Non-Abelian duality, parafermions, and supersymmetry

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Non-Abelian duality in relation to supersymmetry is examined. When the action of the isometry group on the complex structures is nontrivial, extended supersymmetry is realized nonlocally after duality, using path-ordered Wilson lines. Prototype examples considered in detail are, hyper-Kähler metrics with SO(3) isometry and supersymmetric WZW models. For the latter, the natural objects in the nonlocal realizations of supersymmetry arising after duality are the classical non-Abelian parafermions. The canonical equivalence of WZW models and their non-Abelian duals with respect to a vector subgroup is also established. [S0556-2821(96)02314-4]

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I. INTRODUCTION

Target space duality (T duality) [1] interpolates between effective field theories corresponding to backgrounds with different spacetimes and even topological properties. Since strings propagating in T-dual backgrounds are equivalent and since the validity of the corresponding effective field theories is limited, we may use T duality as a way of probing truly stringy phenomena. The latter have to be taken into account in order to resolve paradoxes that appear in attempts to describe various phenomena solely in terms of the local effective field theories. Taking this way of reasoning one step further, we may view some long-standing problems in fundamental physics, for instance, in black holes physics, as nothing but paradoxes of the effective field theory description, which will cease to exist once string theoretical effects are properly taken into account. Though this is a speculation at the moment, it provides the main motivation for this work.

The best ground to test these ideas is in the relation between duality and supersymmetry, in the presence of rotational isometries [2]. In these cases nonlocal world-sheet effects have to be taken into account in order for supersymmetry and Abelian T-duality to reconcile [3]. Various aspects in this interplay between supersymmetry and Abelian T-duality have been considered in [3–6]. This paper is a natural continuation of these works for the cases where the duality group is a non-Abelian one [7–20] (for earlier work see [21] and references therein).

The plan of the paper is as follows. In Sec. II we consider two-dimensional bosonic σ models that are invariant under the action of a non-Abelian group G on the left. We briefly review the canonical transformation that generates the dual σ model in a way suitable for transformations of other geometrical objects. Then, we extend it to models with $N=1$ world-sheet supersymmetry. For cases that admit $N=2$ or $N=4$ extended supersymmetry, we derive the transformation rules of the corresponding complex structures. We show that, when these belong to nontrivial representations of a rotational subgroup of the duality group G, nonlocal world-sheet effects manifested with Wilson lines are necessary to restore

extended supersymmetry at the string theoretical level. Nevertheless, this appears to be lost after non-Abelian duality from a local effective field theory point of view. As examples, four-dimensional hyper-Kähler metrics with SO(3) isometry are considered in detail. The Eguchi-Hanson, Taub-Newman-Unti-Tamburino (NUT), and Atiyah-Hitchin metrics are famous examples among them. Explicit expressions for the three complex structures are given in general, which could be useful for other independent applications.

In Sec. III we consider the dual of a Wess-Zumino-Witten (WZW) model for a group G with respect to the vector action of a non-Abelian subgroup H. In such cases, extended supersymmetry is always realized nonlocally after duality. We show how these realizations become natural using classical non-Abelian parafermions of the G/H coset conformal field theory. We also establish the (so far lacking) canonical equivalence of these models. As an example, we consider the dual, with respect to SU(2), of the WZW model based on SU(2)⊗U(1).

In Sec. IV we present our conclusions, and discuss future directions of this work.

We have also written an appendix where we present in detail the canonical treatment of the models of Sec. III following Dirac's method for constrained Hamiltonian systems.

II. LEFT INVARIANT MODELS

We consider classical string propagation in d-dimensional backgrounds that are invariant under the left action[1] of a group G with dimension $\dim(G) \leq d$. We may split the target space variables as $X^M = \{X^\mu, X^i\}$, where X^μ, $\mu = 1, 2, \ldots, \dim(G)$ parametrize a group element in G and $X^i, i = 1, 2, \ldots, d - \dim(G)$ are some internal coordinates

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[1]In the language of Poisson-Lie T duality [15] we concentrate on cases of semi-Abelian doubles, where the coalgebra is Abelian, or in other words to traditional non-Abelian duality. The reason is that there are no known nontrivial examples of Poisson-Lie T duality where supersymmetry enters also into the game. Nevertheless, we comment in Sec. IV on how Poisson-Lie T duality may be used as a manifest supersymmetry restoration technique, in a string theoretical context.

which are inert under the group action. It will be convenient to think of them as parametrizing a group locally isomorphic to $U(1)^{d-\dim(G)}$. We also introduce a set of representation matrices $\{t^A\}=\{t^a,t^i\}$, with $a=1,2,\ldots,\dim(G)$ and $i=1,2,\ldots,d-\dim(G)$, which we normalize to unity. The components of the left- and right-invariant Maurer-Cartan one-forms are defined as

$$L_M^A=-i\operatorname{Tr}(t^A\hat{g}^{-1}\partial_M\hat{g}),$$

$$R_M^A=-i\operatorname{Tr}(t^A\partial_M\hat{g}\hat{g}^{-1})=C^{AB}(\hat{g})L_M^B, \qquad (2.1)$$

where $\hat{g}=ge^{it_iX^i}$, with $g\in G$ and $C^{AB}(\hat{g})=\operatorname{Tr}(t^A\hat{g}t^B\hat{g}^{-1})$. When we specialize to the internal space, $L_j^i=R_j^i=\delta^i{}_j$, $C^{ij}=\delta^{ij}$ and the corresponding structure constants are zero. The inverses of Eqs. (2.1) will be denoted by L_A^M and R_A^M, respectively.

The most general Lagrangian density which is manifestly invariant under the transformation $g\to\Lambda g$, for some constant matrix $\Lambda\in G$, is given by

$$\mathcal{L}=E_{AB}^+L_M^AL_N^B\partial_+X^M\partial_-X^N, \qquad (2.2)$$

where the couplings E_{MN}^+ can only depend on the X^i's and thus are also invariant under the action of the group G. For later use we also introduce $E_{AB}^-=E_{BA}^+$. An equivalent expression to Eq. (2.2) is

$$\mathcal{L}=E_{ij}^+\partial_+X^i\partial_-X^j+E_{ab}^+L_\mu^aL_\nu^b\partial_+X^\mu\partial_-X^\nu+E_{ai}^+L_\mu^a\partial_+X^\mu\partial_-X^i$$
$$+E_{ia}^+L_\mu^a\partial_+X^i\partial_-X^\mu. \qquad (2.3)$$

The natural time coordinate on the world sheet is $\tau=\sigma^++\sigma^-$, while $\sigma=\sigma^+-\sigma^-$ denotes the corresponding spatial variable. The Poisson brackets of the variable X^μ and its conjugate momentum P_μ are $\{X^\mu(\sigma),P_\nu(\sigma')\}=\delta_\nu^\mu\delta(\sigma-\sigma')$. Since the only dependence of Eq. (2.3) on the variables X^μ is via the combinations $L_\mu^a\partial_\pm X^\mu$, it is convenient to know the Poisson brackets of $L_\mu^a\partial_\sigma X^\mu$ and $L_a^\mu P_\mu$. After a simple computation, we find

$$\{\partial_\sigma X^\mu L_\mu^a(\sigma),L_b^\nu P_\nu(\sigma')\}=f_{bc}^a L_\mu^c\partial_\sigma X^\mu\delta(\sigma-\sigma')$$
$$+\delta^a{}_b\partial_\sigma\delta(\sigma-\sigma'),$$

$$\{L_a^\mu P_\mu(\sigma),L_b^\nu P_\nu(\sigma')\}=f_{ab}^c L_c^\nu P_\nu\delta(\sigma-\sigma'). \qquad (2.4)$$

At this point we perform the transformation $(X^\mu,P_\nu)\to(\widetilde{X}^\mu,\widetilde{P}_\nu)$ defined as [13,16]

$$L_\mu^a\partial_\sigma X^\mu=\widetilde{P}^a, \quad L^{\mu a}P_\mu=\partial_\sigma\widetilde{X}^a-f^{ab}\widetilde{P}_b, \qquad (2.5)$$

where $f^{ab}\equiv f_c^{ab}\widetilde{X}^c$. One can show that it preserves the Poisson brackets (2.4) and hence it is a canonical one. The X^i's remain unaffected by this transformation, so that $\widetilde{X}^i=X^i$. It is then a straightforward procedure to find the Lagrangian density to the dual to Eq. (2.3) σ model by applying the usual rules of canonical transformations in the Hamiltonian formalism. Here we only quote the final result:

$$\widetilde{\mathcal{L}}=E_{ij}^+\partial_+X^i\partial_-X^j+(\partial_+\widetilde{X}^a-E_{ai}^+\partial_+X^i)(M_-^{-1})_{ab}$$
$$\times(\partial_-\widetilde{X}^b+E_{bj}^+\partial_-X^j), \qquad (2.6)$$

with

$$M_-^{ab}=E_{ab}^++f_{ab}. \qquad (2.7)$$

In addition, conformal invariance requires the shift of the dilaton [1] by $\ln\det(M_-)$. The action (2.7) was obtained in [8] in the traditional approach to non-Abelian duality, where one adds to Eq. (2.2) a Lagrange multiplier term and introduces nondynamical gauge fields which are then integrated out using their classical equations of motion.

The transformation (2.5) was first applied to principal chiral models (PCM's), with $G=SU(2)$ in [13] and for general group in [16]. In PCMs there is no internal space and $E_{ab}^+=\delta_{ab}$. Hence, after non-Abelian duality with respect to the left action of the group there are still conserved currents associated with the right action of the group which generate symmetries in the dual model. It is tempting to attribute the success of the canonical transformation (2.5) to the existence of such conserved (local) currents. However, this is not true since Eq. (2.6), which has generically no conserved (local) currents, correctly follows from Eq. (2.5). Instead, what is common in the models (2.2) is the fact that the group action is entirely from the left. As a consequence, in the traditional approach with gauge fields, we can completely fix a unitary gauge as $g=1$, by appropriately choosing the X^μ's. In some sense Eq. (2.5) is a straightforward generalization of the corresponding transformation for Abelian isometry groups [22]. We will see in Sec. III that when the action of the isometry group is not entirely on the left or on the right, the analogue of the canonical transformation (2.5) is radically different.

It is important to know how the world-sheet derivatives of the target space variables $\partial_\pm X^M$ transform under the canonical transformation. It is quite straightforward, and in fact easier than applying the canonical transformation in all its glory, to show that Eq. (2.5) and the fact that the canonical transformation preserves the Lorentz invariance of the two-dimensional σ-model action (2.2), imply the dual model (2.6) as well as[2]

$$L_M^A\partial_\pm X^M=Q_{\pm B}^A\partial_\pm\widetilde{X}^B, \qquad (2.8)$$

where the matrix $Q_\pm$ is defined as

$$(Q_\pm)_B^A=\begin{pmatrix} \pm(M_\pm^{-1})^{ab} & -(M_\pm^{-1})^{ac}E_{jc}^\pm \\ 0 & \delta_j^i \end{pmatrix}, \qquad (2.9)$$

with $M_+^{ab}\equiv M_-^{ba}$. Of course this transformation acts trivially on the internal variables X^i, as it should. Notice that $Q_\pm$ only depends on the dual model variables $\widetilde{X}^M=(\widetilde{X}^a,X^i)$. Also, $A_\pm\equiv t_aQ_{\pm B}^a\partial_\pm\widetilde{X}^B$ can be identified with the on-shell values of the gauge fields introduced in the traditional approach to non-Abelian duality. For later convenience, the inverse matrix to Eq. (2.9) is also given:

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[2]Details on the application of this method, for the case of Abelian T duality, can be found in [23].

$$(Q_\pm^{-1})^A_B = \begin{pmatrix} \pm(M_\pm)^{ab} & \pm E^\pm_{ja} \\ 0 & \delta^i_j \end{pmatrix}. \qquad (2.10)$$

In terms of these matrices the metric corresponding to Eq. (2.6) can be written as

$$\widetilde{G}_{AB} = Q^C_{\pm A} Q^D_{\pm B} G_{CD},$$

$$G_{CD} \equiv \tfrac{1}{2}(E^+_{CD} + E^-_{CD}), \qquad (2.11)$$

where both expressions corresponding to the plus and the minus signs give the same result for $\widetilde{G}_{AB}$, as they should.

Let us consider the transformation (2.8) for the plus sign. It amounts to a nonlocal redefinition of the target space variables X^μ in the group element $g \in G$ associated with the isometry

$$g = P e^{i \int^{\sigma^+} A_+}, \qquad (2.12)$$

where P stands for path ordering of the exponential. The integration is carried out for fixed σ^- and connects a base point with σ^+. Since the equations of motions for Eq. (2.6) imply the vanishing of the field strength associated with A_+, the expression (2.12) for g can be replaced by a similar one using A_- and integration carried out for fixed σ^+. The dual background (2.6) is a local function of $\widetilde{X}^M$ due to the fact that in the original background (2.2) all group dependence was via the left-invariant L^a_μ. However, other geometrical objects are not bound to have such a dependence. In these cases they become nonlocal in the dual picture. We will shortly encounter examples of that kind.

$N=1$ *world-sheet supersymmetry*. Any background can be made $N=1$ supersymmetric [24]. Thus, it is expected that a manifestly supersymmetric version of the non-Abelian duality transformation exists. Indeed, this was found, in the traditional formalism, for the general model (2.3) in [14]. In terms of a canonical transformation there is work for the supersymmetric version of the nonlinear chiral model on O(4) and its dual [25]. Since supersymmetry dictates the form of any transformation compatible with it once the bosonic part is known, it is straightforward to find the canonical transformation for the general supersymmetric model by applying the following procedure: First, one obtains the supersymmetric version of Eq. (2.8) by simply replacing the bosonic fields and world-sheet derivatives by their respective superfields and world-sheet superderivatives

$$L^A_M(Z)D_\pm Z^M = Q^A_{\pm B}(\widetilde{Z})D_\pm \widetilde{Z}^B, \qquad (2.13)$$

where $Z^M = X^M - i\theta_+ \Psi^M_- + i\theta_- \Psi^M_+ - i\theta_+\theta_- F^M$ is a generic $N=1$ superfield, with a similar expression for $\widetilde{Z}^M$, and $D_\pm = \mp i\partial_{\theta_\mp} \mp \theta_\mp \partial_\pm$. We have denoted by $\Psi^M_\pm$ the world-sheet fermions and by $\theta^\pm$ the two Grassman variables. The highest component of the superfield F^M is eliminated by using its equations of motion. It finally assumes the form $F^M = i(\Omega^+)^M_{N\Lambda}\Psi^N_-\Psi^\Lambda_+$. Next, we expand both sides of Eq. (2.13) and read the corresponding transformation rules of the components. We find that the transformation of the bosonic part is given by Eq. (2.8) with the right-hand side modified

by a quadratic term in the world-sheet fermions $\Psi^M_\pm$ of similar chirality as the world-sheet derivative:

$$L^A_M \partial_\pm X^M = Q^A_{\pm B}\partial_\pm \widetilde{X}^B - i\partial_B Q^A_{\pm C}\widetilde{\Psi}^B_\pm \widetilde{\Psi}^C_\pm$$

$$+ \frac{i}{2} f^A_{BC} Q^B_{\pm D} Q^C_{\pm E} \widetilde{\Psi}^D_\pm \widetilde{\Psi}^E_\pm. \qquad (2.14)$$

We note that bosons in the dual model are composites of bosons and fermions of the original model. This boson-fermion symphysis is a common characteristic of duals to supersymmetric σ models and was first observed in [26] for the non-Abelian dual of the supersymmetric extension of the chiral model on O(4) and for Abelian duality in [4]. Accordingly, the redefinition of the group element $g \in G$, though similar to Eq. (2.12), will also involve the quadratic in the fermion terms that appear in the right-hand side of Eq. (2.14). Nevertheless, since they can always be generated from the bosonic first term, we will refrain, in the rest of the paper, from writing them explicitly. The transformation of $L^A_M \Psi^M_\pm$ is similar to Eq. (2.8):

$$L^A_M \Psi^M_\pm = Q^A_{\pm B}\widetilde{\Psi}^B_\pm. \qquad (2.15)$$

The on-shell expression for the highest component of the superfield can be used to find the transformation of the generalized connection:

$$(\widetilde{\Omega}^\pm)^A_{BC} = (Q_\pm^{-1}L)^A_M (L^{-1}Q_\mp)^N_B (L^{-1}Q_\pm)^\Lambda_C [(\Omega^\pm)^M_{N\Lambda}$$

$$- \partial_N L^D_\Lambda L^M_D] + \partial_B (Q_\pm)^D_C (Q_\pm^{-1})^A_D. \qquad (2.16)$$

When the group G is Abelian, the transformations of the world-sheet fermions and of the connections reduce to the corresponding ones in [4].[3] Consider now a field $V^M_\pm$ that transforms under non-Abelian duality similarly to Eq. (2.8). Namely,

$$L^A_M V^M_\pm = Q^A_{\pm B}\widetilde{V}^B_\pm. \qquad (2.18)$$

We will call such a field a $(1_\pm,0)$ tensor, since its transformation under duality resembles that of a vector field under diffeomorphisms. In general, a $(n_+,n_-;m_+,m_-)$ tensor will have $n_\pm$ upper and $m_\pm$ lower indices of the indicated chirality. It is a straightforward computation to prove, using Eqs. (2.16) and (2.18), that

$$\widetilde{D}^\pm_A \widetilde{V}^B_\pm = (L^{-1}Q_\mp)^M_A (Q_\pm^{-1}L)^B_N D^\pm_M V^N_\pm. \qquad (2.19)$$

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[3]There is an alternative expression to Eq. (2.16) in which the right-hand side depends manifestly on variables of the dual model only. It can be easily found using the identity

$$(\Omega^\pm)^M_{N\Lambda} - \partial_\Lambda L^D_N L^M_D = L^M_A L^B_N L^C_\Lambda ((\Omega^\pm)^A_{BC} + \tfrac{1}{2}G^{AD}(f_{C[D}{}^E E^\mp_{B]E} + f_{BD}{}^E E^\mp_{EC})),$$

$$(2.17)$$

where $(\Omega^\pm)^A_{BC}$ are the connections defined using E^+_{AB}.

Hence, the covariant derivative of a $(1_+,0)$ tensor is a $(1_+,1_-)$ tensor. More generally, the covariant derivative D_M^+ on a tensor of type $(n_+,0;m_+,0)$ will transform it into a $(n_+,0;m_+,1_-)$-type tensor. Similarly, the covariant derivative D_M^- on a $(0,n_-;0,m_-)$-type tensor will transform it into a $(0,n_-;1_+,m_-)$-type one. The fact that the action of covariant derivatives on tensors of the type we have indicated preserves their tensor character is not a trivial statement. Any other combination of covariant derivatives on these or more general tensors produces objects that transform anomalously under duality. For instance, the generalized curvature $R_{MNK\Lambda}^+$, though a tensor under diffeomorphisms, is not one under duality [14,4,6]. This is ultimately connected to the nonlocal nature of the duality transformation when the latter is viewed merely as a redefinition of the target space variables [cf. Eq. (2.12)].

Extended world-sheet supersymmetry. Conventionally, extended $N=2$ supersymmetry [27–29] requires that the background is such that an (almost) complex (Hermitian) structure F_{MN}^+ in each sector, associated to the right- and left-handed fermions, exists. Similarly, $N=4$ extended supersymmetry [28–30] requires that, in each sector, there exist three complex structures $(F_I^+)_{MN}$, $I=1,2,3$. The complex structures are covariantly constant, with respect to the generalized connections, and are represented by antisymmetric matrices. In the case of $N=4$ they obey the SU(2) Clifford algebra. If in addition they are integrable they also satisfy the Nijenhuis conditions, though these are not necessary for the existence of extended supersymmetry [31]. The above requirements put severe restrictions on the backgrounds that admit a solution. For instance, in the absence of torsion the metric should be Kahler for $N=2$ and hyper-Kahler for $N=4$ [28].

In order to determine the fate of extended supersymmetry under non-Abelian duality, it is useful to assign the complex structures to representations of the isometry group G. The simplest cases to consider are those with complex structures belonging to the singlet representation, thus remaining invariant under the group action on the left. The most general form of such complex structures is

$$F_{MN}^\pm = F_{AB}^\pm L_M^A L_N^B, \qquad (2.20)$$

where $F_{AB}^\pm$ is an antisymmetric matrix independent of the X^μ's which obeys $(F^2)_B^A = -\delta_B^A$. Its functional dependence on the internal space variables X^i is determined by demanding that Eq. (2.20) is covariantly constant. In order to find how Eq. (2.20) transforms under non-Abelian duality we consider, similarly to the case of Abelian duality [3], the two-form $F^\pm = F_{MN}^\pm dX^M \wedge dX^N$ and its transformation properties induced by Eq. (2.8). The result is

$$\widetilde{F}_{AB}^\pm = Q_{\pm A}^C Q_{\pm B}^D F_{CD}^\pm. \qquad (2.21)$$

Hence, $F_{AB}^\pm$ transforms as a $(0,2_\pm)$ tensor under duality. Then, it follows that $\widetilde{D}_A^\pm \widetilde{F}_{BC}^\pm = 0$. Similarly, one verifies that all properties of the original complex structure are properties of its duals as well. In the case of $N=4$ with three complex structures that are singlets, each one of them is of the form (2.20), with the corresponding $(F_I^\pm)_{AB}$, $I=1,2,3$ obeying the SU(2) Clifford algebra. They transform as in Eq. (2.21) under duality and they similarly define a locally realized $N=4$ in the dual model.

Consider now cases where the complex structures transform in a nontrivial represenation of the duality group G. This is impossible if we only have $N=2$ extended supersymmetry since there should be at least two complex structures to form a nontrivial representation. On the other hand, it is well known that this implies the existence of a third one and thus we are led to consider the case of $N=4$ extended supersymmetry. If the duality group is SO(3)$\approx$SU(2), with structure constants $f_{IJK} = \sqrt{2}\epsilon_{IJK}$ in our normalization, then this implies that the Lie derivative acts as $\pounds_{R_I} F_J^\pm = f_{IJK} F_K^\pm$. Thus, the complex structures $F_I^\pm$, $I=1,2,3$ transfrom in the triplet representation. For bigger groups the same transformation is valid if we restrict it to an appropriate rotational SO(3) subgroup of G. Let us introduce a singlet under the group G matrix $(\Phi_I^\pm)_{AB}$, which satisfies the same properties as the matrix $(F_I^\pm)_{AB}$. The form of the triplet complex structures is then

$$(F_I^\pm)_{MN} = C^{IJ}(g)(\Phi_J^\pm)_{MN}, \qquad (\Phi_J^\pm)_{MN} = (\Phi_J^\pm)_{AB} L_M^A L_N^B, \qquad (2.22)$$

where $I,J=1,2,3$, but note that, as always $A=1,2,\ldots$, dim$(G),\ldots,d$. In order to prove this, it is enough to notice that $\pounds_{R_I} C_{JK} = f_{IJL} C_{LK}$ and $\pounds_{R_I} \Phi_K^\pm = 0$. Consider now the effects of non-Abelian duality on complex structures of the form (2.22). The singlet factor $(\Phi_I^\pm)_{MN}$ remains local and transforms similarly to Eq. (2.21). However, the matrix C_{IJ} involves the group element $g \in G$ explicitly, which then will be given by the path-ordered Wilson line (2.12). Hence, in the dual model

$$(\widetilde{F}_I^\pm)_{AB} = C^{IJ}(g)(\widetilde{\Phi}_J^\pm)_{AB}, \qquad (\widetilde{\Phi}_J^\pm)_{AB} = Q_{\pm A}^C Q_{\pm B}^D (\Phi_J^\pm)_{CD}. \qquad (2.23)$$

The complex structure as a whole is nonlocal precisely due to the attached Wilson line. The question is whether or not it can still be used to define an extended supersymmetry. The nonlocal complex structure (2.23) still satisfies the SU(2) Clifford algebra, but it is no longer covariantly constant. This is similar to the case of Abelian duality, as was first found in [3] and further elaborated in [4]. Instead, they have to satisfy the general conditions for existence of nonlocal complex structures [6],

$$\widetilde{D}_A^\pm (\widetilde{F}_I^\pm)_{BC} \partial_\mp \widetilde{X}^A + \widetilde{\partial}_\mp (\widetilde{F}_I^\pm)_{BC} = 0, \qquad (2.24)$$

where the tilded world-sheet derivative acts only on the nonlocal part of the complex structure. Using Eq. (2.23), we find that Eq. (2.24) implies the following equation for $\widetilde{\Phi}_I^\pm$:

$$C_{IJ}\widetilde{D}_A^\pm (\widetilde{\Phi}_J^\pm)_{BC} + C_{IE} f_{JD}^E Q_{\mp A}^D (\widetilde{\Phi}_J^\pm)_{BC} = 0. \qquad (2.25)$$

Then, the transformations (2.23) and (2.19) imply

$$C_{IJ} D_M^\pm (\Phi_J^\pm)_{N\Lambda} + C_{IA} f_{JB}^A L_M^B (\Phi_J^\pm)_{N\Lambda} = 0, \qquad (2.26)$$

which is nothing but the covariantly constancy equation for the local complex structure (2.22) rewritten as an equation for $(\Phi_I^\pm)_{MN}$. Thus, we have proved that the original local

$N=4$ breaks down to a local $N=1$, whereas the part corresponding to the extended supersymmetry gets realized nonlocally. Nevertheless, in a string setting $N=4$ remains a genuine supersymmetry.

In order to fully illustrate the previous general discussion it will be enough to focus on the special class of four-dim hyper-Kähler metrics with SO(3) symmetry. An additional reason is that hyper-Kähler geometry is an interesting subject by itself, especially in connection with the theory of gravitational instantons, supersymmetric models, and supergravity, and various moduli problems in monopole physics, string theory, and elsewhere. The line element of four-dim hyper-Kähler metrics with SO(3) symmetry, in the Bianchi-type IX formalism, is given by

$$ds^2 = f^2(t)dt^2 + a_1^2(t)\sigma_1^2 + a_2^2(t)\sigma_2^2 + a_3^2(t)\sigma_3^2. \quad (2.27)$$

Here, σ_i, $i=1,2,3$ are the left-invariant one-forms of SO(3).[4] In the parametrization of the group element in terms of Euler angles, $g = e^{(i/2)\,\phi\sigma_3}e^{(i/2)\,\theta\sigma_2}e^{(i/2)\,\psi\sigma_3}$, they assume the form

$$\sigma_1 = \tfrac{1}{2}(\sin\theta\cos\psi d\phi - \sin\psi d\theta),$$

$$\sigma_2 = \tfrac{1}{2}(\sin\theta\sin\psi d\phi + \cos\psi d\theta),$$

$$\sigma_3 = \tfrac{1}{2}(d\psi + \cos\theta d\phi). \quad (2.28)$$

The coordinate t of the metric can always be chosen so that

$$f(t) = \tfrac{1}{2}a_1 a_2 a_3, \quad (2.29)$$

using a suitable reparametrization. It was established some time ago [32] that the second-order differential equations that provide the self-duality condition for the class of metrics (2.27) in the parametrization (2.29) can be integrated once to yield the following first-order system in t:

$$\frac{a_i'}{a_i} = \frac{1}{2}\vec{a}^2 - a_i^2 - 2f\frac{\lambda_i}{a_i}, \quad i=1,2,3, \quad (2.30)$$

where the three parameters λ_i remain undetermined for the moment. The derivatives (denoted by prime) are taken with respect to t. We essentially have two distinct categories of solutions to Eq. (2.30), depending on the values of the parameters λ_1, λ_2, λ_3. The first is described by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and the second by $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The Eguchi-Hanson metric belongs to the first category and the Taub-NUT and the Atiyah-Hitchin metrics to the second. These three cases provide the only nontrivial hyper-Kähler four metrics with SO(3) isometry that are complete and nonsingular [33].

Complex structures. It is known (see, for instance, [33]) that the complex structures for the Eguchi-Hanson metric are singlets under the SO(3) action whereas those for the Taub-

NUT and the Atiyah-Hitchin transform as a triplet. Moreover, for the Eguchi-Hanson and the Taub-NUT metrics, explicit expressions are known [33,34]. For the Atiyah-Hitchin metric the complex structures are only known in the Toda-frame formulation of the metric [35], which was found using the fact that $\partial/\partial\phi$ is a manifest Killing vector field[5] of Eq. (2.27). Recently also, one of the complex structures of the Atiyah-Hitchin metric, in the parametrization (2.27), appeared in [36]. However, the result for the general metric (2.27) is not known, so that we will proceed with its derivation.

We will prove that any hyper-Kähler metric that is SO(3) invariant with line element given by (2.27) and (2.30), has three complex structures given by

$$F_i = \begin{cases} K_i & \text{if} \quad \lambda_1 = \lambda_2 = \lambda_3 = 0 \\ C_{ij}K_j & \text{if} \quad \lambda_1 = \lambda_2 = \lambda_3 = 1 \end{cases}, \quad (2.31)$$

where K_i is given by

$$K_i = 2e_0 \wedge e_i + \epsilon_{ijk}e_j \wedge e_k, \quad (2.32)$$

with the tetrads defined as $e_0 = fdt$ and $e_i = a_i\sigma_i$. In accordance with (2.20) and (2.22) the F_i's for $\lambda_i = 0$ are singlets of SO(3) whereas for $\lambda_i = 1$ transform in the triplet representation. In order to prove Eq. (2.31), let us first note that clearly the K_i's obey the quaternionic algebra. Since $C_{ik}C_{jk} = \delta_{ij}$, it is easy to verify that the F_i's, in general, obey the same algebra as well. Then, it remains to prove that $D_\mu(F_i)_{\nu\rho} = 0$. Since the torsion is zero, it suffices to show that F_i is a closed two-form and that the associated Nijenhuis tensor vanishes.[6] A short computation using Eq. (2.30) to substitute for derivatives with respect to t gives

$$dK_i = -4f\epsilon_{ijk}\lambda_j a_k dt \wedge \sigma_j \wedge \sigma_k. \quad (2.33)$$

Thus, in the cases where $\lambda_i = 0$, we find that indeed $F_i = K_i$ are closed forms. Then, using the property $dC_{ij} = 2C_{im}\epsilon_{mjk}\sigma_k$, we compute that

$$d(CK)_i = -4fC_{ij}\epsilon_{jmk}(\lambda_m - 1)a_k dt \wedge \sigma_m \wedge \sigma_k$$
$$+ 2C_{ij}\epsilon_{jmk}\epsilon_{mln}a_p a_n\sigma_k \wedge \sigma_p \wedge \sigma_n. \quad (2.34)$$

It can be easily seen that the second line in the above equation vanishes identically. Hence also for the cases $\lambda_i = 1$, $F_i = C_{ij}K_j$ are closed forms. Verifying the vanishing of the Nijenhuis tensor is a bit harder task, but nevertheless straightforward, and will not yield any details.

The dual σ model. Non-Abelian duality on Eq. (2.27) with respect to the SO(3) isometry group corresponds to a canonical transformation which for the world-sheet derivatives assumes the form [cf. Eq. (2.8)]

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[4] Since the internal space parametrized by the variable t is one-dimensional, it will not be confusing to use instead of upper case letters I,J,K, lower case ones i,j,k. Also, in order to comply with standard notation in the literature and avoid factors of $\sqrt{2}$ we will use $\sigma_i = (1/\sqrt{2})L^i$ for the left-invariant Maurer-Cartan forms. Then, also $f_{ijk} = \sqrt{2}\epsilon_{ijk}$.

[5] Any hyper-Kähler metric with a rotational Killing symmetry can be formulated in the Toda frame [37], in which case the explicit expressions for the complex structures are known in general [3].

[6] This implies that there exists an atlas such that one of the F_i's is constant. The integrability of the quaternionic structure which would have implied that an atlas existed such that all three F_i's were constant requires much stronger conditions to be satisfied [38]. Nevertheless, for the existence of $N=4$ supersymmetry, this integrability is not needed.

$$\sigma_i^\pm = \pm 2e^{-\widetilde{\Phi}}\left(\frac{4f^2}{a_i^2}\partial_\pm\chi^i + \chi^i\chi\cdot\partial_\pm\chi \pm \epsilon_{ijk}\chi_k a_k^2\partial_\pm\chi^j\right),$$

$$(2.35)$$

where $\sigma_i^\pm$ are the (1,0) and (0,1) components of the decomposition of the one-forms (2.28) on the world sheet and the χ^i's represent the three variables dual to the Euler angles. The dual to the background (2.27) can be obtained by specializing Eq. (2.6) in this case. The explicit form for the fields is [12]

$$d\widetilde{s}^2 = f^2 dt^2 + e^{-\widetilde{\Phi}}\left(\chi_i\chi_j + \delta_{ij}\frac{4f^2}{a_i^2}\right)d\chi_i d\chi_j,$$

$$\widetilde{B}_{ij} = -e^{-\widetilde{\Phi}}\epsilon_{ijk}\chi_k a_k^2,$$

$$e^{\widetilde{\Phi}} = 4(4f^2 + a_i^2\chi_i^2). \qquad (2.36)$$

The dual complex structures. The dual to the two-form (2.32) can be obtained from Eq. (2.23) or by directly transforming it using Eq. (2.35). The result is

$$\widetilde{K}_i^\pm = e^{-\widetilde{\Phi}}\left[\pm 4f dt\wedge\left(\frac{4f^2}{a_i^2}d\chi^i + \chi^i\chi\cdot d\chi \pm \epsilon_{ijk}\chi_k a_k^2 d\chi^j\right)\right.$$
$$\left. \pm\frac{4f}{a_i}\chi\cdot d\chi\wedge d\chi^i + a_i^2\epsilon_{ijk}a_j a_k d\chi^j\wedge d\chi^k\right]. \qquad (2.37)$$

For the cases where the original hyper-Kahler metric corresponds to the choice $\lambda_i=0$ in Eq. (2.30), these are in fact the three complex structures for the dual background (2.36), which has locally realized $N=4$ supersymmetry. It can be shown that the (anti)self-duality conditions of the dilaton-axion field are solved and, therefore, we have found that Eq. (2.36) is a new class of axionic instantons which are related to hyper-Kahler metrics (2.27) via non-Abelian duality. Though not obvious, it can be shown that the metric in Eq. (2.36) is conformally flat [for the case where Eq. (2.27) is the Eguchi-Hanson metric this was observed in [12]], and the conformal factor $e^{-\widetilde{\Phi}}$ satisfies the Laplace equation adapted to the flat metric. This is in agreement with a theorem proved in [39] for four-dim backgrounds with $N=4$ world-sheet supersymmetry and nonvanishing torsion. The particular form of the coordinate change needed to explicitly demonstrate this is complicated and not very illuminating. Here we mention the result for the non-Abelian dual to four-dimensional flat space which corresponds to the choice $a_1=a_2=a_3=(-t)^{-1/2}$ in Eq. (2.27). We found that the dual metric can be written in terms of Cartesian coordinates x_i, as $d\widetilde{s}^2 = e^{-\widetilde{\Phi}}dx_i dx_i$, where $e^{\widetilde{\Phi}}=2r\sqrt{r+x_4}$, with $r^2=x_i x_i$.

For the cases where the original hyper-Kahler metric corresponds to the choice $\lambda_i=1$ in Eq. (2.30) the dual background has nonlocally realized $N=4$ world-sheet supersymmetry. The complex structures are $\widetilde{F}_i^\pm = C_{ij}(g)\widetilde{K}_j^\pm$, with $C_{ij}(g)$ being nonlocal functionals of the dual space variables according to Eq. (2.35).

III. DUALS OF WZW MODELS

We would like to make contact with exact conformal field theoretical results. The hyper-Kahler metrics and their non-Abelian duals we have examined are not appropriate for such an investigation since their description in terms of exact conformal field theories is, at present, unknown. The best examples to consider in this respect are non-Abelian duals of WZW models, since, as it turns out, the nonlocal realizations of supersymmetry that arise after duality can be naturally expressed in terms of non-Abelian parafermions.

The WZW model action, to be denoted by $I_{\mathrm{WZW}}(g)$, for a group element $g\in G$, corresponds to a background with metric and torsion given by

$$G_{MN}=L_M^A L_N^A=R_M^A R_N^A,$$

$$H_{MN\Lambda}=f_{ABC}L_M^A L_N^B L_\Lambda^C=f_{ABC}R_M^A R_N^B R_\Lambda^C. \qquad (3.1)$$

A WZW model for a general group can be made $N=1$ supersymmetric on the world-sheet [40]. If the group is an even-dimensional one the supersymmetry is promoted to an $N=2$ [41]. Moreover, WZW models based on quaternionic groups have actually $N=4$ [41]. The general form of the complex structures is very similar to Eq. (2.20):

$$F_{MN}^+ = F_{AB}^+ L_M^A L_N^B, \qquad F_{MN}^- = F_{AB}^- R_M^A R_N^B, \qquad (3.2)$$

where the *constant* matrices $F_{AB}^\pm$ are Lie-algebra complex structures [41]. The covariant constancy of $F_{MN}^\pm$ follows trivially from the fact that $D_M^+ L_N^A = D_M^- R_N^A = 0$, which are valid for any WZW model. It is obvious that F^+ (F^-) is invariant under the left (right) group action. Thus, under the vector action of a non-Abelian subgroup H of G, i.e., $g\to\Lambda^{-1}g\Lambda$, none of the F^+, F^- is invariant.

The analogue of the canonical transformation (2.5) or (2.8) for the non-Abelian dual of a WZW model with respect to its vector subgroup H will be presented in the next subsection. Here, we proceed traditionally by starting with the usual gauged WZW action [42,43] plus a Langrange multiplier term:

$$S = I_{\mathrm{WZW}}(g) + \frac{k}{\pi}\int \mathrm{Tr}(A_+\partial_- g g^{-1} - g^{-1}\partial_+ g A_-$$
$$+ A_+ g A_- g^{-1} - A_+ A_-) + i\mathrm{Tr}(vF_{+-}), \qquad (3.3)$$

where $A_\pm$ are gauge fields in the Lie-algebra of a subgroup H of G with corresponding field strength $F_{+-} = \partial_+ A_- - \partial_- A_+ - [A_+,A_-]$ and v are some Lie algebra variables in H that play the role of Lagrange multipliers. We also split indices as $A=(a,\alpha)$, where $a\in H$ and $\alpha\in G/H$. Variation of Eq. (3.3) with respect to all fields gives the classical equations of motion

$$\delta A_+: \quad D_- g g^{-1}|_H + iD_- v = 0, \qquad (3.4)$$

$$\delta A_-: \quad g^{-1}D_+ g|_H + iD_+ v = 0, \qquad (3.5)$$

$$\delta g: \quad D_+(D_- g g^{-1}) + F_{+-} = 0, \qquad (3.6)$$

$$\delta v: \quad F_{+-} = 0. \qquad (3.7)$$

To find the dual σ model a unitary gauge should be chosen. This is done by fixing $\dim(H)$ variables among the total number of $\dim(G) + \dim(H)$ ones, thus remaining with a total of $\dim(G)$ variables, which we will denote by X^M. If $H \neq G$ then generically there is no isotropy subgroup and we can gauge fix all $\dim(H)$ variables in the group element $g \in G$. If $H = G$ then the nontrivial isotropy subgroup corresponding to the Cartan subalgebra of G cannot be gauge fixed away. In such cases we gauge fix $\dim(G) - \mathrm{rank}(G)$ parameters in g and the remaining $\mathrm{rank}(G)$ ones among the Lagrange multipliers v^a. Then we eliminate the gauge fields using their classical equations of motion (3.4) and (3.5):

$$A_+^a = +i(C^T - I - f)^{-1}_{ab}(L_\mu^b \partial_+ X^\mu + \partial_+ v^b) \equiv A_{+M}^a \partial_+ X^M,$$

$$A_-^a = -i(C - I + f)^{-1}_{ab}(R_\mu^b \partial_- X^\mu + \partial_- v^b) \equiv A_{-M}^a \partial_- X^M. \tag{3.8}$$

Finally, the dual σ model is given by [8,11]

$$S = I_{\mathrm{WZW}}(g) - \frac{k}{\pi} \int (L_\mu^a \partial_+ X^\mu + \partial_+ v^a)$$
$$\times (C - I + f)^{-1}_{ab}(R_\nu^b \partial_- X^\nu + \partial_- v^b). \tag{3.9}$$

A dilaton $\Phi = \ln\det(C - I + f)$ is also induced in order to preserve conformal invariance at one loop [1].

As in the previous section, it will be convenient to have an explicit expression for the generalized connections of the dual model (3.9). For this we utilize the classical string equation for the dual action (3.9), $D_+(D_- g g^{-1}) = 0$, which follows from Eq. (3.6) after we use Eq. (3.7). In these equations the gauged fields entering the covariant derivatives should be replaced by their on-shell values (3.8). We define

$$\mathrm{Tr}(t^A g^{-1} D_+ g) = i\mathcal{L}_M^A \partial_+ X^M,$$

$$\mathrm{Tr}(t^A D_- g g^{-1}) = i\mathcal{R}_M^A \partial_- X^M. \tag{3.10}$$

Under gauge transformations $\mathcal{L}_M^A$ and $\mathcal{R}_M^A$ are left and right invariant, respectively. Then it is easy to cast the classical equations of motion into the standard form for any two-dimensional σ model,

$$\partial_+ \partial_- X^M + (\Omega^-)^M_{N\Lambda} \partial_+ X^N \partial_- X^\Lambda = 0, \tag{3.11}$$

from which we read off the generalized connection of the dual model:

$$(\Omega^-)^M_{N\Lambda} = \mathcal{L}_A^M \partial_\Lambda \mathcal{L}_N^A + if_{Bc}^A \mathcal{L}_A^M \mathcal{L}_N^B A_{-\Lambda}^c. \tag{3.12}$$

It is convenient to define the following gauge-invariant elements, in the Lie algebra of G,

$$\Psi_+ = -ih_-^{-1} g^{-1} D_+ g h_-, \quad \Psi_- = -ih_+^{-1} D_- g g^{-1} h_+, \tag{3.13}$$

where the group elements $h_\pm \in H$ are given by path-ordered exponentials similar to Eq. (2.12):

$$h_+^{-1} = P e^{-\int \sigma^+ A_+}, \quad h_-^{-1} = P e^{-\int \sigma^- A_-}, \tag{3.14}$$

with the gauge fields $A_\pm$ determined by Eq. (3.8). They obey $A_\pm = \partial_\pm h_\pm h_\pm^{-1}$. Using the classical equations of motion (3.4)–(3.7), it can be shown that Ψ_+ and Ψ_- are chiral

$$\partial_- \Psi_+ = 0, \quad \partial_+ \Psi_- = 0. \tag{3.15}$$

We will also denote $\Psi^A = \Psi_{\pm M}^A \partial_\pm X^M$, where

$$\Psi_{+M}^A = C^{BA}(h_-)\mathcal{L}_M^B, \quad \Psi_{-M}^A = C^{BA}(h_+)\mathcal{R}_M^B. \tag{3.16}$$

Because they have Wilson lines attached to them, $\Psi_\pm$ are nonlocal. Since, the action we started with Eq. (3.3) contains the standard gauged WZW action corresponding to the coset G/H, it is expected that $\Psi_\pm$ will be related to the classical non-Abelian parafermions [44,45]. The precise relationship will be uncovered in the next subsection.

We are now in the position to examine the fate of world-sheet supersymmetry under non-Abelian duality. We will show that the dual action (3.9) has nonlocally realized extended supersymmetry with complex structures, corresponding to Eq. (3.2), given by

$$\widetilde{F}_{MN}^+ = F_{AB}^+ \Psi_{+M}^A \Psi_{+N}^B, \quad \widetilde{F}_{MN}^- = F_{AB}^- \Psi_{-M}^A \Psi_{-N}^B. \tag{3.17}$$

It is obvious that the dual complex structures (3.17) obey all properties of their counterparts (3.2) except that they are not covariantly constant. Being nonlocal, they should satisfy instead, the equation [6]

$$\widetilde{D}_M^\pm(\widetilde{F}^\pm)_{N\Lambda} \partial_\mp \widetilde{X}^M + \widetilde{\partial}_\mp(\widetilde{F}^\pm)_{N\Lambda} = 0, \tag{3.18}$$

where the tilded derivative acts only on the nonlocal part of the complex structures contained in $h_\pm$, which are given by the path-ordered exponentials (3.14). For this it is enough to prove that

$$\widetilde{D}_M^\pm \Psi_\pm^A \partial_\mp X^M + \widetilde{\partial}_\mp \Psi_\pm^A = 0, \tag{3.19}$$

where, similar to Eq. (3.18), the tilded world-sheet derivative acts only on the nonlocal part of $\Psi_\pm^A$. This becomes a straightforward computation after we use the expression for the generalized connection of the dual model (3.12).

Thus, we have shown that as long as H is non-Abelian, T duality breaks all local extended supersymmetries which are then realized nonlocally with complex structures given by Eq. (3.17). Our treatment is equally applicable to the cases where H is an Abelian subgroup of G. However, in such cases T duality preserves one extended supersymmetry. In order to see that, let us recall [41] that for any even-dimensional WZW model the nonvanishing elements of the matrix $F_{AB}^\pm$ in the Cartan basis are $F_{\alpha\bar\alpha}^\pm = i$ and $F_{ij}^\pm$, where i, j here are labels in the Cartan subalgebra of G and α ($\bar\alpha$) is a positive (negative) root label. Since the group H is Abelian, we have $C_{ij}(h_\pm) = \delta_{ij}$. Using the fact that $C_\beta^\alpha(h_\pm)C_{\bar\gamma}^{\bar\alpha}(h_\pm) = \delta_{\beta\bar\gamma}$ and Eq. (3.16), we find that the complex structures (3.17) are local functions of the target space variables and assume the form

$$\widetilde{F}_{MN}^+ = i\mathcal{L}_{[M}^\alpha \mathcal{L}_{N]}^{\bar\alpha} + F_{ij}^+ \mathcal{L}_M^i \mathcal{L}_N^j,$$

$$\widetilde{F}_{MN}^- = i\mathcal{R}_{[M}^\alpha \mathcal{R}_{N]}^{\bar\alpha} + F_{ij}^- \mathcal{R}_M^i \mathcal{R}_N^j. \tag{3.20}$$

We conclude that, if H is Abelian, T duality preserves the local $N=2$ of the even-dimensional supersymmetric WZW models. However, this is not the case for the two additional complex structures present in WZW models based on quaternionic groups, which actually have $N=4$ extended supersymmetry. These cannot be written in a form similar to Eq. (3.20) and remain genuinely nonlocal. More details for the case of the WZW model based on $SU(2) \otimes U(1)$ can be found in [3,35,46] and for a general quaternionic group in [6].

Non-Abelian parafermions

We will now find the precise relation of $\Psi_\pm$ to the non-Abelian classical parafermions of the coset theory G/H [45]. Moreover, we will show that their Poisson brackets obey the same algebra as the currents of the original WZW model. This provides the (so far lacking) canonical equivalence between a WZW model for G and its dual with respect to a vector subgroup H as it is given by Eq. (3.9). In retrospect the emergence of parafermions is not a surprise since the non-Abelian duals of WZW models are related to gauged WZW models, as it was shown in [11,12].

Since we are interested in the computation of Poisson brackets, our treatment here will be completely classical. Hence, the nontrivial Jacobians arising from changing variables inside the functional path integral [43] will be ignored. Let us define the gauge-invariant analogue of g,h,v as

$$f=h_-^{-1}gh_- \in G, \quad h=h_+^{-1}h_- \in H, \widetilde{v}=h_-^{-1}vh_- \in H, \tag{3.21}$$

and introduce a group element $\lambda \in H$ such that the $i\partial_-\widetilde{v}=-\partial_-\lambda\lambda^{-1}$. With these definitions the gauge field strength $F_{+-}=h_-\partial_-(h^{-1}\partial_+h)h_-^{-1}$. Then with the help of the Polyakov-Wiegman formula, the action (3.3) assumes the form

$$S=I_{\text{WZW}}(hf)-I_{\text{WZW}}(h\lambda)+I_{\text{WZW}}(\lambda). \tag{3.22}$$

The form of Ψ_-, defined in Eq. (3.13), in terms of gauge-invariant quantities, is $\Psi_-=-ih\partial_-ff^{-1}h^{-1}$. The latter expression contains h_+ whose definition (3.14) involves a timelike integral, when we regard σ^+ as "time." This makes the computation of the corresponding Poisson brackets very difficult to perform. Thus, as in [44,45], we make use of the equation of motion $F_{+-}=0$ to replace h_+ by h_- in the definition of Ψ_- in Eq. (3.13) or equivalently to consider Poisson brackets of [7]

$$\Psi=\frac{ik}{\pi}\partial_-ff^{-1}, \tag{3.23}$$

where for notational convenience we have modified the normalization factor and have dropped the minus sign as a subscript. We should point out that the on-shell condition $\partial_+\Psi=0$ is still obeyed. The computation of the Poisson brackets using directly the action (3.22) will be done systematically in the appendix using Dirac's canonical approach to

constrained systems. Here we follow a shortcut which enables us to make direct contact with the parafermions. We rewrite the action (3.22) by shifting $h \rightarrow h\lambda^{-1}$, as [12]

$$S=I_{\text{WZW}}(h\lambda^{-1}f)-I_{\text{WZW}}(h)+I_{\text{WZW}}(\lambda). \tag{3.24}$$

The first two terms correspond to the gauged WZW action for the coset G/H and the third to an additional WZW action. Parafermions are introduced, similar to [44,45], by defining

$$\Psi^{G/H}=\frac{ik}{\pi}\partial_-(\lambda^{-1}f)f^{-1}\lambda, \tag{3.25}$$

where the superscript emphasizes that they are valued in the coset G/H. Their Poisson brackets have been computed in [44,45]:

$$\{\Psi_\alpha^{G/H}(x),\Psi_\beta^{G/H}(y)\}$$
$$=-\frac{k}{\pi}\delta_{\alpha\beta}\delta'(x-y)-f_{\alpha\beta\gamma}\Psi_\gamma^{G/H}(y)\delta(x-y)$$
$$-\frac{\pi}{2k}f_{c\alpha\gamma}f_{c\beta\delta}\epsilon(x-y)\Psi_\gamma^{G/H}(x)\Psi_\delta^{G/H}(y), \tag{3.26}$$

where the antisymmetric step function $\epsilon(x-y)$ equals $+1(-1)$ if $x>y$ ($x<y$). The last term in Eq. (3.26) is responsible for their nontrivial monodromy properties and unusual statistics. The currents corresponding to the WZW model action $I_{\text{WZW}}(\lambda)$ in Eq. (3.24) are defined as

$$J=\frac{ik}{\pi}\partial_-\lambda\lambda^{-1}=\frac{k}{\pi}\partial_-\widetilde{v}, \tag{3.27}$$

with $\partial_+J=0$ on shell. Using the basic Poisson brackets for a WZW model [47],

$$\{\text{Tr}(t^a\lambda^{-1}\delta\lambda)(x), \quad \text{Tr}(t^b\lambda^{-1}\delta\lambda)(y)\}=-\frac{\pi}{2k}\epsilon(x-y)\delta^{ab}, \tag{3.28}$$

and the variation under infinitesimal transformations,

$$\delta J_a=\frac{ik}{\pi}C_{ab}(\lambda)\text{Tr}[t^b\partial_-(\lambda^{-1}\delta\lambda)], \tag{3.29}$$

one proves that the following current algebra is obeyed [47]:

$$\{J_a(x),J_b(y)\}=-\frac{k}{\pi}\delta_{ab}\delta'(x-y)-f_{abc}J_c(y)\delta(x-y). \tag{3.30}$$

In addition, due to the "decoupling" in Eq. (3.24) we have $\{\Psi_\alpha^{G/H},J_b\}=0$. In order to compute the Poisson brackets of Eq. (3.23) we first note that $\Psi_a=J_a$, due to Eq. (3.4). Hence, the brackets $\{\Psi_a,\Psi_b\}$ are the same as in Eq. (3.30). On the other hand, $\Psi_\alpha=C_{\alpha\beta}(\lambda)\Psi_\beta^{G/H}$. To determine $\{\Psi_\alpha,\Psi_\beta\}$ and $\{\Psi_\alpha,J_b\}$ we need the variation

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[7]From now on we concentrate on one chiral sector only. We will use x or y to denote the world-sheet coordinate σ^-.

$$\delta\Psi_\alpha = C_{\alpha\beta}(\lambda)\delta\Psi_\beta^{G/H} + i\text{Tr}(t^b\lambda^{-1}\delta\lambda)f_{b\gamma\delta}C_{\alpha\delta}(\lambda)\Psi_\gamma^{G/H}.\tag{3.31}$$

Then, using Eqs. (3.26), (3.28), and (3.31), we find

$$\{\Psi_\alpha(x),\Psi_\beta(y)\} = -\frac{k}{\pi}\delta_{\alpha\beta}\delta'(x-y) - [f_{\alpha\beta\gamma}\Psi_\gamma(y)$$
$$+ f_{\alpha\beta c}J_c(y)]\delta(x-y),\tag{3.32}$$

and

$$\{J_a(x),\Psi_\beta(y)\} = -f_{a\beta\gamma}\Psi_\gamma(y)\delta(x-y).\tag{3.33}$$

Thus, the closed algebra obeyed by $\Psi_A = \{J_a,\Psi_\alpha\}$ is given by Eqs. (3.30), (3.32), and (3.33), which is the current algebra for G. We emphasize the fact that, even though the Ψ_α's are related to the coset parafermions $\Psi_\alpha^{G/H}$'s, they are not parafermions themselves since in their Poisson brackets (3.32) there is no term similar to the third term in Eq. (3.26). The reason is precisely the ''dressing'' provided by the extra fields (Lagrange multipliers). This is equivalent to the well-known realizations of current algebras in conformal field theory using parafermions. Hence, we have shown a canonical equivalence between a WZW model for a general group G and its dual with respect to a vector subgroup H in the sense that the algebras obeyed by the natural (equivalently, symmetry-generating) objects in the two models are the same.

Non-Abelian dual to SU(2) ⊗ U(1)

The corresponding WZW action is given by

$$S = \frac{k}{4\pi}\int \partial_+\phi\partial_-\phi + \partial_+\theta\partial_-\theta + \partial_+\psi\partial_-\psi + 2\cos\theta\partial_+\phi\partial_-\psi$$
$$+ \partial_+\rho\partial_-\rho.\tag{3.34}$$

This is the most elementary nontrivial model with $N=4$ world-sheet supersymmetry. The three complex structures in the right sector are given by

$$F_i^+ = 2d\rho\wedge\sigma_i - \epsilon_{ijk}\sigma_j\wedge\sigma_k,\tag{3.35}$$

where the left-invariant Maurer-Cartan forms of SU(2), defined in Eq. (2.28), have been used. The complex structures for the left sector can be similarly written down:

$$F_i^- = 2d\rho\wedge\tilde{\sigma}_i - \epsilon_{ijk}\tilde{\sigma}_j\wedge\tilde{\sigma}_k,\tag{3.36}$$

where $\tilde{\sigma}_i$ are the right-invariant Maurer-Cartan forms of SU(2). Their explicit expressions can be obtained from Eq. (2.28) by letting $(\phi,\theta,\psi)\to(-\psi,-\theta,-\phi)$, up to an overall minus sign. We can readily see that Eqs. (3.35) and (3.36) are of the general form (3.2).

Under SU(2) transformations the variable ρ is inert. The non-Abelian dual of Eq. (3.34) with respect to a vector SU(2) was found in [8], and we will not repeat all the steps of the derivation here. We only mention that a proper unitary gauge choice is $\phi=\psi=0$ among the variables of the SU(2) group element and $v_3=0$ among the Lagrange multipliers. The latter choice becomes necessary because, according to our discussion after Eq. (3.7), there is a nontrivial isotropy

group in this case. After we make the shift $v_2\to v_2-\theta$, the classical solutions for the gauge fields $A_\pm = (i/2)\vec{A}_\pm\cdot\vec{\sigma}_{\text{Pauli}}$ are

$$\vec{A}_\pm = \frac{\mp 1}{2v_1^2\sin^2\frac{\theta}{2}}\Bigg[v_1^2\partial_\pm v_1 + v_1(\sin\theta-\theta+v_2)\partial_\pm v_2, v_1$$
$$\times(\sin\theta-\theta+v_2)\partial_\pm v_1 + \left(4\sin^4\frac{\theta}{2} + (\sin\theta-\theta+v_2)^2\right)$$
$$\times\partial_\pm v_2, \pm 2v_1\sin^2\frac{\theta}{2}\partial_\pm v_2\Bigg],\tag{3.37}$$

and the background fields of the dual model are found to be

$$ds^2 = d\rho^2 + d\theta^2 + \frac{1}{v_1^2\sin^2\frac{\theta}{2}}\Bigg(4\sin^4\frac{\theta}{2}dv_2^2$$
$$+ [v_1dv_1 + (v_2-\theta+\sin\theta)dv_2]^2\Bigg),$$

$$\Phi = \ln\left(v_1^2\sin^2\frac{\theta}{2}\right),\tag{3.38}$$

with zero antisymmetric tensor. Note that, even though the torsion vanishes, the Ricci tensor is not zero due to the presence of a nontrivial dilaton. This means that the manifold is not hyper Kähler, as the latter property implies Ricci flatness [28]. The reason for this apparent paradox is, of course, the fact that the original local $N=4$ world-sheet supersymmetry is realized in the dual model (3.38) nonlocally, except for the $N=1$ part. In the right sector the expressions for the nonlocal complex structures are given by

$$\tilde{F}_i^+ = C_{ji}(h_-)(2d\rho\wedge\mathcal{L}_j - \epsilon_{jkl}\mathcal{L}_k\wedge\mathcal{L}_l),\tag{3.39}$$

where $\mathcal{L}_i = \mathcal{L}_\mu^i dX^\mu$, $X^\mu = \{\theta,v_1,v_2\}$, and

$$(\mathcal{L}_\mu^i) = \begin{pmatrix} 0 & -1 & -\dfrac{v_2-\theta}{v_1} \\[2mm] 1 & 0 & 0 \\[2mm] 0 & -\cot\dfrac{\theta}{2} & -\dfrac{2+\cot\dfrac{\theta}{2}(v_2-\theta)}{v_1} \end{pmatrix}.\tag{3.40}$$

In the left sector, the nonlocal complex structures are

$$\tilde{F}_i^- = C_{ji}(h_+)(2d\rho\wedge\mathcal{R}_j - \epsilon_{jkl}\mathcal{R}_k\wedge\mathcal{R}_l),\tag{3.41}$$

where $\mathcal{R}_i = \mathcal{R}_\mu^i dX^\mu$ and

$$(\mathcal{R}^i_\mu) = \begin{pmatrix} 0 & -1 & -\dfrac{v_2 - \theta}{v_1} \\[2ex] 1 & 0 & 0 \\[2ex] 0 & \cot\dfrac{\theta}{2} & \dfrac{2 + \cot\dfrac{\theta}{2}(v_2 - \theta)}{v_1} \end{pmatrix} . \qquad (3.42)$$

The group elements $h_\pm \in$ SU(2) are given by the path-ordered Wilson lines (3.14), with gauge fields (3.37).

IV. DISCUSSION AND CONCLUDING REMARKS

In this paper we examined the behavior of supersymmetry under non-Abelian T duality. We considered models that are invariant under the left action of a general semi-simple group. We gave the general form of the corresponding σ models as well as of the complex structures, in cases that admit extended world-sheet supersymmetry, and found their transformation rules under non-Abelian duality by utilizing a canonical transformation. Although, as a general rule, $N=1$ world-sheet supersymmetry is preserved under duality, whenever the action of the group on the complex structures is nontrivial, extended supersymmetry seems to be incompatible with non-Abelian duality. However, this is only an artifact of the description in terms of an effective field theory, since nonlocal world-sheet effects restore supersymmetry at the string level. As examples, SO(3)-invariant hyper-Kähler metrics which include the Eguchi-Hanson, the Taub-NUT, and the Atiyah-Hitchin metrics were considered in detail. Explicit expressions for the three complex structures were given which should be useful in moduli problems in monopole physics. We have also considered WZW models and their non-Abelian duals with respect to the vector action of a subgroup. The canonical equivalence of these models was shown by explicitly demonstrating that the algebra obeyed by the Poisson brackets of chiral currents of the WZW model is preserved under the non-Abelian duality transformation. The effect of non-Abelian duality is that the currents are represented in terms coset parafermions. The latter are nonlocal and have non-trivial braiding properties due to Wilson lines attached to them. We believe that this type of canonical equivalence is not restricted to just WZW models and their duals but to other models with vector action of the isometry group.

Non-Abelian duality destroys manifest target space supersymmetry as well, in the sense that the standard Killing spinor equations do not have a solution. In fact, the breaking of manifest target space supersymmetry occurs hand-in-hand with the breaking of local $N=4$ extended world-sheet supersymmetry. This is attributed to the relation between Killing spinors and complex structures [48], using $F_{\mu\nu} = \bar{\xi}\Gamma_{\mu\nu}\xi$. The situation is similar to the case of Abelian duality [2,49,4–6] with the difference that the nonlocal Killing spinors arising after duality do not define a local $N=2$ world-sheet supersymmetry using the above relation between Killing spinors and complex structures. The lowest-order effective field theory is not enough at all to understand the fate of target space supersymmetry under non-Abelian duality, since one has to generate the whole supersymmetry algebra and not just its truncated part corresponding to the Killing spinor equations. In the realization of the supersymmetry algebra after non-Abelian duality, massive string modes play a crucial role and a complete truncation to only the massless modes is inconsistent. This becomes apparent by making contact with the work of Scherk and Schwarz [50] on coordinate-dependent compactifications. The arguments are similar to the case of Abelian duality and were presented in [46].

This investigation is part of a program whose goal is to use nontrivial stringy effects occurring in duality symmetries, in physical situations that seem paradoxical in the effective field theory approach. In particular, we would like to view T duality as a mechanism of restoring various symmetries, such as supersymmetry, in a manifest way. An example of how this works is based on the background corresponding to SU(2)$_k$/U(1) $\otimes$ SL(2,IR)$_{-k}$/U(1). This has $N=4$ world-sheet supersymmetry which, however, is not manifest and is realized using parafermions [51]. An appropriate Abelian duality transformation leads to an axionic instanton background with manifest $N=4$ and target space supersymmetry restored [6]. An equivalent model where target space supersymmetry was restored by making a moduli parameter dynamical was considered in [5]. In order to advance these ideas and use non-Abelian duality as the symmetry restoration mechanism, one has to relax the condition that an isometry group exists at all, since in any case this is being destroyed by non-Abelian duality. The notion of non-Abelian duality in the absence of isometries is now well defined and is under the name Poisson-Lie T duality [15] and the closely related quasi-axial-vector duality which was initiated in [17], explicitly constructed in [18], whereas its relation to the Poisson-Lie T duality was investigated in [19]. The idea is to search in various backgrounds of interest in black hole physics or cosmology for ''noncommutative conservation laws'' that generalize [15] the usual conservation laws. The hope is that in the dual description, various properties, which were hidden, become manifest and possibly resolve certain paradoxes with field theoretical origin. We hope to report work in this direction in the future.

APPENDIX: DERIVATION OF POISSON BRACKETS

In this appendix we derive the Poisson brackets of Sec. III in a more systematic way. Because of the gauging procedure, it turns out that we are dealing with constrained Hamiltonian systems. A consistent way of implementing the constraints was provided by Dirac (see, for instance, [52]). For our purposes the relevant part of his analysis is that, given a set of second class constraints $\{\varphi_a\}$, one first computes the matrix generated by their Poisson brackets:

$$D_{ab} = \{\varphi_a, \varphi_b\}. \qquad (A1)$$

In this and in similar computations we are free to use the constraints only after calculating their Poisson brackets. When D_{ab} is invertible one simply postulates that the usual Poisson brackets are replaced by the so-called Dirac brackets, defined as

$$\{A,B\}_D = \{A,B\} - \{A,\varphi_a\}D_{ab}^{-1}\{\varphi_b,B\}, \qquad (A2)$$

for any two phase space variables A and B. Then, the constraints can be strongly set to zero since they have vanishing Dirac brackets among themselves and with anything else.

As a very elementary application of this method, consider an arbitrary action that is first order in time derivatives:

$$S = \int dt\, A_a(X)\dot{X}^a. \qquad (A3)$$

The conjugate momentum to X^a is given by $P_a = A_a$ and, therefore, we cannot solve for the velocity $\dot{X}^a$ in terms of the momentum P_a. Hence, we impose the constraint

$$\varphi_a = P_a - A_a \approx 0 \qquad (A4)$$

and follow Dirac's procedure. Using the basic Poisson brackets $\{X^a, P_b\} = \delta_b^a$, we find that the matrix (A1) is given by $D_{ab} \equiv M_{ab} = \partial_a A_b - \partial_b A_a$. Assuming that it is invertible and after using Eq. (A2), we obtain that for the general phase space variables A, B the corresponding Dirac brackets are given by

$$\{A,B\}_D = \frac{\partial A}{\partial X^a} M_{ab}^{-1} \frac{\partial B}{\partial X^b}, \quad M_{ab} = \frac{\partial A_b}{\partial X^a} - \frac{\partial A_a}{\partial X^b}. \quad (A5)$$

These Dirac brackets coincide with the Poisson brackets postulated in [47] for the action Eq. (A3). In practice, we read off the matrix M_{ab} by simply considering the variation of Eq. (A3):

$$\delta S = \int dt\, M_{ab}\delta X^a \dot{X}^b. \qquad (A6)$$

In the rest of this appendix, as well as in the bulk of the paper, we will call the Dirac brackets (A5) simply Poisson brackets in order to comply with standard terminology in the literature.

The models we encountered in Sec. III belong to the general type (A3) where σ^+ is considered as the time variable, whereas σ^- is treated as a continuous index. In that respect our treatment differs from the one in [53] where $\tau = \sigma^+ + \sigma^-$ was taken as the time variable and computation of brackets of parafermions was not considered.

Gauged WZW models. The purpose is to reproduce Eq. (3.26) in a straightforward way compared to that in [44,45] and mainly to be able to compare with the analogous derivations of Eqs. (3.30), (3.32), and (3.33) which will follow.

Using the definitions (3.21), we can write the gauged WZW action as

$$S = I_{\text{WZW}}(hf) - I_{\text{WZW}}(h). \qquad (A7)$$

A general variation of the action gives

$$\delta S = \frac{k}{\pi}\int h^{-1}\delta h[f\partial_-(f^{-1}\partial_+f)f^{-1}$$
$$+ f\partial_-(f^{-1}h^{-1}\partial_+hf)f^{-1} - \partial_-(h^{-1}\partial_+h)]$$
$$+ f^{-1}\delta f[\partial_-(f^{-1}\partial_+f) + \partial_-(f^{-1}h^{-1}\partial_+hf)].$$
$$(A8)$$

Using for notational convenience the definition

$$Z_I = (\text{Tr}(t^a h^{-1}\delta h), \text{Tr}(t^A f^{-1}\delta f)), \qquad (A9)$$

we compute the basic Poisson brackets (cf. footnote 7)

$$\{Z_I(x), Z_J(y)\} = -\frac{\pi}{2k}M_{IJ}^{-1}(x,y)\epsilon(x-y), \qquad (A10)$$

where the matrix $M(x,y)$ is defined as

$$M(x,y) = \begin{pmatrix} C_{ab}(f(x)f^{-1}(y)) - \delta_{ab} & C_{aB}(f(x)) \\ C_{bA}(f(y)) & \delta_{AB} \end{pmatrix}. \qquad (A11)$$

Inverting the above matrix and explicitly writing out Eq. (A10), we obtain

$$\{\text{Tr}(t^A f^{-1}\delta f)(x), \text{Tr}(t^B f^{-1}\delta f)(y)\}$$
$$= \frac{\pi}{2k}\epsilon(x-y)[C_{cA}(f(y))C_{cB}(f(x)) - \delta_{AB}],$$

$$\{\text{Tr}(t^a h^{-1}\delta h)(x), \text{Tr}(t^b h^{-1}\delta h)(y)\}$$
$$= \frac{\pi}{2k}\epsilon(x-y)\delta_{ab},$$

$$\{\text{Tr}(t^a h^{-1}\delta h)(x), \text{Tr}(t^B f^{-1}\delta f)(y)\}$$
$$= -\frac{\pi}{2k}\epsilon(x-y)C_{aB}(f(x)). \qquad (A12)$$

We would like to compute Poisson brackets of the gauged-invariant quantities $\Psi = (ik/\pi)\partial_-ff^{-1}$, obeying $\partial_+\Psi = 0$ on shell, and $H = -(ik/\pi)\partial_-hh^{-1}$. Using the variations

$$\delta\Psi_A = \frac{ik}{\pi}C_{AB}(f)\text{Tr}[t^B\partial_-(f^{-1}\delta f)],$$

$$\delta H_a = -\frac{ik}{\pi}C_{ab}(h)\text{Tr}[t^b\partial_-(h^{-1}\delta h)], \qquad (A13)$$

and Eq. (A12), we obtain

$$\{\Psi_\alpha(x), \Psi_\beta(y)\} = -\frac{k}{\pi}\delta_{\alpha\beta}\delta'(x-y) - [f_{\alpha\beta\gamma}\Psi_\gamma(y)$$
$$+ f_{\alpha\beta c}\Psi_c(y)]\delta(x-y)$$
$$- \frac{\pi}{2k}f_{c\alpha\gamma}f_{c\beta\delta}\Psi_\gamma(x)\Psi_\delta(y)\epsilon(x-y),$$
$$(A14)$$

$$\{\Psi_a(x),\Psi_b(y)\}=f_{abc}\Psi_c(y)\delta(x-y)$$
$$-\frac{\pi}{2k}f_{cad}f_{cbe}\Psi_d(x)\Psi_e(y)\epsilon(x-y),$$
(A15)

$$\{\Psi_a(x),\Psi_\beta(y)\}=-\frac{\pi}{2k}f_{cad}f_{c\beta\gamma}\Psi_d(x)\Psi_\gamma(y)\epsilon(x-y),$$
(A16)

and

$$\{H_a(x),H_b(y)\}=\frac{k}{\pi}\delta_{ab}\delta'(x-y)-f_{abc}H_c(y)\delta(x-y),$$
(A17)

$$\{H_a(x),\Psi_b(y)\}=\frac{k}{\pi}C_{ab}(h(x))\delta'(x-y),$$
(A18)

$$\{H_a(x),\Psi_\beta(y)\}=0.$$
(A19)

The form of the action (A7) suggests that the equation of motion corresponding to the gauge field A_+ has to be imposed as a constraint, i.e. $\varphi_1^a(x)=\Psi^a(x)\approx0$. Then, the on-shell condition $F_{+-}=0$ implies the constraint $\varphi_2^a=H^a\approx0$ (or $h\approx1$). However, due to Eqs. (A17) and (A18), we observe that these cannot be imposed strongly. They are second class constraints and the matrix (A1), in the basis $\varphi_a(x)=\{\varphi_1^a(x),\varphi_2^a(x)\}$, is given by

$$D_{ab}(x,y)=\frac{k}{\pi}\begin{pmatrix}0 & \delta_{ab}\\ \delta_{ab} & \delta_{ab}\end{pmatrix}\delta'(x-y),$$
(A20)

whereas its inverse is

$$D_{ab}^{-1}(x,y)=\frac{\pi}{2k}\begin{pmatrix}-\delta_{ab} & \delta_{ab}\\ \delta_{ab} & 0\end{pmatrix}\epsilon(x-y).$$
(A21)

Then, using Eq. (A2), we can compute the Dirac brackets of the Ψ_α's. It turns out that $\{\Psi_\alpha,\Psi_\beta\}_D\approx\{\Psi_\alpha,\Psi_\beta\}$, hence obtaining the result (3.26).

Non-abelian duals of WZW Models In this case the starting point is the action (3.22). Its general variation is given by

$$\delta S=\frac{k}{\pi}\int h^{-1}\delta h[f\partial_-(f^{-1}\partial_+f)f^{-1}$$
$$+f\partial_-(f^{-1}h^{-1}\partial_+hf)f^{-1}-\lambda\partial_-(\lambda^{-1}h^{-1}\partial_+h\lambda)\lambda^{-1}$$
$$-\lambda\partial_-(\lambda^{-1}\partial_+\lambda)\lambda^{-1}]+f^{-1}\delta f[\partial_-(f^{-1}\partial_+f)$$
$$+\partial_-(f^{-1}h^{-1}\partial_+hf)]-\lambda^{-1}\delta\lambda\partial_-(\lambda^{-1}h^{-1}\partial_+h\lambda).$$
(A22)

Similar to Eq. (A9), we define

$$Z_I=[\mathrm{Tr}(t^ah^{-1}\delta h),\mathrm{Tr}(t^a\lambda^a\delta\lambda),\mathrm{Tr}(t^Af^{-1}\delta f)].$$
(A23)

These obey Eq. (A10) with the matrix $M(x,y)$ now defined as

$$M(x,y)=\begin{pmatrix}C_{ab}(f(x)f^{-1}(y))-C_{ab}(\lambda(x)\lambda^{-1}(y)) & -C_{ab}(\lambda(x)) & C_{aB}(f(x))\\ -C_{ba}(\lambda(y)) & 0 & 0\\ C_{bA}(f(y)) & 0 & \delta_{AB}\end{pmatrix}.$$
(A24)

Inverting this matrix, we find that the nonzero basic Poisson brackets are given by

$$\{\mathrm{Tr}(t^Af^{-1}\delta f)(x),\mathrm{Tr}(t^Bf^{-1}\delta f)(y)\}=-\frac{\pi}{2k}\delta_{AB}\epsilon(x-y),$$

$$\{\mathrm{Tr}(t^a\lambda^{-1}\delta\lambda)(x),\mathrm{Tr}(t^b\lambda^{-1}\delta\lambda)(y)\}=-\frac{\pi}{2k}\delta_{ab}\epsilon(x-y),$$

$$\{\mathrm{Tr}(t^a\lambda^{-1}\delta\lambda)(x),\mathrm{Tr}(t^Bf^{-1}\delta f)(y)\}$$
$$=-\frac{\pi}{2k}C_{aB}(\lambda^{-1}(x)f(x))\epsilon(x-y),$$

$$\{\mathrm{Tr}(t^ah^{-1}\delta h)(x),\mathrm{Tr}(t^b\lambda^{-1}\delta\lambda)(y)\}$$
$$=\frac{\pi}{2k}C_{ab}(\lambda(y))\epsilon(x-y).$$
(A25)

Using them and the variations (A13) and (3.29), we calculate the Poisson brackets

$$\{\Psi_A(x),\Psi_B(y)\}=-\frac{k}{\pi}\delta_{AB}\delta'(x-y)-f_{ABC}\Psi_C(y)\delta(x-y),$$
(A26)

$$\{J_a(x),J_b(y)\}=-\frac{k}{\pi}\delta_{ab}\delta'(x-y)-f_{abc}J_c(y)\delta(x-y),$$
(A27)

$$\{J_a(x),\Psi_b(y)\}=-\frac{k}{\pi}\delta_{ab}\delta'(x-y)-f_{abc}J_c(y)\delta(x-y),$$
(A28)

$$\{J_a(x),\Psi_\beta(y)\}=0,$$
(A29)

and

$$\{H_a(x),J_b(y)\}=-\frac{k}{\pi}C_{ab}(h(x))\delta'(x-y)$$

$$-f_{dbc}C_{ad}(h(y))J_c(y)\delta(x-y),$$

$$(A30)$$

$$\{H_a(x),H_b(y)\}=\{H_a(x),\Psi_A(y)\}=0. \qquad (A31)$$

As in the case of gauged WZW models, we have to impose the equation of motion corresponding to A_+ as a constraint, i.e., $\varphi_1^a(x)=\Psi^a(x)-J^a(x)\approx0$, as well as $\varphi_2^a(x)=H^a(x)\approx0$ corresponding to $F_{+-}=0$. Since they cannot be imposed strongly, we again follow Dirac's procedure. We first compute the matrix (A1):

$$D_{ab}(x,y)$$

$$=\frac{k}{\pi}\begin{pmatrix} 0 & C_{ab}(\lambda(x)\lambda^{-1}(y)) \\ C_{ba}(\lambda(x)\lambda^{-1}(y)) & 0 \end{pmatrix}\delta'(x-y),$$

$$(A32)$$

and its inverse

$$D_{ab}^{-1}(x,y)$$

$$=\frac{\pi}{2k}\begin{pmatrix} 0 & C_{ab}(\lambda(y)\lambda^{-1}(x)) \\ C_{ba}(\lambda(y)\lambda^{-1}(x)) & 0 \end{pmatrix}\epsilon(x-y).$$

$$(A33)$$

Then, using Eq. (A2), we obtain that the Dirac brackets $\{\Psi_A,\Psi_B\}_D$ coincides with the corresponding Poisson bracket (A26). As a consistency check, the Dirac brackets of the J_a's should coincide with the Dirac brackets of the Ψ_a's because the constraint φ_1^a is imposed strongly. This can be verified using Eq. (A2) and the explicit form of the matrix D_{ab}^{-1} in Eq. (A33). We note that this is not the case for the corresponding Poisson brackets as one can see from Eq. (A28) and (A29).

Finally, let us mention that the conclusion we have reached about WZW models and their non-Abelian duals, would have, of course, been the same even if we had worked, within Dirac's general framework, with the action (3.24) instead of Eq. (3.22).

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