Cyclic cohomology of Hopf algebras *

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Abstract

We give a construction of Connes-Moscovici's cyclic cohomology for any Hopf algebra equipped with a character. Furthermore, we introduce a non-commutative Weil complex, which connects the work of Gelfand and Smirnov with cyclic cohomology. We show how the Weil complex arises naturally when looking at Hopf algebra actions and invariant higher traces, to give a non-commutative version of the usual Chern-Weil theory.

Keywords: Cyclic cohomology, Hopf algebras, X-complex, characteristic classes, Weil complex

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INTRODUCTION

In the transversal index theorem for foliation (cohomological form), the characteristic classes involved are apriori cyclic cocycles on an algebra $\mathcal{A}$ associated to the foliation. In their computation, A. Connes and H. Moscovici [8] have discovered that the action of the operators appearing from the non-commutative index formula can be organized in an action of a Hopf algebra $\mathcal{H}_T$ on $\mathcal{A}$, and that the cyclic cocycles are made out just by combining the action with a certain invariant trace $\tau : \mathcal{A} \longrightarrow \mathbf{C}$. In other words, they define a cyclic cohomology $HC^*_\delta(\mathcal{H}_T)$, in such a way that the cyclic cocycles involved are in the target of a characteristic map $k_\tau : HC^*_\delta(\mathcal{H}_T) \longrightarrow HC^*(\mathcal{A})$, canonically associated to the pair $(\mathcal{A}, \tau)$. When computed, $HC^*_\delta(\mathcal{H}_T)$ gives the Gelfand-Fuchs cohomology and the characteristic map k_τ is a non-commutative version of the classical [1] characteristic map $k : H^*(WO_q) \longrightarrow H^*(M/\mathcal{F})$ for codimension q foliations $(M, \mathcal{F})$ (see [2, 10] for the relation between $HC^*(\mathcal{A})$ and $H^*(M/\mathcal{F})$). The definition of $HC^*_\delta(\mathcal{H})$ given in [8] applies to any Hopf algebra $\mathcal{H}$ endowed with a character δ, satisfying certain conditions (see the end of 2.4). In the context of $\mathcal{H}_T$ this provides a new beautiful relation of cyclic cohomology with Gelfand-Fuchs cohomology, while, in general, it can be viewed as a non-commutative extension of the Lie algebra cohomology (see Theorem 5.6).

Our first goal is to show that Connes-Moscovici formulas can be used under the minimal requirement $S^2_\delta = Id$ (which answers a first question raised in [8]), and to give a new definition/interpretation of $HC^*_\delta(\mathcal{H})$ in the spirit of Cuntz-Quillen's formalism.

Independently, in the work of Gelfand and Smirnov on universal Chern-Simons classes, there are implicit relations with cyclic cohomology ([18]). Our second goal is to make these connections explicit. This leads us to a noncommutative Weil complex $W(\mathcal{H})$ associated to a coalgebra, which extends the constructions from [18, 30].

Our third goal is to show that $W(\mathcal{H})$ is intimately related to the cyclic cohomology $HC^*_\delta(\mathcal{H})$ and to the construction (see 7.4, 7.6) of characteristic homomorphisms k_τ associated to invariant higher traces τ (which is a second problem raised in [8]). The construction of k_τ is inspired by the construction of the usual Chern-Weil homomorphism (see e.g [15]), and of the secondary characteristic classes for foliations ([1]).

This work is strongly influenced by the Cuntz-Quillen approach to cyclic cohomology ([12, 13, 29] etc).

Here is an outline of the paper. In Section 1 we bring together some basic results about characters δ and the associated twisted antipodes S_δ on Hopf algebras. In section 2 we present some basic terminology, describe the problem (see 2.4) of defining $HC^*_\delta(\mathcal{H})$, and explain why the case $S^2_\delta = Id$ is better behaved (see Proposition 2.5). Under this requirement, we define a cyclic cohomology $HC^*_{\delta-inv}(R)$ for any $\mathcal{H}$-algebra R, and we indicate how Cuntz-Quillen machinery can be adapted to this situation (Theorem 2.7). The relevant information which is needed for the cyclic cohomology of Hopf algebras, only requires a small part of this machinery. This is captured by a localized X-complex (denoted by $X_\delta(R)$); in Section 3, after recalling the X-complex interpretation of S-operations (see 3.1), we introduce $X_\delta(R)$ (see 3.2) and compute it in the case where $R = T(V)$ is the tensor algebra of an $\mathcal{H}$-module V (see Proposition 3.4).

In Section 4 we prove that $HC^*_\delta(\mathcal{H})$ can be defined under the minimal requirement $S^2_\delta = Id$ (see Proposition 4.4); also, starting with the question "which is the target of characteristic maps $k_\tau : HC^*_\delta(\mathcal{H}) \longrightarrow HC^*(A)$, associated to pairs (A, τ) consisting of a $\mathcal{H}$-algebra A

and a δ-invariant trace τ (see 2.4)?", we explain/interpret the definition of $HC^*_\delta(\mathcal{H})$ in terms of localized X-complexes (see 4.6, and Theorem 4.7). This interpretation is the starting point in constructing the characteristic maps associated to higher traces (Section 7). We also recall Connes-Moscovici's recent proposal to extend the definition to the non-unimodular case.

In Section 5 we give some examples, including $\mathcal{H} = U_q(sl_2)$, and a detailed computation of the fundamental example where $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$ (see Theorem 5.6).

In section 6 we introduce the non-commutative Weil complex (by collecting together 'forms and curvatures' in a non-commutative way). We show that there are two relevant types of cocycles involved (which, in the case considered by Gelfand and Smirnov, correspond to Chern classes, and Chern-Simons classes, respectively), we describe the Chern-Simons transgression, and prove that it is an isomorphism between these two types of cohomologies (Theorem 6.7). In connection with cyclic cohomology, we show that the non-commutativity of the Weil complex naturally gives rise to an S-operator, and to cyclic bicomplexes computing our cohomologies (see 6.8-6.10).

In section 7 we come back to Hopf algebra actions, and higher traces, and we show how the non-commutative Weil complex can be used to construct the characteristic map k_τ associated to higher traces (see 7.4, 7.6). To prove the compatibility with the S-operator (Theorems 7.5 and 7.7), we also show that the truncations of the Weil complex can be expressed in terms of relative X-complexes (Theorem 7.9). When $\mathcal{H} = \mathbf{C}$, we re-obtain the cyclic cocycles (and their properties) described by Quillen [29]. In general, the truncated Weil complexes still compute $HC^*_\delta(\mathcal{H})$, as explained by Theorem 7.3. Section 8 is devoted to the proof of this theorem, and the construction of characteristic maps associated to equivariant cycles.

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1 Preliminaries on Hopf Algebras

In this section we review some basic properties of Hopf algebras (see [31]) and prove some useful formulas on twisted antipodes.

Let $\mathcal{H}$ be a Hopf algebra. As usual, denote by S the antipode, by ϵ the counit, and by $\Delta(h) = \sum h_0 \otimes h_1$ the coproduct. Recall some of the basic relations they satisfy:

$$\sum \epsilon(h_0)h_1 = \sum \epsilon(h_1)h_0 = h, \tag{1}$$

$$\sum S(h_0)h_1 = \sum h_0 S(h_1) = \epsilon(h)\cdot 1, \tag{2}$$

$$S(1) = 1, \epsilon(S(h)) = \epsilon(h), \tag{3}$$

$$S(gh) = S(h)S(g), \tag{4}$$

$$\Delta S(h) = \sum S(h_1)\otimes S(h_0). \tag{5}$$

Throughout this paper, the notions of $\mathcal{H}$-module and $\mathcal{H}$-algebra have the usual meaning, with $\mathcal{H}$ viewed as an algebra. The tensor product $V\otimes W$ of two $\mathcal{H}$-modules is an $\mathcal{H}$-module with the diagonal action:

$$h(v\otimes w) = \sum h_0(v)\otimes h_1(w). \tag{6}$$

A character on $\mathcal{H}$ is any non-zero algebra map $\delta : \mathcal{H} \longrightarrow \mathbf{C}$. Characters will be used for 'localizing' modules: for an $\mathcal{H}$-module V, define V_δ as the quotient of V by the space of co-invariants (linear span of elements of type $h(v) - \delta(h)v$, with $h \in \mathcal{H}, v \in V$). In other words,

$$V_\delta = \mathbf{C}_\delta \otimes_{\mathcal{H}} \mathbf{V},$$

where $\mathbf{C}_\delta = \mathbf{C}$ is viewed as an $\mathcal{H}$-module via δ. Before looking at very simple localizations (see 1.3), we need to discuss the 'twisted antipode' $S_\delta : = \delta * S$ associated to a character δ (recall that $*$ denotes the natural product on the space of linear maps from the coalgebra $\mathcal{H}$ to the algebra $\mathcal{H}$, [31]). Explicitly,

$$S_\delta(h) = \sum \delta(h_0)S(h_1), \quad \forall h \in \mathcal{H}.$$

Lemma 1.1 *The following identities hold:*

$$\sum S_\delta(h_0)h_1 = \delta(h) \cdot 1, \tag{7}$$

$$S_\delta(1) = 1, \epsilon(S_\delta(h)) = \delta(h), \tag{8}$$

$$\Delta S_\delta(h) = \sum S(h_1) \otimes S_\delta(h_0), \tag{9}$$

$$S_\delta(gh) = S_\delta(h)S_\delta(g), \tag{10}$$

$$\sum S^2(h_1)S_\delta(h_0) = \delta(h) \cdot 1. \tag{11}$$

proof: These follow easily from the previous relations. For instance, the first relation follows from the definition of S_δ, (2), and (1), respectively:

$$\sum S_\delta(h_0)h_1 = \sum \delta(h_0)S(h_1)h_2 = \sum \delta(h_0)\epsilon(h_1) \cdot 1 = \delta(\sum h_0\epsilon(h_1)) \cdot 1 = \delta(h) \cdot 1.$$

The other relations are proved in a similar way. $\square$

Lemma 1.2 . *For any two $\mathcal{H}$-modules V, W:*

$$h(v) \otimes w \equiv v \otimes S_\delta(h)(w) \quad \textit{mod co-invariants}$$

proof: From the definition of S_δ, $v \otimes S_\delta(h)w = \sum \delta(h_0)v \otimes S(h_1)w$, so, modulo coinvariants, it is $\sum h_0(v) \otimes h_1S(h_2)(w) = \sum \epsilon(h_1)h_0(v) \otimes w = h(v) \otimes w$, where for the last two equalities we have used (2) and (1), respectively. $\square$

It follows easily that:

Corollary 1.3 . *For any $\mathcal{H}$-module V, there is an isomorphism:*

$$(\mathcal{H} \otimes V)_\delta \equiv V, \qquad (h, v) \mapsto S_\delta(h)v.$$

There is a well known way to recognize Hopf algebras with $S^2 = Id$ (see [31], pp. 74). We extend this result to twisted antipodes:

Lemma 1.4 . *For a character δ, the following are equivalent:*

(i) $S_\delta^2 = Id$,

(ii) $\sum S_\delta(h_1)h_0 = \delta(h)\cdot 1, \ \forall h \in \mathcal{H}$

proof: The first implication follows by applying S_δ to (7), using (10), and *(i)*. Now, assume *(ii)* holds. First, remark that $S \circ S_\delta = \delta$. Indeed,

$$(S * (S \circ S_\delta))(h) = \sum S(h_0)S(S_\delta(h_1)) = \sum S(S_\delta(h_1)h_0) = \delta(h)\cdot 1,$$

(where we have used the definition of $*$, (5), and (ii), respectively.) Multiplying this relation by Id on the left, we get $S \circ S_\delta = Id * \delta$. Using the definition of S_δ, (9), and the previous relation, respectively,

$$S_\delta^2(h) = \sum \delta(S_\delta(h)_0)S(S_\delta(h)_1) = \sum \delta(S(h_1)S(S_\delta(h_0)) = \sum \delta(S(h_2))h_0\delta(h_1),$$

which is (use that δ is a character, and the basic relations again):

$$\delta(\sum h_1 S(h_2))h_0 = \sum \delta(\epsilon(h_1))h_0 = \sum \epsilon(h_1)h_0 = h. \ \Box$$

2 Invariant traces

In this section we present some basic terminology like invariant traces, $\mathcal{H}$-algebras. For such an algebra R, the non-commutative differential forms on R can be localized, under the hypothesis $S_\delta^2 = Id$, and a cyclic cohomology $HC^*_{\delta-inv}(R)$ shows up. For completeness, we indicate how the Cuntz-Quillen machinery [12] can be adapted to this context (see Theorem 2.7); this extends, in particular, the usual correspondence ([12]) between δ-invariant cyclic cocycles and δ-invariant higher traces (with equivariant linear splitting).

2.1 Flat algebras: Let A be an algebra, not necessarily unital. An action $\mathcal{H}\otimes A \longrightarrow A$ of $\mathcal{H}$ (viewed as an algebra) on A is called flat (and say that A is a $\mathcal{H}$-algebra) if:

$$h(ab) = \sum h_0(a)h_1(b), \quad \forall h \in \mathcal{H}, \ a, b \in A \tag{12}$$

The motivation for the terminology is that, in our interpretations (see 4.6), it plays a role similar to the usual flat connections in geometry.

2.2 Invariant traces: Let $\mathcal{H}$ be a Hopf algebra endowed with a character δ, and A a $\mathcal{H}$-algebra. A trace $\tau : A \longrightarrow \mathbf{C}$ is called δ-invariant if:

$$\tau(ha) = \delta(h)\tau(a), \quad \forall h \in \mathcal{H}, \ a \in A.$$

If $\delta = \epsilon$ is the counit, we simply call τ invariant.

Recall [29] that an even (n dimensional) higher trace on an algebra R is given by an extension $0 \longrightarrow I \longrightarrow L \longrightarrow R \longrightarrow 0$ and a trace on L/I^{n+1}, while an odd higher trace is given by an extension as before, and an I- adic trace, i.e. a linear functional on I^{n+1} vanishing on $[I^n, I]$. Starting with an extension of $\mathcal{H}$-algebras, and a δ-invariant trace τ, we talk about equivariant (or δ-invariant) higher traces.

2.3 Examples: If $\mathcal{H} = \mathbf{C}[\Gamma]$ is the group algebra of a discrete group Γ (recall that $S(\gamma) = \gamma \otimes \gamma, \epsilon(\gamma) = 1$ if $\gamma = 1$, and 0 otherwise), $\mathcal{H}$-algebras are precisely Γ-algebras.

If G is a connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra, then $\mathcal{H}$-algebras are precisely infinitesimal G-algebras; that is, algebras A endowed with linear maps (Lie derivatives) $L_v : A \longrightarrow A$, linear on $v \in \mathfrak{g}$, such that $L_{[v,w]} = L_v L_w - L_w L_v$, $L_v(ab) = L_v(a)b + aL_v(b)$. If $\Delta : G \longrightarrow \mathbf{C}$ is a character, it induces an infinitesimal character δ on $U(\mathfrak{g})$: $\delta(v) = (\frac{d}{dt})_{t=0}\Delta(exp(tv))$, $v \in \mathfrak{g}$. If A is a topological (locally convex) G-algebra, δ-invariance of traces on A is equivalent to $\tau(ga) = \Delta(g)\tau(a)$, $\forall g \in G, a \in A$.

Remark that, in general, the action of a Hopf algebra on itself is not flat. A basic example of flat action is the diagonal action of $\mathcal{H}$ on its tensor algebra $T\mathcal{H} = \oplus_{n \geq 1}\mathcal{H}^{\otimes n}$ (see (6)). Another basic example is the algebra $\Omega^*(R)$ of noncommutative differential forms on a $\mathcal{H}$-algebra R. Recall that:

$$\Omega^n(R) = \tilde{R} \otimes R^{\otimes n},$$

where $\tilde{R}$ is R with a unit adjoined. Extending the action of $\mathcal{H}$ to $\tilde{R}$ by $h \cdot 1 := \epsilon(h)1$, we have an action of $\mathcal{H}$ on $\Omega^*(R)$ (the diagonal action). To check the flatness condition: $h(\omega\eta) = \sum h_0(\omega)h_1(\eta)$, $\forall \omega, \eta \in \Omega(R)$, remark that one can formally reduce to the case where ω and η are degree 1 forms, in which case the computation is easy.

Recall also the usual operators d, b, B, k acting on $\Omega^*(R)$ (see [11], paragraph 3 of [12]): $d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$, $b(\omega da) = (-1)^{deg(\omega)}[\omega, a]$, $k = 1 - (bd + db)$, $B = (1 + k + \ldots + k^n)d$ on $\Omega^n(R)$.

2.4 The Problem: Let δ be a character on a Hopf algebra $\mathcal{H}$. The problem of defining a cyclic cohomology $'HC^*_\delta(\mathcal{H})'$, should answer the question: which are the nontrivial cyclic cocycles on a $\mathcal{H}$-algebra A, arising from a δ-invariant trace τ, and the action of $\mathcal{H}$ on A. In particular, for any pair (A, τ) one should have an associated characteristic map:

$$k_\tau : HC^*_\delta(\mathcal{H}) \longrightarrow HC^*(A) \ ,$$

compatible with the S-operation on cyclic cohomology. There is a similar problem for invariant *higher* traces.

In [8], Connes and Moscovici have introduced $HC^*_\delta(\mathcal{H})$ under the hypothesis that there is an algebra $\mathcal{A}$, endowed with an action of $\mathcal{H}$, and with a δ-invariant *faithful* trace $\tau : \mathcal{A} \to \mathbf{C}$. As pointed out in [8], this requirement is quite strong; a more natural hypothesis would be the weaker condition $S^2_\delta = Id$.

Proposition 2.5 . *If $S^2_\delta = Id$, then for any $\mathcal{H}$-algebra R, the operators d, b, k, B, acting on $\Omega^*(R)$ (see 2.3), descend to $\Omega^*(R)_\delta$.*

proof: Since d commutes with the action of $\mathcal{H}$, and k, B (and all the other operators appearing in paragraph 3 of [12]) are made out of d, b, it suffices to prove that, modulo co-invariants,

$$b(h \cdot \eta) \equiv b(\delta(h)\eta), \quad \forall h \in \mathcal{H}, \ \eta \in \Omega^*(R).$$

For $\eta = \omega da$, one has$(-1)^{|\omega|}b(h \cdot \omega da) = \sum h_0(\omega)h_1(a) - \sum h_1(a)h_0(\omega)$.
Using Lemma 1.2 and (7), $\sum h_0(\omega)h_1(a) \equiv \sum \omega \cdot S_\delta(h_0)h_1 a = \delta(h)\omega a$.
Using Lemma 1.2, and (ii) of Lemma 1.4, $\sum h_1(a)h_0(\omega) \equiv \sum a \cdot S_\delta(h_1)h_0\omega = \delta(h)a\omega$, which ends the proof. $\square$

Definition 2.6 *($S_\delta^2 = Id$) Define the localized cyclic cohomology $HC^*_{\delta-inv}(R)$ of R as the cyclic cohomology of the mixed complex $\Omega^*(R)_\delta$. Similarly for Hochschild and periodic cyclic cohomologies, and also for homology.*

This cohomology is not used in the next sections, but it fits very well in our discussion of higher traces. Recall that, via a certain notion of homotopy, higher traces correspond exactly to cyclic cocycles on R (for the precise relations, see pp. 417- 419 in [12]). Using $HC^*_{\delta-inv}(R)$ instead of $HC^*(R)$, this relation extends to the equivariant setting (provided one restricts to higher traces which admit an equivariant linear splitting). The main ingredient is the following theorem which we include for completeness. It is analogous to one of the main results in [12] (Theorem 6.2). The notation $T(R)$ stands for the (non-unital) tensor algebra of R, and $I(R)$ is the kernel of the multiplication map $T(R) \to R$. Recall also that if M is a mixed complex [22], θM denotes the associated Hodge tower of M, which represents the cyclic homology type of the mixed complex (for more details on the notations and terminology see [12]).

Theorem 2.7 *There is a homotopy equivalence of towers of super-complexes:*

$$\mathcal{X}_\delta(TR, IR) \simeq \theta(\Omega^*(R)_\delta).$$

proof: The proof from [12] can be adapted. For this, one uses the fact that the projection $\Omega^*(R) \longrightarrow \Omega^*(R)_\delta$ is compatible with all the structures (with the operators, with the mixed complex structure). All the formulas we get for free, from [12]. The only thing we have to do is to take care of the action. For instance, in the computation of $\Omega^1(TR)_\natural$ (pp. $399 - 401$ in [12]), the isomorphism $\Omega^1(TR)_\natural \cong \Omega^-(R)$ is not compatible with the action of $\mathcal{H}$, but, using the same technique as in 1.2, it descends to localizations (which means that we can use the natural (diagonal) action we have on $\Omega^-(R)$). With this in mind, the analogous of Lemma 5.4 in [12] holds, that is, $\mathcal{X}_\delta(TR, IR)$ can be identified (without regarding the differentials) with the tower $\theta(\Omega^*(R)_\delta)$. Denote by k_δ the localization of k. The spectral decomposition with respect to k_δ is again a consequence of the corresponding property of k ([12], pp $389 - 391$ and pp. $402 - 403$), and the two towers are homotopically concentrated on the nillspaces of k_δ, corresponding to the eigenvalue 1. Lemma 6.1 of [12] identifies the two boundaries corresponding to this eigenvalues, which concludes the theorem. $\Box$

3 S-operations and X-complexes

In this section we recall Quillen's interpretation of a certain degree two cohomology operation ('S-operators') in terms of X-complexes, and describe a localized version (to be used in sections 4 and 6). As before, $\mathcal{H}$ is a Hopf algebra endowed with a character δ such that $S_\delta^2 = Id$.

3.1 S-operations: If R is a DG algebra, denote by $R_\natural = R/[R, R]$ the complex obtained dividing out by the linear span of graded commutators. In examples like tensor algebras, the algebras considered by Gelfand, Smirnov etc (see [18] and references therein), the noncommutative Weil complex of Section 6, and, in general when R is 'free', one encounters a very interesting degree two operation in the cohomology of $R_\natural$, $S : H^*(R_\natural) \longrightarrow H^{*+2}(R_\natural)$. This phenomenon, due to the non-commutativity of R, has been very nicely explained by Quillen ([29, 30]). In general, for any algebra R, there is a sequence:

$$0 \longrightarrow R_{\natural} \xrightarrow{d} \Omega^1(R)_{\natural} \xrightarrow{b} R \xrightarrow{\natural} R_{\natural} \longrightarrow 0, \tag{13}$$

Here $\Omega^1(R)_{\natural} = \Omega^1(R)/[\Omega^1(R), R]$, $b(xdy) = [x, y]$, and π is the projection. In our graded setting, one uses graded commutators, and (13) is a sequence of complexes. In general, it is exact in the right. When it is exact (and this happens in our examples), it can be viewed as an Ext^2 class, and induces a degree 2 operator $S : H^*(R_{\natural}) \longrightarrow H^{*+2}(R_{\natural})$, explicitly described by the following diagram chasing ([30], pp. 120). Given $\alpha \in H^k(R_{\natural})$, we represent it by a cocycle c, and use the exactness to solve successively the equations:

$$c = \natural(u),\ \partial(u) = b(v),\ \partial(v) = d(w).$$

where ∂ stands for the vertical boundary. Then $S(\alpha) = [\,\natural(w)] \in H^{k+2}(R_{\natural})$.

Equivalently, pasting together (13), we get a resolution, usually denoted by $X^+(R)$:

$$0 \longrightarrow R_{\natural} \xrightarrow{d} \Omega^1(R)_{\natural} \xrightarrow{b} R \xrightarrow{d} \Omega^1(R)_{\natural} \xrightarrow{b} R \longrightarrow \ldots$$

Emphasize that, when working with bicomplexes with anti-commuting differentials, one has to introduce a '$-$' sign for the even vertical boundaries (i.e. for those of R). So, one can use the cyclic bicomplex $X^+(R)$ to compute the cohomology of $R_{\natural}$, and then S is simply the shift operator.

The X-complex of R is simply the full version of $X^+(R)$, that is, the super-complex:

$$X(R) : \quad R \underset{d}{\overset{b}{\leftrightarrows}} \Omega^1(R)_{\natural} \quad , \tag{14}$$

where $b(xdy) = [x, y]$, $d(x) = dx$. It is defined in general, for any algebra, and it can be viewed as the degree one level of the Hodge tower associated to $\Omega^*(R)$. In our graded setting, it is a cyclic bicomplex.

3.2 The localized X-complex: When $R = T\mathcal{H}$ is the tensor DG algebra of $\mathcal{H}$, then $T\mathcal{H}_{\natural}$ computes the cyclic cohomology of $\mathcal{H}$, viewed as a coalgebra (cf. Theorem 4.2), and our previous discussion describes the usual S-operation in cyclic cohomology. We need a similar construction for $T\mathcal{H}_{\natural,\delta}$. Here, if R is a DG algebra endowed with a flat action of $\mathcal{H}$ compatible with the differentials (a $\mathcal{H}$-DG algebra on short), $R_{\natural,\delta} := R/[R, R] +$(coinvariants) denotes the complex obtained dividing out R by the linear span of graded commutators and coinvariants (i.e. elements of type $h(x) - \delta(h)x$, with $h \in \mathcal{H}$, $x \in R$).

Since $S_{\delta}^2 = Id$, we know (cf. Proposition 2.5) that b, d descend, and we define the localized X-complex of R as the degree one level of the Hodge tower associated to $\Omega^*(R)_{\delta}$. In other words, this is simply the super-complex (a cyclic bicomplex in our graded setting):

$$X_{\delta}(R) : \quad R_{\delta} \underset{d}{\overset{b}{\leftrightarrows}} \Omega^1(R)_{\natural,\delta} \quad ,$$

where:

$$\Omega^1(R)_{\natural,\delta} := \ \Omega^1(R)_{\delta}/b\Omega^2(R)_{\delta} = \ \Omega^1(R)/[\Omega^1(R), R] + \text{(coinvariants)},$$

and the formulas for b, d are similar to the ones for $X(R)$. There is one remark about the notation: $\Omega^1(R)_{\natural,\delta}$ is not the localization of $\Omega^1(R)_{\natural}$; in general, there is no natural action of $\mathcal{H}$ on it.

3.3 Example. Before proceeding, let's look at a very important example: the (non-unital) tensor algebra $R = T(V)$ of an $\mathcal{H}$-module V. Adjoining a unit, one gets the unital tensor algebra $\tilde{R} = \widetilde{T}(V) = \oplus_{n\geq 0} V^{\otimes n}$. The computation of $X(R)$ was carried out in [29], Example 3.10. One knows that ([12], pp. 395) $R = T(V)$, $\Omega^1(R)_\natural = V \oplus \widetilde{T}(V) = T(V)$, and also the description of the boundaries: $d = \sum_{i=0}^{i=n} t^i$, $b = (t-1)$ on $V^{\otimes(n+1)}$, where t is the backward-shift cyclic permutation. The second isomorphism is essentially due to the fact that, since V generates $T(V)$, any element in $\Omega^1(T(V))$ can be written in the form $xd(v)y$, with $x, y \in \tilde{T}(V), v \in V$ (see also the proof of the next proposition). To compute $X_\delta(R)$, one still has to compute its odd part. The final result is:

Proposition 3.4 . *For $R = T(V)$:*

$$X^0_\delta(R) = T(V)_\delta, \quad X^1_\delta(R) = T(V)_\delta,$$

where the action of $\mathcal{H}$ on $T(V)$ is the usual (diagonal), and the boundaries have the same description as the boundaries of $X(R)$: they are $(t-1)$, N (which descend to the localization). The same holds when V is a graded $\mathcal{H}$-module, provided we replace the the backward-shift cyclic permutation t by its graded version.

proof: One knows ([12], pp. 395):

$$\tilde{R} \otimes V \otimes \tilde{R} \xrightarrow{\sim} \Omega^1(R), \quad x \otimes v \otimes y \mapsto x(dv)y,$$

which, passing to commutators, gives (compare to [12], pp. 395):

$$R = V \otimes \tilde{R} \xrightarrow{\sim} \Omega^1(R)_\natural, \quad v \otimes y \longrightarrow \natural(dvy),$$

and the projection map $\natural : \Omega^1(R) \longrightarrow \Omega^1(R)_\natural$ identifies with:

$$\natural : \tilde{R} \otimes V \otimes \tilde{R} \longrightarrow V \otimes \tilde{R}, \quad x \otimes v \otimes y \longrightarrow v \otimes yx.$$

So $X^1_\delta(R)$ is obtained from $T(V)$, dividing out by the linear subspace generated by elements of type:

$$\natural(h \cdot x \otimes v \otimes y - \delta(h)x \otimes v \otimes y) = \sum h^1(v) \otimes h^2(y)h^0(x) - \delta(h)v \otimes yx \in T(V).$$

Now, for $y = 1$, this means exactly that we have to divide out by coinvariants (of the diagonal action of $\mathcal{H}$ on $T(V)$). But this is all, because modulo these coinvariants we have (from Lemma 1.2):

$$\sum h_1(v) \otimes h_2(y)h_0(x) \equiv \sum v \otimes S_\delta(h_1) \cdot (h_2(y)h_0(x)),$$

while, from (9), (2) and (ii) of Lemma 1.2, (1):

$$\sum S_\delta(h_1) \cdot (h_2(y)h_0(x)) = \sum S(h_2)h_3(y)S_\delta(h_1)h_0(x) = \sum \epsilon(h_1)\delta(h_0)yx = \delta(h)yx. \ \Box$$

4 Cyclic Cohomology of Hopf Algebras

In this section we introduce the cyclic cohomology of Hopf algebras (endowed with a character δ as before). First we prove that Connes-Moscovici's formulas can be used under the minimal condition $S_\delta^2 = Id$ (see 4.4). Next (see 4.6) we present a second approach to defining $HC_\delta^*(\mathcal{H})$ as the natural solution to our problem 2.4. The two approaches coincide, and this leads us to a X-complex interpretation of our cohomology (see Theorem 4.7). This interpretation is also the starting point in dealing with higher traces (section 7).
Let $\mathcal{H}$ be a Hopf algebra endowed with a character δ.

4.1 Cyclic cohomology of coalgebras: Looking first just at the coalgebra structure of $\mathcal{H}$, one defines the cyclic cohomology of $\mathcal{H}$ by duality with the case of algebras. As in [8], we define the Λ-module ([6]), denoted $\mathcal{H}^\natural$, which is $\mathcal{H}^{\otimes(n+1)}$ in degree n, whose co-degeneracies are:

$$d^i(h^0, \ldots, h^n) = \begin{cases} (h^0, \ldots, h^{i-1}, \Delta h^i, h^{i+1}, \ldots, h^n) & \text{if } 0 \leq i \leq n \\ \sum (h^0_{(1)}, h^1, \ldots, h^n, h^0_{(0)}) & \text{if } i = n+1 \end{cases} .$$

and whose cyclic action is:

$$t(h^0, \ldots, h^n) = (h^1, h^2, \ldots, h^n, h^0).$$

Denote by $HC^*(\mathcal{H})$ the corresponding cyclic cohomology, by $C_\lambda^*(\mathcal{H})$ the cyclic complex, and by $CC^*(\mathcal{H})$ the cyclic (upper plane) bicomplex (Quillen-Loday-Tsygan's) computing it. Recall that the DG tensor algebra of $\mathcal{H}$, denoted $T(\mathcal{H})$, is $\mathcal{H}^{\otimes n}$ in degrees $n \geq 1$ and 0 otherwise, and has the differential $b' = \sum_0^n (-1)^i d^i$. The following proposition shows that the S-operator acting on $HC^*(\mathcal{H})$ (apriori described by the shift on $CC^*(\mathcal{H})$), is the S-operator described by an X-complex:

Proposition 4.2 *Up to a shift on degrees, the cyclic bicomplex of $\mathcal{H}$, $CC^*(\mathcal{H})$ coincides with the X-complex of the DG algebra $T(\mathcal{H})$, and the cyclic complex $C_\lambda^*(\mathcal{H})$ is isomorphic to $T(\mathcal{H})_\natural$. This is true for any coalgebra.*

proof: It follows from the computation in the proof of Lemma 3.4, or by dualizing the analogous result for algebras (Theorem 4 and Lemma 2.1 of [29]).
□

Let us be more precise about the shifts. In a precise way, the proposition identifies $CC^*(\mathcal{H})$ with the super-complex of complexes:

$$\ldots \longrightarrow X^1(T\mathcal{H})[-1] \longrightarrow X^0(T\mathcal{H})[-1] \longrightarrow X^1(T\mathcal{H})[-1] \longrightarrow \ldots \quad ,$$

and gives an isomorphism: $C_\lambda^*(\mathcal{H}) \cong T(\mathcal{H})_\natural[-1]$.

4.3 Cyclic cohomology of Hopf algebras: Localizing the cyclic module $\mathcal{H}^\natural$, we obtain a new object, denoted $\mathcal{H}_\delta^\natural$. By Lemma 1.3, it is $\mathcal{H}^{\otimes n}$ in degree n, and the projection becomes:

$$\pi : \mathcal{H}^\natural \longrightarrow \mathcal{H}_\natural^\natural, \ \pi(h^0, h^1, \ldots, h^n) = S_\delta(h^0) \cdot (h^1, \ldots, h^n),$$

where '$\cdot$' stands for the diagonal action of $\mathcal{H}$ (cf. Section 2, (6)).

It is not true in general that the structure maps of $\mathcal{H}^\natural$ descend to maps $d_\delta^i, s_\delta^i, t_\delta$ on $\mathcal{H}_\delta^\natural$, but the compatibility with π forces the following formulas, which make sense in general (compare to [8], formulas (37) – (40)):

$$d_\delta^i(h^1, \dots, h^n) = \begin{cases} (1, h^1, \dots, h^n) & \text{if } i = 0 \\ (h^1, \dots, h^{i-1}, \Delta h^i, h^{i+1}, \dots, h^n) & \text{if } 1 \le i \le n \\ (h^1, \dots, h^n, 1) & \text{if } i = n+1 \end{cases} .$$

$$s_\delta^i(h^1, \dots, h^n) = (h^1, ..., \epsilon(h^{i+1}), \dots, h^n), \quad 0 \le i \le n-1,$$

$$t_\delta(h^1, \dots, h^n) = S_\delta(h_1) \cdot (h^2, \dots, h^n, 1)$$

Apriori $\mathcal{H}_\delta^\natural$ is just an ∞-cyclic [16] module (in the sense that the cyclic relation $t_\delta^{n+1} = 1$ is not necessarily satisfied). As pointed out by Connes and Moscovici, checking directly the cyclic relation $t_\delta^{n+1} = Id$ (which forces $S_\delta^2 = Id$) is not completely trivial. They have proved it in [8] under the assumption mentioned in 2.4.

Proposition 4.4 *Given a Hopf algebra $\mathcal{H}$ and a character δ, the previous formulas make $\mathcal{H}_\delta^\natural$ into a cyclic module if and only if $S_\delta^2 = Id$. More precisely:*

$$t_\delta^{n+1}(h^1, h^2, \dots, h^n) = (S_\delta^2(h^1), \dots, S_\delta^2(h^n)) .$$

Proof: Dualizing the construction for algebras (see [10, 16, 25]), to any coalgebra homomorphism $\theta : \mathcal{H} \longrightarrow \mathcal{H}$ one associates a ∞-cyclic module $\mathcal{H}^\natural(\theta)$. It is a slight modification of $\mathcal{H}^\natural$ of 4.1, obtained by replacing d^{n+1}, t in 4.1 by:

$$d^{n+1}(h^0, \dots, h^n) = \sum (h^0_{(1)}, h^1, \dots, h^n, \theta(h^0_{(0)})),$$

$$t(h^0, \dots, h^n) = (h^1, h^2, \dots, h^n, \theta(h^0)).$$

Choosing $\theta := S_\delta^2$, since $\pi : \mathcal{H}^\natural(\theta) \longrightarrow \mathcal{H}_\delta^\natural$ is surjective, it suffices to show that π is compatible with the structure maps. The non-trivial formulas are $\pi d_\theta^{n+1} = d_\theta^{n+1}\pi$, $\pi t_\theta = t_\theta \pi$. We prove the last one. We need the following two relations which follow easily from (5), (9):

$$\Delta^{n-1} S_\delta(h) = \sum S(h_{(n)}) \otimes \dots \otimes S(h_{(2)}) \otimes S_\delta(h_{(1)}), \tag{15}$$

$$\Delta^{n-1} S_\delta S(h) = \sum S^2(h_{(1)}) \otimes \dots \otimes S^2(h_{(n-1)}) \otimes S_\delta S(h_{(n)}). \tag{16}$$

(where the sums are over $\Delta^{n-1} h = \sum h_{(0)} \otimes ... \otimes h_{(n)}$.) We have :

$$\begin{aligned} & t_\delta \pi(h^0, h^1, \dots, h^n) = \\ & = \sum t_\delta((S(h^0_{(n)}), \dots, S(h^0_{(2)}), S_\delta(h^0_{(1)})) \star (h^1, \dots, h^n)) \\ & = \sum S_\delta(h^1) S_\delta S(h^0_{(n)}) \cdot (S(h^0_{(n-1)}), \dots, S(h^0_{(2)}), S_\delta(h^0_{(1)}), 1) \star (h^2, \dots, h^n, 1) . \end{aligned}$$

where $\star$ stands for the componentwise 'product' on $\mathcal{H}^{\otimes n}$. We want to prove it equals to $\pi t_\theta(h^0, h^1, ..., h^n) = S_\delta(h^1) \cdot (h^2, ..., h^n, S_\delta^2(h^0)) = S_\delta(h^1) \cdot (1, ..., 1, S_\delta^2(h^0)) \star (h^2, ..., h^n, 1)$, so it suffices to show that for any $h^0 = h \in \mathcal{H}$:

$$\sum S_\delta S(h_{(n)}) \cdot (S(h_{(n-1)}), \dots, S(h_{(2)}), S_\delta(h_{(1)}), 1) = (1, \dots, S_\delta^2(h)). \tag{17}$$

Using (16), the left hand side is:

$$\sum (S^2(h_{(n)})S(h_{(n-1)}), \ldots, S^2(h_{(2n-3)})S(h_{(2)}), S^2(h_{(2n-2)})S(h_{(1)}), S_\delta S(h_{(2n-1)})) \tag{18}$$

Using successively (11) for $\delta = \epsilon$, (1), and the coassociativity of Δ, this is:

$$\sum (1, ..., 1, \epsilon(h_{(2)}), S^2(h_{(3)})S_\delta(h_{(1)}), S_\delta S(h_{(4)})) =$$
$$= \sum (1, ..., 1, S^2(h_{(2)})S_\delta(h_{(1)}), S_\delta S(h_{(3)})) = \sum (1, ..., 1, \delta(h_{(1)}), S_\delta S(h_{(2)})) \ ,$$

by (11). Since $\sum \delta(h_{(1)})S_\delta S(h_{(2)}) = S_\delta(\sum \delta(h_{(1)})S(h_{(2)})) = S_\delta^2(h)$, we obtain the right hand side of (17). $\square$

Definition 4.5 *If $S_\delta^2 = Id$, define $HC_\delta^*(\mathcal{H})$ as the cohomology defined by the cyclic module $\mathcal{H}_\delta^\natural$; denote by $C_{\lambda,\delta}^*(\mathcal{H})$ the associated cyclic complex, and by $CC_\delta^*(\mathcal{H})$ the associated cyclic bicomplex.*

In connection to our problem 2.4, to any δ-invariant trace τ on a $\mathcal{H}$-algebra A one associates a characteristic map $k^\tau : HC_\delta^*(\mathcal{H}) \longrightarrow HC^*(A)$,

$$k^\tau(h_1, \ldots, h_n)(a_0, a_1, \ldots, a_n) = \tau(a_0 h_1(a_1) \ldots h_n(a_n)), \tag{19}$$

which is compatible with the S-operator (since it exists at the level of cyclic modules). Next we interpret/motivate this characteristic map, as well as the cohomology under discussion.

4.6 The (localized) characteristic map: Let A be a $\mathcal{H}$-algebra, and let $\tau : A \longrightarrow \mathbf{C}$ be a trace on A. There is an obvious map induced in cyclic cohomology (which uses just the coalgebra structure of $\mathcal{H}$):

$$\gamma^\tau : HC^*(\mathcal{H}) \longrightarrow HC^*(A), \ (h^0, \ldots, h^n) \longrightarrow \gamma(h^0, \ldots, h^n), \tag{20}$$
$$\gamma(h^0, \ldots, h^n)(a_0, \ldots, a_n) = \tau(h^0(a_0) \cdot \ldots \cdot h^n(a_n)).$$

In order to find the relevant complexes in the case of invariant traces, we give a different interpretation of this simple map. We can view the action of $\mathcal{H}$ on A, as a linear map:

$$\gamma_0 : \mathcal{H} \longrightarrow Hom(B(A), A)^1 = Hom_{lin}(A, A)$$

where $B(A)$ is the (DG) bar coalgebra of A. Recall that $B(A)$ is $A^{\otimes n}$ in degrees $n \geq 1$ and 0 otherwise, with the coproduct:

$$\Delta(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n),$$

and with the usual b' boundary as differential. Then $Hom(B(A), A)$ is naturally a DG algebra (see [29]), with the product: $\phi * \psi := m \circ (\phi \otimes \psi) \circ \Delta$ (m stands for the multiplication on A). Explicitly, for $\phi, \psi \in Hom(B(A), A)$ of degrees p and q, respectively,

$$(\phi * \psi)(a_1, \ldots, a_{p+q}) = (-1)^{pq} \phi(a_1, \ldots, a_p) \psi(a_{p+1}, \ldots, a_{p+q}) \ ,$$

The map γ_0 uniquely extends to a DG algebra map:

$$\tilde{\gamma} : T(\mathcal{H}) \longrightarrow Hom(B(A), A). \tag{21}$$

This can be viewed as a characteristic map for the flat action (see Proposition 6.2). Recall also ([29]) that the norm map N can be viewed as a closed cotrace $N : C_*^{\lambda}(A)[1] \to B(A)$ on the DG coalgebra $B(A)$, that is, N is a chain map with the property that $\Delta \circ N = \sigma \circ \Delta \circ N$, where σ is the graded twist $x \otimes y \mapsto (-1)^{deg(x)deg(y)} y \otimes x$. A formal property of this is that, composing with N and τ, we have an induced trace:

$$\tau_{\natural} : Hom(B(A), A) \longrightarrow C_{\lambda}^*(A)[1], \quad \tau_{\natural}(\phi) = \tau \circ \phi \circ N. \tag{22}$$

Composing with $\tilde{\gamma}$, we get a trace on the tensor algebra:

$$\tilde{\gamma}^{\tau} : T(\mathcal{H}) \longrightarrow C_{\lambda}^*(A)[1], \tag{23}$$

and then a chain map:

$$\gamma^{\tau} : T(\mathcal{H})_{\natural} \longrightarrow C_{\lambda}^*(A)[1]. \tag{24}$$

Via Proposition 4.2, it induces (20) in cohomology.

Let's now start to use the Hopf algebra structure of $\mathcal{H}$, and the character δ. First of all remark that the map $\tilde{\gamma}$ is $\mathcal{H}$-invariant, where the action of $\mathcal{H}$ on the right hand side of (21) comes from the action on A: $(h \cdot \phi)(a) = h\phi(a), \ \forall \ a \in B(A)$. To check the invariance condition: $\tilde{\gamma}(hx) = h\tilde{\gamma}(x), \ \forall \ x \in T(\mathcal{H})$, remark that the flatness of the action reduces the checking to the case where $x \in \mathcal{H} = T(\mathcal{H})^1$, and that is obvious. Secondly, remark that if the trace τ is δ-invariant, then so is (22). In conclusion, $\tilde{\gamma}^{\tau}$ in (23) is an invariant trace on the tensor algebra, so our map (24) descends to a chain map:

$$\gamma_{\delta}^{\tau} : T(\mathcal{H})_{\natural,\delta} \longrightarrow C_{\lambda}^*(A)[1],$$

So $H^*(TH_{\natural,\delta})$ naturally appears as the solution of our problem 2.4; also, using the localized X-complex $X_{\delta}(T\mathcal{H})$ (see 3.2), we have a short exact sequence:

$$0 \longrightarrow T\mathcal{H}_{\natural,\delta} \xrightarrow{N} T\mathcal{H}_{(\delta)} \xrightarrow{1-t} T\mathcal{H}_{(\delta)} \longrightarrow T\mathcal{H}_{\natural,\delta} \longrightarrow 0 \tag{25}$$

describing an S-operation (cf 3.1) in our cohomology $H^*(TH_{\natural,\delta})$. These new objects are related to 4.5 by the following (compare to Proposition 4.2):

Theorem 4.7 *Given a Hopf algebra $\mathcal{H}$ and a character δ such that $S_{\delta}^2 = Id$, one has isomorphisms:*

$$C_{\lambda,\delta}^*(\mathcal{H}) \cong TH_{\natural,\delta}, \quad CC_{\delta}^*(\mathcal{H}) \cong X_{\delta}(T\mathcal{H}),$$

up to the same degree shift as in Proposition 4.2.

proof: We have seen in Proposition 3.4:

$$\Omega^1(T\mathcal{H})_{\natural} \cong T\mathcal{H}, \quad \Omega^1(T\mathcal{H})_{\natural,\delta} \cong (T\mathcal{H})_{\delta}.$$

The first isomorphism is the one which gives the identification $X(T\mathcal{H}) \cong CC^*(\mathcal{H})$ of Proposition 4.2. The second isomorphism, combined with the isomorphism (cf. Lemma 1.3):

$$(T\mathcal{H})_{\delta}^{n+1} \cong \mathcal{H}^{\otimes n}, \ [h_0 \otimes h_1 \otimes \ldots \otimes h_n] \mapsto S_{\delta}(h_0) \cdot (h_1 \otimes \ldots \otimes h_n),$$

(with the inverse $h_1 \otimes \ldots \otimes h_n \mapsto [1 \otimes h_1 \otimes \ldots \otimes h_n]$), gives the identification $X_{\delta}(T\mathcal{H}) \cong CC_{\delta}^*(\mathcal{H})$. $\square$

4.8 The uni-modular case: Motivated by examples like quantum groups, compact matrix groups [33] and their duals, Connes and Moscovici have recently proposed [9] an extension of $HC^*_\delta(\mathcal{H})$ to the more general case where S_δ is not necessarily involutive, but there exists an invertible group-like element $\sigma \in \mathcal{H}$ such that:

$$S_\delta^2(h) = \sigma h \sigma^{-1} \ \ \forall\, h \in \mathcal{H}, \quad \delta(\sigma) = 1. \tag{26}$$

In the terminology of [9], one says that (δ, σ) is a modular pair. For any such pair (δ, σ), one defines a cyclic module $\mathcal{H}^\sharp_{\delta,\sigma}$ by the same formulas as in 4.3 except for:

$$d^{n+1}_{\delta,\sigma}(h^1, \ldots, h^n) = (h^1, \ldots, h^n, \sigma) \ ,$$

$$t_{\delta,\sigma}(h^1, \ldots, h^n) = S_\delta(h_1)(h^2, \ldots, h^n, \sigma) \ .$$

Let $C^*_{\lambda,\delta,\sigma}(\mathcal{H})$, $CC^*_{\delta,\sigma}(\mathcal{H})$ be the associated cyclic complex, and cyclic bicomplex, respectively. The resulting cohomology is denoted by $HC^*_{\delta,\sigma}(\mathcal{H})$, and appears as the target of characteristic maps associated to pairs (A, τ) with τ a δ-invariant σ-trace (i.e. $\tau(ab) = \tau(b\sigma(a))$).

Our interpretations extend to this setting. For any $\mathcal{H}$-algebra R, we define the following localized complex:

$$X_{\delta,\sigma}(R): \quad R_\delta \ \underset{d}{\overset{b_\sigma}{\rightleftarrows}} \ \Omega^1(R)_{\natural,\delta} \ \ ,$$

where, this time, $b_\sigma(dxy) = -[x,y]_\sigma$, where $[x,y]_\sigma$ is the twisted commutator $xy - y\sigma(x)$, and $\Omega^1(R)_{\natural,\delta}$ is the quotient of $\Omega^1(R)$ by the subspace linearly spanned by coinvariants and twisted commutators $[x,\omega]_\sigma$ ($x \in R$, $\omega \in \Omega^1(R)$). Similarly one defines $R_{\natural,\delta}$, which fits into a sequence (exact on the right):

$$0 \longrightarrow R_{\natural,\delta} \xrightarrow{d} \Omega^1(R)_{\natural,\delta} \xrightarrow{b_\sigma} R_\delta \xrightarrow{\natural} R_{\natural,\delta} \longrightarrow 0.$$

As for $X_\delta(R)$, there is an obvious extension to the graded case.

Theorem 4.9 *Let (δ,σ) be as before. Then, for any $\mathcal{H}$ (DG) algebra R, $X_{\delta,\sigma}(R)$ is a well defined complex. For $R = T\mathcal{H}$:*

$$T\mathcal{H}_{\natural,\delta} \cong C^*_{\lambda,\delta,\sigma}(\mathcal{H}), \quad , X_{\delta,\sigma}(T\mathcal{H}) \cong CC^*_{\delta,\sigma}(\mathcal{H}) \ ,$$

up to the same degree shift as in Proposition 4.2.

proof: The first part follows from the fact that $d : R \longrightarrow \Omega^1(R)$, and $b_\sigma : \Omega^1(R)$ map coinvariants into coinvariants (with the same proof as for 2.5), b_σ kills the twisted commutators, $b_\sigma d = 0$ modulo coinvariants (straightforward), and $db_\sigma = 0$ in $\Omega^1(R)_{\natural,\delta}$. The last assertion follows from $\delta(\sigma) = 1$, and the relation:

$$db_\sigma(dxy) = [x, dy]_\sigma - [\sigma^{-1}(y), dx]_\sigma + (\sigma^{-1}(\omega) - \omega),$$

where $\omega = yd(\sigma(x))$. The second part is a straightforward extension of 3.4, 4.7. $\square$

One can also extend our interpretations 4.6 of the characteristic map.

5 Some Examples

In this section we compute the cohomology under discussion in several examples. Unless specified, (δ, σ) is a pair consisting of a character, and an invertible group-like element as in 4.8 (i.e. satisfying $S_\delta^2(h) = \sigma h \sigma^{-1}$). In most of our examples, $\sigma = 1$.
As a technical tool, let's remark that the complex computing $HH^*_{\delta,\sigma}(\mathcal{H})$ depends just on the coalgebra structure of $\mathcal{H}$, and the group like elements $1, \sigma \in \mathcal{H}$. More precisely, denoting by $\mathbf{C}_\sigma$ the (left/right) one-dimensional $\mathcal{H}$ comodule induced by the group-like element σ, and by $\mathbf{C}$ the one corresponding to $\sigma = 1$, we have:

Lemma 5.1 *There are isomorphisms:*

$$HH^*_{\delta,\sigma}(\mathcal{H}) \cong Cotor^*_{\mathcal{H}}(\mathbf{C}, \mathbf{C}_\sigma) \ ,$$

proof: For any group-like element σ one has a standard resolution $\mathbf{C}_\sigma \stackrel{\sigma}{\longrightarrow} B(\mathcal{H}, \mathbf{C}_\sigma)$ of $\mathbf{C}_\sigma$ by (free) left $\mathcal{H}$ comodules. Here $B(\mathcal{H}, \mathbf{C}_\eta)$ is $\mathcal{H}^{\otimes(n+1)}$ in degree n, and has the boundary:

$$d_\sigma{}'(h^0, \ \ldots, h^n) = \sum_{i=0}^{n} (-1)^i (h^0, \ \ldots, \Delta(h^i), \ \ldots, h^n) + (-1)^{n+1}(h^0, \ \ldots, h^n, \sigma). \quad (27)$$

Hence $Cotor_{\mathcal{H}}(\mathbf{C}, \mathbf{C}_\sigma)$ is computed by the chain complex $\mathbf{C}\Box_{\mathcal{H}}\mathbf{B}(\mathcal{H}, \mathbf{C}_\sigma)$, that is, by the Hochschild complex of $\mathcal{H}^\natural_\delta$. $\Box$

5.2 Example (group-algebras): If $\mathcal{H} = \mathbf{C}[\Gamma]$ is the group algebra of a discrete group Γ (see 2.3), we have:

$$HP^0_\epsilon(\mathbf{C}[\Gamma]) \cong \mathbf{C}, \quad HP^1_\epsilon(\mathbf{C}[\Gamma]) \cong 0$$

($\epsilon =$ the counit, $\sigma = 1$).

proof: We have the following periodic resolution I^* of $\mathbf{C}$ by free $\mathbf{C}[\Gamma]$-comodules:

$$0 \longrightarrow \mathbf{C} \stackrel{\eta}{\longrightarrow} \mathbf{C}[\Gamma] \stackrel{\alpha}{\longrightarrow} \mathbf{C}[\Gamma] \stackrel{\beta}{\longrightarrow} \mathbf{C}[\Gamma] \stackrel{\alpha}{\longrightarrow} \mathbf{C}[\Gamma] \longrightarrow \ldots$$

where $\eta(1) = 1$, $\alpha(g) = g$ for $g \neq 1$ and $\alpha(1) = 0$, $\beta(g) = 0$ for $g \neq 1$ and $\beta(1) = 1$. Hence $HH_\epsilon(\mathcal{H}) = Cotor_{\mathcal{H}}(\mathbf{C}, \mathbf{C})$ is computed by $\mathbf{C}\Box_{\mathcal{H}} I^*$, that is, by $0 \longrightarrow \mathbf{C} \stackrel{0}{\longrightarrow} \mathbf{C} \stackrel{id}{\longrightarrow} \mathbf{C} \stackrel{0}{\longrightarrow} \mathbf{C} \stackrel{id}{\longrightarrow} \ldots$. So $HH^*_\epsilon(\mathcal{H}) = \mathbf{C}$ if $n = 0$ and 0 otherwise, and the statement follows from the SBI sequence. $\Box$

5.3 Example (algebras with Haar integrals): Recall that a left Haar integral for the Hopf algebra $\mathcal{H}$ is a linear map $\tau : \mathcal{H} \longrightarrow \mathbf{C}$ with the property $\tau(1) = 1$, $\sum \tau(h_0)h_1 = \tau(h) \cdot 1$ for all $h \in \mathcal{H}$. Basic Hopf algebras which admit Haar integral are: finite dimensional Hopf algebras (by 5.1.6 of [31]), group-algebras, algebras of smooth functions on a compact quantum group G (by the fundamental Theorem 4.2 of [33]). We recall that in the case of compact matrix groups there is a preferred choice of the character δ, namely the modular character f_{-1} of Theorem 5.6 [33]. One has:

Proposition 5.4 *If $\mathcal{H}$ admits a left Haar measure then:*

$$HP^0_\delta(\mathbf{C}[\Gamma]) \cong \mathbf{C}, \quad HP^1_\delta(\mathbf{C}[\Gamma]) \cong 0$$

proof: Use the SBI sequence and the fact that the left integral τ induces a contraction $(h^0, \ldots, h^n) \mapsto \tau(h^0)(h^1, \ldots, h^n)$ of the Hochschild complex. $\square$

5.5 Example (enveloping algebras): Following [8] (Theorem 6.(i)), we present now a detailed computation for the case where $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$. Let δ be a character of $\mathfrak{g}$ (i. e. $\delta : \mathfrak{g} \to \mathbf{C}$ linear, with $\delta|_{[\mathfrak{g},\mathfrak{g}]} = 0$), and extend it to $U(\mathfrak{g})$. Denote by $\mathbf{C}_\delta$ the $\mathfrak{g}$ module $\mathbf{C}$ with the action induced by δ. Since $S_\delta(v) = -v + \delta(v)$ for all $v \in \mathfrak{g}$, we are in the uni-modular case ($\sigma = 1$). The final result of our computation is:

Theorem 5.6 *For any Lie algebra $\mathfrak{g}$, and any $\delta \in \mathfrak{g}^*$:*

$$HP^*_\delta(U(\mathfrak{g})) \cong \bigoplus_{i \equiv * \mathrm{mod}\ 2} H_i(\mathfrak{g}; \mathbf{C}_\delta).$$

As a first step in the proof of 5.6, let's look at the symmetric (Hopf) algebra $S(V)$ on a vector space V. Recall that the coproduct is defined on generators by $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\forall v \in V$.

Lemma 5.7 [1] *For any vector space V, the maps $A : \Lambda^n(V) \longrightarrow S(V)^{\otimes n}$, $v_1 \wedge \ldots \wedge v_n \mapsto (\sum_\sigma sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!$ induce isomorphisms:*

$$HH^*_\delta(S(V)) \cong \Lambda^*(V).$$

proof: We will use a Koszul type resolution for the left $S(V)$ comodule $\mathbf{C}_\delta$. Let $e_1, \ldots, e_k$ be a basis of V, and $\pi^i \in V^*$ the dual basis. The linear maps π^i extend uniquely to derivations $\pi^i : S(V) \longrightarrow S(V)$. Remark that each of the π^i's are maps of left $S(V)$ comodules. Indeed, to check that $(1 \otimes \pi^i) \circ \Delta = \Delta \circ \pi^i$, since both sides satisfy the Leibniz rule, it is enough to check it on the generators $e_i \in S(V)$, and that is easy. Consider now the co-augmented complex of left $S(V)$ comodules:

$$0 \longrightarrow \mathbf{C}_\eta \stackrel{\eta}{\longrightarrow} \mathbf{S(V)} \otimes \mathbf{\Lambda^0(V)} \stackrel{\mathbf{d}}{\longrightarrow} \mathbf{S(V)} \otimes \mathbf{\Lambda^1(V)} \stackrel{\mathbf{d}}{\longrightarrow} \ldots,$$

with the boundary $d = \sum \pi^i \otimes e_i$, that is:

$$d(x \otimes v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{k} \pi^i(x) \otimes e_i \wedge v_1 \wedge \ldots \wedge v_n.$$

Point out that the definition does not depend on the choice of the basis, and it is dual to the Cartan boundary on the Weil complex of V, viewed as a commutative Lie algebra. This also explains the exactness of the sequence. Alternatively, one can use a standard 'Koszul argument', or, even simpler, remark that $(S(V) \otimes \Lambda^*(V)) \otimes (S(W) \otimes \Lambda^*(W)) \equiv (S(V \oplus W) \otimes \Lambda^*(V \oplus W))$ as chain complexes (for any two vector spaces V and W), which reduces the

[1] recently it has been pointed out to me that a version of this is due to P. Cartier, and appears also in [22], pp. 435- 442

assertion to the case where $dim(V) = 1$. So we get a resolution $\mathbf{C}_\eta \longrightarrow \mathbf{S(V)} \otimes \mathbf{\Lambda^*(V)}$ by free (hence injective) left $S(V)$ comodules. Then 5.1 implies that $HH^*_\delta(S(V))$ is computed by $\mathbf{C}_\eta \Box_{\mathbf{S(V)}} (\mathbf{S(V)} \otimes \mathbf{\Lambda^*(V)})$, that is, by $\Lambda^*(V)$ with the zero differential. This proves the second part of the theorem.

To show that the isomorphism is induced by A, we have to compare the previous resolution with the standard bar resolution $B(S(V), \mathbf{C}_\eta)$ (see the proof of 5.1). We define a chain map of left $S(V)$ comodules:

$$P : B(S(V), \mathbf{C}_\eta) \longrightarrow \mathbf{S(V)} \otimes \mathbf{\Lambda^*(V)} \ ,$$

$$P(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = x_0 \otimes pr(x_1) \wedge \ldots \wedge pr(x_n),$$

where $pr : S(V) \longrightarrow V$ is the obvious projection map. We check now that it is a chain map, i.e.:

$$dP(x_0 \otimes x_1 \otimes ... \otimes x_n) = Pd(x_0 \otimes x_1 \otimes \ldots \otimes x_n).$$

First of all, we may assume $x_1, \ldots, x_n \in V$ (otherwise, both terms are zero). The left hand side is then:

$$\sum_{i=1}^{k} \pi^i(x_0) \otimes e_i \wedge x_1 \wedge \ldots \wedge x_n,$$

while the right hand side is:

$$P(\Delta(x_0) \otimes x_1 \otimes \ldots \otimes x_n) = (id \otimes pr)(\Delta(x_0)) \wedge x_1 \wedge \ldots \wedge x_n \ .$$

So we are left with proving that:

$$(id \otimes pr)\Delta(x) = \sum_{i=1}^{k} \pi^i(x) \otimes e_i, \quad \forall\ x \in S(V),$$

and this can be checked directly on the linear basis $x = e_{i_1} \ldots e_{i_n} \in S(V)$. In conclusion, P is a chain map between our free resolutions of $\mathbf{C}_\eta$ (in the category of left $S(V)$ comodules). By the usual homological algebra, the induced map $\bar{P}$ obtained after applying the functor $\mathbf{C}_\eta \Box_{\mathbf{S(V)}}$- , induces isomorphism in cohomology. From the explicit formula:

$$\bar{P}(x_1 \otimes \ldots \otimes x_n) = pr(x_1) \wedge \ldots \wedge pr(x_n),$$

we see that $\bar{P} \circ A = Id$, so our isomorphism is induced by both $\bar{P}$ and A. $\Box$

proof of 5.6: Consider the mixed complex [22]:

$$\Lambda : \ \Lambda^0(\mathfrak{g}) \underset{0}{\overset{d_{Lie}}{\rightleftarrows}} \Lambda^1(\mathfrak{g}) \underset{0}{\overset{d_{Lie}}{\rightleftarrows}} \Lambda^2(\mathfrak{g}) \underset{0}{\overset{d_{Lie}}{\rightleftarrows}} \ \cdots \ ,$$

where d_{Lie} stands for the usual boundary in the Chevalley-Eilenberg complex computing $H_*(\mathfrak{g})$. Denote by $\mathcal{B}$ the mixed complex associated to the cyclic module $\mathcal{H}^\natural$, and by $\mathcal{B}_\delta$ its localization, i.e. the mixed complex associated to the cyclic module $\mathcal{H}^\natural_\delta$ (so they are the mixed complexes computing $HC^*(\mathcal{H})$, and $HC^*_\delta(\mathcal{H})$, respectively). Here $\mathcal{H} = U(\mathfrak{g})$. Let $\pi : \mathcal{B} \to \mathcal{B}_\delta$ be the projection map, which, after our identifications (see 4.3), is degree-wise given by:

$$\pi : \mathcal{H}^{\otimes(n+1)} \longrightarrow \mathcal{H}^{\otimes n},\ \pi(h_0 \otimes \ldots \otimes h_n) = S_\delta(h_0) \cdot (h_1 \otimes \ldots \otimes h_n).$$

Denote by B and B_δ the usual (degree (-1)) 'B- boundaries' of the two mixed complexes $\mathcal{B}, \mathcal{B}_\delta$. Recall that $B = N\sigma_{-1}\tau$, where:

$$\sigma_{-1}(h_0, \ldots, h_n) = \epsilon(h_0)(h_1, \ldots, h_n),\ \tau(h_0, \ldots, h_n) = (-1)^n(h_1, \ldots, h_n, h_0)\ ,$$

and $N = 1 + \tau + \ldots + \tau^n$ on $\mathcal{H}^{\otimes(n+1)}$.

We will show that Λ and $\mathcal{B}_\delta$ are quasi-isomorphic mixed complexes (which easily implies the theorem), but for the computation we have to use the mixed complex $\mathcal{B}$, where explicit formulas are easier to write. We define the map:

$$A : \Lambda^n(\mathfrak{g}) \longrightarrow \mathcal{H}^{\otimes n},\ A(v_1 \wedge \ldots \wedge v_n) = (\sum_\sigma sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!.$$

The fact that the (localized) Hochschild boundary depends just on the coalgebra structure of $U(\mathfrak{g})$ and on the unit, which are preserved by the Poincare-Birkhoff-Witt Theorem (see e.g. [32]), together with Lemma 5.7, shows that A is a quasi-isomorphism of mixed complexes, provided we prove its compatibility with the degree (-1) boundaries, that is:

$$B_\delta(A(x)) = A(d_{Lie}(x)),\quad \forall\ x = v_1 \wedge \ldots \wedge v_n \in \Lambda^n(\mathfrak{g}). \tag{28}$$

Using that $A(x) = \pi(y)$, where $y = (\sum sign(\sigma) 1 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!$, we have

$$\begin{aligned} B_\delta A(x) &= \pi(B(y)) = \\ &= \pi N \sigma_{-1}(\sum sign(\sigma)\ (1 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} - (-1)^n v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \otimes 1))/n! \\ &= \pi(N(\sum sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}))/n! \\ &= \pi(\sum sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/(n-1)!\ . \end{aligned}$$

But

$$\pi(v \otimes v_1 \otimes \ldots \otimes v_n) = \delta(v) v_1 \otimes \ldots \otimes v_n - \sum_{i=1}^n v_1 \otimes \ldots \otimes v v_i \otimes \ldots v_n\ ,$$

and, with these, it is straightforward to see that $B_\delta A(x)$ equals to:

$$A(\sum_{i=1}^n (-1)^{i+1} \delta(v_1) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_n + \sum_{i<j} (-1)^{i+j} [v_i, v_j] \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_n),$$

i.e. with $A(d_{Lie}(x))$. $\square$

5.8 Example (the quantum enveloping algebra of sl_2): We look now at the simplest example of a quantized envelopping algebra, namely $U_q(sl_2)$. As an algebra, it is generated by the symbols E, F, K, K^{-1}, subject to the relations: $KE = q^2 EK, KF = q^{-2} FK, KK^{-1} = K^{-1}K = 1, [E, F] = (K - K^{-1})/(q - q^{-1})$. The co-algebra structure is given by:

$$\Delta(K) = K \otimes K,\ \Delta(E) = 1 \otimes E + E \otimes K,\ \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\epsilon(K) = 1,\ \epsilon(E) = \epsilon(F) = 0,$$

while for the antipode: $S(K) = K^{-1}$, $S(E) = -EK^{-1}$, $S(F) = -KF$. One has $S^2(h) = KhK^{-1}$ for all $h \in U_q(sl_2)$, hence this is a first example with $\sigma \neq 1$.

Proposition 5.9 *$HP^0(U_q(sl_2)) = 0$, and $HP^1(U_q(sl_2)) \cong \mathbf{C}^2$ with the generators represented by E and KF.*
(we have ommited the indices $\delta = \epsilon$, $\sigma = K$ from the notation)

proof: Denote $E = x$, $K = \sigma$, $KF = y$. Clearly x, and y define cyclic cocycles. Using the SBI sequence, it suffices to prove the similar statement for Hochschild cohomology. We first prove that, for any n,

$$Cotor_{\mathcal{H}}(\mathbf{C}, \mathbf{C}_{\sigma^a}) = 0, \quad \forall\ a < 0 \ . \tag{29}$$

We use induction over n. It is obvious for $n = 0$; let's assume it is true for any $k < n$. Remark that, by the proof of 5.1, $Cotor(\mathbf{C}_\alpha, \mathbf{C}_\beta)$ (α, β- group-like elements) is computed by the complex $B(\mathcal{H}; \mathbf{C}_\alpha, \mathbf{C}_\beta)$, which is $\mathcal{H}^{\otimes n}$ in degree n and has the boundary $u \mapsto (\alpha, u) - d_\beta{}'(u)$ (see (27)). Denote $B_a = B(\mathcal{H}; \mathbf{C}, \mathbf{C}_{\sigma^a})$, and d_a its boundary.
One has the following basis of $\mathcal{H}$:

$$\{x^m y^k \sigma^p : m, k, p \ - \text{integers} \ \ , m, k \geq 0\} \ . \tag{30}$$

Let '$\leq$' be the order $(m, k) \leq (m', k')$ iff $m \leq m'$, or $m = m'$ and $k \leq k'$. For any pair (m, k) of positive integers, denote by $L_{m,k}$, and $L^{<}_{m,k}$ the subcomplexes of B_a linearly spanned by elements of type $(x^i y^j, \ldots)$, with $(i, j) \leq (m, k)$, and $(i, j) < (m, k)$, respectively.
For the proof of (29), let $[z] \in Cotor^n(\mathbf{C}, \mathbf{C}_{\sigma^a})$, represented by a cocycle $z \in B_a$. We claim that:

$$\exists m, k \geq 0, \ \exists u \in L_{m,k} \ : \ \ [z] = [u] \ . \tag{31}$$

Indeed, defining $\tau : \mathcal{H} \longrightarrow \mathbf{C}$ on the basis (30) by $\tau(1) = 1$ and 0 otherwise, and $\theta = \tau \otimes Id_{\mathcal{H}} \otimes \ldots \otimes Id_{\mathcal{H}} : B_a \longrightarrow B_a$, we have $d_a\theta + \theta d_a = Id - \phi$ where ϕ is identity on elements of type $(x^m y^k, \ldots)$, $(m, k) \neq (0, 0)$, and 0 otherwise. Choose (m, k) minimal such that (31) holds. Hence we find $v \in B_a^{n-1}$ such that:

$$u \equiv (x^m y^k, v) \ \ \mathrm{mod} \quad L^{<}_{m,k} \ .$$

Assume first $(m, k) \neq (0, 0)$. Since $d_a(u) \equiv -(x^m y^k, \sigma^{m+k}, v) + (x^m y^k, d_a{}'(v))$ mod $L^{<}_{m,k}$, we must have $(\sigma^{m+k}, v) = d_a{}'(v)$, i.e. v represents a $(n-1)$-cocycle in the standard complex computing $Cotor(\mathbf{C}_{\sigma^{m+k}}, \mathbf{C}_{\sigma^a})$. This complex is isomorphic (by the multiplication by σ^{-m-k}) to the standard complex computing $Cotor(\mathbf{C}, \mathbf{C}_{\sigma^{a-m-k}})$, hence, by the induction hypothesis, $v = (\sigma^{m+k}, w) - d_a{}'(w)$ for some w. Choosing $u' = u + d_a(x^m y^k, w)$, we then have $[z] = [u']$, and $u' \in L^{<}_{m,k}$, which contradicts the minimality of (m, k). We are left with the case $(m, k) = (0, 0)$, when, since $\phi(u) = 0$, one gets $[u] = 0$.
A completely similar argument shows that:

$$Cotor^n(\mathbf{C}, \mathbf{C}) = 0 \ \ \forall\ n \geq 1 \ , \quad Cotor^n(\mathbf{C}, \mathbf{C}_\sigma) = 0 \ \ \forall\ n \geq 2 \ ,$$

while clearly $Cotor^0(\mathbf{C}, \mathbf{C}_\sigma) = 0$. Let $[z] \in Cotor^1(\mathbf{C}, \mathbf{C}_\sigma)$. As above, we find a minimal (m, k) such that (31) holds, and let $\eta \in \mathbf{C}$ such that $u \equiv \eta x^m y^k$ mod $L^{<}_{m,k}$. Again, since $d_1(u) \equiv \eta((x^m y^k, \sigma^{m+k}) - (x^m y^k, \sigma))$ mod $L^{<}_{m,k}$, we must have $\eta\sigma^{m+k} = \eta\sigma$, hence $\eta = 0$, or $m + k = 1$. In other words, $Cotor^1(\mathbf{C}, \mathbf{C}_\sigma) \cong \mathbf{C}^2$, with the generators $[x], [y]$. $\square$

6 A non-commutative Weil complex

In this section we introduce/describe a non-commutative Weil complex associated to a coalgebra, which extends/explains some results in [18, 30], and will naturally appear in the construction of characteristic maps associated to higher traces (section 7). We describe the relevant cohomologies (analogues of Chern, Chern/Simons classes), and (using section 3) the S-operators acting on them.

Let $\mathcal{H}$ be a coalgebra. Define its Weil algebra $W(\mathcal{H})$ as the (non-commutative, non-unital) DG algebra freely generated by the symbols h of degree 1, $\omega(h)$ of degree 2, linear on $h \in \mathcal{H}$. The differential of $W(\mathcal{H})$ is similar to the b' differential of $T(\mathcal{H})$ (see 4.1). It is denoted by ∂, and is the unique derivation which acts on generators by:

$$\partial(h) = \omega_h - \sum h_0 h_1,$$

$$\partial(\omega_h) = \sum \omega_{h_0} h_1 - \sum h_0 \omega_{h_1}.$$

Example 6.1 This algebra is intended to be a non-commutative analogue of the usual Weil complex of a Lie algebra (see [15]). Particular cases have been used in the study of universal Chern-Simons forms. When $\mathcal{H} = \mathbf{C}\rho$ (i.e. $\mathbf{C}$, with 1 denoted by ρ), with $\Delta(\rho) = \rho \otimes \rho$, it is the algebra introduced in [30]; for $\mathcal{H} = \mathbf{C}\rho_1 \bigoplus ... \bigoplus \mathbf{C}\rho_n$ with $\Delta(\rho_i) = \rho_i \otimes \rho_i$, we obtain one of the algebras studied on [18].

We discuss now its 'universal property'. Given a DG algebra Ω^*, and a linear map:

$$\phi : \mathcal{H} \longrightarrow \Omega^1,$$

define its curvature:

$$\omega_\phi : \mathcal{H} \longrightarrow \Omega^2, \ \omega_\phi(h) = d\phi(h) + \sum \phi(h_0)\phi(h_1).$$

Alternatively, using the natural DG algebra structure of $Hom(\mathcal{H}, \Omega^*)$,

$$\omega_\phi := d(\phi) - 1/2[\phi, \phi] \in Hom(\mathcal{H}, \Omega)^2.$$

There is a unique algebra homomorphism (the characteristic map of ϕ):

$$k(\phi) : W(\mathcal{H}) \longrightarrow \Omega^*,$$

sending h to $\phi(h)$ and ω_h to $\omega_\phi(h)$.

One can easily see that (compare with the usual Weil complex of a Lie algebra):

Proposition 6.2 . *The previous construction induces a $1-1$ correspondence between linear maps $\phi : \mathcal{H} \longrightarrow \Omega^1$ and DG algebra maps $k : W(\mathcal{H}) \longrightarrow \Omega^*$. In particular, there is a $1-1$ correspondence between flat linear maps $\phi : \mathcal{H} \longrightarrow \Omega^1$ (i. e. with the property that $\omega_\phi = 0$), and DG algebra maps $k : T(\mathcal{H}) \longrightarrow \Omega^*$.*

An immediate consequence is that $W(\mathcal{H})$ does not depend on the co-algebra structure of $\mathcal{H}$. Actually one can see directly that $(W(\mathcal{H}), \partial) \cong (W(\mathcal{H}), d)$, where d is the derivation on $W(\mathcal{H})$ defined on generators by $d(h) = \omega_h, \ \ d(\omega_h) = 0$ (i.e. the differential corresponding to $\mathcal{H}$ with the trivial co-product). An explicit isomorphism sends h to h and ω_h to $\omega_h + \sum h_0 h_1$.

Corollary 6.3 . *The Weil algebra $W(\mathcal{H})$, and the complex $W(\mathcal{H})_\natural$ are acyclic.*

6.4 Extra-structure on $W(\mathcal{H})$**:** Now we look at the extra-structure of $W(\mathcal{H})$. First of all, denote by $I(\mathcal{H})$ the ideal generated by the curvatures ω_h. The powers of $I(\mathcal{H})$, and the induced truncations are denoted by:

$$I_n(\mathcal{H}) := I(\mathcal{H})^{n+1}, \quad W_n(\mathcal{H}) := W(\mathcal{H})/I(\mathcal{H})^{n+1}.$$

Remark that $W_0(\mathcal{H}) = T(\mathcal{H})$ is the tensor (DG) algebra of $\mathcal{H}$ (up to a minus sign in the boundary, which is irrelevant, and will be ignored). Dual to even higher traces, we introduce the complex:

$$W_n(\mathcal{H})_\natural := W_n(\mathcal{H})/[W_n(\mathcal{H}), W_n(\mathcal{H})]$$

obtained dividing out the (graded commutators). In the terminology of [18] (pp. 103), it is the space of 'cyclic words'. Dual to odd higher traces:

$$I_n(\mathcal{H})_\natural := I_n(\mathcal{H})/[I(\mathcal{H}), I_{n-1}(\mathcal{H})].$$

It is interesting that all these complexes compute the same cohomology (independent of n !), namely the cyclic cohomology of $\mathcal{H}$ viewed as a coalgebra. This is the content of Theorem 6.7, Proposition 4.2, and Section 8.

Secondly, we point out a bi-grading on $W(\mathcal{H})$: defining ∂_0 such that $\partial = \partial_0 + d$, then $W(\mathcal{H})$ has a structure of bigraded differential algebra, with $deg(h) = (1,0), deg(\omega_h) = (1,1)$. Actually $W(\mathcal{H})$ can be viewed as the tensor algebra of $\mathcal{H}^{(1,0)} \bigoplus \mathcal{H}^{(1,1)}$ (two copies of $\mathcal{H}$ on the indicated bi-degrees). With this bi-grading, q in $W^{p,q}$ counts the number of curvatures. The boundary d increases q, while ∂_0 increases p.

6.5 Example: Let's have a closer look at $\mathcal{H} = \mathbf{C}\rho$ with $\Delta(\rho) = h\rho \otimes \rho$, for which the computations were carried out by D. Quillen [30], recalling the main features of our complexes:

(1) ω^n are cocycles of $W(\mathcal{H})_\natural$ (where $\omega = \omega_{h\rho}$). They are trivial in cohomology (cf. Corollary 6.3).

(2) the place where ω^n give non-trivial cohomology classes is $I_{m,\natural}$, with m sufficiently large.

(3) the cocycles ω^n (trivial in $W(\mathcal{H})_\natural$) transgress to certain (Chern-Simons) classes. The natural complex in which these classes are non-trivial (in cohomology) is $W_m(\mathcal{H})_\natural$.

(4) there are striking 'suspensions' (by degree 2 up) in the cohomology of all the complexes $W_n(\mathcal{H})_\natural$, $I_n(\mathcal{H})_\natural$, $\tilde{I}_n(\mathcal{H})_\natural$.

Our intention is also to explain these phenomena (in our general setting).

6.6 'Chern-Simons contractions'. Starting with two linear maps:

$$\rho_0, \rho_1 : \mathcal{H} \longrightarrow \Omega^1,$$

we form:

$$t\rho_0 + (1-t)\rho_1 := \rho_0 \otimes t + \rho_1 \otimes (1-t) : \mathcal{H} \longrightarrow (\Omega^* \otimes \Omega(1))^1,$$

where $\Omega(1)$ is the algebraic DeRham complex of the line: $\mathbf{C[t]}$ in degree 0, and $\mathbf{C[t]dt}$ in degree 1, with the usual differential. Composing its characteristic map $W(\mathcal{H}) \longrightarrow \Omega \otimes \Omega(1)$, with the degree -1 map $\Omega \otimes \Omega(1) \longrightarrow \Omega$ coming from the integration map $\int_0^1 : \Omega(1) \longrightarrow \mathbf{C}$

(emphasize that we use the graded tensor product, and the integration map has degree -1), we get a degree -1 chain map:

$$k(\rho_0, \rho_1) : W(\mathcal{H}) \longrightarrow \Omega.$$

As usual, $[k(\rho_0, \rho_1), \partial] = k(\rho_1) - k(\rho_0)$.
The particular case where $\Omega = W(\mathcal{H}), \rho_0 = 0, \rho_1 = Id_{\mathcal{H}}$ gives a contraction of $W(\mathcal{H})$:

$$H := k(Id_{\mathcal{H}}, 0) : W(\mathcal{H}) \longrightarrow W(\mathcal{H}).$$

We point out that H preserves commutators, and induces a chain map

$$CS : I_n(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural[1], \quad [x] \mapsto [H(x)], \tag{32}$$

to which we will refer as the Chern Simons map. The formulas for the contraction H resemble the usual ones ([18, 29, 30]). For instance, at the level of $W(\mathcal{H})_\natural$, one has:

$$H(\frac{\omega_h^{n+1}}{(n+1)!}) = \int_0^1 \frac{1}{n!} h(t\omega_h + (t^2 - t) \sum_{(h)} h_0 h_1)^n dt \tag{33}$$

Theorem 6.7 *The Chern-Simons map is an isomorphism* $H^*(I_n(\mathcal{H})_\natural) \tilde{\longrightarrow} H^{*-1}(W_n(\mathcal{H})_\natural)$ *(compatible with the* S*-operator described below).*

proof: We consider the following slight modification of $I_n(\mathcal{H})$:

$$\tilde{I}_n(\mathcal{H})_\natural := I_n(\mathcal{H}) / I_n(\mathcal{H}) \cap [W(\mathcal{H}), W(\mathcal{H})].$$

One has a short exact sequence:

$$0 \longrightarrow \tilde{I}_n(\mathcal{H})_\natural \longrightarrow W(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural \longrightarrow 0,$$

and, using Corollary 6.3, the boundary of the long exact sequence induced in cohomology gives an isomorphism $\tilde{\partial} : H^{*-1}(W_n(\mathcal{H})_\natural) \tilde{\longrightarrow} H^*(\tilde{I}_n(\mathcal{H})_\natural)$. The same formula (32) defines a chain map $\tilde{CS} : \tilde{I}_n(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural[1]$, and one can easily check that $\tilde{CS} \circ \tilde{\partial} = Id$. Now, since CS is the composition of $\tilde{CS}$ with the canonical projection $I_n(\mathcal{H})_\natural \twoheadrightarrow \tilde{I}_n(\mathcal{H})_\natural$, it suffices to show that the last map induces isomorphism in cohomology. We prove this after describing the S- operator. $\square$

6.8 The S-operator: The discussion in 3.1 applies to the Weil complex $W(\mathcal{H})$, explaining the 'suspensions' (by degree 2 up) in the various cohomologies we deal with. It provides cyclic bicomplexes computing our cohomologies, in which S can be described as a shift. As in cyclic cohomology, one introduce these bicomplexes directly, and prove all the formulas in a straightforward manner. Here we prefer to apply the formal constructions of 3.1 to $W(\mathcal{H})$ and to compute its X-complex. This computation can be carried out exactly as in the case of the tensor algebra (see Example 3.3), and this is done in the proof of Theorem 7.9. We end up with the following exact sequence of complexes (which can be taken as a definition):

$$\ldots \longrightarrow W^b(\mathcal{H}) \xrightarrow{t-1} W(\mathcal{H}) \xrightarrow{N} W^b(\mathcal{H}) \xrightarrow{t-1} W(\mathcal{H}) \longrightarrow \ldots,$$

where we have to explain the new objects. First of all, $W^b(\mathcal{H})$ is the same as $W(\mathcal{H})$ but with a new boundary $b = \partial + b_t$ with b_t described below. The t operator is the backward cyclic permutation:

$$t(ax) = (-1)^{|a||x|}xa,$$

for $a \in H$ or of type ω_h. This operator has finite order in each degree of $W(\mathcal{H})$: we have $t^p = 1$ on elements of bi-degree (p,q), so $t^{k!} = 1$ on elements of total degree k. The norm operator N is $N := 1 + t + t^2 + ... + t^{p-1}$ on elements of bi-degree (p,q). The boundary b of $W^b(\mathcal{H})$ is $b = \partial + b_t$,

$$b_t(ax) = t(\partial_0(a)x),$$

for $a \in \mathcal{H}$ or of type ω_h. For all the operators involved, see also section 8. Obviously, the powers $I(\mathcal{H})^{n+1}$ are invariant by $b, t-1, N$, so we get similar sequences for $I_n(\mathcal{H})$, $W_n(\mathcal{H})$. For reference, we conclude:

Corollary 6.9 *There are exact sequences of complexes:*

$$CC(W_n(\mathcal{H})): \quad \dots \longrightarrow W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow \dots \tag{34}$$

$$0 \longrightarrow W_n(\mathcal{H})_\natural \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow \dots \tag{35}$$

$$0 \longrightarrow I_n(\mathcal{H})_\natural \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \longrightarrow \dots \tag{36}$$

Corollary 6.10 *There are short exact sequences of complexes:*

$$0 \longrightarrow W_n(\mathcal{H})_\natural \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow W_n(\mathcal{H})_\natural \longrightarrow 0 \tag{37}$$

$$0 \longrightarrow I_n(\mathcal{H})_\natural \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \longrightarrow I_n(\mathcal{H})_\natural \longrightarrow 0 \tag{38}$$

In particular, (35), (36), give bicomplexes which compute the cohomologies of $W_n(\mathcal{H})_\natural$, $I_n(\mathcal{H})_\natural$. They are similar to the (first quadrant) cyclic bicomplexes appearing in cyclic cohomology, and come equipped with an obvious shift operator, which defines our S- operation:

$$S : H^*(W_n(\mathcal{H})_\natural) \longrightarrow H^{*+2}(W_n(\mathcal{H})_\natural),$$

(and similarly for $I_n(\mathcal{H})_\natural$). Alternatively, one can obtain S as cup-product by the Ext^2 classes arising from Corollary 6.10.

End of proof of theorem 6.7: Denote for simplicity by $\underline{CC}^*(I_n), \underline{CC}^*(W), \underline{CC}^*(W_n)$ the (first quadrant) cyclic bicomplexes (or their total complexes) of I_n, W, and W_n, respectively. We have a map of short exact sequences of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{I}_n(\mathcal{H})_\natural & \longrightarrow & W(\mathcal{H})_\natural & \longrightarrow & W_n(\mathcal{H})_\natural & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \\ 0 & \longrightarrow & \underline{CC}^*(I_n) & \longrightarrow & \underline{CC}^*(W) & \longrightarrow & \underline{CC}^*(W_n) & \longrightarrow & 0 \end{array}$$

where we have used the fact that $N : I_n(\mathcal{H})_\natural \longrightarrow I_n(\mathcal{H})$ factors through the projection $I_n(\mathcal{H})_\natural \twoheadrightarrow \tilde{I}_n(\mathcal{H})_\natural$ (being defined on the entire $W(\mathcal{H})_\natural$). Applying the five lemma to the exact sequences induced in cohomology by the previous two short exact sequences, the statement follows. $\square$

6.11 Example: There are canonical Chern and Chern-Simons classes induced by any group-like element $\rho \in \mathcal{H}$ (i.e. with the property $\Delta(\rho) = \rho \otimes \rho$). Denote by ω its curvature. Since $\partial(\omega^n) = [\,\omega^n, \rho]$ is a commutator, ω^n define cohomology classes:

$$ch_{2n}(\rho) : = [\,\natural(\frac{1}{n!}\omega^n)] \in H^{2n}(I_m(\mathcal{H})_\natural) \ , \tag{39}$$

for any $n \geq m$. The associated Chern-Simons class $cs_{2n-1}(\rho) := CS(ch_{2n}(\rho))$ is given by the formula (see (33)):

$$cs_{2n-1}(\rho) = [\,\natural\{\frac{1}{(n-1)!}\int_0^1 \rho(t\partial(\rho) + t^2\rho^2)^{n-1}dt\}] \in H^{2n-1}(W_m(\mathcal{H})_\natural).$$

To compute $S(ch_{2n}(\rho))$, we have to solve successively the equations:

$$\begin{cases} \partial(\frac{1}{n!}\omega^n) & = & (t-1)(v) \\ b(v) & = & N(w) \end{cases}$$

and then $S(ch_{2n}(\rho)) = [\,\natural(w)]$. The first equation has the obvious solution $v = \frac{1}{n!}\rho\omega^n$, whose $b(v) = \frac{1}{n!}\omega^{n+1}$, so the second equation has the solution $w = \frac{1}{(n+1)!}\omega^{n+1}$. In conclusion,

$$S(ch_{2n}(\rho)) = ch_{2(n+1)}(\rho), \quad S(cs_{2n-1}(\rho)) = cs_{2n+1}(\rho). \tag{40}$$

(where the second relation follows from the first one and Theorem 6.7.)

7 The Weil complex and higher traces

We explain now how the Weil complex introduced in the previous section appears naturally in the case of higher traces, and Hopf algebra actions. The main reason that $HC^*_\delta(\mathcal{H})$ is still the target of these characteristic maps is that it can be computed by the truncation of the Weil complex (see Theorem 7.3, whose proof is postponed until the next section). To prove the compatibility with the S-operator, we first have interpret the complexes introduced in the previous section in terms of Cuntz-Quillen's (tower of) relative X-complexes. We will obtain in particular the case of usual traces discussed in Section 4. Also, for $\mathcal{H} = \mathbf{C}\rho$ (example 6.1), we re-obtain the results , and interpretations of some of the computations of [29] (see Example 7.11 below).

In this section $\mathcal{H}$ is a Hopf algebra, δ is a character on $\mathcal{H}$, and A is a $\mathcal{H}$-algebra. We assume for simplicity that $S_\delta^2 = Id$.

7.1 Localizing $W(\mathcal{H})$: First of all remark that the Weil complex $W(\mathcal{H})$ is naturally an $\mathcal{H}$ DG algebra. By this we mean a DG algebra, endowed with a (flat) action, compatible with the grading and with the differentials. The action is defined on generators by:

$$g \cdot i\,(h) : = i\,(gh), \quad g \cdot \omega_h : = \omega_{gh}, \quad \forall\ g, h \in \mathcal{H}.$$

and extended by $h(xy) = \sum h_0(x)h_1(y)$. Here, to avoid confusions, we have denoted by $i : \mathcal{H} \longrightarrow W(\mathcal{H})$ the inclusion. Remark that the action preserves the bi-degree (see 6.4), so $W(\mathcal{H})_\delta$ has an induced bi-grading. We briefly explain how to get the localized version for the constructions and the properties of the previous section. First of all one can localize with respect to δ as in Section 3, and (with the same proof as of Proposition 2.5), all the operators descend to the localized spaces. The notation $I_n(\mathcal{H})_{\natural,\delta}$ stands for $I_n(\mathcal{H})$ divided out by commutators and co-invariants. For Theorem 6.7, remark that the contraction used there is compatible with the action. To get the exact sequences from Corollary 6.9 and 6.10, we may look at them as a property for the cohomology of finite cyclic groups, acting (on each fixed bi-degree) in our spaces. Or we can use the explicit map $\alpha : W(\mathcal{H}) \to W(\mathcal{H})$ defined by $\alpha := (t + 2t^2 + ... + (p-1)t^{p-1})$ on elements of bi-degree (p, q), which has the properties: $(t-1)\alpha + N = pId$, $\alpha(I(\mathcal{H})^{n+1}) \subseteq I(\mathcal{H})^{n+1}$, and α descends (because t does). So, also the analogue of Theorem 6.7 follows. In particular $H^*_{\delta}(W_n(\mathcal{H})_{\natural})$ is computed either by the complex $W_n(\mathcal{H})_{\natural,\delta}$, or by the (localized) cyclic bicomplex $CC^*_{\delta}(W_n(\mathcal{H}))$ (analogous to (34)). Similarly, we consider the S operator, and the periodic versions of these cohomologies. Due to the shift in the degree already existent in the case of the tensor algebra (see 4.2), we re-index these cohomologies:

Definition 7.2 *Define $HC^*_{\delta}(\mathcal{H}, n) := H^{*+1}(W_n(\mathcal{H})_{\natural,\delta})$, and denote by $CC^*_{\delta}(\mathcal{H}, n)$ the cyclic bicomplex computing it, that is, $CC^*_{\delta}(W_n(\mathcal{H}))$ shifted by one in the vertical direction.*

Remark that for $n = 0$ we obtain Connes-Moscovici's cyclic cohomology and:

$$CC^*(\mathcal{H}, 0) = CC^*(\mathcal{H})\ ,\ \ CC^*_{\delta}(\mathcal{H}, 0) = CC^*_{\delta}(\mathcal{H}),$$

while, in general, there are obvious maps:

$$\ldots \xrightarrow{\pi_3} HC^*_{\delta}(\mathcal{H}, 2) \xrightarrow{\pi_2} HC^*_{\delta}(\mathcal{H}, 1) \xrightarrow{\pi_1} HC^*_{\delta}(\mathcal{H}, 0) \cong HC^*_{\delta}(\mathcal{H}). \tag{41}$$

In the next section we will prove:

Theorem 7.3 *$HC^*_{\delta}(\mathcal{H}, n) \cong HC^{*-2n}_{\delta}(\mathcal{H})$, and the tower (41) is the S operation tower for $HC^*_{\delta}(\mathcal{H})$. More precisely, there are isomorphisms $\beta : HC^*_{\delta}(\mathcal{H}, n) \xrightarrow{\sim} HC^{*-2}_{\delta}(\mathcal{H}, n-1)$ such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[dr, "\pi"] & \cdots \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \\
\cdots \arrow[r] \arrow[dr, "\pi"] & HC^*_{\delta}(\mathcal{H}, 2) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^{*+2}_{\delta}(\mathcal{H}, 2) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^{*+4}_{\delta}(\mathcal{H}, 2) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \\
\cdots \arrow[r] \arrow[dr, "\pi"] & HC^{*-2}_{\delta}(\mathcal{H}, 1) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^*_{\delta}(\mathcal{H}, 1) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^{*+2}_{\delta}(\mathcal{H}, 1) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \\
\cdots \arrow[r] & HC^{*-4}_{\delta}(\mathcal{H}, 0) \arrow[r, "S"] & HC^{*-2}_{\delta}(\mathcal{H}, 0) \arrow[r, "S"] & HC^*_{\delta}(\mathcal{H}, 0) \arrow[r, "S"] & \cdots
\end{tikzcd}$$

7.4 The case of even equivariant traces: Consider now an equivariant even trace over A, i.e. an extension:

$$0 \longrightarrow I \longrightarrow R \xrightarrow{u} A \longrightarrow 0 \tag{42}$$

and a δ-invariant trace $\tau : R \longrightarrow \mathbf{C}$ vanishing on I^{n+1}. To describe the induced characteristic map, we choose a linear splitting $\rho : A \longrightarrow R$ of (42). As in the case of the usual Weil complex, there is a unique equivariant map of DG algebras:

$$\tilde{k} : W(\mathcal{H}) \longrightarrow Hom(B(A), R),$$

sending $1 \in \mathcal{H}$ to ρ. This follows from Proposition 6.2 and from the equivariance condition (with the same arguments as in 4.6). Here, the action of $\mathcal{H}$ on $Hom(B(A), R)$ is induced by the action on R. Since ρ is a homomorphism modulo I, $\tilde{k}$ sends $I(\mathcal{H})$ to $Hom(B(A), I)$, so induces a map $W_n(\mathcal{H}) \longrightarrow Hom(B(A), R/I^{n+1})$. As in 4.6, composing with the δ-invariant trace:

$$\tau_\natural : Hom(B(A), R/I^{n+1}) \longrightarrow C^*_\lambda(A)[1], \quad \phi \mapsto \tau \circ \phi \circ N \ ,$$

we get a δ-invariant trace on $W_n(\mathcal{H})$, so also a chain map:

$$k^{\tau,\rho} : W_n(\mathcal{H})_{\natural,\delta} \longrightarrow C^*_\lambda(A)[1] \ . \tag{43}$$

Denote by the same letter the map induced in cohomology:

$$k^{\tau,\rho} : HC^*_\delta(\mathcal{H}, n) \longrightarrow HC^*(A), \tag{44}$$

or, using the isomorphism of Theorem 7.3:

$$k^{\tau,\rho} : HC^{*-2n}_\delta(\mathcal{H}) \longrightarrow HC^*(A), \tag{45}$$

Theorem 7.5 *The characteristic map (45) of the even higher trace τ does not depend on the choice of the splitting ρ and is compatible with the S-operator.*

proof:(compare to [29]) We use 6.6. If ρ_0, ρ_1 are two liftings, form $\rho = t\rho_0 + (1-t)\rho_1 \in Hom(A, R[t])$, viewed in the degree one part of the DG algebra $Hom(B(A), R \otimes \Omega(1))$. It induces a unique map of $\mathcal{H}$DG algebras $\tilde{k}_\rho : W(\mathcal{H}) \longrightarrow Hom(B(A), R \otimes \Omega(1))$, sending 1 to ρ, which maps $I(\mathcal{H})$ to the DG ideal $Hom(B(A), I \otimes \Omega(1))$ (since ω_ρ belongs to the former). Using the trace $\tau \otimes \int : R/I^{n+1} \otimes \Omega(1) \longrightarrow \mathbf{C}$, and the universal cotrace on $B(A)$, it induces a chain map:

$$k_{\rho_0,\rho_1} : W_n(\mathcal{H}) \longrightarrow C^*_\lambda(A)[1],$$

which kills the coinvariants and the commutators. The induced map on $W_n(\mathcal{H})_{\natural,\delta}$ is a homotopy between k^{τ,ρ_0} and k^{τ,ρ_1}. The compatibility with S follows from the fact that the characteristic map (43) can be extended to a map between the cyclic bicomplexes $CC^*_\delta(\mathcal{H}, n)$ and $CC^*(A)$. We will prove this after shortly discussing the case of odd higher traces. $\square$

7.6 The case of odd equivariant traces: A similar discussion applies to the case of odd equivariant traces on A, i.e. extensions (42) endowed with a δ-invariant linear map $\tau : I^{n+1} \longrightarrow \mathbf{C}$, vanishing on $[I^n, I]$. The resulting map $H^{*+1}(I_n(\mathcal{H})_{\natural,\delta}) \longrightarrow HC^{*-1}(A)$, combined with Corollary 6.7, the comments in 7.1, and Theorem 7.3, give the characteristic map:

$$k^{\tau,\rho} : HC^{*-2n-1}_\delta(\mathcal{H}) \cong HC^{*-1}_\delta(\mathcal{H}, n) \longrightarrow HC^*(A), \tag{46}$$

which has the same properties as in the even case:

Theorem 7.7 *The characteristic map (46) of the odd higher trace τ does not depend on the choice of the splitting ρ and is compatible with the S-operator.*

7.8 The localized tower $\mathcal{X}_\delta(R, I)$: Recall that given an ideal I in the algebra R, one has a tower of super-complexes $\mathcal{X}_\delta(R, I)$ given by ([12], pp. 396):

$$\mathcal{X}^{2n+1}(R, I) : R/I^{n+1} \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural / \natural(I^{n+1}dR + I^n dI) \quad ,$$

$$\mathcal{X}^{2n}(R,I) : R/(I^{n+1}+[I^n,R]) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural/\natural(I^n dR) ,$$

where $\natural : \Omega^1(R) \to \Omega^1(R)_\natural$ is the projection. The structure maps $\mathcal{X}^n(R,I) \to \mathcal{X}^{n+1}(R,I)$ of the tower are the obvious projections. We have a localized version of this, denoted by $\mathcal{X}_\delta(R,I)$, and which is defined by:

$$\mathcal{X}_\delta^{2n+1}(R,I) : R/(I^{n+1}+\mathrm{coinv}) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}/\natural(I^{n+1}dR + I^n dI) \quad ,$$

$$\mathcal{X}_\delta^{2n}(R,I) : R/(I^{n+1}+[I^n,R]+\mathrm{coinv}) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}/\natural(I^n dR) ,$$

where, this time, $\natural$ denotes the projection $\Omega^1(R) \to \Omega^1(R)_{\natural,\delta}$.

Remark that the construction extends to the graded case, and each $\mathcal{X}^n(R,I)$ is a supercomplex of complexes.

Theorem 7.9 *The cyclic bicomplex $CC^*_\delta(\mathcal{H},n)$ is isomorphic to the bicomplex $\mathcal{X}_\delta^{2n+1}(W(\mathcal{H}),I(\mathcal{H}))$.*

proof: The computation is similar to the one of $X(T\mathcal{H})$ (see Example 3.3 and Proposition 3.4). Denote $W = W(\mathcal{H})$, $I = I(\mathcal{H})$, and let $V \subset W(\mathcal{H})$ be the linear subspace spanned by h's and ω_h's. Remark that W, as a graded algebra, is freely generated by V. This allows us to use exactly the same arguments as in 3.3, 3.4 to conclude that $\Omega^1 W \cong \tilde{W} \otimes V \otimes \tilde{W}$, $\Omega^1(W)_\natural \cong V \otimes \tilde{W} = W$, $\Omega^1(W)_{\natural,\delta} \cong W_\delta$; the projection $\natural : \Omega^1(W) \to \Omega^1(W)_\natural$ identifies with:

$$\natural : \Omega^1(W) \longrightarrow V \otimes \tilde{W} = W, \quad x\partial_u(v)y \mapsto (-1)^\mu vyx , \tag{47}$$

for $x, y \in \tilde{W}, v \in V$. Here $\mu = deg(x)(deg(v) + deg(y))$ introduces a sign, due to our graded setting, and $\partial_u : W \to \Omega^1(W)$ stands for the universal derivation of W. Using this, we can compute the new boundary of W, coming from the isomorphism $W \cong \Omega^1(W)_\natural$, and we end up we the b-boundary of W, defined in Section 5. For instance, if $x = hx_0 \in W$ with $h \in \mathcal{H}$, since $\natural(\partial_u(h)x_0) = x$ by (47), its boundary is:

$$\begin{aligned}
&\natural(\partial_u(\partial(h)x_0) - \partial_u(h)\partial(x_0)) = \\
&\quad = \natural(\partial_u(\omega_h - \sum h_0h_1)x_0 - \partial_u(h)\partial(x_0)) = \\
&\quad = \natural(\partial_u(\omega_h)x_0 - \sum \partial_u(h_0)h_1x_0 - \sum h_0\partial_u(h_1)x_0 - \partial_u(h)\partial(x_0)) = \\
&\quad = \omega_h x_0 - \sum h_0h_1x_0 - \sum (-1)^{deg(x)} h_1x_0h_0 - h\partial(x_0) = \\
&\quad = \partial(h)x_0 - \sum t(h_0h_1x_0) - h\partial(x_0) = \\
&\quad = \partial(hx_0) + t(\partial_0(h)x_0) = b(hx_0)
\end{aligned}$$

Remark also that our map (47) has the property:

$$\natural(I^n\partial_u I + I^{n+1}\partial_u W) = I^{n+1}. \tag{48}$$

These give the identification $\mathcal{X}^{2n+1}(W,I) \cong CC^*(\mathcal{H},n)$. The localized version of this is just a matter of checking that the isomorphism $\Omega^1(W)_{\natural,\delta} \cong W_\delta$ already mentioned, induces $\Omega^1(W)_{\natural,\delta}/\natural(I^n\partial_u I + I^{n+1}\partial_u W) \cong (W/I^{n+1})_\delta$, which follows from (48). $\square$

7.10 Proof of the S-relation: We freely use the dual constructions for (DG) coalgebras B, such as the universal coderivation $\Omega_1(B) \to B$, the space of co-commutators $B^{\natural} = Ker(\Delta - \sigma \circ \Delta : B \to B \otimes B)$, and the X-complex $X(B)$ (see [29]). Denote $B = B(A)$, $L = Hom(B, R)$, $J = Hom(B, I)$. Our goal is to prove that the characteristic map (43) can be defined at the level of the cyclic bicomplexes. Consider first the case of even traces τ. Since the $\mathcal{H}$ DG algebra map $\tilde{k} : W(\mathcal{H}) \to L$ maps $I(\mathcal{H})$ inside J, there is an induced map $\mathcal{X}_{\delta}^{2n+1}(W(\mathcal{H}), I(\mathcal{H})) \to \mathcal{X}_{\delta}^{2n+1}(L, J)$, extending $W_n(\mathcal{H})_{\natural,\delta} \to (L/J^{n+1})_{\natural,\delta}$. So, it suffices to show that the map $(L/J^{n+1})_{\natural,\delta} \to Hom(B^{\natural}, (R/I^{n+1})_{\natural,\delta})$ (constructed as (43)), lifts to a map of super-complexes (of complexes)

$$\mathcal{X}_{\delta}^{2n+1}(L, J) \to Hom(X(B), (R/I^{n+1})_{\natural,\delta}) \tag{49}$$

Indeed, using Theorem 7.9, the (similar) computation of $X(B)$ (as the cyclic bicomplex of A), the interpretation of the norm map N as the universal cotrace of B (see [29]), and the fact that any τ as above factors through $(R/I^{n+1})_{\natural,\delta} \to \mathbf{C}$, the map (49) is 'universal' for our problem. The construction of (49) is quite simple. The composition with the universal coderivation of B is a derivation $L \to Hom(\Omega_1(B), R)$ on L, so it induces a map $\chi : \Omega^1(L) \to Hom(\Omega_1(B), R)$. Since χ is a L-bimodule map, and it is compatible with the action of $\mathcal{H}$, it induces a map $\Omega^1(L)_{\natural} \to Hom(\Omega_1(B)^{\natural}, (R/I^{n+1})_{\natural,\delta})$, which kills $\natural(J^n dJ + J^{n+1} dL + coinv)$. This, together with with the obvious $(L/J^{n+1})_{\delta} \to Hom(B, (R/I^{n+1})_{\natural,\delta})$, give (49). For the case of odd higher traces we proceed similarly, and use the remark that (49) was apriori defined at the level of L, and $\Omega^1(L)$, and one can restrict to the ideals (instead of dividing out by them). $\square$

7.11 Examples: Choosing $\rho = 1 \in \mathcal{H}$ (the unit of $\mathcal{H}$) in Example 6.11, and applying the characteristic map to the resulting classes, we get the Chern/Chern-Simons classes (in the cyclic cohomology of A), described in [29]. Remark that our proof of the compatibility with the S operator consists on two steps: the first one proves the universal formulas (40) at the level of the Weil complex, while the second one shows, in a formal way, that the characteristic map can be defined at the level of the cyclic bicomplexes. This allows us to avoid the explicit cochain computations.

Another interesting example is when $\mathcal{H} = U(\mathfrak{g})$ as in Example 2.3, $\delta =$ the counit. Via the computation of Theorem 5.6, our construction associates to any G-algebra A, and any G-invariant higher trace τ on A, of parity i, a $\mathbf{Z}/2\mathbf{Z}$ graded characteristic maps:

$$k_{\tau} : H_*(\mathfrak{g}) \longrightarrow HP^{*+i}(A). \tag{50}$$

When τ is a usual invariant trace $\tau : A \longrightarrow \mathbf{C}$, we have the following formula (use (19) and the map A used in the proof of Theorem 5.6):

$$\begin{aligned} &k_{\tau}(v_1 \wedge \ldots \wedge v_n)(a_0, a_1, \ldots, a_n) = \\ &= \frac{1}{n!} \sum_{\sigma} sign(\sigma) \tau(a_0 v_{\sigma(1)}(a_1) \ldots v_{\sigma(n)}(a_n)) \ , \end{aligned}$$

(where $v(a) := L_v(a)$ is the Lie derivative). This coincides with the characteristic map described in [5].

8 Proof of Theorem 7.3, and equivariant cycles

This section is devoted to the proof of Theorem 7.3. At the end we illustrate how the new complexes computing $HC_{\delta}^{*}(\mathcal{H})$ which arise during the proof can be used to construct characteristic maps associated to equivariant cycles.

We first concentrate on the non-localized version, whose proof uses explicit formulas which can be easily localized. So, we construct isomorphisms

$$\beta : H^*(W_n(\mathcal{H})_\natural) \xrightarrow{\sim} H^{*-2}(W_{n-1}(\mathcal{H})_\natural)$$

(and explicit inverses) such that the following diagram is commutative:

$$\begin{tikzcd} H^*(W_{n,\natural}) \arrow[r, "S"] \arrow[d, "\beta"'] \arrow[dr, "\pi"] & H^{*+2}(W_{n,\natural}) \arrow[d, "\beta"] \\ H^{*-2}(W_{n-1,\natural}) \arrow[r, "S"] & H^*(W_{n-1,\natural}) \end{tikzcd}$$

Let's start by fixing some notations. Denote $W_n(\mathcal{H}) = W_n$, $I(\mathcal{H}) = I$, $I^{(n)} = I^{n+1}/I^n$ viewed as the subspace of W spanned by elements having exactly n curvatures. The only grading we consider is the total grading (with $deg(h) = 1, deg(\omega_h) = 2$); notations like $(W_n, \partial), (W_n, b)$ are used to specify the complexes we are working with. In general, if the (signed) cyclic permutation acts on a vector space X, denote $X_\natural = X/Im(1-t)$.
We review now the various operators we have. First of all,

$$\partial = \partial_0 + d,$$

where ∂_0, d are the degree 1 derivations given on generators by:

$$\partial_0(h) = -\sum h_0 h_1, \quad \partial_0(\omega_h) = -\sum \omega_{h_0} h_1 - h_0 \omega_{h_1},$$

$$d(h) = \omega_h, \quad d(\omega_h) = 0.$$

Secondly, the operator $b = \partial + b_t$, where:

$$b_t(hx) = \sum (-1)^{degx} h_1 x h_0, \quad b_t(\omega_h x) = \sum h_1 x \omega_{h_0} - (-1)^{degx} \omega_{h_1} x h_0.$$

Define also $b_0 = \partial_0 + b_t$. It is straightforward to check:

$$b = b_0 + d, \quad b_0^2 = d^2 = [b_0, d] = 0.$$

Point out that d commutes with t.
For the construction of β we need the following degree -1 operator:

$$\theta : W \longrightarrow W, \quad \theta(hx) = 0, \quad \theta(\omega_h x) = hx.$$

For constructing the inverse of β, we will use the degree 0 operators $\phi_i : I \longrightarrow W$, $i = 0, 1$. On $I^{(n)}$,

$$\phi_1(hx) = 0, \quad n\phi_1(\omega_h x) = x\omega_h.$$

For $y = x_0 x_1 \ldots x_p \in I^{(n)}$, where each of the x_i's are of type h or ω_h, we put $\lambda_i(y) = \#\{j \leq i : x_j \text{ is of type } \omega_h\}$, and define $n\phi_0(y) = \sum_1^{n-1} \lambda_i(y) t^i(y)$. For a conceptual motivation, see the next proof. We can actually forget about these formulas, and just keep their relevant properties:

Lemma 8.1 *(i)* $[\theta, b_0] = 0,\ [\theta, \partial_0] = 0,\ [\theta, d] = 1,\ \theta^2 = 0,$
(ii) $\phi_1 N - (1-t)\phi_0 = 1, N\phi_1 - \phi_0(1-t) = 1,$
(iii) $\phi_1 b_0 = \partial_0 \phi_1$ *modulo* $Im(1-t)$, $\phi_1\theta = 0,$
(iv) $\phi_0\partial_0 = b_0\phi_0$ *modulo* $Im\theta$.

Proof: (i) and (iii) follow by direct computation. For instance, for the first part of (iii) one has $\phi_1 b_0 = 0 = \partial_0\phi_1$ on elements of type hx, while on elements of type $\omega_h x \in I^{(n)}$:

$$\begin{aligned}\phi_1 b_0(\omega_h x) = \ & \phi_1((\omega_{h_0} h_1 - h_0\omega_{h_1})x + \omega_h \partial_0(x) + h_1 x \omega_{h_0} - (-1)^{deg(x)}\omega_{h_1} x h_0) = \\ & = n(h_1 x \omega_{h_0} + \partial_0(x)\omega_h - (-1)^{deg(x)} x h_0 \omega_{h_1})\end{aligned}$$

$$\partial_0\phi_1(\omega_h x) = n\partial_0(x\omega_h) = n(\partial_0(x)\omega_h + (-1)^{deg(x)} x\omega_{h_0} h_1 - (-1)^{deg(x)} x h_0 \omega_{h_1}),$$

and the two expressions are clearly the same modulo $Im(1-t)$.
One can check directly also (ii). Instead, let's explain that ϕ_0, ϕ_1 have been constructed in such a way that (ii) holds. On the graded algebra $W = \oplus I^{(n)}$, we have a Goodwillie [19] type derivation: multiplication by the number of curvatures. Since W is a tensor algebra, it comes equipped with a canonical connection (see the end of section 3 in [13]). We know that the X- complex of W is the cyclic bicomplex, and the general Cartan homotopy formula of [12] for our derivation D, gives precisely the homotopy $(n\phi_0, n\phi_1)$ on $I^{(n)}$. For (iv), remark first that $Im\theta = Ker\theta$, so it suffices to show that $A := (\theta\phi_0)\partial_0 - b_0(\theta\phi_0)$ is zero. From the first formula of (iii), the second of (ii), and (i), it follows that $A(1-t) = 0$. So, it is enough to check $A = 0$ on homogeneous monomials having a curvature as first element. Such an element can be written as $X = \omega(h^1)X^1 \dots \omega(h^n)X^n \in I^{(n)}$, where $X^i \in \widetilde{T}(\mathcal{H})$, $\omega(h) = \omega_h$. On X, $\theta\phi_0(X) = \sum_1^n \epsilon_i (i-1) h^i X^i \omega(h^{i+1})X^{i+1} \dots \omega(h^n)X^n \omega(h^1)X^1 \dots \omega(h^{i-1})X^{i-1}$ (ϵ_i are corresponding signs), and the proof becomes a lengthy straightforward computation. $\square$

To define β, we need the right complexes computing $H^*(W_{n,\natural})$. One of them is given by the following:

Lemma 8.2 *(i) There are isomorphisms* $p : H^*(W_{n,\natural}) \xrightarrow{\sim} H^*(I_\natural^{(n)}/Imd, \partial_0)$, *compatible with the* S*-operations.*
(ii) One has short exact sequences:

$$0 \longrightarrow (I_\natural^{(n-1)}/Imd, \partial_0) \xrightarrow{d} (I_\natural^{(n)}, \partial_0) \longrightarrow (I_\natural^{(n)}/Imd, \partial_0)[1] \longrightarrow 0, \tag{51}$$

whose induced boundary in cohomology identifies, via p, *with* π*:*

$$\begin{tikzcd} H^*(W_{n,\natural}) \arrow[r, "\pi"] \arrow[d, "p"'] & H^*(W_{n-1,\natural}) \arrow[d, "p"'] \\ H^*(I_\natural^{(n)}/Imd, \partial_0) \arrow[r, "\delta"] & H^*(I_\natural^{(n-1)}/Imd, \partial_0) \end{tikzcd}$$

Here, we view $(W_{n,\natural}, \partial)$ as the total complex of the double complex:

$$0 \longrightarrow (I_\natural^{(0)}, \partial_0) \xrightarrow{d} (I_\natural^{(1)}, \partial_0) \xrightarrow{d} \dots \xrightarrow{d} (I_\natural^{(n)}, \partial_0) \longrightarrow 0 \longrightarrow \dots$$

and p is induced by the obvious augmentation sending $[\sum_0^n x_i]$ into $[x_n]$ ($x_i \in I^{(i)}$). The S operation on $H^*(I_\natural^{(n)}/Imd, \partial_0)$ of (i) is defined by the cyclic bicomplex which is augmentation of (35), or, similar to (37), by the Ext^2 class defined by the extension:

$$0 \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \xrightarrow{N} (I^{(n)}/Imd, b_0) \xrightarrow{t-1} (I^{(n)}/Imd, \partial_0) \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \longrightarrow 0 \quad (52)$$

Using that W is contractible along d (cf. 8.1 (i)), (i) is clear. Using that $d{\circ}t = t{\circ}d$, and that taking invariants under the action of a finite group does not affect exactness, also the first part of (ii) follows, while the last part is a routine spelling out of the boundary of long exact sequences.

There is a slight modification of (52) which can be used to compute $H^*(W_{n,\natural})$, obtained as follows: (52) splits into two short exact sequence:

$$0 \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \xrightarrow{N} (I^{(n)}/Imd, b_0) \longrightarrow (I^{(n)}/Imd + ImN, b_0) \longrightarrow 0, \quad (53)$$

$$0 \longrightarrow (I^{(n)}/Imd + ImN, b_0) \xrightarrow{t-1} (I^{(n)}/Imd, \partial_0) \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \longrightarrow 0 \quad (54)$$

Since the middle complex of (54) is acyclic, e.g. by using the contraction $s_{-1}(hx) = \epsilon(h)x$, $s_{-1}(\omega_h x) = 0$ (which commutes with d), we get a quasi-isomorphism (which, in cohomology, is independent of the contraction):

$$s_{-1}(1-t) : (I^{(n)}/Imd + ImN, b_0) \xrightarrow{q.i.} (I_\natural^{(n)}/Imd, \partial_0)[1] \quad (55)$$

Via this, the S operator is simply the boundary of the long exact sequence induced by (53). Now, our map is defined as the chain map:

$$\beta : (I_\natural^{(n)}/Imd, \partial_0) \longrightarrow (I^{(n-1)}/Imd + ImN, b_0))[1] \quad (56)$$

induced by $-\theta{\circ}N$. To understand our choice of complexes, let's just mention that (56) is an isomorphism when $n = 1$. Note also that β, as well as the chain map α bellow describing its homotopical inverse, do not depend on the structure of $\mathcal{H}$, other then the vector space structure.

Now, to see that β is compatible with the S operation, and to construct its inverse (in cohomology), we make use of the following diagram with exact rows and columns:

$$\begin{tikzcd}
(I_\natural^{(n-1)}/Imd, \partial_0)[1] \arrow[r, "d"] \arrow[d, "N"'] & (I_\natural^{(n)}, \partial_0) \arrow[r, "p_5"] \arrow[d, shift right, "\tilde{N}"'] \arrow[dl, "-\tilde{\beta}"'] & (I_\natural^{(n)}/Imd, \partial_0) \arrow[d, "N"] \\
(I^{(n-1)}/Imd, b_0)[1] \arrow[r, shift left, "\tilde{d}"] \arrow[d] & (I^{(n)}, b_0) \arrow[l, shift left, "\tilde{\theta}"] \arrow[r, shift left, "p_1"] \arrow[u, shift right, "\tilde{\phi}_1"'] \arrow[d, shift right, "p_2"'] & (I^{(n)}/Imd, b_0) \arrow[l, shift left, "r"] \arrow[d, "p_3"] \arrow[dl, "p_2 \circ r"] \\
(I^{(n-1)}/Imd + ImN, b_0)[1] \arrow[r, "d"'] & (I^{(n)}/ImN, b_0) \arrow[r, "p_4"'] \arrow[u, shift right, "s"'] & (I^{(n)}/Imd + ImN, b_0)
\end{tikzcd}$$

Here $\tilde{N}, \tilde{\phi}_1, \tilde{d}, \tilde{\theta}$ are induced by N, ϕ_1, d, θ, respectively, p_1, p_2 are the obvious projections, r is the map induced by θd, s is the one induced by $\phi_0(1-t)$, and $\tilde{\beta}$ is the one induced by $-\theta N$. From 8.1 (ii), (iii), $\tilde{\phi}_1, s$ are chain maps with:

$$\tilde{\phi}_1 N = Id, \ p_2 s = Id, \ N\tilde{\phi}_1 + s p_2 = Id. \quad (57)$$

Also, from (i) of the same Lemma, $\tilde{\theta}, r$ are chain maps with:

$$\tilde{\theta}\tilde{d} = Id,\ p_1 r = Id,\ r p_1 + \tilde{d}\tilde{\theta} = Id. \tag{58}$$

Since $-\tilde{\beta}d = \tilde{\theta}Nd = \tilde{\theta}\tilde{d}N = N$, $\tilde{\beta}$ induces a map between the Cokernels of d and N, and this is precisely our map (56). Moreover, $\tilde{\beta}$ induces a map between the left vertical short exact sequence, and the upper horizontal one. The boundaries of the long exact sequences induced in cohomology are, by the previous remarks and by (ii) of Lemma 8.2, the $-S$ operator, and π, respectively (the '$-$' sign in front of S is due to the fact that, given a short exact sequence, and shifting by one, the boundary induced in cohomology is '$-$ the initial boundary'; it also explains the '$-$' sign in our definition of β). Hence, by naturality, $\pi = (-S)(-\beta) = S\beta$. Similar arguments show that $p_2 r$ induces a chain map β' between the kernels of p_3 and p_4. By a diagram chasing and (58), we have $d\circ\beta'\circ p_5 = d\circ\beta\circ p_5$, hence $\beta' = \beta$. Using the naturality of the long exact sequences induced in cohomology by the right vertical and the bottom horizontal, we find $\pi = \beta S$.
Hence we are left with proving that β induces isomorphism in cohomology. We define now a new map on our diagram:

$$\tilde{\alpha} := -\tilde{\phi}_1\tilde{d} : (I^{(n-1)}/Imd, b_0)[1] \longrightarrow (I_\natural^{(n)}, \partial_0).$$

First of all, $-\tilde{\alpha}N = \tilde{\phi}_1\tilde{d}N = \tilde{\phi}_1 Nd = d$ by (57), so $\tilde{\alpha}$ induces a map between the Cokernels of N and d:

$$\alpha : (I^{(n-1)}/Imd + ImN, b_0)[1] \longrightarrow (I_\natural^{(n)}/Imd, \partial_0).$$

Since $\phi_1 d\theta N = \phi_1(1-\theta d)N$, $\phi_1 N \equiv 1$ modulo $Im(1-t)$, and $\phi_1\theta = 0$ (cf. 8.1), we have:

$$\alpha\circ\beta\ =\ 1\ .$$

Now we show that $\beta\alpha = 1$ in cohomology. Since $\theta N\phi_1 d = \theta(1+\phi_0(1-t))d \equiv 1 + \theta\phi_0(1-t)d$ modulo $Im(d)$, it suffices to show that:

$$\theta\phi_0 d(1-t) : (I^{(n-1)}/Imd + ImN, b_0) \longrightarrow (I^{(n-1)}/Imd + ImN, b_0)$$

is trivial in cohomology. For this we remark that our map factors as:

$$(I^{(n-1)}/Imd + ImN, b_0) \xrightarrow{1-t} (I^{(n-1)}/Imd, \partial_0) \xrightarrow{\theta\phi_0 d} (I^{(n-1)}/Imd + ImN, b_0),$$

where the second map is a chain map by the non-trivial 8.1 (iv), and the middle complex is contractible (by the usual s_{-1}).
Now, using Lemma 1.2, and Lemma 1.4, it is easy to see that all the formulas and arguments localize without any problem; this concludes the proof of Theorem 7.3.

8.3 Example (Equivariant cycles): We point out that the new complexes computing $HC_\delta^*(\mathcal{H})$, arising from Lemma 8.2, appear naturally in the construction of characteristic maps associated to equivariant cycles. Recall [7] that a chain of dimension n is a triple $(\Omega, d, \int)$ where $\Omega = \oplus_{j=0}^n \Omega^n$ is a DG algebra, and $\int : \Omega^n \longrightarrow \mathbf{C}$ is a graded trace on Ω. It is a cycle if $\int$ is closed. If (Ω, d) is a $\mathcal{H}$ DG algebra, and $\int$ is δ-invariant, we say that $(\Omega, d, \int)$ is a $\mathcal{H}$-chain. For instance, if a Lie group G acts smoothly on the Ω^j's, $d(g\omega) = gd(\omega)$, $\int g\omega = \int \omega$, for $g \in G$, $\omega \in \Omega$ (i. e. $(\Omega, d, \int)$ is a G- equivariant chain), then, with the induced infinitesimal action , $(\Omega, d, \int)$ is an $U(\mathfrak{g})$-chain.

A $\mathcal{H}$-cycle over an algebra A is given by such a cycle, together with an algebra homomorphism $\rho : A \longrightarrow \Omega^0$. We view ρ as an element of (total) degree 1 on the bigraded differential algebra $L = Hom(BA, \Omega)$. The structure on L is the one induced by the graded structures on $B(A)$, and Ω, respectively. That is, the bigrading: $L^{p,q} = Hom(A^{\otimes p}, \Omega^q)$, the differentials $d(f) = d \circ f$, $\partial_0(f) = -(-1)^{deg(f)} f \circ b'$, the product: $(\phi * \psi)(a_1, \ldots, a_{p+q}) = (-1)^{pdeg(\psi)} \phi(a_1, \ldots, a_p)\ \psi(a_{p+1}, \ldots, a_{p+q})$. As in 4.6, there is a unique $\mathcal{H}$ DG algebra map $k : W(\mathcal{H}) \longrightarrow Hom(BA, \Omega)$, sending 1 to ρ; it is compatible with the bigraded differential structure. On the other hand, using $\int : \Omega^n \longrightarrow \mathbf{C}[n]$, we have a graded trace $\int^{\natural} : Hom(BA, \Omega) \longrightarrow C_{\lambda}(A)[n]$, similar to (22), $\int^{\natural}(f) = \int \circ f \circ N$, whose composition with k is still denoted by $k : W(\mathcal{H}) \longrightarrow C_{\lambda}(A)[n]$.

Now, since $\int$ is closed, δ-invariant, and supported on degree n, the relevant target of k is the complex $(I^{(n)}_{\natural,\delta}/Imd, \partial_0)$, hence it induces a characteristic map $k : HC_{\delta}^{*-2n-1}(\mathcal{H}) \cong H^*(I^{(n)}_{\natural,\delta}/Imd, \partial_0) \longrightarrow H^*(C_{\lambda}(A)[n+1]) \cong HC^{*-n-1}(A)$. As in the proof of 7.5, it is compatible with the S-operator.

Let's assume now that $(\Omega, d, \int)$ is cobordant to the trivial cycle, that is, there is an n-dimensional chain $(\tilde{\Omega}, \tilde{d}, \tilde{\int})$, $\tilde{\rho} : A \longrightarrow \tilde{\Omega}^0$, and an equivariant chain map $r : \tilde{\Omega} \longrightarrow \Omega$ such that $\tilde{\int} \circ \tilde{d} = \int \circ r$, $r \circ \tilde{\rho} = \rho$. As before, we have an induced map $\tilde{k} : (I^{(n+1)}_{\natural,\delta}/Imd, \partial_0)[-1] \longrightarrow C_{\lambda}(A)[n+1]$. Since $\tilde{\int} \circ \tilde{d} = \int \circ r$, we have $\tilde{k} \circ d = k$, and we are on the localized version of the short exact sequence (51). Since its boundary identifies with π, hence with the S operator cf. 7.3, we deduce that, after stabilizing by S, k is trivial (in cohomology). We summarize our discussion:

Corollary 8.4 *Let A be an algebra. The previous construction associates to any n-dimensional equivariant cycle $(\Omega, d, \int)$ over A a characteristic map compatible with the S-operator:*

$$k : HC_{\delta}^{*}(\mathcal{H}) \longrightarrow HC^{*+n}(A)\ .$$

If k_0, k_1 are associated to cobordant cycles, then $S \circ k_0 = S \circ k_1$.

For instance, for $\mathcal{H} = U(\mathfrak{g})$ previously mentioned, using Theorem 5.6 we get:

Corollary 8.5 *Given a Lie group G, and a smooth n-dimensional G-cycle over an algebra A, we have induced ($\mathbf{Z}/2\mathbf{Z}$ graded) characteristic maps:*

$$k : H_*(\mathfrak{g}) \longrightarrow HP^{*+n}(A)\ .$$

Cobordant G-cycles induce the same map.

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