An Algorithm for the Decomposition of Semisimple Lie Algebras *

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Abstract

We consider the problem of decomposing a semisimple Lie algebra defined over a field of characteristic zero as a direct sum of its simple ideals. The method is based on the decomposition of the action of a Cartan subalgebra. An implementation of the algorithm in the system ELIAS is discussed at the end of the paper.

Keywords: Lie algebra, Cartan subalgebra, Direct sum decomposition, Algorithm.

1 Introduction

In this paper we describe an algorithm that helps to determine the structure of a semisimple Lie algebra. It is implemented in a general library of Lie algebra algorithms, called ELIAS (for Eindhoven LIe Algebra System) which will be part of the computer algebra system GAP4. The library ELIAS is part of a bigger project, called ACELA, which aims at an interactive book on Lie algebras (cf. [1]).

One of the fundamental structure theorems on semisimple Lie algebras over a field of characteristic zero characterizes these Lie algebras as direct sums of simple Lie algebras (see [4], p. 71). In this paper we address the algorithmic problem of computing such a direct sum decomposition.

All simple Lie algebras (and hence all semisimple Lie algebras) have been classified (see [3], [4]). A simple Lie algebra over an algebraically closed field of characteristic zero is isomorphic either to an element of one

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of the "great" classes of simple Lie algebras (A_n, B_n, C_n, D_n) or to one of the exceptional Lie algebras (G_2, F_4, E_6, E_7, E_8). The proof of this classification uses a distinguished subalgebra, called *Cartan subalgebra*. This is a nilpotent subalgebra that equals its own normalizer in the Lie algebra L. It can be shown that Cartan subalgebras exist if the field is of characteristic zero. Via the adjoint representation (sending an element $x \in L$ to the transformation corresponding to the left multiplication with x in L) these Cartan subalgebras are viewed as Lie algebras of linear transformations in L. And if the Lie algebra is semisimple, then these subalgebras are toral (i.e., the induced transformations are simultaneously diagonalisable). As a consequence, L can be decomposed into a direct sum of common eigenspaces called *root spaces*. Furthermore, it can be shown that these root spaces are all one dimensional.

In order to arrive at a "splitting" of the Cartan subalgebra (i.e., a simultaneous diagonalisation), in general the gound field needs to be algebraically closed. On a computer however, such fields are not feasible. So we have to restrict our attention to the field $\mathbf{Q}$ of rational numbers and algebraic extensions (of low degree) of that field. In particular we are not able to use a root space decomposition of our semisimple Lie algebra. In Section 2, we therefore describe a near root space decomposition with respect to a Cartan subalgebra and use it to decompose the Lie algebra into a direct sum of simple ideals. We note that there exist effective methods to compute a Cartan subalgebra (see [2]).

In Section 3 we illustrate the algorithm in two examples. Finally in Section 4 we compare our algorithm to a more general one described in [5].

2 The algorithm

First we transcribe a result from Jacobson's book ([4]).

Lemma 2.1 *Let A, B be linear transformations in a finite dimensional vector space V satisfying*

$$A^n B = [A, [A, \ldots, [A, B] \ldots]] = 0 \quad (n \text{ factors } A)$$

for some n. Let p be a polynomial and let $V_{p(A)} = \{v \in V \mid p(A)^m v = 0$ for some $m > 0\}$. Then $V_{p(A)}$ is invariant under B.

Proof. See [4], p. 40.

Let L be a semisimple Lie algebra with Cartan subalgebra H. Since the $\mathrm{ad}h$ for $h \in H$ are semisimple transformations they all have a squarefree minimum polynomial. Therefore, using Lemma 2.1 (H is nilpotent) we can compute a decompositon

$$L = L_1 \oplus \cdots \oplus L_s \oplus H$$

of L such that the restriction of every basis element of H to L_i has an irreducible minimum polynomial ($1 \leq i \leq s$). (The Cartan subalgebra H itself is an invariant subspace corresponding to the polynomial X.)

Proposition 2.2 *Let L be a semisimple Lie algebra over a field k of characteristic 0. Let H be a Cartan subalgebra of L with basis $\{h_1, \ldots, h_n\}$. Suppose that we have a decomposition*

$$L = L_1 \oplus \cdots \oplus L_s \oplus H$$

of L such that the minimum polynomial of every $\mathrm{ad}h_i$ restricted to L_j is irreducible. Then for every $h \in H$, the minimum polynomial of $\mathrm{ad}h$ restricted to L_j is irreducible.

Proof. Suppose that there is an $h \in H$ such that the minimum polynomial of $\mathrm{ad}h$ restricted to L_j is reducible. Then it is the product of two distinct polynomials because $\mathrm{ad}h$ is semisimple. Since H is a nilpotent Lie algebra, we can apply Lemma 2.1. It follows that there is a decomposition $L_j = V \oplus W$ where V and W are invariant under $\mathrm{ad}h_i$ for $1 \leq i \leq n$. But if we tensor with the algebraic closure of k, then L splits into a direct sum of common eigenspaces for the action of H. These eigenspaces are already determined by the common action of the basis elements h_i and they are one-dimensional (see [3], Proposition 8.4). But since the restriction of every $\mathrm{ad}h_i$ to V has the same minimum polynomial as the restriction to W, this is not possible (for every eigenvalue there is an eigenvector in V, but also in W). $\square$

The next theorem states that the decomposition of Proposition 2.2 is compatible with the direct sum decomposition of L.

Theorem 2.3 *Let L and H be the same as in Proposition 2.2 and let*

$$L = L_1 \oplus \cdots \oplus L_s \oplus H$$

be a decomposition of L as in Proposition 2.2. Suppose that L decomposes as a direct sum of ideals, $L = I_1 \oplus I_2$. Then every L_i is contained in I_1 or in I_2.

Proof. First we note that H decomposes as $H = H_1 \oplus H_2$ where H_l is a Cartan subalgebra of I_l. By Proposition 2.2 there is an element $h \in H_1 \cup H_2$ such that ad$h$ is nonsingular on L_i. (Else the minimum polynomial of every element of $H_1 \cup H_2$ is X, forcing $L_i \subset H$, since $N_L(H) = H$.)

Now, without loss of generality we may assume that $h \in H_1$. In that case we also have that h is an element of I_1, and hence ad$h$ maps L_i into $L_i \cap I_1$. The conclusion is that $L_i = \mathrm{ad}h L_i \subset L_i \cap I_1$. □

This theorem leads to the following algorithm.

Algorithm Decompose
Input: A semisimple Lie algebra L.
Output: A list of bases of the ideals of L.

Step 1 Compute a Cartan subalgebra H of L (see [2]).

Step 2 Let $\{h_1, \dots, h_n\}$ be a basis of H. Compute a decomposition

$$L = L_1 \oplus \cdots L_s \oplus H$$

of L such that the restriction of adh_i to L_j has an irreducible minimum polynomial (for $1 \le i \le n$ and $1 \le j \le s$).

Step 3 For $1 \le j \le s$ compute a basis of the ideal generated by L_j and delete multiple instances from the list.

Remark. If L is a Lie algebra arising "in nature", then it is easy to check whether L is semisimple. Let $\{x_1, \dots, x_n\}$ be a basis of L and let K be the matrix $(\mathrm{Tr}(\mathrm{ad}x_i \cdot \mathrm{ad}x_j))$. Then L is semisimple if and only if $\det K \neq 0$ ([4], p. 69).

3 Examples

In this section we show how the method works in two examples. For the input we suppose that a Lie algebra L of dimension n is given by an array of n^3 structure constants (c_{ij}^k) for $1 \le i, j, k \le n$ such that the Lie multiplication is described by

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k,$$

where $\{x_1, \dots, x_n\}$ is a basis of L.

Example 1. Let L be a 6 dimensional Lie algebra with basis

$$\{h_1, x_1, y_1, h_2, x_2, y_2\}.$$

The structure constants of L are specified in Table 1.

$$\begin{array}{lcrlcr}
[h_1, x_1] & = & 2x_1 & [h_2, x_1] & = & 2x_1 \\
[h_1, y_1] & = & -2y_1 & [h_2, y_1] & = & -2y_1 \\
[h_1, x_2] & = & 2x_2 & [h_2, x_2] & = & -2x_2 \\
[h_1, y_2] & = & -2y_2 & [h_2, y_2] & = & 2y_2 \\
[x_1, y_1] & = & \frac{1}{2}h_1 + \frac{1}{2}h_2 & [x_2, y_2] & = & \frac{1}{2}h_1 - \frac{1}{2}h_2.
\end{array}$$

Table 1: Nonzero products of the basis elements of a 6 dimensional Lie algebra.

Brackets that are not present are assumed to be 0. The determinant of the matrix K (remark at the end of Section 2) is 2^{16}, hence L is semisimple. As is easily verified, $H = \langle h_1, h_2 \rangle$ is a Cartan subalgebra. The minimum polynomial of $\mathrm{ad}h_1$ is $X(X-2)(X+2)$. The decomposition of L relative to $\mathrm{ad}h_1$ is

$$L = \langle x_1, x_2 \rangle \oplus \langle y_1, y_2 \rangle \oplus \langle h_1, h_2 \rangle.$$

These subspaces are stable under $\mathrm{ad}h_2$. The minimum polynomial of $\mathrm{ad}h_2$ restricted to $\langle x_1, x_2 \rangle$ is $(X-2)(X+2)$. So this subspace decomposes under the action of $\mathrm{ad}h_2$ as $\langle x_1 \rangle \oplus \langle x_2 \rangle$. We have a similar decomposition for $\langle y_1, y_2 \rangle$. Hence the decomposition (as discussed in Section 2) of L is

$$L = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \langle h_1, h_2 \rangle.$$

Now the ideal generated by x_1 is spanned by $\{x_1, y_1, (h_1 + h_2)/2\}$. Similarly, the ideal generated by x_2 is spanned by $\{x_2, y_2, (h_1 - h_2)/2\}$. It follows that we have found the decomposition of L into simple ideals.

Example 2. Let L be a Lie algebra with basis $\{x_1, \ldots, x_6\}$ and multiplication table as shown in Table 2.

The determinant of the matrix K is -2^{20} and therefore L is semisimple. A Cartan subalgebra of L is spanned by $\{x_1, x_2\}$. The transformations $\mathrm{ad}x_1$ and $\mathrm{ad}x_2$ have minimum polynomials $X(X^2+4)$ and $X(X^2-4)$ respectively. The decomposition of Section 2 is

$$L = L_{3,4} \oplus L_{5,6} \oplus L_{1,2},$$

where $L_{i,j}$ is the subspace spanned by $\{x_i, x_j\}$. From the multiplication table it follows that the ideals generated by $L_{3,4}$ and $L_{5,6}$ are both equal to L. Hence, by Theorem 2.3 we have that L is a simple Lie algebra.

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	0	$2x_4$	$-2x_3$	$-2x_6$	$2x_5$
x_2	0	0	$2x_3$	$2x_4$	$-2x_5$	$-2x_6$
x_3	$-2x_4$	$-2x_3$	0	0	x_2	x_1
x_4	$2x_3$	$-2x_4$	0	0	x_1	$-x_2$
x_5	$2x_6$	$2x_5$	$-x_2$	$-x_1$	0	0
x_6	$-2x_5$	$2x_6$	$-x_1$	x_2	0	0

Table 2: Multiplication table of a 6 dimensional Lie algebra.

4 Evaluation

In [5] a more general method for decomposing a Lie algebra as a direct sum of ideals is described. This method consists of finding idempotents in the centralizer of ad$L$ in the full matrix algebra $M_{\dim L}(k)$. Here we compare this general method with the special method that we propose. The general method has of course the advantage of being more general. Furthermore, with this method it can also be decided whether L is "absolutely indecomposable" (i.e., whether L decomposes over the algebraic closure of k). However, a disadvantage of this method is the fact that it computes the centralizer of ad$L$ in the matrix algebra $M_{\dim L}(k)$. From the computational point of view this is a very difficult task.

We have tested [1] both methods on some direct sums of $\mathfrak{sl}_2$, $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ over the field $\mathbf{Q}$. First we take the standard Chevalley basis to produce the structure constants of L. The results are shown in Table 3. From this table we see that the general method is faster for the small examples. However, the running times of this method increase rapidly as the dimension grows.

Lie algebra	dimension	general method	special method
$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	6	12	19
$\mathfrak{sl}_2 \oplus \mathfrak{sl}_3$	11	35	52
$\mathfrak{sl}_3 \oplus \mathfrak{sl}_3$	16	127	84
$\mathfrak{sl}_2 \oplus \mathfrak{sl}_4$	18	205	109

Table 3: Running times (in seconds) of the general and the special method.

In this example the structure constants are all "nice" numbers (i.e., small integers). For the next test example we take L to be $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and we increase

[1]The computations were performed on a SUN SPARC workstation

the complexity of the input by taking a a random 6×6 matrix M and then computing a basis of L corresponding to M^i for $i = 1, 2, \ldots, 5$. The results are diplayed in Table 4.

basis	general method	special method
M	32	12
M^2	78	15
M^3	123	15
M^4	173	16
M^5	253	20

Table 4: Running times (in seconds) of the general and the special method applied to structure constants of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ of increasing complexity.

It is seen that the special method has almost no problems dealing with the increase of complexity. The general method, however, experiences considerable difficulties.

The conclusion is that the special method is better behaved in practice whereas the general method is theoretically more interesting (it can be applied in all cases, and it can decide whether L is absolutely indecomposable).

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