Z}_{q-1}\times\mathcal{U}_1$ we have $[k^*\colon(k^*)^2]=4$. Therefore k admits exactly three non-isomorphic quadratic field extensions. Denoting an extension by $k_\nu=k(\sqrt{\nu})$, one easily determines the following possibilities for ν: $\nu=\varepsilon$ (unramified extension), $\nu=\pi$, $\nu=\varepsilon\pi$ (ramified extensions). For any ν let N_ν denote the norm of k_ν with respect to k. The set $N_\nu(k_\nu^*)$ is a subgroup of k^*, contains $(k^*)^2$ and satisfies: $[N_\nu(k_\nu^*) : (k^*)^2] = 2$, $[k^* : N_\nu(k_\nu^*)] = 2$. Notice that $N_\nu(k_\nu^*)$ is an open subgroup of k^*.

7. *'Iwasawa decomposition' of $PGL(2, k)$.*

Let k be a $\mathfrak{p}$-adic field. Throughout this chapter G will be the projective linear group over k of rank 2, i.e. the group $GL(2, k)$ divided by its centre. G is in a natural way a locally compact group. G is unimodular since $GL(2, k)$ and its centre are.

Let us first consider the group $GL(2, k)$. Choose ν as in 6. We introduce three subgroups:

U_0 consisting of the matrices $\begin{pmatrix} \alpha & \beta \\ \nu\beta & \alpha \end{pmatrix}$, where $\alpha, \beta \in k$, $\alpha^2 - \nu\beta^2 \in k^*$:

A consisting of the matrices $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, where $\lambda \in k^*$;

N consisting of the matrices $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, where $\mu \in k$.

(7.1) Lemma. *Every element $g \in GL(2, k)$ can be written uniquely in the form $g = a \cdot n \cdot u$ ($a \in A$, $n \in N$, $u \in U_0$). The projections $g \mapsto a$, $g \mapsto n$ and $g \mapsto u$ are continuous.*

Proof. Consider the group $B = A \cdot N$. Clearly every element $b \in B$ can be written uniquely in the form $b = a \cdot n$; the mappings $b \mapsto a$, $b \mapsto n$ are continuous.

Put
$$g = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad b = \begin{pmatrix} \lambda & \mu \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} \alpha & \beta \\ \nu\beta & \alpha \end{pmatrix}.$$

Then we have $g = b \cdot u$, where
$$\alpha = t, \ \nu\beta = z, \ \lambda = -\nu \cdot \frac{\det g}{z^2 - \nu t^2}, \ \mu = \frac{xz - \nu yt}{z^2 - \nu t^2}.$$

Moreover $g \mapsto b$ and $g \mapsto u$ are continuous. So the lemma follows.

We shall write $GL(2, k) = A \cdot N \cdot U_0$. By passing to G, $B = A \cdot N$ may be considered as unaltered, U_0 becomes a subgroup U of G. We write $G = A \cdot N \cdot U$ or $G = B \cdot U$. Observe that (7.1) remains valid on G. Since U depends on the choice of ν we also write U_ν. One easily checks that U is a compact subgroup of G. The 'Iwasawa decomposition' of G, introduced above, leads to a decomposition of Haar measure on G. We have $dg = db \cdot du$, provided each measure is well normalized. The subgroup B is parametrized by $\lambda \in k^*$, $\mu \in k$. B is not unimodular. A left Haar measure on B will be denoted db, a right Haar measure $d_r(b)$. One verifies that one can take
$$db = \frac{d\lambda d\mu}{|\lambda|^2}, \ d_r(b) = \frac{d\lambda d\mu}{|\lambda|}.$$

8. *Commutativity of $L_\tau(G, U)$.*

The continuous irreducible unitary representations τ of U are one-dimensional and are identified with (unitary) characters of U. We shall prove that the algebras $L_\tau(G, U)$, introduced in 1, are commutative if we take G and U as in 7.

First we will have a suitable set of representatives for the double cosets with respect to U in G. Since $G = B \cdot U$, every double coset contains an element of B.

(8.1) Lemma. *Any double coset with respect to U_0 in $GL(2, k)$ contains an element of the form $\begin{pmatrix} x & y \\ \nu y & t \end{pmatrix}$ where $x, y, t \in k$, $xt - \nu y^2 \in k^*$.*

Proof. It suffices to show that for given $\lambda \in k^*$, $\mu \in k$ there exist $\alpha, \beta, x, y, t \in k$ such that

$$\begin{pmatrix} \alpha & \beta \\ \nu\beta & \alpha \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ \nu y & t \end{pmatrix},$$

where $\alpha^2 - \nu\beta^2 \neq 0$, $xt - \nu y^2 \neq 0$. Consequently one has to solve the equation $\beta\lambda = \alpha\mu + \beta$ by $\alpha, \beta \in k$, satisfying $\alpha^2 - \nu\beta^2 \neq 0$. But this is a triviality.

For $\quad g \in GL(2, k)$, $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ put ${}^*g = \begin{pmatrix} x & \nu^{-1} z \\ \nu y & t \end{pmatrix}$.

Since
$$ {}^*g = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot {}^t g \cdot \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix},$$

the map $g \mapsto {}^*g$ is an anti-automorphism of $GL(2, k)$, which leaves the centre fixed. Therefore $g \mapsto {}^*g$ induces an anti-automorphism of G, which we also denote by $g \mapsto {}^*g$.

For any complex function f on G we define ${}^*f(g) = f({}^*g)$ $(g \in G)$. Let f_1, f_2 be continuous functions with compact support on G. The following relation is easily verified: ${}^*(f_1 \star f_2) = {}^*f_2 \star {}^*f_1$.

(8.2) Theorem. *The algebras $L_\tau(G, U)$ are commutative.*

Proof. For $g \in GL(2, k)$ write $g = u \cdot s \cdot u'$ with s of the form $\begin{pmatrix} x & y \\ \nu y & t \end{pmatrix}$ and $u, u' \in U_0$ (8.1).

Then ${}^*g = u' \cdot s \cdot u$. Pass to G. In the same notation we have for all $f \in L_\tau(G, U)$:

$$ {}^*f(g) = f({}^*g) = \tau(uu') f(s) = f(g) \quad (g \in G).$$

Hence $f_1 \star f_2 = {}^*({}^*f_2 \star {}^*f_1) = {}^*(f_2 \star f_1) = f_2 \star f_1$ for all $f_1, f_2 \in L_\tau(G, U)$. This completes the proof.

9. *Commutativity of $L_\sigma(G, K)$.*

Let K be a maximal compact subgroup of G and let σ be a continuous irreducible unitary representation of K on a finite dimensional vector space E_σ. Let d_σ be the dimension of E_σ. We shall prove that the algebra $L_\sigma(G, K)$, defined in 1, is commutative. The notations, introduced in 1, will be used in this section.

It is known that (modulo conjugation in G) exactly two maximal compact subgroups in G exist. They are open in G ([1(c)], § 6). One subgroup is well-known: the group $PGL(2, \mathcal{O})$, consisting of the integer matrices, whose determinant is a unit. A representative K' for the other conjugate class is generated by the image in G of the following set of matrices in $GL(2, k)$:

$$\left\{ k = \begin{pmatrix} k_{11} & k_{12} \\ \pi k_{21} & k_{22} \end{pmatrix} : \begin{array}{r} k_{ij} \in \mathcal{O},\ 1 \leqslant i,\ j \leqslant 2 \\ \det(k) \in \mathcal{U} \end{array} \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} \right\}.$$

Clearly U_ε is contained in $PGL(2, \mathcal{O})$ and U_π, $U_{\varepsilon\pi}$ are contained in K'.

(9.1) Proposition. *Any maximal compact subgroup of G contains some conjugate of certain U_ν.*

For arbitrary $g \in G$, $S \subset G$ put ${}^gS = gSg^{-1}$. If f is a mapping defined on S, put ${}^gf(s) = f(g^{-1}sg)$. Then ${}^gf$ is defined on ${}^gS$.

(9.2) Lemma. *The map $f \mapsto {}^gf$ defines an isomorphism of $L_\tau(G, U)$ onto $L_{{}^g\tau}(G, {}^gU)$.*

The proof follows 'par transport de structure'.

To prove that $L_\sigma(G, K)$ is commutative, we may assume by (9.1) and (9.2) that $U = U_\nu$ (for some ν) is contained in K.

If we restrict σ to U, σ can be decomposed into irreducible parts: an orthonormal basis exists in E_σ such that $\sigma(u)$ is diagonal with respect to this basis for all $u \in U$. Denote the diagonal elements of $\sigma(u)$ by

$$\tau_1^\sigma(u), \ldots, \tau_{d_\sigma}^\sigma(u) \quad (u \in U).$$

The functions $u \mapsto \tau_i^\sigma(u)$ ($u \in U$, $1 \leqslant i \leqslant d_\sigma$) are characters of U. Since for all characters τ of U the algebra $L_\tau(K, U)$ is commutative (K is open in G), we obtain by (4.2) that the characters τ_i^σ ($1 \leqslant i \leqslant d_\sigma$) are mutually distinct. So we have

(9.3) Theorem. *Let K be a maximal compact subgroup of G containing a conjugate U of some U_ν. Let σ be an irreducible continuous finite-dimensional representation of K. Then the restriction of σ to U decomposes into inequivalent characters.*

Let $\mathbf{f}$ be in $L_\sigma(G, K)$. Denote by $\mathbf{f}_{ij}(g)$ the matrix coefficients of $\mathbf{f}(g)$ with respect to the basis, chosen above ($1 \leqslant i$, $j \leqslant d_\sigma$, $g \in G$). Simple matrix calculus yields: $\mathbf{f}_{ij}(ugu') = \tau_i^\sigma(u)\mathbf{f}_{ij}(g)\tau_j^\sigma(u)$ for all u, $u' \in U$, $g \in G$. Therefore we have: $\mathbf{f}_{ii} \in L_{\tau_i^\sigma}(G, U)$ for all $\mathbf{f} \in L_\sigma(G, K)$ ($1 \leqslant i \leqslant d_\sigma$). By (1.1) the mapping $\mathbf{f} \mapsto d_\sigma \mathbf{f}_{ii}$ ($\mathbf{f} \in L_\sigma(G, K)$) is a *-isomorphism onto the subalgebra $L_{e_i}(G, K)$ of $L_{\tau_i^\sigma}(G, U)$. Hence $L_\sigma(G, K)$ is commutative, since $L_{\tau_i^\sigma}(G, U)$ is (cf. 8).

So we have proved:

(9.4) Theorem. *For each continuous irreducible finite-dimensional representation σ of K, the algebra $L_\sigma(G, K)$ is commutative.*

(9.5) Corollary. *The restriction to K of a continuous irreducible unitary representation of G decomposes into inequivalent irreducible unitary representations of K.*

Partial results are obtained by MAUTNER [8], BRUHAT (for $SL(2, k)$) ([1(a)], in particular Théorème 3) and KIRILLOV [7] (for $GL(2, k)$). The techniques used in the case of a semi-simple linear Lie group for proving results like the above cannot be applied here. In particular the 'principal series' of G does not form a complete set of representations of G ([1(a)], [8]).

10. *Characters.*

In this section we only deal with the maximal compact subgroup $K = PGL(2, \mathcal{O})$. 'Reduction modulo π^n' defines a homomorphism of $PGL(2, \mathcal{O})$ onto $G_n = PGL(2, \mathcal{O}/P_n)$, $n \geqslant 1$. G_n is a finite group. Let us denote the kernel of this homomorphism by K_n. Then $\{K_n\}_{n\geqslant 1}$ is a fundamental system of neighbourhoods of the identity in G, consisting of open compact subgroups. Let $\mathscr{S}(G)$ be the convolution algebra of locally constant complex-valued functions on G with compact support. To any $f \in \mathscr{S}(G)$ there exists a natural number n such that f is constant modulo K_n. Therefore, if supp (f) is contained in K, f can be considered as a function on the finite group G_n. Observe that $\mathscr{S}(G)$ is dense in $L^p(G)$ $(1 \leqslant p < \infty)$, the space of p^{th}-power integrable functions on G.

For any compact set $C \subset G$ and any $n \geqslant 1$, let $\mathscr{S}_n(C)$ be the space of all $f \in \mathscr{S}(G)$, with supp $(f) \subset C$, which are constant modulo K_n. Provided with the sup-norm, $\mathscr{S}_n(C)$ becomes a Banach space. One has $\mathscr{S}_n(C) \subset \mathscr{S}_{n'}(C')$ whenever $n \leqslant n'$, $C \subset C'$. Clearly $\mathscr{S}(G) = \bigcup_{n,C} \mathscr{S}_n(C)$. Put on $\mathscr{S}(G)$ the inductive limit topology with respect to the $\mathscr{S}_n(C)$. Then $\mathscr{S}(G)$ is a complete locally convex Hausdorff space, the so-called space of Schwartz functions on G. A sequence $\{f_m\}$ $(f_m \in \mathscr{S}(G))$ converges to zero if and only if there are n, C such that $f_m \in \mathscr{S}_n(C)$ for all m and $\{f_m\}$ tends to zero in $\mathscr{S}_n(C)$.

$\mathscr{S}'(G)$, the (topological) dual of $\mathscr{S}(G)$, is called the space of *distributions* on G. A linear function on $\mathscr{S}(G)$ is continuous if and only if it is continuous on each subspace $\mathscr{S}_n(C)$.

The following theorem is a special case of [1(b)] (Proposition 16).

(10.1) Theorem. *Let π be a continuous irreducible unitary representation of G. Then for all $f \in \mathscr{S}(G)$, $\pi(f)$ is an operator of finite rank (and hence of trace class). Moreover the character of π, $f \mapsto tr\, \pi(f)$ $(f \in \mathscr{S}(G))$ is a distribution on G.*

The foregoing results imply that G is a type I group (in the sense of the Von-Neumann classification) and hence G admits a Plancherel formula (cf. [3], 15.5.2, 18.8.2).

It seems to be true that the characters of the irreducible unitary representations of G are given by locally integrable functions which are locally constant on the set of regular elements. This result was announced by P. J. Sally Jr. and J. A. Shalika for $SL(2, k)$. For semi-simple Lie groups this was proved by Harish-Chandra several years ago.

Chapter III: Some harmonic analysis on $L(G, U)$.

11. *Summary of harmonic analysis on k.*

For details we refer to [9] (§ 2).

Let χ be a continuous character of k^+ which is trivial on $\mathcal{O}$ but not trivial on P^{-1}. Each continuous character of k^+ can be written in the form χ_u $(u \in k)$, where $\chi_u(x) = \chi(ux)$ $(x \in k)$. The mapping $u \mapsto \chi_u$ establishes an isomorphism between k^+ and $\hat{k}^+$. The character χ is called a *basic* character. Fix a basic character χ of k^+. The Fourier transform of a function $f \in L^1(k^+)$ is defined by $\hat{f}(u) = \int_k f(x)\chi(ux)dx$ $(u \in k)$. Due to the normalization of the Haar measure on k^+, the Fourier inversion formula takes the form $f(x) = \int_k \hat{f}(u)\chi(-ux)du$ $(\hat{f} \in L^1(\hat{k}^+))$.

Let $\mathscr{S}(k)$ be the space of locally constant functions on k with compact support. To any $f \in \mathscr{S}(k)$ an integer n exists such that f is constant on the cosets of P^n in k^+. Similar to the process described in 10 for $\mathscr{S}(G)$, $\mathscr{S}(k)$ is endowed with an inductive limit topology such that $\mathscr{S}(k)$ is a complete locally convex Hausdorff space. $\mathscr{S}(k)$ is called the space of Schwartz functions on k. $\mathscr{S}(k)'$, the topological dual of $\mathscr{S}(k)$ is called the space of distributions on k. The Fourier transform yields a (topological) isomorphism of $\mathscr{S}(k)$ onto itself. More explicitly: $f \in \mathscr{S}(k)$ is supported on P^n and is constant on the cosets (in k^+) of P^m if and only if $\hat{f}$ is supported on P^{-m} and is constant on the cosets (in k^+) of P^{-n} (cf. [9], § 2, lemma (A_1)). Observe that $\mathscr{S}(k)$ is dense in $L^p(k^+)$ $(1 \leqslant p < \infty)$.

The map dual to the Fourier transform on $\mathscr{S}(k)$ defines a (topological) isomorphism of $\mathscr{S}(k)'$. The image of a distribution ϕ under this map is called the Fourier transform of ϕ and is denoted $\hat{\phi}$. One has $\hat{\phi}(f) = \phi(\hat{f})$ for all $f \in \mathscr{S}(k)$.

Let t_n be the characteristic function of P^n. Then $t_n \in \mathscr{S}(k)$ and $\hat{t}_n = q^{-n}t_{-n}$ for all integers n.

As to the characters of k^*, any continuous (unitary) character c of k^* is trivial on certain subgroup $\mathscr{U}_n$ $(n \geqslant 0)$. If c is trivial on $\mathscr{U}$, c is called unramified or of ramification degree zero. If c is trivial on $\mathscr{U}_n$ $(n \geqslant 1)$, but not on $\mathscr{U}_{n-1}$, c is said to be ramified with ramification degree n. The same definition applies to the characters of $\mathscr{U}$. Since $k^* \simeq \mathbf{Z} \times \mathbf{Z}_{q-1} \times \mathscr{U}_1$ (cf. 6), the character group of k^* takes the form $\hat{k}^* \simeq \mathbf{T} \times \mathbf{Z}_{q-1} \times \hat{\mathscr{U}}_1$. where $\mathbf{T}$ denotes the group of the complex numbers of modulus 1. For

$\lambda \in k^*$ we write $\lambda = \pi^n \cdot \tilde{\lambda}$, where n is determined by $|\lambda| = q^n$ and $\tilde{\lambda}$ by requiring $\tilde{\lambda} \in \mathscr{U}$. The characters of k^* can be written in the form

$$c(\lambda) = |\lambda|^{-s}\, \tilde{c}(\tilde{\lambda}) \text{ where } s = it, \tag{1}$$

$$-\frac{\pi}{\log q} < t \leqslant \frac{\pi}{\log q}$$

and $\tilde{c}$ a character of $\mathscr{U}$. – The reader will notice the different meanings of the symbol π. – Since there are only finitely many characters of $\mathscr{U}$ of a given ramification degree, we see that $\hat{\mathscr{U}}$ is a countable set. Let us consider now *all* (not necessarily unitary) continuous characters of k^*. Usually they are called quasi-characters. One obtains them by replacing the purely imaginary s in (1) by an arbitrary complex number. Then s is uniquely determined by c, modulo $\frac{2\pi i}{\log q}$. Observe that $\tilde{c}$ is uniquely determined by c.

By the remarks made above one checks easily that one can view $\hat{k}^*$ as a countable discrete collection of circles $\mathbf{T}_{\tilde{c}}$ $(\tilde{c} \in \hat{\mathscr{U}})$. A Haar measure dc on $\hat{k}^*$ is given by integrating over each circle $\mathbf{T}_{\tilde{c}}$ with respect to the usual Haar measure and then summing up over $\tilde{c}$. Thus

$$\int_{\hat{k}^*} f(c)\,dc = \sum_{\tilde{c} \in \hat{\mathscr{U}}} \frac{q' \log q}{2\pi} \int_{-(\pi/\log q)}^{\pi/\log q} f(\tilde{c}\,|\lambda|^{i\alpha})\,d\alpha \tag{2}$$

where $\frac{1}{q'} = 1 - \frac{1}{q}$, $c = \tilde{c} \cdot |\lambda|^{i\alpha}$ and the normalizing constant $\frac{q' \log q}{2\pi}$ has been chosen to make that Plancherel's theorem works. The space of Schwartz functions on k^*, denoted $\mathscr{S}(k^*)$, is the space of locally constant functions with compact support on k^*. So for all $f \in \mathscr{S}(k^*)$ an integer $m \geqslant 0$ exists such that f is constant on the cosets (in k^*) of $\mathscr{U}_m$. $\mathscr{S}(k^*)$ is dense in $L^p(k^*)$, $1 \leqslant p < \infty$. The restriction to k^* of a function $f \in \mathscr{S}(k)$ belongs to $\mathscr{S}(k^*)$ if and only if $f(0) = 0$.

Let us denote by $\hat{\mathscr{S}}(k^*)$ the space of complex-valued functions f on $\hat{k}^*$ with compact support (vanishing outside a finite number of circles) which are on each circle a trigonometric polynomial:

$$f(c) = f(\tilde{c}\,|\lambda|^{i\alpha}) = \sum_{r=-m}^{m} a_r(\tilde{c}, f)\, q^{ir\alpha}$$

Both $\mathscr{S}(k^*)$ and $\hat{\mathscr{S}}(k^*)$ can be endowed with a 'test-space' topology (inductive limit topology).

The Fourier transform on k^* and $\hat{k}^*$ is called Mellin transform. If $f \in L^1(k^*)$ then

$$\hat{f}(c) = \int_{k^*} f(\lambda)\, c(\lambda)\, d^*\lambda \quad (c \in \hat{k}^*).$$

The Fourier inversion formula takes the form

$$f(\lambda)=\int_{\hat{k}^*} \hat{f}(c)c^{-1}(\lambda)\,dc, \text{ provided } \hat{f}\in L^1(\hat{k}^*, dc),\ dc$$

being chosen in (2).

The Mellin transform yields a (topological) isomorphism of $\mathscr{S}(k^*)$ onto $\hat{\mathscr{S}}(k^*)$.

12. *Double cosets and sections.*

We want a description of the set of double cosets of G with respect to $U(=U_\nu)$, denoted $X=U\backslash G/U$. Each double coset contains an element of B. Therefore it suffices to investigate when two elements of B are in the same coset. Obviously we may restrict ourselves to $GL(2,k)$ and ask for the equivalence of two elements in B with respect to U_0. It is not hard to see that

$$b=\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix} \text{ and } b'=\begin{pmatrix}\lambda' & \mu'\\ 0 & 1\end{pmatrix}$$

are in the same coset with respect to U_0 if and only if the following relation holds:

$$\frac{\lambda^2-\nu\mu^2+1}{\lambda}=\frac{\lambda'^2-\nu\mu'^2+1}{\lambda'}.$$

The proof is a simple exercise and is left to the reader. Put

$$\Phi\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix}=\frac{\lambda^2-\nu\mu^2+1}{\lambda}\quad (\lambda\in k^*,\ \mu\in k).$$

Then we have:

(12.1) Proposition. *Two elements $b, b'\in B$ are in the same coset with respect to U if and only if $\Phi(b)=\Phi(b')$.*

So X may be identified with the range of Φ, which we denote by X again. Φ is a continuous function on B with values in X, a subset of k. We want a continuous section $\Psi\colon X\to B$ (such that $\Phi\circ\Psi=$identity on X).

(12.2) Proposition. *There exists a continuous section $\Psi\colon X\to B$.*

Proof. We shall point out the unramified case: $\nu=\varepsilon$. The ramified case is then easy and is left to the reader. So assume $\nu=\varepsilon$. Let us solve the equation

$$\frac{\lambda^2-\varepsilon\mu^2+1}{\lambda}=2\varrho\quad (\lambda\in k^*,\ \mu\in k,\ 2\varrho\in X).$$

It can be reduced to $(\lambda-\varrho)^2-\varepsilon\mu^2=\varrho^2-1$. So we have a solution (λ,μ) if and only if $\varrho^2-1\in N_\varepsilon(k_\varepsilon^*)$ or $\varrho=\pm 1$.

We recall that the map $x \mapsto x^2$ induces on $\mathscr{U}_1$ an automorphism of $\mathscr{U}_1$. So we can speak of *the* square root (denoted $\sqrt{\ }$) of any element in $\mathscr{U}_1$. Due to the choice of the quadratic extension k_ε we have that every unit in k^* is in $N_\varepsilon(k_\varepsilon^*)$. Indeed, solving an equation of the following type: $x^2-\varepsilon y^2=\zeta$ ($\zeta \in \mathscr{U}$ given; $x, y \in k$) is a simple exercise if we pass to the finite residue class field. So we have $x, y \in k$ satisfying $x^2-\varepsilon y^2=\zeta\cdot\delta$ where $\delta \in \mathscr{U}_1$. Therefore $\delta \in (k^*)^2 \subset N_\varepsilon(k_\varepsilon^*)$, hence $\zeta \in N_\varepsilon(k_\varepsilon^*)$. If $|\varrho|>1$, put $\Psi(2\varrho)=\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix}$, where $\lambda=\varrho+\varrho\sqrt{1-\dfrac{1}{\varrho^2}}$, $\mu=0$. If $|\varrho|<1$, we have $\varrho^2-1=$ $=-1(1-\varrho^2)$, hence again $\varrho^2-1 \in N_\varepsilon(k_\varepsilon^*)$.

Let $-1=a^2-\varepsilon b^2$ for suitable $a, b \in k$. Now, put

$$\Psi(2\varrho)=\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix}, \text{ where } \lambda=\varrho+a\sqrt{1-\varrho^2},\ \mu=b\sqrt{1-\varrho^2}.$$

It remains to consider $\varrho=\pm 1$ and $\{\varrho\colon \varrho^2-1 \in N_\varepsilon(k_\varepsilon^*),\ |\varrho|=1\}$. The set $\{\varrho^2-1\colon \varrho^2-1 \in N_\varepsilon(k_\varepsilon^*),\ |\varrho|=1\}$ is a disjoint union of the sets $\pi^{2n}\cdot\varepsilon^k\cdot\mathscr{U}_1$ ($n>0$, $0\leqslant k<q-1$) and the sets $(\varepsilon^{2l}-1)\,\mathscr{U}_1$, where l is such that $\varepsilon^l \neq \pm 1$, $0<l<q-1$. Observe that π is not in $N_\varepsilon(k_\varepsilon^*)$, hence only even powers of π occur. Let us consider the sets $\pi^{2n}\cdot\varepsilon^k\cdot\mathscr{U}_1$ ($n>0$, $0\leqslant k<q-1$). Put

$$\varrho^2-1=\pi^{2n}\,\varepsilon^k\ F(\varrho)=\pi^{2n}\,\varepsilon^k\ (\sqrt{F(\varrho)})^2.$$

Then we write

$$\Psi(2\varrho)=\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix}, \text{ where } \lambda=\varrho+\pi^n\,\varepsilon^{k/2}\sqrt{F(\varrho)},\ \mu=0$$

if k is even (or zero); $\lambda=\varrho+b\pi^n\,\varepsilon^{(k+1)/2}\sqrt{F(\varrho)}$, $\mu=a\pi^n\,\varepsilon^{(k-1)/2}\sqrt{F(\varrho)}$ if k is odd, ϱ being chosen such that $\varrho^2-1 \in \pi^{2n}\cdot\varepsilon^k\cdot\mathscr{U}_1$.

Similar procedures can be applied to the remaining sets. We leave this to the reader. Finally put

$$\Psi(2)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \Psi(-2)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

From the explicit formulas derived here it easily follows that Ψ is continuous (in particular at the points ± 2) and $\Phi\circ\Psi=id.$ on X. So Ψ is a continuous section.

13. *The transformation 'F_f'* (1).

For $f \in L_\tau(G, U)$, put $F_f(\lambda)=|\lambda|^{-\frac{1}{2}}\int_k f(b_{\lambda,\mu})\,d\mu$, where

$$\lambda \in k^*,\ b_{\lambda,\mu}=\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix}.$$

Clearly F_f is a continuous function with compact support on k^*.

(13.1) Proposition. *The transformation* $f \mapsto F_f$ *enjoys the following properties:*

(i) $f \mapsto F_f$ *is linear,*

(ii) $F_{f_1 \star f_2} = F_{f_1} \star F_{f_2}$, *where the second star means convolution in* $L(k^*)$,

(iii) $F_{f^*}(\lambda) = \overline{F_f(\lambda^{-1})}$ *for all* $\lambda \in k^*$,

(iv) $F_f(\lambda) = F_f(\lambda^{-1})$ *for all* $\lambda \in k^*$.

Proof. The properties (i), (ii) and (iii) are easily verified. Let us prove (iv). By (12.1) $b_{\lambda^{-1},\mu}$ and $b_{\lambda,\lambda\mu}$ are in the same double coset with respect to U. So there exists $\alpha, \beta, \gamma, \delta \in k$ such that

$$\begin{pmatrix} \alpha & \beta \\ \nu\beta & \alpha \end{pmatrix} \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ \nu\delta & \gamma \end{pmatrix} = \begin{pmatrix} \lambda & \lambda\mu \\ 0 & 1 \end{pmatrix}, \quad \alpha^2 - \nu\beta^2 \neq 0, \ \gamma^2 - \nu\delta^2 \neq 0.$$

Simple calculation shows that

$$\begin{pmatrix} \alpha & \beta \\ \nu\beta & \alpha \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ \nu\delta & \gamma \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

provided $(\lambda, \mu) \neq (-1, 0)$. Hence we have for all $f \in L_\tau(G, U)$ $f(b_{\lambda,\lambda\mu}) = f(b_{\lambda^{-1},\mu})$ for every $\lambda \in k^*$, $\mu \in k$.

Therefore:

$$F_f(\lambda^{-1}) = |\lambda|^{\frac{1}{2}} \int_k f(b_{\lambda^{-1},\mu}) d\mu = |\lambda|^{\frac{1}{2}} \int_k f(b_{\lambda,\lambda\mu}) d\mu = |\lambda|^{-\frac{1}{2}} \int_k f(b_{\lambda,\mu}) d\mu = F_f(\lambda)$$

for all $\lambda \in k^*$. This completes the proof.

If we want to distinguish between different τ, we shall write F_f^τ in stead of F_f. For the moment we keep τ fixed.

There is a connection (well-known to specialists) between the transformation $f \mapsto F_f$ and the so-called *principal series* of representations of G. Let us briefly describe this connection for the convenience of the reader (cf. [6], p. 239–243).

Let c be any quasi-character of k^*, written in the form $c(\lambda) = |\lambda|^s \tilde{c}(\tilde{\lambda})$ where s is a complex number,

$$-\frac{\pi}{\log q} < Im\ s \leqslant \frac{\pi}{\log q}.$$

Put

$$\alpha(b) = c(\lambda)|\lambda|^{-\frac{1}{2}} \text{ for } b = \begin{pmatrix} \lambda & \mu \\ 0 & 1 \end{pmatrix} \quad (\lambda \in k^*, \ \mu \in k).$$

Clearly α is a one-dimensional continuous representation of B, which is trivial on N. So $\alpha(b) = \alpha(a)$ if $b = an$ ($b \in B$, $a \in A$, $n \in N$). Let H^α be the complex vector space of the continuous functions f on G satisfying $f(bx) = \alpha(b)f(x)$ for all $x \in G$, $b \in B$. Restricting f to U, we obtain a bijective linear map of H^α onto the space $L(U)$ of all continuous complex-valued functions θ on U (by (7.1)). On H^α a representation T^α of G is defined by $(T_g^\alpha f)(x)=f(xg)$ $(g, x \in G)$.

We pass to U and write $\pi_\alpha(g)$ in stead of T_g^α.

We have

$$\pi_\alpha(g)\theta(u)=\theta(w(u, g))\cdot\alpha(a(u, g)) \quad (u \in U,\ g \in G)$$

where $ug=a(u, g)\cdot n(u, g)\cdot w(u, g)$; $a(u, g) \in A$, $n(u, g) \in N$, $w(u, g) \in U$.

We state some properties of the mappings $g \mapsto w(u, g)$ and $g \mapsto a(u, g)$.

(i) $w(u, g_1g_2)=w(w(u, g_1), g_2)$

(ii) $a(u, g_1g_2)=a(u, g_1)a(w(u, g_1), g_2)$

(iii) $a(u, u')=1$

(iv) $a(u, u^{-1}g)=a(e, g)$.

According to the decomposition $G=ANU$ we choose a Haar measure on G: $dx=da\,dn\,du$.

Let $g \in G$ be fixed. For $a=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}$, put $\beta(a)=|\lambda|$.

We have

$$xg=a\cdot n\cdot u\cdot g=a\cdot n\cdot a(u, g)n(u, g)w(u, g)=$$
$$=a\cdot a(u, g)(a(u, g)^{-1}\cdot n\cdot a(u, g)\cdot n(u, g))w(u, g).$$

Since $d(a^{-1}na)=\beta(a)dn$ for all $a \in A$ and since G is unimodular, we obtain

$$\int_G f(x)dx=\int_G f(xg)dx=\int_A\int_N\int_U f(a\cdot n\cdot w(u, g))\beta(a(u, g))\,da\,dn\,du$$

for all suitable functions f on G. This yields:

$$\int_U \theta(u)du=\int_U \beta(a(u, g))\,\theta(w(u, g))du \text{ for all } \theta \in L(U),\ g \in G.$$

If one applies these formulas one easily proves the following proposition.

(13.2) Proposition.

(i) *For each $g \in G$, $\pi_\alpha(g)$ is a continuous linear operator in $L(U)$, provided with the topology of $L^2(U, du)$.*
Extend $\pi_\alpha(g)$ to the whole of $L^2(U)$ for all $g \in G$.

(ii) *π_α is a continuous representation of G on the Hilbert space $L^2(U)$.*

(iii) *π_α is unitary if and only if $Re\ s=0$.*

The representations $\pi_\alpha=\operatorname*{ind}_{B\uparrow G}\alpha$, realized on $L^2(U)$, coincide on U with the right regular representation of U on $L^2(U)$ for every α. From other realizations we know that π_α is irreducible for all α such that π_α is unitary. (cf. e.g. [8], Theorem 3.2).

Now let f be a function in $L_\tau(G, U)$. Take the function in $L^2(U)$ which transforms under π_α according to $\bar{\tau}$, having length 1:

$$u \mapsto \overline{\tau(u)} \quad (u \in U).$$

We obtain

$$(\pi_\alpha(f)\bar{\tau}|\bar{\tau}) = \int_G f(g)\,(\pi_\alpha(g)\bar{\tau}|\bar{\tau})\,dg =$$

$$= \int_G \int_U f(g)\,\overline{\tau(w(u,g))}\,\alpha(a(u,g))\,\tau(u)\,du\,dg = \int_G f(g)\,\overline{\tau(w(e,g))}\,\alpha(a(e,g))\,dg =$$

$$= \int_B f(b)\,\alpha(b)\,db = \int_k \int_{k^*} f\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix} \alpha\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} d\mu \frac{d^*\lambda}{|\lambda|} = \int_{k^*} F_f^\tau(\lambda)\,c(\lambda)|\lambda|^{-1}\,d^*\lambda.$$

We could write

$$tr\,\pi_\alpha(f) = \int_{k^*} F_f^\tau(\lambda) c(\lambda)|\lambda|^{-1}\,d^*\lambda \quad (f \in L_\tau(G, U)).$$

14. *An incomplete system.*

Let us recall that a collection of representations π of a complex associative algebra $\mathscr{A}$ on complex vector spaces is said to form a complete system, if $\pi(f)=0$ for all π implies $f=0$ $(f \in \mathscr{A})$. It is known that the principal series of G does not form a complete system for the algebra $L(G)$ (in full contrast with the real Lie group case). For certain algebras $L_\sigma(G, K)$ however, where $K = PGL(2, \mathcal{O})$, the principal series does form a complete system (cf. [1(a)], [8]). Let us consider the algebra $L_\tau(G, U)$ for $\tau = 1$, also denoted $L(G, U)$. We shall briefly prove that the principal series of G does *not* form a complete set of representations for $L(G, U_\varepsilon)$. Clearly this is equivalent to the assertion that $f \mapsto F_f$ $(f \in L(G, U_\varepsilon))$ is not injective (13). It suffices to consider this problem on $K = PGL(2, \mathcal{O})$ and (after reduction modulo π) even on the finite group $PGL(2, q)$. Now consult a character table for the representations of $PGL(2, q)$ (e.g. R. Steinberg, The representations of $GL(3, q)$, $GL(4, q)$, $PGL(3, q)$ and $PGL(4, q)$, Can. J. 3, p. 225, 226 (1951). Observe that the group $U_\varepsilon \subset K$ is mapped on the matrices of the form $\begin{pmatrix}\sigma^a & 0\\ 0 & \sigma^{aq}\end{pmatrix}_{a+\mathrm{mult.}\,q+1}$.

After a few computations one sees that the representation with character $\chi_{q-1}^{(n)}$ (q odd, $n = 1, 2, \ldots, \frac{1}{2}(q-1)$) in Steinberg's notation, possesses a matrix coefficient f satisfying

$$f(ugu') = f(g) \text{ and } \sum_{\mu \in \mathbf{F}_q} f\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix} = 0 \text{ for all } \lambda \in \mathbf{F}_q^*.$$

Let us consider the real group $G = PSL(2, \mathbf{R})$ with $U = PSO(2, \mathbf{R})$. Denote by $\mathscr{D}(G)$ the algebra of all C^∞-functions on G with compact support. Let $\mathscr{D}_+$ be the algebra of all C^∞-functions on $\mathbf{R}_+^*$ with compact support satisfying $f(\lambda) = f(\lambda^{-1})$. Both algebras are provided with the usual inductive limit topologies. Then $f \mapsto F_f$ is an isomorphism (both algebraical and topological) of $\mathscr{D}(G) \cap L(G, U)$ onto $\mathscr{D}_+$. This can easily be deduced from the results obtained by Takahashi ([10], I, § 4).

On a $\mathfrak{p}$-adic group (like k^* and $G = PGL(2, k)$, k a $\mathfrak{p}$-adic field) the role of $\mathscr{D}(G)$ is played by the algebra of Schwartz functions. From now on we restrict ourselves to the subgroup G_ν of G, being the image of the subgroup of $GL(2, k)$ consisting of the matrices with determinant in $N_\nu(k_\nu^*)$, under the canonical map $GL(2, k) \to G$. Let $\mathscr{S}_+$ be the subalgebra of $\mathscr{S}(k^*)$ consisting of all functions f satisfying $f(\lambda) = f(\lambda^{-1})$ $(\lambda \in k^*)$ and supp $(f) \subset (k^*)^2$. Then $f \mapsto F_f|_{(k^*)^2}$ is a *surjective* continuous homomorphism from $\mathscr{S}(G_\nu) \cap L(G, U_\nu)$ onto $\mathscr{S}_+$. This will be proved in the next section.

15. *The transformation 'F_f'* (2).

The results of this section do hold even for the real groups $PSL(2, \mathbf{R})$ and $SL(2, \mathbf{R})$ and then they yield a proof without differential equations and the application of Abel's integral equation (cf. [5(b)], [10]). However, we shall be concerned with the $\mathfrak{p}$-adic case only.

Let $\mathscr{S}(G_\nu, U_\nu)$ be the algebra of Schwartz functions on G, bi-invariant with respect to U_ν and with support contained in G_ν. Each function $f \in \mathscr{S}(G_\nu, U_\nu)$ is completely determined by its restriction to B, which is clearly a Schwartz function on B, satisfying

$$f(b_{\lambda,\mu}) = \phi\left(\frac{\lambda^2 - \nu\mu^2 + 1}{\lambda}\right) \quad (\lambda \in N_\nu(k_\nu^*),\ \mu \in k) \tag{1}$$

for a suitable function ϕ on k. By (12.2) ϕ can be chosen of Schwartz type. Observe that ϕ is not uniquely determined by f: ϕ may be changed (at least) on an open set, as is easily seen in view of (12.2). On the other hand, any Schwartz function ϕ on k yields a function $f \in \mathscr{S}(G_\nu, U_\nu)$ in the obvious way, such that (1) holds. Now consider

$$F_f(\lambda) = |\lambda|^{-\frac{1}{2}} \int_k f(b_{\lambda,\mu})\, d\mu \quad (\lambda \in (k^*)^2) \text{ for } f \in \mathscr{S}(G_\nu, U_\nu).$$

Then clearly $F_f \in \mathscr{S}_+$. One checks immediately the continuity of this transformation (considered as a homomorphism $\mathscr{S}(G_\nu, U_\nu) \to \mathscr{S}_+$).

(15.1) Theorem. *The transformation $f \mapsto F_f$ yields a continuous homomorphism of $\mathscr{S}(G_\nu, U_\nu)$ onto $\mathscr{S}_+$.*

The greater part of this theorem follows from (13.1) and the considerations from above. We have still to prove that the transformation is surjective. We first state a lemma.

(15.2) Lemma. *Let χ be a basic character of k^+. Then for all $\lambda \in k^*$ one has: $\int_{|\mu| \leqslant q^n} \chi(\lambda\mu^2)\, d\mu$ does not depend on n if $n \geqslant \dfrac{\log |\lambda|}{2 \log q}$, and, assuming this, equals $|\lambda|^{-\frac{1}{2}} \gamma(\lambda)$, where $\gamma(\lambda)$ is a complex number of modulus 1.*

γ is constant on the cosets of $(k^*)^2$ in k^*. Moreover one has

$$\gamma(1) = 1,\ \gamma(\varepsilon) = 1,\ \gamma(\varepsilon\pi) = -\gamma(\pi);$$
$$\gamma(\pi) = \pm 1 \text{ if } -1 \text{ is a square in } k,$$
$$\gamma(\pi) = \pm i \text{ if } -1 \text{ is not a square in } k.$$

Proof. We use some results about the Fourier transform of quadratic characters due to A. Weil, in the setting of CARTIER [2]. Applying [2] (Satz 1) to the quadratic character $\mu \mapsto \chi(\lambda\mu^2)$ ($\mu \in k$, $\lambda \in k^*$ fixed) we obtain:

$$\int_{|\lambda\mu| \leqslant q^m} \chi(\lambda\mu^2)\,d\mu = \gamma(\lambda)|\lambda|^{-\frac{1}{2}} q^{-m} \int_{|\mu| \leqslant q^{-m}} \overline{\chi(\lambda\mu^2)}\,d\mu,$$

where $\gamma(\lambda)$ is a complex number of modulus 1. For m such that $|\lambda| q^{-2m} \leqslant 1$ we have

$$q^{-m} \int_{|\mu| \leqslant q^{-m}} \chi(\lambda\mu^2)\,d\mu = 1.$$

Therefore

$$\int_{|\lambda\mu| \leqslant q^m} \chi(\lambda\mu^2)\,d\mu = \gamma(\lambda)|\lambda|^{-\frac{1}{2}} \text{ for all } m \geqslant \frac{\log|\lambda|}{2\log q}.$$

Let α be a square in k^*. Then $\gamma(\alpha\lambda)\ |\alpha\lambda|^{-\frac{1}{2}} = |\alpha|^{-\frac{1}{2}}\, \gamma(\lambda)|\lambda|^{-\frac{1}{2}}$ for all $\lambda \in k^*$. Hence $\gamma(\alpha\lambda) = \gamma(\lambda)$ for all $\lambda \in k^*$. So γ is constant on the cosets of $(k^*)^2$ in k^*.

For $\lambda = 1$ or $\lambda = \varepsilon$, take $m = 0$. Then $\gamma(\lambda) = \int_{\mathcal{O}} \chi(\lambda\mu^2)\,d\mu = 1$.

For $\lambda = \pi$ or $\lambda = \varepsilon\pi$, take $m = 0$ again. Then $\gamma(\lambda) = q^{-\frac{1}{2}} \sum_{k=0}^{q-1} \chi(\tilde{\lambda}\pi^{-1}\varepsilon_k^2)$, where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{q-1}$ is a set of representatives for $\mathcal{O}$ modulo P and $\tilde{\lambda} = 1$ if $\lambda = \pi$, $\tilde{\lambda} = \varepsilon$ if $\lambda = \varepsilon\pi$. Clearly we now have $\gamma(\varepsilon\pi) = -\gamma(\pi)$. If -1 is a square in k we have $\overline{\gamma(\pi)} = \gamma(-\pi) = \gamma(\pi)$. Hence $\gamma(\pi) = \pm 1$. If -1 is not a square in k we have $\gamma(-\pi) = \gamma(\varepsilon\pi)$. Therefore $\gamma(\pi) = \overline{-\gamma(\pi)}$ and hence $\gamma(\pi) = \pm i$.

Proof of the theorem.

First we shall modify the expression for F_f.

Let f be a function in $\mathscr{S}(G_\nu, U_\nu)$. Take any $\phi \in \mathscr{S}(k)$ such that (1) holds. Let χ be a basic character of k^+. By Fourier's inversion formula we have

$$\phi(\xi) = \int_k \chi(-\xi\eta)\hat{\phi}(\eta)\,d\eta \quad (\xi \in k).$$

Inserting this expression in

$$F_f(\lambda) = |\lambda|^{-\frac{1}{2}} \int_k \phi\left(\frac{\lambda^2 - \nu\mu^2 + 1}{\lambda}\right) d\mu,$$

we obtain

$$F_f(\lambda) = |\lambda|^{-\frac{1}{2}} \int_k \int_k \chi\left(-\eta\frac{\lambda^2 - \nu\mu^2 + 1}{\lambda}\right) \hat{\phi}(\eta)\,d\eta\,d\mu \quad (\lambda \in (k^*)^2).$$

Since both η and μ actually run over compact sets, we may change the order of integration:

$$(2) \qquad F_f(\lambda) = |\lambda|^{-\frac{1}{2}} \int_k \chi\left(-\eta\frac{\lambda^2+1}{\lambda}\right)\hat{\phi}(\eta)\,d\eta \cdot \int_S \chi\left(\frac{\nu\eta\mu^2}{\lambda}\right) d\mu,$$

where S denotes some compact set, such that $f(b_{\lambda,\mu}) = 0$ for all $\lambda \in (k^*)^2$ and all $\mu \notin S$.

By the remarks made above, we may assume that ϕ satisfies

$$\hat{\phi}(0) = \int_k \phi(\xi)\, d\xi = 0.$$

So the integrals in (2) may be taken over η outside some neighbourhood of $\eta = 0$.

For large S we have by (15.2):

$$\int_S \chi\left(\frac{\nu\eta\mu^2}{\lambda}\right) d\mu = |\nu|^{-\frac{1}{2}} |\lambda|^{\frac{1}{2}} |\eta|^{-\frac{1}{2}} \gamma(\nu\eta) \quad (\lambda \in (k^*)^2,\ \eta \in k^*).$$

We obtain:

$$F_f(\lambda) = |\nu|^{-\frac{1}{2}} \int_k \chi\left(-\eta \frac{\lambda^2+1}{\lambda}\right) |\eta|^{-\frac{1}{2}} \gamma(\nu\eta)\, \hat{\phi}(\eta)\, d\eta \quad (\lambda \in (k^*)^2).$$

Let us now prove the surjectivity.

Let h be any function in $\mathscr{S}_+$. We can find $H \in \mathscr{S}(k)$ satisfying:

$$h(\lambda) = H\left(\frac{\lambda^2+1}{\lambda}\right)\ (\lambda \in (k^*)^2),\ \int_k H(\xi)\, d\xi = 0.$$

This follows easily from our considerations for the formula

$$f(b_{\lambda,\mu}) = \phi\left(\frac{\lambda^2 - \nu\mu^2 + 1}{\lambda}\right)$$

from above, taking $\mu = 0$.

Put

$$\phi(\xi) = |\nu|^{\frac{1}{2}} \int_k \chi(-\eta\xi)\, \hat{H}(\eta) |\eta|^{\frac{1}{2}} \overline{\gamma(\nu\eta)}\, d\eta \quad (\xi \in k).$$

Then $\phi \in \mathscr{S}(k)$, $\hat{\phi}(0) = 0$.

Put

$$f(g) = f(b_{\lambda,\mu}) = \phi\left(\frac{\lambda^2 - \nu\mu^2 + 1}{\lambda}\right),$$

where $g = b_{\lambda,\mu} \cdot u$

$$(g \in G_\nu,\ b_{\lambda,\mu} \in B,\ u \in U_\nu,\ \lambda \in N_\nu(k_\nu^*),\ \mu \in k).$$

Then $f \in \mathscr{S}(G_\nu, U_\nu)$ and

$$F_f(\lambda) = \int_k \chi\left(-\eta \frac{\lambda^2+1}{\lambda}\right) |\eta|^{-\frac{1}{2}}\, \gamma(\nu\eta)\, \hat{H}(\eta) |\eta|^{\frac{1}{2}}\, \overline{\gamma(\nu\eta)}\, d\eta =$$

$$= H\left(\frac{\lambda^2+1}{\lambda}\right) = h(\lambda) \text{ for all } \lambda \in (k^*)^2.$$

This completes the proof.

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