Foliation groupoids and their cyclic homology*

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Introduction

The purpose of this paper is to prove two theorems which concern the position of étale groupoids among general smooth (or "Lie") groupoids. Our motivation comes from the non-commutative geometry and algebraic topology concerning leaf spaces of foliations. Here, one is concerned with invariants of the holonomy groupoid of a foliation [4, 34], such as the cohomology of its classifying space [14], the cyclic homology of its smooth convolution algebra [2, 7], or the K-theory of the C^*-convolution algebras. Many results here depend on the fact that such a holonomy groupoid can be "reduced" to what is called a complete transversal of the foliation, giving rise to an equivalent étale groupoid . For étale groupoids (sometimes called r-discrete groupoids in the literature [30, 33]), the cyclic homology, sheaf theory and classifying spaces are each well understood, as is the relation between these.

Our first theorem provides a criterion for determining whether a given Lie groupoid is equivalent to an étale one. We prove that this is the case iff all the isotropy groups of the groupoid are discrete, or equivalently, iff the anchor map of the associate Lie algebroid is injective. These conditions are often easy to check in examples.

We recall that the Lie algebroid of a Lie groupoid is an infinitesimal structure which plays the same role as the Lie algebra of a Lie group. Lie algebroids with injective anchor map are the same things as foliations, so another way of phrasing our first theorem is by saying that a Lie groupoid is equivalent to an étale one, exactly when it integrates a foliation. For this reason, we have decided to refer to these groupoids as "foliation groupoids". It is not a surprise to see that much of the standard literature on foliations deals with foliation groupoids; for instance, an overall assumption in [24] is the discreteness of the isotropy groups. Our first theorem can also be seen as a general "slice theorem", which generalizes the reduction to transversals for foliations and the slice theorem for infinitesimally free actions of compact Lie groups. This slice theorem is expected to be a special case of a more general slice theorem conjectured by A. Weinstein. We also prove that, among the Lie groupoids which integrate a given foliation, the holonomy and monodromy groups are extreme examples. Results of this

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kind, but formulated in terms of microdifferentiable groupoids, go back to [32, 3, 31].

Our second theorem concerns the invariance of cylic type homologies under equivalence. We prove that equivalent foliation groupoids have isomorphic Hochschild, cyclic and periodic cyclic homology groups. This invariance is perhaps not really surprising, especially since analogous results for étale groupoids [9, 26] , and for the K-theory of C*-algebras associated to groupoids [25, 16] are well known (see also [12]). Nonetheless, we believe our second theorem has some relevance. The theorem implies that the cyclic type homologies of leaf spaces are totally independent of the particular model of the holonomy groupoid, and its proof provides explicit isomorphisms (summarized in the Remark at the end). The theorem also completes the computation for algebras associated to Lie group actions with discrete stabilizers. Moreover, this second theorem may in fact be an intermediate step toward a similar result for (more) general Lie groupoids. (Observe in this context that some parts of the proof, such as the H-unitality of the convolution algebra, apply to general Lie groupoids.)

The plan of this paper is as follows. In the first section we have collected the preliminary definitions concerning Lie groupoids, their Lie algebroids, and their cyclic homology. In the second section we state the main results. Since our motivation partly came from a better understanding of (the relation between different approaches to) the longitudinal index theorem for foliations (see [5, 6, 27, 15]), we have added a few brief comments at the end of this section. Section 3 contains the proof of the first theorem and the related results, and Section 4 contains the proof of the second theorem. We also mention that the part concerned with the theory of Lie groupoids (namely Theorem 1 and Proposition 1, and their proofs in Section 3) can be read independently of the preliminaries on cyclic homology in Section 1.

1 Preliminaries

We begin by recalling the necessary definitions and notation concerning groupoids and cyclic homology. Standard references include [19, 30, 14] for groupoids, and [6, 17, 18] for cyclic homology.

Groupoids: A *groupoid* G is a (small) category in which every arrow is invertible. We will write G_0 and G_1 for the set of objects and the set of arrows in G, respectively. The source and target maps are denoted by $s, t : G_1 \longrightarrow G_0$, while $m(g, h) = g \circ h$ is the composition, and $i(g) = g^{-1}$ denotes the inverse of g. One calls G a *smooth groupoid* if G_0 and G_1 are smooth manifolds, all the structure maps are smooth, and s and t are submersions. Basic examples include Lie groups, manifolds, crossed products of manifolds by Lie groups, the holonomy and the monodromy groupoids of a foliation, Haefliger's groupoid Γ^q, and groupoids associated to orbifolds.

If G is a smooth groupoid and $X, Y \subset G_0$, we write $G_X = s^{-1}(X)$, $G^Y = t^{-1}(Y)$, $G_X^Y = s^{-1}(X) \cap t^{-1}(Y)$. Note that G_X^X has the structure of a groupoid (the restriction of G to X). When $X = Y = \{x\}$, $x \in G_0$, we simplify the notations to G_x, G^x, G_x^x; these are submanifolds of G_1, and G_x^x is a Lie group, called the *isotropy group* of G at x.

The tangent spaces at 1_x of G_x form a bundle $\mathfrak{g}$ over G_0, of "s-vertical" tangent vectors on G_1; it is the restriction along $u : G_0 \hookrightarrow G_1$ of the vector bundle $T^s(G_1) = Ker(ds : TG_1 \to TG_0)$. The differential $dt : TG_1 \to TG_0$ of the target map induces a map of vector bundles over G_0,

$$\alpha : \mathfrak{g} \to TG_0 ,$$

called the *anchor map.* Moreover, the space of sections $\Gamma\mathfrak{g}$ is equipped with a Lie bracket $[\cdot\,,\cdot]$. This bracket makes α into a Lie algebra homomorphism $\Gamma(\alpha) : \Gamma\mathfrak{g} \to \mathcal{X}(G_0)$ into the vectorfields on G_0, satisfying the identity $[X, fY] = f[X,Y] + \alpha(X)(f)\cdot Y$ for any $X, Y \in \Gamma\mathfrak{g}$ and $f \in C^\infty(G_0)$. This structure

$$(\mathfrak{g},\, [\cdot\,,\cdot]\,, \alpha)$$

is called *the Lie algebroid* of G, and briefly denoted $\mathfrak{g}$ in this paper.

A *homomorphism* $\varphi : G \to H$ between two smooth groupoids is a smooth functor. Thus, it is given by two smooth maps (both) denoted $\varphi : G_0 \to H_0$ and $\varphi : G_1 \to H_1$, commuting with all the structure maps ($\varphi \circ s = s \circ \varphi$, etc.). Such a homomorphism is called an *essential equivalence* if the map $s\pi_2 : K_0 \times_{G_0} G_1 \longrightarrow G_0$, defined on the space of pairs $(y, g) \in K_0 \times G_1$ with $t(g) = \varphi(y)$, is a surjective submersion, and the square

$$\begin{tikzcd} K_1 \arrow[r, "\varphi_1"] \arrow[d, "(s,t)"'] & G_1 \arrow[d, "(s,t)"] \\ K_0 \times K_0 \arrow[r, "\varphi_0 \times \varphi_0"] & G_0 \times G_0 \end{tikzcd} \tag{1}$$

is a pullback. Two groupoids G_i are said to be *Morita equivalent* if there exists a third groupoid G, and essential equivalences $\varphi_i : G \longrightarrow G_i$ as above ($i \in \{1, 2\}$).

If $f : X \longrightarrow G_0$ is a smooth map, one defines the pullback of G along f as the groupoid $f^*(G)$ whose space of objects is X, and whose arrows between $x, y \in X$ are the arrows of G between $f(x)$ and $f(y)$. When the map $s\pi_2 : X \times_{G_0} G_1 \longrightarrow G_0$ is a surjective submersion, the groupoid $f^*(G)$ is smooth and the obvious smooth functor $f^*(G) \longrightarrow G$ is a Morita equivalence. For instance, given a family $\mathcal{U} = \{U_i\}$ of opens in G_0, we define the groupoid $G_{\mathcal{U}}$ as the pullback along $f : \coprod_i U_i \longrightarrow G_0$. If $\mathcal{U}$ is a covering, then $G_{\mathcal{U}}$ is Morita equivalent to G. Also, if $G = Hol(M, \mathcal{F})$ is the holonomy groupoid of a foliation $(M, \mathcal{F})$, and $i_T : T \longrightarrow M$ is a *transversal* for $\mathcal{F}$ (recall that this means that T intersects each leaf transversally), then $i_T^*(Hol(M, \mathcal{F})) = Hol_T(M, \mathcal{F})$ is the reduced holonomy groupoid of $\mathcal{F}$. If T is a *complete transversal* (i.e. intersects each leaf at least once), then $Hol_T(M, \mathcal{F})$ is the standard étale groupoid (see below) which is Morita equivalent to $Hol(M, \mathcal{F})$.

A smooth groupoid G is called *étale* (or r-discrete) if the source map $s : G_1 \longrightarrow G_0$ is a local diffeomorphism. This implies that all other structure maps are also local diffeomorphisms. Basic examples are discrete groups, manifolds, crossed products of manifolds by (discrete) groups, the reduced holonomy groupoid of a foliation, Haefliger's groupoid Γ^q, groupoids associated to orbifolds.

The category *Etale* of étale groupoids (with generalized homomorphisms) plays an essential role in the study of leaf spaces of foliations. It should be viewed as an enlargement of the category of smooth manifolds

$$Top \subset Etale \tag{2}$$

to which many of the classical constructions from algebraic topology extend: homotopy, sheaves, cohomology, compactly supported cohomology, Leray spectral sequences, Poincaré duality, principal bundles, characteristic classes etc. See [9, 11, 14, 22, 26].

In extending these constructions, one often uses the following property, typical of étale groupoids. Any arrow $g : x \longrightarrow y$ induces a (canonical) germ $\sigma_g : (U, x) \tilde{\longrightarrow} (V, y)$ from a neighborhood U of x in G_0 to a neighborhood V of y. Indeed, we can define $\sigma_g = t \circ \sigma$, where $x \in U \subset G_0$ is so small that $s : G_1 \longrightarrow G_0$ has a section $\sigma : U \longrightarrow G_1$ with $\sigma(x) = g$.

Convolution algebras and cyclic homology: Let G be a smooth groupoid. To define its smooth convolution algebra $C_c^\infty(G)$, one uses the convolution product, defined for functions ϕ, ψ on G and $g \in G_1$, by

$$(\phi * \psi)(g) = \int_{g_1 g_2 = g} \phi(g_1)\psi(g_2). \tag{3}$$

We assume for simplicity that G is Hausdorff. (For general groupoids, possibly non-Hausdorff, the construction of the convolution algebra is slightly more involved [10].) If G is étale, then the integration is simply summation, but, in general, one has to give a precise meaning to the integration in the previous formula. For this, some choices have to be made. If one wants to work with complex-valued functions $\phi, \psi \in C_c^\infty(G_1)$, then one has to fix a smooth Haar system for G (we refer to [33] for precise definitions). Instead, it is possible to use a line bundle $\mathcal{L}$ of "densities" which is isomorphic to the trivial bundle (in a non-canonical way), and to work with compactly supported smooth sections of $\mathcal{L}$, $\phi, \psi \in C_c^\infty(G; \mathcal{L})$. Fixing a trivialisation of $\mathcal{L}$ induces a Haar system on G, and gives an isomorphism $C_c^\infty(G; \mathcal{L}) \cong C_c^\infty(G)$.

Let us recall Connes' choice of $\mathcal{L}$ [4]. Let $\mathfrak{g}$ be the Lie algebroid of G. Denote by $\mathcal{D}^{1/2}$ the line bundle on G_0 consisting of transversal half-densities. Writing $p = dim(\mathfrak{g})$, the fiber of $\mathcal{D}^{1/2}$ over $x \in G_0$ consists of maps ρ from the exterior power $\Lambda^p \mathfrak{g}_x$ to $\mathbb{C}$ such that $\rho(\lambda v) = |\lambda|^{1/2}\rho(v)$ for all $\lambda \in \mathbb{R}$, $v \in \Lambda^p \mathfrak{g}_x$. There is a similar bundle $\mathcal{D}^r$ for any r. The bundle of densities ($r = 1$) is usually denoted by $\mathcal{D}$. We put $\mathcal{L} = t^*\mathcal{D}^{1/2} \otimes s^*\mathcal{D}^{1/2}$. Then (3) makes sense for $\phi, \psi \in C_c^\infty(G; \mathcal{L})$. Indeed, looking at the variable $g_2 = h$, one has to integrate $\phi(gh^{-1})\psi(h) \in \mathcal{D}_x^{1/2} \otimes \mathcal{D}_z \otimes \mathcal{D}_y^{1/2}$ with respect to $z \xleftarrow{h} y$ varying in G_y. But $\mathcal{D}_z$ is canonically isomorphic to the fiber at h of the bundle of densities on the manifold G_y, hence the integration makes sense and gives an element $(\phi * \psi)(g) \in \mathcal{D}_x^{1/2} \otimes \mathcal{D}_y^{1/2} = \mathcal{L}_g$.

In the sequel we will omit $\mathcal{L}$ from the notation $C_c^\infty(G; \mathcal{L})$.

Given an algebra A, recall the definition of Connes' cyclic complex $C_*^\lambda(A)$, and of Hochschild's complex $C_*(A)$. The latter has $C_n(A) = A^{\otimes(n+1)}$, with boundary b given by

$$\begin{aligned} b(a_0, a_1, \ldots, a_n) &= b'(a_0, a_1, \ldots, a_n) + (-1)^n(a_n a_0, a_1, \ldots, a_{n-1}), \\ b'(a_0, a_1, \ldots, a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n), \end{aligned}$$

while the cyclic complex is the quotient $C_n^{\lambda}(A) := A^{\otimes(n+1)}/Im(1-\tau)$ with boundary induced by b. Here τ is the signed cyclic permutation:

$$\tau(a_0, a_1, \ldots, a_n) = (-1)^n(a_n, a_0, \ldots, a_{n-1}) \ .$$

Recall that the cyclic homology groups $HC_*(A)$ of A are computed by the complex $C_*^{\lambda}(A)$. Also, the Hochschild homology groups $HH_*(A)$ are computed by $C_*(A)$, provided A is *H-unital*. Recall that H-unitality means that $(C_*(A), b')$ is acyclic, and it plays a crucial role in the excison theorems for cyclic homology [35]. For instance, (smooth) convolution algebras of étale groupoids have local units, and this implies H-unitality; actually we will show that $C_c^{\infty}(G)$ is H-unital for any smooth groupoid G.

In the present context, the algebra A we work with is endowed with a locally convex topology, and the relevant homology groups are obtained by replacing the algebraic tensor products by topological ones. One has many topological tensor products available, but the appropriate choice is often dictated by the type of algebras under consideration and by the desire to have a computable target for Chern characters. For instance, when $A = C_c^{\infty}(M)$ for a manifold M, one recovers (compactly supported) DeRham cohomology and the classical Chern character, provided one uses the inductive tensor product of locally convex algebras. The same product is relevant for convolution algebras, and, in the sequel, $\otimes$ will denote this topological tensor product. Actually, the only thing the reader needs to know about it is that $C_c^{\infty}(M) \otimes C_c^{\infty}(N) \cong C_c^{\infty}(M \times N)$ for any two manifolds M, N (and our results apply to any tensor product with this property).

2 Main results

In this section we present our main results concerning smooth groupoids which appear in foliation theory. The first one is the characterisation theorem already mentioned in the introduction:

Theorem 1 *For a smooth groupoid G, the following are equivalent:*
(i) G is Morita equivalent to a smooth étale groupoid;
(ii) The Lie algebroid $\mathfrak{g}$ of G has an injective anchor map;
(iii) All isotropy Lie groups of G are discrete.

We will refer to groupoids with this property as *foliation groupoids*. For instance, the action groupoid $M \rtimes G$ associated to the action of a Lie group on a manifold M (which models the orbit space M/G) is a foliation groupoid, provided all the isotropy groups $G_x = \{g \in G : xg = x\}$ are discrete. Also, if G is a foliation groupoid, then so is any pull-back of G (e.g. the groupoid $G_{\mathcal{U}}$ associated to any cover $\mathcal{U}$ of G_0). The motivating examples are, however, the holonomy and the monodromy groupoids $Hol(M, \mathcal{F})$ and $Mon(M, \mathcal{F})$ of any foliation $(M, \mathcal{F})$ (note that the monodromy groupoid appears in literature also under the name of "the homotopy groupoid" [31]). The construction of the holonomy along longitudinal paths (paths inside leaves) can be viewed as a morphism

$$hol : Mon(M, \mathcal{F}) \longrightarrow Hol(M, \mathcal{F}) \tag{4}$$

which is the identity on M (i.e., it is morphism of groupoids over M).

Note that any foliation groupoid G defines a foliation $\mathcal{F}$ on G_0, and G can be viewed as an integration of $\mathcal{F}$. In many examples one actually starts with a foliation $(M, \mathcal{F})$, and then chooses a convenient foliation groupoid G integrating $\mathcal{F}$. It is generally accepted that the holonomy and the monodromy groupoids are actually extreme examples of such integrations. The following proposition gives a precise formulation of this principle. For simplicity we restrict ourselves to *s-connected* groupoids, i.e. groupoids G with the property that all its s-fibers are connected. Recall [19] that, if G is arbitrary, one can allways find an open $s-$connected subgroupoid of G by taking the connected components of the units in the s-fibers.

Proposition 1 *Let $(M, \mathcal{F})$ be a foliation. For any s-connected smooth groupoid G integrating $\mathcal{F}$, there is a natural factorization of the holonomy morphism (4) into homomorphisms h_G, hol_G of groupoids over M,*

$$Mon(M, \mathcal{F}) \xrightarrow{h_G} G \xrightarrow{hol_G} Hol(M, \mathcal{F}).$$

The maps h_G and hol_G are surjective local diffeomorphisms. Moreover, G is s-simply connected (i.e. has simply connected s-fibers) if and only if h_G is an isomorphism.

We will give explicit constructions of h_G and hol_G later. However, we should remark that the first of these homomorphisms is a consequence of integrability results for Lie algebroids in [20]; see also [23].

We next turn to the cyclic homology of convolution algebras of foliation groupoids. Since the étale case is well understood [2, 7, 9], our aim is to show that the homology doesn't change when one passes from a given foliation groupoid to a Morita equivalent étale groupoid. Thus, one of our main results is the following:

Theorem 2 *If G and H are Morita equivalent foliation groupoids, then*

$$HC_*(C_c^\infty(G)) \cong HC_*(C_c^\infty(H)) \ ,$$

and similarly for Hochschild and periodic cyclic homology.

We emphasize that, due to the applications we have in mind, our aim is to prove the previous theorem by means of explicit formulas (see the remark at the end). As said in the introduction, we conjecture that this theorem in fact holds for smooth groupoids generally. Note also that some of our lemmas are proved in this generality. For instance, since H-unitality is usually relevant to excision theorems [35], and since convolution algebras appear in the short exact sequences given by the pseudo-differential calculus [29], the following result which is independent interest:

Proposition 2 *The convolution algebra $C_c^\infty(G)$ of any smooth groupoid G is H-unital.*

Note that Theorem 2, combined with the results of [2, 7, 9] concludes the computation of the cyclic homology for various foliation groupoids. Apart from the holonomy and the monodromy groupoids, we mention the groupoids modeling orbifolds, and the groupoids associated to Lie group actions with discrete stabilizers.

Remarks: Before turning to the proofs in the next section, we make some further remarks:

(i) The holonomy groupoid of a foliation $(M, \mathcal{F})$ appears as the right model for the leaf space $M/\mathcal{F}$. Proposition 1 shows that it is the minimal *smooth* "desingularization" of the leaf space. We want to point out, however, that the holonomy groupoid may not be the most appropriate model when looking at problems whose primarly interest is not the leaf space. One can find many examples where other foliation groupoids integrating $\mathcal{F}$ are equally good, and sometimes even more suitable. This applies, for example, to the results of [15] which can be obtained using any Hausdorff groupoid integrating the given foliation (all that matters is that the groupoid has the property stated in Lemma 3 below). Regarding the Hausdorffness, we remark that there is no relation between the Hausdorffness of $Mon(M, \mathcal{F})$ and of $Hol(M, \mathcal{F})$, and there are foliations $\mathcal{F}$ whose monodromy and holonomy groupoids are both non-Hausdorff, but which admit Hausdorff integrations G.

(ii) In the longitudinal index theory for foliations $(M, \mathcal{F})$ of a compact manifold M, the analytic index of a longitudinal elliptic operator D can again be defined using any foliation groupoid G integrating $\mathcal{F}$. First of all one lifts D to an operator along the s-fibers of G, and then the pseudodifferential calculus on G (namely the short exact sequence given by the symbol map of Theorem 8 in [29], and the boundary map of the long exact sequence it induces in K-theory) gives a precise meaning to the index $Ind_G(D) \in K_0(C_c^{\infty}(G))$ depending just on the symbol of D (actually just on the induced class in $K^1(S^*\mathcal{F})$). Classically, this construction is applied to the holonomy groupoid, but Theorem 1 shows that the best choice is the monodromy groupoid of $(M, \mathcal{F})$, where $Ind_G(D)$ provides the maximal information. Since the monodromy groupoid of the foliation by one leaf is (Morita equivalent to) the fundamental group of M, our remark agrees also with the framework of the L^2-index theorem of Atiyah [1] and the higher versions of Connes and Moscovici [5] (see also [28]).

Now, the general Chern character in cyclic homology [6], combined with our Theorem 2, and with the computations at units given in Theorem 4.1.3. of [7], give a Chern character localized at units $Ch^1 : K_0(C_c^{\infty}(G)) \longrightarrow H_c^*(G)$ (in order to restrict to units, we do have to assume G to be Hausdorff) . The cohomology groups $H_c^*(G)$ are the re-indexed homology groups of [9] applied to any etale groupoid equivalent to G. The longitudinal index formula for foliations (non-commutative approach) gives a topological interpretation for $Ch^1(Ind_G(D))$. More general formulas should correspond to other localizations (cf 4.1.2 in [7]) of the Chern character.

(iii) Following a different route (in the spirit of Bismut's approach to the families index theorem), Heitsch- Lazarov [15] define certain cohomology classes $\bar{C}h_{\mathcal{E}}(D) \in H^*_{c,bas}(M/\mathcal{F})$ playing the role of "the Chern character of the index bundle". Here $H^*_{c,bas}(M/\mathcal{F})$ are the basic cohomology groups of Haefliger [13]. The connection with Connes approach (conjectured in [15]) can be described as follows. For any integration G of $\mathcal{F}$ there is a tautological map $j_b : H_c^*(G) \longrightarrow H_{c,bas}^*(M/\mathcal{F})$, which combined with Ch^1 previously described, induces a basic Chern character at units $Ch_{bas}^1 : K_0(C_c^\infty(G)) \longrightarrow H_{c,bas}^*(M/\mathcal{F})$. For a longitudinal elliptic operator D one gets $Ch_{bas}^1(Ind_G(D)) \in H_{c,bas}^*(M/\mathcal{F})$ independent of the choice of the Hausdorff integration G. Comparing the two longitudinal index theorems of [6] and [15], one sees that (with the proper normalizations) $Ch_{bas}^1(Ind_G(D)) = \bar{C}h_{\mathcal{E}}(D)$. Of course, an interesting question is to give a direct argument for this equality between the basic Chern character of the analytical index, and the Chern character of the index bundle. In this context we remark that, in contrast with Ch^1, it is possible to describe the basic Chern character Ch_{bas}^1 by relatively simple explicit formulas (with the help of connections), using Haefliger's integration [13] along leaves and the non-commutative version [17] of the Chern-Weil construction (see [8] for details).

3 Proof of the characterisation theorem

In this section we present the proofs of Theorem 1 and of Proposition 1.

Proof of Theorem 1:

$(ii) \Leftrightarrow (iii)$: immediate because the Lie algebra of the isotropy group G_x^x is the kernel of the anchor map $\alpha : \mathfrak{g}_x \to T_x(G_0)$.

$(i) \Rightarrow (iii)$: since the isotropy groups of an étale groupoid are clearly discrete, it suffices to remark that this property is invariant under Morita equivalence. Indeed, since the pullback square (1) has a surjective submersion on the bottom, the fibers of the left hand vertical map are discrete iff those of the right hand vertical map are. Thus, the isotropy groups of G are discrete iff those of H are.

$(ii) \Rightarrow (i)$: suppose the anchor map

$$\alpha : \mathfrak{g} \to T(G_0)$$

of the Lie algebroid $\mathfrak{g}$ of G is injective. Write $\mathcal{F} \subseteq T(G_0)$ for the image $\alpha(\mathfrak{g})$. Then $\mathcal{F}$ is an involutive subbundle of $T(G_0)$, hence defines a foliation $\mathcal{F}$ of G_0. On the other hand, the submersion $s : G_1 \to G_0$ (source) defines a foliation $\tilde{\mathcal{F}}$ on G_1, whose leaves are the connected components of the fibers of $s : G_1 \to G_0$. Denote by p the dimension of $\mathcal{F}$, by q its codimension. From the hypothesis, the dimension of $\tilde{\mathcal{F}}$ is $p = dim(G_x) = rank(\mathfrak{g})$, while its codimension n is equal to the codimension of G_x in G, so that $n = p + q$.

Lemma 1 *The target map $t : (G_1, \tilde{\mathcal{F}}) \longrightarrow (G_0, \mathcal{F})$ maps leaves into leaves, and its restriction to each leaf is a local diffeomorphism. If G is s-connected, then, for any point $x \in G_0$, the space $t(G_x) = L_x$ is the leaf through x, and*

$$t : G_x \longrightarrow L_x \tag{5}$$

is a smooth covering projection with structure group G_x^x.

Proof: For any $g : x \longrightarrow y$ in G, one has a commutative diagram

$$\begin{tikzcd}
T_y(G_y) = \mathfrak{g}_y \arrow[d, "(dR_g)_y"', "\sim"] \arrow[dr, "\alpha_y"] & \\
T_g(G_x) = \tilde{\mathcal{F}}_g \arrow[r, "(dt)_g"] & T_y(G_0)
\end{tikzcd}$$

where α_y maps $\mathfrak{g}_y$ isomorphically into $\mathcal{F}_y$, and $R_g : (G_y, y) \longrightarrow (G_x, g)$ is the right multiplication by g. Thus, the target map induces an isomorphism

$$(dt)_g : \tilde{\mathcal{F}}_g \xrightarrow{\sim} \mathcal{F}_y. \tag{6}$$

This shows that the target map $t : (G_1, \tilde{\mathcal{F}}) \longrightarrow (G_0, \mathcal{F})$ maps leaves to leaves, and that its restriction to each leaf is a local diffeomorphism. Hence, for any $y \in G_0$, and any connected component C of G_y, the map $t|_C$ is a local diffeomorphism of C into some leaf of $\mathcal{F}$. To prove it is onto, it suffices to remark that $\{t(C) : C$ is a connected component of $G_y, y \in G_0\}$ is a partition of G_0. Indeed, if $C_i \subset G_{y_i}$ are connected components so that $t(C_1) \cap t(C_2)$ is non-empty, we find $g_i \in C_i$ with $t(g_1) = t(g_2) = y$. Since $R_{g_i} : G_y \longrightarrow G_{y_i}$ are diffeomorphisms, the $R_{g_i}^{-1}(C_i)$ will be connected components of G_y, both containing 1_y, hence $R_{g_1}^{-1}(C_1) = R_{g_2}^{-1}(C_2)$, which shows that $t(C_1) = t(C_2)$. $\square$

The following Lemma will complete the proof of the Theorem:

Lemma 2 *For any transversal T of $\mathcal{F}$, the groupoid G_T^T is étale. If T is complete, then G is Morita equivalent to G_T^T.*

Proof: First we claim that the source map restricts to a local diffeomorphism

$$s : G^T \longrightarrow G_0 \ . \tag{7}$$

Since t is a submersion (hence, in particular, it is transversal to T), $G^T = t^{-1}(T)$ is a submanifold of G_1 of codimension equal to the codimension of T in G_0 (i.e. to p), whose tangent space at $g : x \longrightarrow y$ consists of vectors $\xi \in T_g(G_1)$ with the property that $(dt)_g(\xi) \in T_y(T)$. By counting dimensions, it suffices to prove that the map above is an immersion, i.e., since $ker(ds)_g = \tilde{\mathcal{F}}_g$, that

$$\tilde{\mathcal{F}}_g \cap (dt)_g^{-1}(T_y(T)) = \{0\}. \tag{8}$$

But this is immediate from the isomorphism (6) and the fact that T is transversal to $\mathcal{F}$.

Since (7) is a local diffeomorphism, the inverse image G_T^T of T is a submanifold, and the restriction $s : G_T^T \longrightarrow T$ is a local diffeomorphism. Thus G_T^T is étale. Moreover, if the transversal T is complete, then $s : G^T \longrightarrow G$ is a surjection, and hence the obvious functor $G_T^T \longrightarrow G$ is an essential equivalence. This proves the lemma. $\square$

For the proof of Proposition 1 we need the following Lemma. We first recall some terminology. Given a submersion $\pi : U \longrightarrow T$, the connected components of its fibers define a foliation on U. Denote by $U \times_T U$ the fibered product $\{(x, y) \in U \times U : \pi(x) = \pi(y)\}$. We say that π is a *trivializing submersion* of $\mathcal{F}$ if its domain U is open in G_0, the fibers of π are contractible, and they coincide with the plaques of $\mathcal{F}$ in U.

Lemma 3 *Let G be a foliation groupoid, and let $\mathcal{F}$ be the induced foliation on G_0. For any trivializing submersion $\pi : U \longrightarrow T$ of $\mathcal{F}$, there exists a unique open subgroupoid $G(U) \subset G_1$ such that the map $(t,s) : G_1 \longrightarrow G_0 \times G_0$ restricts to an isomorphism of smooth groupoids:*

$$(t,s) : G(U) \xrightarrow{\sim} U \times_T U \ .$$

Proof: First note that any such open subgroupoid of G is contained in the s-connected component of G_U^U. Hence it suffices to show that this s-connected component, denoted $G(U)$, has the desired property. In other words, it suffices to prove that if $U = G_0$ and if G is s-connected, the map $(t,s) : G_1 \longrightarrow U \times_T U$ is a diffeomorphism. Remark that (6) implies that $(t,s) : G_1 \longrightarrow U \times U$ is an immersion. By counting the dimensions, it follows that $(t,s) : G_1 \longrightarrow U \times_T U$ is a local diffeomorphism. It is also bijective because, by Lemma 1, for any $x \in U$, the map $t : G_x \longrightarrow \pi^{-1}(\pi(x))$ is a covering projection with connected total space, and contractible base space, hence it is a diffeomorphism. □

Proof of Proposition 1: Of course one can use Lemma 1 to define h_G. We indicate a slightly different description, which immediately implies the smoothness of h_G. Let $\alpha : [0,1] \longrightarrow L$ be a longitudinal path with $\alpha(0) = x$, $\alpha(1) = y$. By the local triviality of $\mathcal{F}$ and the compactness of $\alpha[0,1]$ we find a sequence U_i of domains of trivializing submersions $\pi_i : U_i \longrightarrow T_i$, and real numbers t_i, so that:

$$0 = t_0 < t_1 < \ \ldots \ < t_k = 1, \quad \alpha([t_i, t_{i+1}]) \subset U_i, \quad 0 \leq i \leq k-1 \ .$$

From Lemma 3 we find unique arrows $g_{i+1} : \alpha(t_i) \longrightarrow \alpha(t_{i+1})$ in $G(U_i)$; we put

$$h_G(\alpha) := g_k \, g_{k-1} \ \ldots \ g_1 \in G \ .$$

This definition closely resembles the construction of the holonomy, and, by the same arguments, $h_G(\alpha)$ depends just on the homotopy class of α. The smoothness of h_G is immediate now, since, near α, the smooth structure of $Mon(M, \mathcal{F})$ is defined precisely using such chains $\{U_i\}$ covering α. That h_G is surjective if G is s-connected follows from the fact that on the s-fibers it is precisely the projection $\tilde{L}_x \longrightarrow G_x$ induced by the covering projection of Lemma 1.

We now construct hol_G. Actually, since h_G is surjective and we want $hol_G \circ h_G = hol$, we only have to show that the holonomy class of α is determined by $h_G(\alpha)$. For this, we remark that the holonomy germ of α can be defined directly in terms of the arrow $h_G(\alpha)$. More precisely, giving any arrow $g : x \longrightarrow y$, and any transversal T containing x and y, one obtains an induced germ $\sigma_g^T : (T,x) \longrightarrow (T,y)$ due to the fact that G_T^T is étale (see our preliminaries on groupoids). We claim that when $g = h_G(\alpha)$, this germ σ_g^T coincides with the holonomy germ of α. This is clear when α is contained in the domain of a trivializing submersion. In general, we use that σ_g^T is functorial in g, and that $\sigma_g^T = \sigma_g^S$ whenever S is another transversal containing T. Choosing any transversal T containing x, y and all the $\alpha(t_i)$'s above, it follows that the germ associated to $g = h_G(\alpha)$, which is

$$\sigma_g^T = \sigma_{g_k}^T \, \sigma_{g_{k-1}}^T \ \ldots \ \sigma_{g_1}^T : (T,x) \longrightarrow (T,y),$$

coincides with the holonomy germ of α. The last part of the Corollary follows from Lemma 1. □

4 Proof of the invariance theorem

In this section we present the proofs of Theorem 2 and of Proposition 2. We will assume throughout that G is Hausdorff. However, we point out that our proofs also apply to the non-Hausdorff case, provided proper care is taken of compact supports [10].

Proof of Proposition 2: We first need some remarks about the extension of compactly supported smooth functions. Let M be a manifold and let A be a closed subset of M. Write $\mathcal{S}_{M,A}$ for the fine sheaf of smooth functions on M which vanish on A. For a closed submanifold $N \subset M$, there is an obvious restriction

$$\Gamma_c(\mathcal{S}_{M,A}) \longrightarrow \Gamma_c(\mathcal{S}_{N,N\cap A}) \ . \tag{9}$$

We will be concerned with the surjectivity of this map, for specific M, N, and A. Note that this surjectivity is a local property: if each $x \in N$ has a neighborhood U in M such that $\Gamma_c(\mathcal{S}_{U,A\cap U}) \longrightarrow \Gamma_c(\mathcal{S}_{N\cap U,N\cap A\cap U})$ is surjective, then it follows (by a partition of unity or a Mayer-Vietoris argument) that (9) is surjective.

If $x \in N$, one can always choose a neighborhood U of x in M and a retraction $r : U \longrightarrow U \cap N$. If, for any x, these U and r can be chosen such that $r(A \cap U) \subset A$, we say that A is *locally retractible* to N in M. Note that this implies the surjectivity of (9). Indeed, since both properties are local, we may assume that there exists a retraction $r : M \longrightarrow N$ such that $r(A) \subset A$; then, for any $\phi \in C_c^\infty(N)$ vanishing on $A \cap N$, $\tilde{\phi}(x) = \theta(x)\phi(r(x))$ defines an extension of ϕ to M vanishing on A, provided we choose $\theta \in C_c^\infty(M)$ with $\theta \equiv 1$ on the support of ϕ.

An easy argument based on the canonical local form of a submersion shows that:

Lemma 4 *Let $s : X \longrightarrow Z$ be a submersion, let $f : Y \longrightarrow Z$ be a smooth map, and let $B \subset Y$ be a closed subset. Then $A = X \times B$ is locally retractible in $M = X \times Y$ to $N = X \times_Z Y$. In particular, (9) is surjective.*

For the proof of the proposition, we have to prove that if $\psi \in C_c^\infty(G_1^p)$ is a cycle with respect to b' (i.e. $b'(\psi) = 0$), then it is b'-homologous to zero (i.e. is of type $b'(\tilde{\psi})$ for some $\tilde{\psi} \in C_c^\infty(G_1^{p+1})$). Recall that $b'(\psi) = \sum_1^{p-1}(-1)^i d_i(\psi)$, where

$$d_i(\psi)(g_1, \ldots, g_{p-1}) = \int_{uv=g_i} \psi(g_1, \ldots, g_{i-1}, u, v, \ldots, g_{p-1})$$

We will first show that, for each $k = 1, \ldots p$, there exists a cycle ψ_k homologous to ψ such that

$$\psi_k(g_1, \ldots, g_p) = 0 \text{ if } s(g_i) = t(g_{i+1}) \text{ for some } i < k. \tag{10}$$

Notice that for such a cycle ψ_k, we have $d_i(\psi_k) = 0$ for $i < k$. We construct ψ_k by induction on k. For $k = 1$ the condition (10) is vacuous, and we can take $\psi_1 = \psi$. Suppose $\psi_1, \ldots, \psi_k$ have been defined. Let $K = supp(\psi_k)$, which is a compact subset of G_1^p. Let $L = \{x : \exists (g_1, \ldots, g_p) \in K \text{ s.t. } x = t(g_k)\}$, and let $\theta \in C_c^\infty(G_1)$ be a function such that $\int_{t^{-1}(x)} \theta = 1$ for all $x \in L$. Now define a function ϕ on the submanifold $N \subset M = G_1^{p+1}$ consisting of those $(g_1, \ldots, g_{p+1})$ for which $g_k g_{k+1}$ is defined, by

$$\phi(g_1, \ldots, g_{p+1}) = \theta(g_k)\psi_k(g_1, \ldots, g_k g_{k+1}, \ldots, g_{p+1})$$

Thus $\phi(g_1, \ldots, g_{p+1}) = 0$ as soon as $s(g_i) = t(g_{i+1})$ for some $1 \le i < k$. By Lemma 4, we can find $\tilde{\phi} \in C_c^\infty(G_1^{p+1})$ such that $\tilde{\phi}(g_1, \ldots, g_{p+1})$ equals zero if $s(g_i) = t(g_{i+1})$ for some $1 \le i < k$, and equals $\theta(g_k)\psi_k(g_1, \ldots, g_k g_{k+1}, \ldots, g_{p+1})$ if $s(g_k) = t(g_{k+1})$. Then we have $d_i(\tilde{\phi}) = 0$ for $1 \le i < k$, $d_k(\tilde{\phi}) = \psi_k$, while for $i > k$ and $s(g_k) = t(g_{k+1})$, $d_i(\tilde{\psi})(g_1, \ldots, g_p) = \theta(g_k)d_{i-1}(\psi_k)(g_1, \ldots, g_k g_{k+1}, \ldots, g_p)$. So, still assuming $s(g_k) = t(g_{k+1})$,

$$\begin{aligned} b'(\tilde{\phi})(g_1, \ldots, g_p) &= (-1)^k\psi_k(g_1, \ldots, g_p) + \sum_{j=k}^{p}(-1)^{j+1}\theta(g_k)d_j(\psi_k)(g_1, \ldots, g_k g_{k+1}, \ldots, g_p) \\ &= (-1)^k\psi_k(g_1, \ldots, g_p) - \theta(g_k)b'(\psi_k)(g_1, \ldots, g_k g_{k+1}, \ldots, g_p) \\ &= (-1)^k\psi_k(g_1, \ldots, g_p) \end{aligned} \tag{11}$$

Thus we can put $\psi_{k+1} = \psi_k - (-1)^k b'(\tilde{\psi})$ to obtain the desired property. Having thus defined $\psi_1, \ldots, \psi_p$, the construction of $\tilde{\phi}$ for $k = p$ gives a function $\tilde{\phi}$ with $d_i(\tilde{\phi}) = 0$ for $i < p$, and $d_p(\tilde{\phi}) = \psi_p$. Thus $b'(\tilde{\phi}) = (-1)^p\psi_p$, showing that ψ_p is a boundary. This proves Proposition 2. □

To prove Theorem 2 we need some preliminary lemmas. We first compare the convolution algebra of G, with the one of the groupoid $G_{\mathcal{U}}$ induced by G and an open covering $\mathcal{U} = \{U_i\}$ of G_0 (cf. our preliminaries). The elements of $C_c^\infty(G_{\mathcal{U}}) = \oplus_{i,j}C_c^\infty(G_{U_j}^{U_i})$ can be written as matrices $\varphi = (\varphi_{i,j})_{i,j}$. We will also use the following left/right action of $C_c^\infty(G_0)$ on $C_c^\infty(G)$:

$$(f\phi)(g) = f(t(g))\phi(g), \quad (\phi f)(g) = \phi(g)f(s(g)) \ .$$

Lemma 5 *For any smooth groupoid G, and for any family $\{\lambda_i \in C_c^\infty(G_0)\}$ so that $\{\lambda_i^2\}$ is a partition of unity subordinated to a locally finite open covering $\mathcal{U} = \{U_i\}$ of G_0, the map*

$$\lambda : C_c^\infty(G) \longrightarrow C_c^\infty(G_{\mathcal{U}}), \ \lambda(\phi) = (\lambda_i\phi\lambda_j)_{i,j}. \tag{12}$$

is an algebra homomorphism whose induced maps in cyclic type homologies are injective.

Proof: One has an obvious inclusion $i : C_c^\infty(G_{\mathcal{U}}) \hookrightarrow M_\infty(C_c^\infty(G))$, already suggested by the notation for elements of $C_c^\infty(G_{\mathcal{U}})$. It suffices to prove that the composition $i^\lambda = i\circ\lambda$ induces isomorphism in cyclic homologies. Using the algebras $A = C_c^\infty(G)$, $A_0 = C_c^\infty(G_0)$, we find ourselves in the abstract situation where we have a triple

$$(A, A_0, \lambda), \quad \lambda = \{\lambda_i\}$$

where A is an H-unital algebra, A_0 is an algebra which acts on both sides on A, and the $\lambda_i \in A_0$ are elements so that, for any given $a \in A$ or $a \in A_0$, the products $\lambda_i a$ and $a\lambda_i$ are nonzero for only finitely many i, and $\sum \lambda_i^2 a = \sum a\lambda_i^2 = a$. We prove that, in this situation, the algebra homomorphism $i^\lambda : A \longrightarrow M_\infty(A)$, $i^\lambda(a) = (\lambda_i a\lambda_j)_{i,j}$ induces isomorphisms in the cyclic type homologies.

Let us first consider the special case where A_0 is a subalgebra of A. Recall from [18] that the trace map $Tr_* : C_*(M_\infty(A)) \longrightarrow C_*(A)$,

$$Tr_*(a^0, a^1, \ldots, a^n) = \sum_{i_0,\ldots,i_n}(a^0_{i_0i_1}, a^1_{i_1i_2}, \ldots, a^n_{i_ni_0}), \ a^i \in M_\infty(A),$$

has this property. Using the SBI argument, it now suffices to show that $Tr_* i_*^{\lambda}$, acting on the Hochschild complex $(C_*(A), b)$, is homotopic to the identity. For this, we construct the homotopy

$$\begin{aligned} h(a^0, a^1, \ldots, a^n) &= \sum_{i_0,\ldots,i_n} (a^0\lambda_{i_0}, \lambda_{i_0}a^1\lambda_{i_1}, \ldots, \lambda_{i_{n-1}}a^n\lambda_{i_n}, \lambda_{i_n}) - \\ &- \sum_{i_0,\ldots,i_{n-1}} (a^0\lambda_{i_0}, \lambda_{i_0}a^1\lambda_{i_1}, \ldots, \lambda_{i_{n-2}}a^{n-1}\lambda_{i_{n-1}}, \lambda_{i_{n-1}}, a^n) + \ldots + \\ &+ (-1)^{n-1}\sum_{i_0,i_1}(a^0\lambda_{i_0}, \lambda_{i_0}a^1\lambda_{i_1}, \lambda_{i_1}, a^2, \ldots, a^n) + \\ &+ (-1)^n\sum_{i_0}(a^0\lambda_{i_0}, \lambda_{i_0}, a^1, \ldots, a^n). \end{aligned}$$

In the general case, we use the new algebra $A_0 \rtimes A$ which is $A_0 \oplus A$ with the product

$$(\lambda, a)(\eta, b) = (\lambda\eta, \lambda b + a\eta + ab),\ \lambda, \eta \in A_0, a, b \in A,$$

Remark that $A_0 \rtimes A$ contains A_0 as a subalgebra (with the inclusion $\rho(\lambda) = (\lambda, 0)$), and the map i^{λ} lifts to a map between short-exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "i^{\lambda}"] & A_0 \rtimes A \arrow[r, "\pi"] \arrow[d, "i^{\lambda}"] & A_0 \arrow[r] \arrow[d, "i^{\lambda}"] & 0 \\ 0 \arrow[r] & M_{\infty}(A) \arrow[r, "\tilde{i}"] & M_{\infty}(A_0 \rtimes A) \arrow[r, "\tilde{\pi}"] & M_{\infty}(A_0) \arrow[r] & 0 \end{tikzcd}$$

($i(a) = (0, a)$, $\pi(\lambda, a) = \lambda$, and $\tilde{i}, \tilde{\pi}$ are induced by i, π). Note that ρ is an algebra splitting of π. By the previous discussion, the statement is true for $A_0 \rtimes A$ and A_0; to deduce it for A, it suffices to use Wodzicki's excision [35] for Hochschild/cyclic homology. $\square$

Next, let R be the pair groupoid $\mathbb{R}^p \times \mathbb{R}^p$ over $\mathbb{R}^p$. For foliation groupoids G we will see that, for suitable choices of coverings $\mathcal{U}$, the groupoid $G_{\mathcal{U}}$ becomes isomorphic to $\Gamma \times R$ for some étale groupoid Γ. Therefore, we state and prove the following lemma only for such groupoids Γ (we mention however that, using Lemma 2, one can actually prove it for general smooth groupoids).

Lemma 6 *For any étale groupoid* Γ*, there is an isomorphism*

$$\tau_* : HC_*(C_c^{\infty}(\Gamma \times R)) \xrightarrow{\sim} HC_*(C_c^{\infty}(\Gamma)),$$

and similarly for Hochschild and periodic cyclic homology (see below for explicit formulas).

Proof: The convolution algebra $\mathcal{R}$ of R consists of compactly supported smooth functions on $k(x, y)$ on $\mathbb{R}^p \times \mathbb{R}^p$, with the product

$$(k_1k_2)(x, z) = \int k_1(x, y)k_2(y, z)dy\ .$$

One has the usual trace τ on $\mathcal{R}$,

$$\tau : \mathcal{R} \longrightarrow \mathbb{C}, \ \tau(k) = \int k(x,x)dx, \tag{13}$$

and an induced chain map

$$\begin{aligned} \tau_* : C_*(C_c^\infty(\Gamma \times R)) \longrightarrow C_*(C_c^\infty(\Gamma)), \\ \tau_*(a^0 \otimes k^0, \dots, a^n \otimes k^n) := \tau(k^0 \dots k^n)(a^0, \dots, a^n), \end{aligned} \tag{14}$$

We choose u, and then define α as in

$$u \in C_c^\infty(I^p), \quad \int u(x)^2 dx = 1 \ , \ \alpha := u \otimes u \in \mathcal{R} \tag{15}$$

(where $I^p = (-1,1)^p$) and consider the algebra homomorphism

$$j_\alpha : C_c^\infty(\Gamma) \longrightarrow C_c^\infty(\Gamma \times R), \ j_\alpha(a) = a \otimes \alpha$$

Since $\tau_* j_\alpha = Id$, it suffices to show that

$$j_\alpha \tau_* : C_*(C_c^\infty(\Gamma \times R)) \longrightarrow C_*(C_c^\infty(\Gamma \times R)),$$

$$(a^0 \otimes k^0, \dots, a^n \otimes k^n) \mapsto \tau(k^0 \dots k^n)(a^0 \otimes \alpha, \dots a^n \otimes \alpha)$$

induces the identity in Hochschild homology (hence, by the usual SBI-argument, in all cyclic homologies). Let us first assume that Γ_0 is compact. We then have the following homotopy:

$$h(a^0 \otimes x^0 \otimes y^0, a^1 \otimes x^1 \otimes y^1, \ \dots, a^n \otimes x^n \otimes y^n) =$$
$$= \sum_{k=0}^{n} (-1)^k \tau(y^0 \otimes x^1)\tau(y^1 \otimes x^2) \dots \tau(y^{k-1} \otimes x^k)\psi_k$$

where ψ_k is the element

$$(a^0 \otimes x^0 \otimes u, a^1 \otimes u \otimes u, \ \dots, a^k \otimes u \otimes u, 1 \otimes u \otimes y^k, a^{k+1} \otimes x^{k+1} \otimes y^{k+1}, \ \dots, a^n \otimes x^n \otimes y^n)$$

for all $a^i \in C_c^\infty(\Gamma)$, $x^i \otimes y^i \in \mathcal{R}$ (it is straightforward to write the corresponding formula for the general elements in $C_c^\infty(\Gamma \times \mathbb{R}^p \times \mathbb{R}^p)$). When Γ_0 is not compact, we have to replace the unit $1 \in C_c^\infty(\Gamma_0) \subset C_c^\infty(\Gamma)$ appearing in the previous formula, by local units (compactly supported smooth functions on Γ_0, which are constantly 1 on compacts which exhaust Γ_0). $\square$

Proof of Theorem 2: Since the theorem is known for étale groupoids [7, 9], and since any foliation groupoid G is Morita equivalent to an etale one (e.g. G_T^T of Lemma 2), it suffices to find, for a given foliation groupoid G, a complete transversal T for which we can prove that $HC_*(C_c^\infty(G)) \cong HC_*(C_c^\infty(G_T^T))$. Let $\mathcal{U} = \{U_1, U_2, \dots\}$ be a locally finite cover of G_0 by foliation charts, say $\varphi_i : \mathbb{R}^p \times \mathbb{R}^p \cong U_i \subset G_0$, and write $T_i = \varphi_i(\{0\} \times \mathbb{R}^q) \subset U_i$ for the transversals, and $\pi_i : U_i \longrightarrow T_i$ for the evident projections. Furthermore, let $G_{\mathcal{U}}$ be the groupoid induced by the cover $\mathcal{U}$ as described in the preliminaries. Now observe that, by Lemma 3, there are isomorphisms $H_i : U_i \times_{T_i} U_i \xrightarrow{\sim} G(U_i)$. Each such H_i, H_j induce a map

$$h_{i,j} : G_{U_j}^{U_i} \longrightarrow G_{T_j}^{T_i}$$

$$h_{i,j}(g) = H_i(\pi_i(t(g)), t(g)) \circ g \circ H_j(s(g), \pi_j(s(g)). \tag{16}$$

If we write $T = \coprod T_i$ for the complete transversal and R for the pair groupoid of Lemma 6, we then obtain an isomorphism (compare to [16])

$$h : G_{\mathcal{U}} \xrightarrow{\sim} G_T^T \times R, \quad h(i, g, j) = (h_{i,j}(g), p_i(t(g)), p_j(s(g)). \tag{17}$$

which can be described in terms of the $h_{i,j}$ and the projections $p_i : U_i \longrightarrow \mathbb{R}^p$ on the first coordinate, by

$$h(i, g, j) = (h_{i,j}(g), p_i(t(g)), p_j(s(g)).$$

The isomorphism h, combined with the map j_α of the proof of Lemma 6, gives a map

$$hj_\alpha : C_c^\infty(G_T^T) \longrightarrow C_c^\infty(G_{\mathcal{U}}) \tag{18}$$

which induces isomorphisms in cyclic type homologies. Now consider a sequence of smooth functions $\lambda_i \in C_c^\infty(G)$ such that the λ_i^2 form a partition of unity subordinate to $\mathcal{U}$. We can choose the U_i and λ_i in such a way (see the proof of the preliminary lemma of [16]) that for the open sets $V_i = \varphi_i(I^p \times I^q)$ with transversals $S_i = \varphi_i(\{0\} \times I^q)$ (recall that $I = (-1, 1)$) one has that $\bar{V}_i \cap \bar{V}_j = \emptyset$ whenever $i \neq j$, while $\lambda_i|_{V_i} = 1$ and each leaf of $\mathcal{F}$ meets at least one S_i.

There is an obvious analogue of (18) associated to the family $\mathcal{V}$ and to the complete transversal $S = \coprod S_i$, and we obtain a commutative square:

$$\begin{tikzcd} C_c^\infty(G_S^S) \arrow[r, "hj_\alpha"] \arrow[d, "e'"'] & C_c^\infty(G_{\mathcal{V}}) \arrow[d, "e"] \\ C_c^\infty(G_T^T) \arrow[r, "hj_\alpha"] & C_c^\infty(G_{\mathcal{U}}) \end{tikzcd}$$

where the vertical e and e' are given by extension by zero. In this diagram, the maps hj_α have been shown to induce isomorphisms in cyclic type homologies, while the map e' does so by Morita invariance for etale groupoids [7, 9]. Hence the map e also induces such isomorphisms. We also have a commutative diagram:

$$\begin{tikzcd} C_c^\infty(G_{\mathcal{U}}) \arrow[r, "e''"] \arrow[dr, "e"'] & C_c^\infty(G) \arrow[d, "\lambda"] \\ & C_c^\infty(G_{\mathcal{U}}) \end{tikzcd}$$

where λ is the map defined in Lemma 5, and e'' is again defined by extension by zero. This diagram and the previous remark on e imply that the maps induced by λ in the cyclic homologies are surjective. Using Lemma 5, it then follows that all the maps in the last two diagrams induce isomorphisms in the cyclic type homologies. $\square$

Remark: Let G be a foliation groupoid, and let T, S be the complete transversals previously constructed. There is a commutative diagram

$$\begin{tikzcd} HC_*(C_c^\infty(G)) \arrow[r, "B"] & HC_*(C_c^\infty(G_T^T)) \\ & HC_*(C_c^\infty(G_S^S)) \arrow[u, "e"'] \arrow[ul, "A"] \end{tikzcd}$$

where A, B, e are isomorphisms described as follows:

(i) e is induced by the extension by zero map;

(ii) A is induced by the algebra homomorphism $A : C_c^\infty(G_S^S) \longrightarrow C_c^\infty(G)$,

$$A(\phi_{i,j})(g) = u_i(t(g))\phi(h_{i,j}(g))u_j(s(g))$$

for all $\phi_{i,j} \in C_c^\infty(G_{S_i}^{S_j})$. Here $h_{i,j}$ is given by the formula (16), and $u_i = u \circ p_i \in C_c^\infty(G_0)$, with u chosen as in (15), and $p_i : U_i \longrightarrow \mathbb{R}^p$ the projection;

(iii) B is induced by the composition

$$C_*(C_c^\infty(G)) \xrightarrow{\lambda} C_*(C_c^\infty(G_{\mathcal{U}})) \xrightarrow{h} C_*(C_c^\infty(G_T^T \times R)) \xrightarrow{\tau_*} C_*(C_c^\infty(G_T^T))$$

where λ is given by (12), h is induced by the isomorphism (17), and τ_* is given by the formula (14).

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