SPINFORMATION:
DOMAIN WALLS, SPIN WAVES AND
ENTANGLEMENT

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SPINFORMATION:
DOMAIN WALLS, SPIN WAVES AND
ENTANGLEMENT

SPINFORMATIE:
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(met een samenvatting in het Nederlands)

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LIST OF PUBLICATIONS

The main chapters of this thesis are based on the following works:

- Chapter 3:
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- Chapter 4:
D. M. F. Hartmann, A. Rückriegel and R. A. Duine, "Spin-accumulation induced magnetic texture in a metal-insulator bilayer", *Phys. Rev. B* **101**, 214440 (2020).
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- Chapter 5:
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Other publications:

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- H. Y. Yuan, A. Kamra, D. M. F. Hartmann, R. A. Duine, "Electrically switchable entanglement channel in van der Waals magnets", *arXiv:2103.15899* (2021), *currently under review at Phys. Rev. App.*

GENERAL INTRODUCTION AND SUMMARY IN ENGLISH

Here we give an introduction to and summary of this thesis for laymen. Chapter 9 is a translation of this chapter in Dutch. An introduction on academic level is given in chapter 2.

In my research I have zoomed in on magnets, like the ones you may have on your fridge. I consider specific shapes and configurations of magnetic and non-magnetic materials and attempt to model the behavior of the system in response to a given input.

The purpose of making these theoretical models is first and foremost to keep myself entertained. I enjoy the process of puzzling and discovering. Second, the new insights generated in my research might lead to new information technologies. Magnetism already plays a big role in certain computer devices, such as data storage, and is prospected to further push the boundaries of our computing capabilities. Third, I pursue the development of quantum computing and quantum sensing methods by exploring the entanglement in magnetic and elementary systems.

1.1 MAGNETISM AND RELATIVITY

Why is that magnet on your fridge magnetic? Our current understanding leans on both special relativity and quantum mechanics. Here, we elaborate on the relativistic part. The quantum part is explored in the academic introduction section 2.2.

To understand magnetism, we have to zoom in on the electron. Electrons are very light weight negatively charged elementary particles that are generally found whirling around the nucleus of an atom. When a charged particle moves, it generates a magnetic field, which follows from electrostatic forces combined with the principle of length contraction from special relativity. The first ingredient, electrostatics, might be familiar from high school physics: Opposite charges attract each other and like charges repel.

The second ingredient is a bit more involved and we have to think about how we measure the length of a moving object. Suppose a rod moves with a speed v . When the front of the rod reaches some set position x , I start my stopwatch and measure the time t it takes for the end of the rod to reach that position,

Figure 1.1: The time I measure for your flying rod to pass point x (a) is different from the time you measure for me passing your rod (b). A moving charge does not feel any electric force as the positive and negative charges in the wire cancel out (c). But in the reference frame of the charge (d) the electrons in the wire are more contracted resulting in a net electric force on the charge q .

see figure 1.1 a. Then the length of the rod is given by v times t . Similarly, you could hop on the flying rod and travel with it. You would see me and my measuring position x move at you with the speed v , see figure 1.1b. Likewise, you measure the time t' it takes for x to pass your rod and hence in your frame of reference, the length of the rod is v times t' . The main result of special relativity is that we do not measure the same time and thus also measure different lengths! In fact, your measurement would take longer (this is called time dilatation, explained as an example in section 1.5) and as a consequence I observe a shorter rod.

Now let's put the two together. Suppose I have a straight conducting wire in my lab with a steady electric current passing through it. There is no net charge as the wire consists of equal amounts of positive charges from the atoms and negative charges from the free electrons. So when I place a charged particle q near the wire, it experiences no net force, see figure 1.1c. But when q starts moving in the opposite direction of the current, it will experience an electrostatic force: If we travel along in q 's frame of reference, we would observe a faster moving current, hence the distance between the individual electrons in the wire would contract and thereby become more dense as in figure 1.1d. Thus, in this frame of reference the wire does have a net negative charge which generates a force on q . In the lab frame we do not observe a net charge of the wire, but we would still see that a force is working on q . And that apparent force we call magnetic and we can effectively describe it with a magnetic field, so that we don't have to switch reference frames all the time to determine these relativistic electrostatic effects.

That briefly summarizes why moving electric charges generate and experience magnetic fields and forces. Even when the magnetic bar that sticks to your fridge is perfectly still. The electrons in it are moving around to generate its magnetism. Not only in orbit around the nucleus, but most effectively they are spinning around their own axis. Details on how the electron spins are in the regime of quantum mechanics, which are presented in the following chapter 2.

The second contribution from quantum mechanics to our understanding of magnetism is the mechanism that ensures that these electrons are all spinning around axes that are parallel, so that their tiny magnetic fields all add up, giving the material a net magnetic field. This mechanism is called the exchange interaction and is also elaborated on in chapter 2. Without this mechanism the tiny magnetic contributions would be oriented randomly and add up to zero.

1.2 DOMAIN WALLS AND CREEP MOTION

It is still possible to have a magnetic material, such as iron, that does not have a net magnetic field. When we zoom in, we see that this is because the material is split up into many domains. In each domain all electron spins align, yet the domains themselves do not align with their neighbors. You could make the material magnetic by exposing it to an external magnetic field, for instance from an adjacent magnet, which you may have experienced with a paper clip or a needle.

From a technological point of view, the coexistence of differently oriented magnetic domains is interesting because these can be used to store data. A very thin material has perpendicular magnetic anisotropy: A mechanism that favors electron spins to point perpendicular to the surface. This leaves only two allowed domains, up and down, which then correspond to 0 and 1 as data. In fact, this principle is used in data storage technologies such as magnetic random access memory (MRAM) and hard disk drives.

Between these domains is a small region where the orientation of the magnetization gradually changes direction from up to down, see figure 1.2. This region is called the domain wall (DW) and its size is governed by two competing effects: The exchange interaction, which wants neighboring spins to align and thus favors a very broad DW; And the anisotropy which wants spins to be either up or down and not in between, thereby preferring an abrupt transition, i.e. a small DW. The DW profile is further studied in chapter 2 and also plays a crucial role in my recent research as presented in chapters 3 to 5.

Novel concepts for data storage have raised the interest in control over the motion of DWs. In a typical magnetic storage device the domains represent bits of the data by their orientation and a reading and writing device close by establishes the interface with your computer. In MRAM all magnetic bits have

Figure 1.2: Two types of domain walls: Bloch (left) and Néel (right). Textures anywhere in between these are also possible. As well as the counter clockwise variants. The way the orientation flips is called chirality.

their own reading and writing device, which allows for high speeds, but less storage capacity per volume. In a hard disk drive, the disk is full of magnetic bits which whirl around past one reading and writing head, yielding a larger storage capacity, but a lower speed of data access.

The novel concept of the racetrack memory is of the latter type, but instead of physically moving the magnet around, only the magnetic domains move on a long track, racing past one reading and writing device [1]. This has several advantages over conventional memory types, as domains can move very fast and the tracks can make use of three dimensional space (whereas a hard disk has to be flat).

Aside from the benefits there still are some challenges to overcome in order to have a properly functioning storage device. For instance, you want all DWs moving in the same direction at the same speed ensuring that all domains remain of the same size, otherwise data would get lost. The speed of a DW depends on its chirality (see figure 1.2) which isn't necessarily the same for all DWs on a racetrack. Fortunately, there is an effect called the Dzyaloshinskii-Moriya-interaction (DMI) that ensures that at the DW the magnetization switches its orientation in the same way.

Hence, the search is on for materials with a suitable DMI. As we cannot just look at a material and see how strong the DMI is, an experiment has to be designed from which the strength can be deduced. A typical experiment measures the DW velocity as a function of an applied external magnetic field that is perpendicular to the orientation of the magnetic domains so that it strongly affects the DW chirality. The velocity vs magnetic field curve is then characterized by the strength of the DMI.

This precise characterization is the topic of chapter 3. In that chapter I derive a theoretical model for the velocity of DWs under the influence of DMI and external magnetic fields. By tuning these parameters in the model to fit experimental observations we quantize the DMI strength of the considered materials. The model describes the DW as a line that is pinned due to impurities in the material. A small push in one direction should not be able to move the DW as it is not strong enough to overcome the pinning force. But because of the non-zero temperature the line fluctuates between different configurations and the small pushing force causes the DW to slowly creep in the direction of the push. The elasticity of the line affects the net creep velocity as a more flexible DW is easily reconfigured. This is where the effects of the external magnetic field and the DMI come into play: they affect the elasticity of the DW.

The concept of creep motion extends beyond our system. You may have witnessed it already without knowing it. For example cracks in materials such as paper or stone are creeping through the material as well as the front of water being absorbed up a piece of porous material such as paper. You can see an animation of a moving DW passing by in the bottom right corner of this book by quickly flipping through the pages.

1.3 SPIN WAVES

If you consider the information transport of the racetrack memory it is rather energy inefficient as the moving of the magnetic domains involves a lot of spins flipping up and down. Therefor the racetrack is not optimal for sending a signal along a wire. Conventionally electricity is used for sending signals where a conducting wire transmits *on* when there is a voltage of say 5 Volts and *off* when there is no voltage over the wire. However, with the ever increasing demand for computational capacity, these electric wires keep getting smaller and denser. One of the challenges this creates is that these signals heat up the computer chip, as a smaller wire means more resistance (a phenomenon you may also have recognized from squeezing your straw while drinking) which is converting those Volts to heat.

The stage is thus set for an alternative signal transporter: spin waves. Instead of flipping the entire spin like in the racetrack, a spin wave is a small perturbation of the orientation of the spin. Because of the exchange interaction (which wants neighboring spins to align) such a perturbation is passed on spin by spin causing a collective wave through the magnet. You can compare the process with *the wave* in a soccer stadium. The wave itself moves around the tribune, but the soccer fans remain in their place and the speed of the wave is determined by how fast neighbors react. Now you can imagine sending information using *the wave* is much more efficient, moreover faster, than asking everyone to stand up and run around. In chapters 4 to 6 these spin waves are studied and employed in several systems and corresponding theoretical models. You can see the spin wave in action in the bottom right flip-book animation of this book after the DW has passed.

In a stadium *the wave* comes into existence when people are excited enough. Alternatively, a wave can be started by locally forcing or biasing it at some place. Similarly a magnet can be excited to form spin waves. However, apart from exciting a spin wave, which is a perturbation of the spin orientation around its ground state axis, such a local biasing can also exert a torque on the ground state axis itself, altering its orientation slightly. This alteration again affects how efficient waves are inserted into the medium. This effect of how the local bias affects the magnetic texture is the main topic of chapter 4. It turns out that the magnetic texture is affected such that it resembles a small part of the DW. Extrapolating the DW texture, the center of the DW however would always lie on the other side of the interface, hence we call it a virtual DW.

To make devices that employ spin waves, we would require magnetic wires to guide the signals. However, DWs provide an alternative for guiding some spin waves: If we have a magnetic slab with a DW stretching across it, low energy spin waves can be easily excited that are restricted to the DW itself. By controlling the configuration of the DW the transport of information is directed. I follow this avenue of solid-state reconfigurable devices in chapter 5.

1.4 ENTANGLEMENT

Spin waves are further studied in chapter 6 living on the spins of an antiferromagnetic lattice. An antiferromagnet is also comprised of a spin lattice, but now the exchange interaction favors all neighboring spins to anti-align. So when passing through the lattice you would first encounter an up spin, then a down spin, and then up again and so forth. The lattice is separable in two sublattices, one containing all spins pointing up, and one with all down spins. Remarkably, spin waves on these two sublattices cannot be regarded as independent: They are entangled. That is to say if I want to model the system, I have to treat the spin waves on both lattices as one whole. If I want to extract information about the spin wave on one lattice, I would have to disentangle it from the other thereby altering the state of both. You might have heard about the quantum mechanical principle of the observation affecting the observed (Schroedingers cat might ring a bell).

To clarify the concept of entanglement, consider the following thought experiment: Suppose I have a pair of gloves and a machine that takes the gloves and randomly puts each in a separate box and then closes it (the same kind of box Schroedingers cat was entrapped in, but in this experiment, no animals will be harmed). The gloves in the boxes are in a quantum mechanical state, which is a superposition of the two classical states being either left-handed or right-handed. Even if I know the precise details of this wave function I cannot predict with absolute certainty what I will observe when I open one of the boxes.

Now I give you one of the boxes and send you on a first class trip to the moon 1.3 light seconds away (that is 384.400km, enjoy the benefits of a thought experiment). When I open my box and see that there is a left-handed glove in there, I immediately know that you have a right-handed glove there. So far nothing special has happened you might think, and we could easily do this experiment in real life (except for the trip to the moon). But something very exciting has happened: As soon as I opened my box, the state of the contents of your box are also determined. This happens instantly, not 1.3 seconds later, otherwise you would be able to observe a left-handed glove as well. So the information of my observation travels way faster than the speed of light. This is exactly what troubled Einstein, so he argued that in fact the information is already present at the moment the gloves are separated and slowly travels with you on your trip to the moon. Even if the machine is truly random, nature would still assign some hidden variables which determine the outcome: There was never a chance that I would observe the right-handed glove in my box. The quantum mechanical description of probability then would just be an effective model to cope with our lack of complete knowledge.

This theory of hidden variables does not hold up in so called Bell experiments which are designed to distinguish between quantum and hidden variable randomness [2]. This does not answer Einsteins problem of causality and a variety

of interpretations is still being pursued, including parallel universes. A review of these is beyond the scope of this introduction.

The entanglement between the spin waves of an antiferromagnet might be exploited in quantum information applications such as quantum computers, however, the entanglement that I study in chapter 7 is on a more fundamental level, concerning two intervals of elementary particles. Remarkably, such elementary systems of tiny particles are theoretically identical to models of gravity in terms of curved space time. As a consequence, results in either theoretical model can be translated to implications for the other. Furthermore, the degree of entanglement can be a good indicator for transitions in the state of a material.

1.5 THEORETICAL RESEARCH

You might have the image that research in physics is always conducted in labs. This is not the case for theoretical research. So what do I do? How do I study natural phenomena by not looking at these phenomena, but by looking at my paper and screen instead?

My efforts stem from the ontology that our universe is mathematically consistent. Throughout the history of science this has been a self-fulfilling prophecy as new mathematics are continually developed explaining newly discovered physical phenomena. In plain English: we assume that everything that happens can be consistently described by formulas. This allows us to make predictions with good certainty. Hence the attempt of science is to define some set of starting assumptions and from that derive what is true or false about the universe.

This naive vision has one major problem: Mathematical logic is incomplete. That is, there is no consistent set of axioms covering all possible statements. Find out yourself by trying to prove Gödel's statement: "This statement cannot be proved". Nonetheless, restricting ourselves to a subset of mathematical theories and of the universe, the approach bears many fruits.

Starting from some physical understanding and intuition of what processes govern the behavior of the system, I translate these ideas to mathematical formulas. Furthermore, I add a notion of physical parameters and observables in a mathematical form. Then the puzzle of expressing one observable into another starts. When the result makes sense and I have not made any mistakes I know I am on the right track. But it gets exciting when the result does not make sense, yet I have not made any mistakes. I want to finish this chapter by giving you some taste of that excitement, discovering Einstein's theory of special relativity.

Einstein started from two simple postulates: In each reference frame all laws of physics are the same; In each reference frame the speed of light is the same. Let's see what this implies, because there seems to be something paradoxical going on. Suppose I am cycling towards you with 10km/h and I throw a

tennis ball at you with a speed that I observe to be 20km/h. Then you would observe the speed of the ball to be 30km/h right? Which is different from my measurement of 20km/h as I am traveling on my bicycle. However, Einsteins second postulate states that if we would do the experiment with light, we should both observe the same speed.

We can investigate what consequences this will have for our perception of time. Suppose on my bike I have a clock that measures time as follows: it emits a light pulse and detects it 29,979cm directly above where it was emitted, see figure 1.3. The time elapsed between emission and detection is one nanosecond. When I am standing still on my bike you and I both agree with the accuracy of my clock. When I start riding my bike I take the clock with me and observe it is working perfectly. The light pulse is still detected right above where it was emitted. But to you it does not look that way: You see the emission of the light pulse, but as my clock is moving relative to you, you see the light pulse has to travel a slightly longer distance as the path of that light is not completely upwards. Because the speed of light has to be the same, you would see my clock running slower. In order to make sense of this Einstein had to abandon the notion that time is absolute. Our perception of time is in fact relative to our frame of reference.

Moreover, from this simple example we can obtain a mathematical expression of how much the time dilates. Say on my bike I measure the time t it takes for the light to travel over the length L to the detector. Then with c the speed of light $L = ct$. In your frame of reference you measure the time t' it takes for the light to travel the slightly longer path $L' = ct'$. Suppose the speed of my bike is v , then using Pythagoras' theorem the distance the light has to travel in your frame of reference is $L' = \sqrt{L^2 + v^2 t'^2}$, see figure 1.3c. Now comes the mathematical consistence: we know $L = ct$ and $L' = ct'$ so by rewriting and rearranging Pythagoras' formula we can express $t = t' \sqrt{1 - v^2/c^2}$. A crazy yet very explicit result. And in fact a very accurate description of reality crucial to our understanding of magnetism as we saw in section 1.1!

Figure 1.3: The light travels a shorter distance in my reference frame (a) than in yours (b). (c) The Pythagorean triangle used to calculate the time dilatation between t and t' .

ADVANCED INTRODUCTION

2.1 MAGNETISM AND SPIN

As Maxwell's equations tell us, magnetism comes from moving charges. The classical picture quickly arises of electrons moving through a conducting wire. But what about magnetic materials such as iron? For macroscopic physics we could treat the magnet as if it is filled with tiny current loops circulating around the direction of the magnetization. But when we zoom in, this picture is no longer adequate. Electrons that are not moving, still have a magnetic moment. At that point physicists usually start talking about spin. Just as the earth spins around its axis and thereby generates a magnetic field, due to the circulating charge in its interior, so do electrons generate a magnetic moment from their spin [3].

In quantum mechanics spin plays a crucial role in our understanding of the fermion. The Dirac equation and the corresponding quantum field theory brings forth a spinorial representation of and a wealth of information about the electron, for instance the quantization of angular momentum in multiples of $\hbar/2$ and an accurate prediction of its gyromagnetic ratio.

Unfortunately, there is a problem with the classical view of a spinning electron: In order to generate the magnetic moment that is measured for an electron, it would have to spin faster than the speed of light. For this reason, the majority of the physics community consider the picture of the physically spinning electron flawed. After both spiritual and scientific contemplation, Pauli was satisfied with spin being a purely quantum mechanical property that has no classical counterpart [4]. Most textbooks and courses in physics convey a similar perspective [5].

But that leaves us with two big questions: If the electron is not spinning, then why does it have angular momentum, and generate a magnetic field? Is there a fundamentally different way of generating magnetic fields other than by moving charge? These questions have a surprisingly simple answer. And what may be most surprising is that it is a relatively unknown answer even though it was established in 1939 by Belinfante [6, 7].

That electrons carry angular momentum and that it is proportional to its magnetic moment is strongly affirmed by the Einstein-de Haas effect (see figure 2.1).

Figure 2.1: In the Einstein-de Haas experiment the initially randomly oriented magnetic moments in a rod are aligned by an external magnetic field. The rod is suspended by a rope and after the alignment of its magnetization, the rod starts rotating. When the direction of the magnetization is inverted, so is the direction of rotation.

In this experiment we take a piece of magnetic material and we suspend it on a rope such that it can freely rotate about its magnetization axis. Then we apply an external magnetic field such that the magnetization reverses its direction. As a consequence, we observe that the magnet starts rotating about its axis with an angular momentum equal to $\hbar$ times the amount of electrons in the material. The effect then is a consequence of conservation of angular momentum.

You might be familiar with other examples where there is no moving mass, yet there is still momentum: Light. A wave packet of light has a quantized energy of h times its oscillation frequency ω . As the wave of light moves through space there is a flow of energy. And this flow of energy carries a proportional momentum p obtained by dividing out the speed of light $c = \lambda\omega$ to obtain $p = h/\lambda$. This momentum is used in solar sails on spacecrafts where light from the sun bounces off a huge reflecting sail, exchanging its momentum and propelling the spacecraft [9].

Figure 2.2: A solar sail. Figure taken from Ref. [8].

It will pay off to investigate the photon a bit more in detail. The flow of energy of a light wave is given by the Poynting vector

$$\mathbf{P} \propto \mathbf{E} \times (\nabla \times \mathbf{A}) \quad (2.1)$$

where $\mathbf{E}$ is the electric field and $\mathbf{A}$ the vector potential of the magnetic field. Circularly polarized light thus has a helical flow of energy, which decomposes into a longitudinal momentum and a spin angular momentum.

Figure 2.3: The energy flow in the Dirac field of a spin up electron. The electron is usually illustrated as a spinning sphere, but the shape of a disk might be more accurate. Colors indicate the flow density.

To understand the angular momentum of an electron, we obtain the Poynting vector from the (symmetrized) energy-momentum tensor of the Dirac field. For a spin up electron with zero net momentum, the flow of energy is illustrated in figure 2.3. Hence, the Dirac field accommodates a circulating energy flow that carries angular momentum.

Moving on to the magnetic moment of the electron we consider the electric current density $\mathbf{j}$ of the Dirac field and use the Gordon decomposition to write

$$\begin{aligned} j_i &\equiv -ec\bar{\psi}\gamma_i\psi \\ &= \frac{e\hbar}{2m} [i\bar{\psi}\partial_i\psi - i(\partial_i\bar{\psi})\psi - \partial_\alpha\bar{\psi}\sigma_i^\alpha\psi], \end{aligned} \quad (2.2)$$

with γ the gamma matrices and σ the Pauli matrices. The first two terms comprise a current density associated with the translational motion of the electron. The last term is a spin current density $\mathbf{j}_s$, which is present even in the rest frame of the electron. By separating the space and time components of the Einstein summation over α , we write

$$\mathbf{j}_s = \nabla \times \mathbf{M} + \partial_t \mathbf{D}. \quad (2.3)$$

Here, $\mathbf{M}$ is a magnetic moment density which arises from the circulating flow of charge as provided by the Dirac field. The dipole moment density $\mathbf{D}$ can be neglected in the nonrelativistic limit.

Hence we arrive at a satisfactory picture of the electron that carries angular momentum not as a spinning solid sphere, but as a wave function with a circular energy flow. The same wave function informs us about a circular current generating the magnetic moment of the electron.

2.2 EXCHANGE INTERACTION

Now that we have a comprehensive picture of the microscopic origin of magnetism in materials, we extend this to a macroscopic perspective of magnetic materials. In a general material the magnetic moments of the electrons are oriented randomly and will cancel out, resulting in a non-magnetic material. So in order to get a magnetic material, we need a mechanism that wants to align the magnetic moments.

Figure 2.4: The ferromagnetic (a) and antiferromagnetic (c) ordering. The different colors indicate the two sublattices of the antiferromagnet. The domain walls (b and d) of the corresponding structures connect two different domains over a finite length λ .

At first instance it seems intuitive that the magnetic moments align by themselves, just as a compass needle aligns itself with the earth magnetic field. This dipole interaction however is not strong enough at the microscopic scale to stabilize the magnetic moments under thermal fluctuations as the Bohr-van Leeuwen theorem stated in 1911 [10–12].

The solution comes from quantum mechanics. The electron is modeled in quantum mechanics as a spin-1/2 fermion. Due to the anti commutation relations of fermionic operators, two fermions cannot occupy the same state as the wavefunction must remain the same under exchange of the two fermions. This is the Pauli exclusion principle and generates an apparent force on the spins called the exchange interaction. The exclusion principle, together with Coulomb repulsion causes the energy of electrons in close proximity to each other to be significantly affected by their relative state (with regards to relative spatial coordinates and spin orientation).

This often makes it energetically favorable for nearby electrons to have spins in opposite directions. Such materials are called anti-ferromagnets. Simple calculations of the exchange energy always seem to favor the opposite alignment of spin. To explain ferromagnetism, where all spins align, it is argued that the microscopic details of such materials favor the neighboring electrons to separate farther apart with parallel spin, allowing them to move closer to the nucleus. The conduction electrons then are an intermediary between the bound electrons inside the shells of the two neighboring lattice sites. These neighboring bound electrons generate the majority of the magnetic moment and both want to be opposite to the conduction electron, and thereby parallel to each other.

When physicists get asked if we have any proof that quantum mechanics is real, they often refer to the double slit experiment or the Stern-Gerlach experiment. But the existence of magnetic materials itself is a purely quantum mechanical phenomenon! Most people don't have a double slit or Stern-Gerlach setup lying in their fridge, but they probably do have a magnet stuck to the fridge door.

In my research I focused on theory where the exchange interaction is modeled phenomenologically:

$$\mathcal{H}_{\text{ex}} = -4 \frac{J_s}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2. \quad (2.4)$$

The phenomenological exchange energy J_s prefers an antiferromagnetic alignment of the neighboring spins $\mathbf{S}_1$ and $\mathbf{S}_2$ when it is negative and a ferromagnetic alignment when it is positive. These two types of alignment are illustrated in figure 2.4a (ferromagnetic) and 2.4c (antiferromagnetic). Assuming a classical approach, let us denote the magnetization in the ferromagnet by the unit vector in spherical coordinates,

$$\mathbf{n} = \frac{2}{\hbar} \mathbf{S} = \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}, \quad (2.5)$$

where the spin is now treated classically, determined by ϕ the azimuthal angle, and θ the polar angle. A misalignment of neighbors thus gives an energy contribution

$$-J_s \mathbf{n}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x} + \mathbf{a}) = -J_s \cos(\delta\zeta), \quad (2.6)$$

where $\mathbf{a}$ is the vector connecting two neighbors and $\delta\zeta$ is the angle between the two magnetizations. In the continuum limit we express this angle in terms of the gradients of θ and ϕ by $\delta\zeta \approx |a| \sqrt{(\nabla\theta)^2 + \sin^2(\theta)(\nabla\phi)^2}$. Now we sum over all nearest neighbor sites, which in the continuum limit becomes an integral. By expanding the cosine to second order and defining the continuum exchange interaction energy density $J = J_s a^{-1}$ with a the lattice spacing we obtain

$$H_{\text{ex}} = \frac{J}{2} \int d^3\mathbf{x} \left((\nabla\theta)^2 + \sin^2(\theta)(\nabla\phi)^2 \right). \quad (2.7)$$

The nature of the exchange interaction is clear: there is an energy cost (or gain, depending on the sign of J) for gradients in the magnetic texture.

2.3 DOMAIN WALLS

Consider a system with a Lagrangian that can be decomposed into a kinetic part and a potential part. When the potential has n discrete degenerate minima, then the system allows a set of $2(n-1)$ solutions called solitary waves [13]. The solitary wave solution connects one minimum to another in its bulk, at the boundaries the structure of one of the minima is attained. Solutions are often called kinks because their energy density is localized in a small region of space called the domain wall (DW). Indeed, this kink can be seen as a wall of

finite thickness that separates two domains which host the minimum solutions. Over this wall thickness the majority of change happens from one minimum to another and the reminiscence of this change exponentially die off away from the DW position.

Polarization DWs have been observed in a system where a one dimensional light wave is given two different boundary conditions with respect to its polarization [14]. Of course, in this thesis magnetic DWs capture our interest. The ground state of the ferromagnet has all magnetic moments aligned and is degenerate with respect to the global orientation of the magnetization. To make this degeneracy discrete, we include an easy axis anisotropy:

$$\mathcal{H}_{\text{ani}} = -4 \frac{K}{\hbar^2} (\mathbf{S} \cdot \hat{\mathbf{e}})^2, \quad (2.8)$$

with K the anisotropic energy bias and $\hat{\mathbf{e}}$ a unitvector along the easy axis. Then the degeneracy becomes twofold: all spins point along $\hat{\mathbf{e}}$ or $-\hat{\mathbf{e}}$. Choosing $\hat{\mathbf{e}} = \hat{\mathbf{z}}$ we obtain the continuum limit

$$H_{\text{ani}} = -K_z \int d^3\mathbf{r} \cos^2 \theta. \quad (2.9)$$

The DW solution is explicitly derived by the Euler-Lagrange formalism

$$\frac{\partial \mathcal{H}}{\partial \theta} = \nabla \frac{\partial \mathcal{H}}{\partial (\nabla \theta)}, \text{ so} \quad (2.10)$$

$$-\frac{K_z}{2} \frac{\partial \cos^2(\theta)}{\partial \theta} = J \nabla^2 \theta. \quad (2.11)$$

We assume homogeneity in the y and z directions so $\nabla \theta = \partial_x \theta$. We set the boundary conditions for $x \rightarrow \pm\infty$ to $\frac{d\theta}{dx} \rightarrow 0$ and $\cos(\theta(x)) \rightarrow \pm 1$. Then by separation of variables we solve equation (2.11) to find

$$\theta(x) = 2 \arctan \left(e^{\frac{x-x_{\text{DW}}}{\lambda}} \right), \quad (2.12)$$

with $\lambda = \sqrt{J/K_z}$ the DW thickness and x_{DW} the DW position, which at this point is a degree of freedom of the soliton, just as the azimuthal angle φ . These are called the collective coordinates that model the dynamics of the kink by enriching them with a time dependence. Indeed, as $x_{\text{DW}} \rightarrow x_0 - vt$ the kink moves longitudinally with a velocity v , leaving its texture intact. Similarly, changes in φ precess the texture. The DW position can also be controlled for instance by external fields or spin currents and the azimuthal angle, i.e. the chirality of the DW, can be fixed for example by an additional anisotropy or the Dzyaloshinskii-Moriya interaction (DMI) [15, 16].

A surface DMI arises when a magnetic material has a contact surface with a material that has a strong spin-orbit coupling: A free electron exchanges

momentum with a lattice electron in the magnetic material. When it then moves over to a neighboring lattice site, its angular momentum rotates as a consequence of the spin-orbit interaction. So when it has reached the neighbor, it exchanges a differently oriented momentum, depending on the direction of its orbit. Thus, the DMI favors a gradient in $\mathbf{n}$ and thereby a certain chirality of the DW.

An illustration of a ferromagnetic DW is given in figure 2.4b. The thickness of the DW λ is finite due to a competition between the exchange energy, which wants the transition to happen slowly, and the anisotropy that favors an immediate transition. In this derivation we considered a ferromagnet, but antiferromagnets also house domain walls (see figure 2.4d), which can be driven much faster because they do not have a demagnetization field.

2.4 CREEP THEORY

In a physical experiment with DWs, the ferromagnet is filled with pinning centers for the DW because real materials contain many impurities. In order to drive the DW, one would have to apply a force large enough to overcome the pinning strength. However, in the presence of fluctuations a sub critical driving force still results in a net motion as the DW hops from one pinning center to the next. Such motion is called creep and the theory of creep motion is applicable to a broad range of systems, for instance the motion of magnetic vortex lines in a type-II superconductor [17].

Creep theory attempts to describe some manifold in a pinning potential, whose barrier height is larger than the driving force, but still exhibits a net macroscopic motion of due to fluctuations. These fluctuations can be either of a thermal or quantum nature. For our purposes we consider thermal fluctuations. The rate at which these fluctuations give rise to net motion of the manifold is estimated from the Arrhenius law [18, 19]; a deformation due to a fluctuation costs elasticity energy, but yields energy from the driving force. These costs and yield scale differently with the size of the fluctuation so that there is a optimal size L_{opt} above which the deformation can grow. At this optimal deformation length the energy cost is maximal and equal to the energy barrier the nucleation problem has to overcome.

Figure 2.5: Illustration of a one dimensional manifold hopping trough a pinning potential V_{pin} . Hopping bends and stretches the DW, but also increases the size of one domain. Figure taken from Ref. [17].

Hence the recipe is clear: Determine the elasticity energy cost and the driving energy gain as a function of the deformation length L . Subtract the two and optimize for L . This will return an effective hopping distance and a corresponding energy barrier, from which the hopping rate is determined. The velocity of the manifold is then immediate. In 1998 Lemerle implemented creep theory to successfully model the DW velocity as a function of the external driving field [20].

The elastic energy cost comes from the exchange interaction introduced earlier (see section 2.2): It favors to align neighboring spins, so a DW texture on itself has an associated energy, which increases linearly with its length (i.e. the transverse length perpendicular to the DW thickness λ). Thus given an exchange energy per DW length $\mathcal{E}$, bending a DW segment of length L over a distance u gives an energy cost of

$$F_{\text{ex}} = \mathcal{E}\delta L \approx 2\mathcal{E}\frac{u^2}{L}, \quad (2.13)$$

assuming the deformation to be triangular shaped. Hence, the exchange energy prefers to keep the DW straight in order to minimize its length. Therefore, $\mathcal{E}$ is also referred to as the exchange stiffness.

The driving energy gain is a Zeeman term which comes from the increase in area of the domain that is aligned with the external magnetic field H :

$$F_{\text{dr}} = \gamma H u L. \quad (2.14)$$

The average hopping distance is a function of L that is determined by the structure and strength of the pinning potential and the elastic energy. The power law depends on the dimensionality of the system, and for one dimensional manifolds is given by $u \propto L^{2/3}\mathcal{E}^{-4/9}$ [21, 22]. Optimizing over L gives the energy barrier $F_b \propto (\mathcal{E}/H)^{1/4}$, which is proportional to $-\ln(v)$ as a result from the Arrhenius law. Hence, we obtain

$$v \propto e^{-(\mathcal{E}/H)^{1/4}}. \quad (2.15)$$

This result is well established, but is inadequate in modeling more complicated DW dynamics. To that end, the creep theory has to be enriched by considering the additional energy cost of bending the DW (which misaligns spins). Furthermore, in the presence of external fields that associate an energy cost to the orientation of the DW, the elasticity is dispersive as it becomes a function of local DW parameters [23].

Investigations into such comprehensive models are key to the development of devices that use dynamical DWs. The dynamics of a DW are an easily accessible observable using the magneto optic Kerr effect, and thereby provide insight into micromagnetic material properties such as the strength of the Dzyaloshinskii-Moriya interaction.

Figure 2.6: Spin waves (b) carry an angular momentum and propagate through a magnetic material by the precession of spins, i.e. without translational movement. In a spin-flip scattering process (a) an itinerant electron in a normal metal (yellow) flips its spin and thereby transfers angular momentum into a ferromagnetic insulator (pink) carried by a magnon.

2.5 MAGNONS

With the controlled movement of magnetic domain walls the promise of novel information storage and transportation methods lie within reach. Before delving into those, we will tease you with another promising magnetic mechanism: spin waves. Suppose I perturb a spin of a ferromagnet, then due to the exchange interaction its neighbors will also tilt and so on, spreading the perturbation like a wave. Such a spin wave is illustrated in figure 2.6b.

As the amount of spin is quantized, my perturbation decreases the up spin in discrete steps of angular momentum $\hbar$. This angular momentum is no longer carried intrinsically by the spins, but by the dynamic precession of the generated spin wave. We regard these spin waves as quasi particles called magnons, which obey bosonic commutation relations.

To benefit the diversity of this introduction, let us consider spin waves on an antiferromagnet. We consider a one-dimensional uniaxial easy-axis antiferromagnet with sublattice A (B) pointing along the $(-)$ z axis described by the Hamiltonian

$$H = \frac{J}{\hbar^2} \sum_{\langle i,j \rangle} \mathbf{S}_A(\mathbf{r}_i) \cdot \mathbf{S}_B(\mathbf{r}_j) - \frac{K}{\hbar^2} \left(\sum_i S_A^{(z)}(\mathbf{r}_i)^2 + \sum_j S_B^{(z)}(\mathbf{r}_j)^2 \right). \quad (2.16)$$

The Holstein-Primakoff transformation rewrites our spin operators in terms of magnon operators

$$S_A^{(z)}(\mathbf{r}_i) = \hbar(S - a_i^\dagger a_i) \quad S_B^{(z)}(\mathbf{r}_j) = \hbar(-S + b_j^\dagger b_j) \quad (2.17)$$

$$S_A^+(\mathbf{r}_i) = \hbar\sqrt{2S - a_i^\dagger a_i} a_i \quad S_B^+(\mathbf{r}_j) = \hbar b_j^\dagger \sqrt{2S - b_j^\dagger b_j} \quad (2.18)$$

$$S_A^-(\mathbf{r}_i) = \hbar a_i^\dagger \sqrt{2S - a_i^\dagger a_i} \quad S_B^-(\mathbf{r}_j) = \hbar\sqrt{2S - b_j^\dagger b_j} b_j \quad (2.19)$$

where $\mathbf{r}_j = \mathbf{r}_i + \mathbf{a}$ and the operators a_i (b_j) are local annihilation operators of a spin flip on the i -th (j -th) site of the A (B) sublattice and satisfy the canonical bosonic commutation relations. Their Fourier transformed counterparts

$$a_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_i a_i e^{i\mathbf{k} \cdot \mathbf{r}_i} \quad b_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_j b_j e^{i\mathbf{k} \cdot \mathbf{r}_j} \quad (2.20)$$

also satisfy the canonical commutation relations and reduce the Hamiltonian up to second order in these ladder operators to

$$H = \sum_{\mathbf{k}} A \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) + C_{\mathbf{k}} \left(a_{\mathbf{k}} b_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger \right); \quad (2.21)$$

$$A = 2S(J + K); \quad C_{\mathbf{k}} = 2JS \cos k. \quad (2.22)$$

As we can see from the second term in the Hamiltonian, the presence of magnons can reduce the energy of the system. So the ground state will contain a combination of a and b magnons. This combination is made explicit by applying a Bogoliubov transformation

$$\begin{aligned} \alpha_{\mathbf{k}} &= u_{\mathbf{k}} a_{\mathbf{k}} + v_{\mathbf{k}} b_{-\mathbf{k}}^\dagger, & \beta_{\mathbf{k}} &= u_{\mathbf{k}} b_{\mathbf{k}} + v_{\mathbf{k}} a_{-\mathbf{k}}^\dagger, \\ u_{\mathbf{k}} &= \sqrt{\frac{A_{\mathbf{k}} + E_{\mathbf{k}}}{2E_{\mathbf{k}}}}, & v_{\mathbf{k}} &= \sqrt{\frac{A_{\mathbf{k}} - E_{\mathbf{k}}}{2E_{\mathbf{k}}}}. \end{aligned} \quad (2.23)$$

This indeed diagonalizes the Hamiltonian

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}} \left(\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^\dagger \beta_{\mathbf{k}} \right), \quad (2.24)$$

where $E_{\mathbf{k}} = \sqrt{A^2 - C_{\mathbf{k}}^2}$.

In a ferromagnet, magnons are excitations and they will dissipate due to damping. There are several mechanisms to create magnons. One way is to create a local oscillating external magnetic field. This is not very effective for antiferromagnets, as they are insensitive to external magnetic fields. The mechanism used in many experiments [24] and in the theoretical models in this thesis use the process of spin-flip scattering, see figure 2.6a. The magnetic material is brought into contact with a normal metal lead that has a bias in its spin population. Due to the exchange interaction, the free electrons in the normal metal can exchange angular momentum with the spins in the magnetic material. Since the free electrons have spin $\hbar/2$, such a process would cause them to completely flip their spin. Because of the bias in the spin population, there is a net transfer of angular momentum into the magnetic material. When the magnetization is aligned with the majority spin population, this angular momentum will be injected in the form of spin waves.

Figure 2.7: (a) Sound waves get localized near the boundaries of the rotunda in the Saint Paul's cathedral. For the source and an animation, see [25]. The same feature emerges for spinwaves on DWs (b): The potential (blue) has a minimum at the DW position. This allows for a spin wave solution that is localized around that minimum (yellow).

Propagating waves can become localized at surfaces due to a local dip in the potential. Due to this dip, these bound waves have a lower energy than the waves propagating through the bulk. A famous example of this phenomenon are the whispering gallery modes, first observed in 1910 in the rotunda of the St. Paul's cathedral in London [26] (see figure 2.7a).

Spin waves also can get localized on the bounds of magnetic domains, i.e. on the DW. To demonstrate this, we employ the Landau Lifschitz Gilbert formalism, where the dynamics of the magnetization texture are captured in terms of an effective magnetic field and torques. For our current purposes, it reduces to:

$$(1 + \alpha \mathbf{n} \times) \partial_t \mathbf{n} = \gamma \mathbf{n} \times \mathbf{H}_{eff}. \quad (2.25)$$

Here γ is the gyromagnetic ratio and α is the damping of the precession motion generated by the effective magnetic field which is obtained from the functional derivative of the energy $M_S \mathbf{H}_{eff} = -\delta \mathcal{H} / \delta \mathbf{n}$, with M_S the saturation magnetization. We consider a thin (in the z direction) film ferromagnet with a DW at $x = x_{DW}$ stretching along the y direction.

To model the spin waves, we define local spherical coordinates where the radial unit vector $\hat{\mathbf{r}}$ points along $\mathbf{n}_{DW}$. Then fluctuations are, in first order, oriented along $\hat{\theta}$ and $\hat{\phi}$. Note that these unit vectors are a function of space,

i.e. they rotate when moving along the y -axis, and are given by the following expressions:

$$\hat{\mathbf{r}} = \mathbf{n}_{DW}; \quad (2.26)$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos \theta \cos \varphi + \hat{\mathbf{y}} \cos \theta \sin \varphi - \hat{\mathbf{z}} \sin \theta; \quad (2.27)$$

$$\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}} \cos \varphi - \hat{\mathbf{x}} \sin \varphi. \quad (2.28)$$

Then the perturbed magnetization orientation is given by

$$\mathbf{n}(x, y, t) = \mathbf{n}_{DW}(y) + \delta \mathbf{n}(x, y, t), \quad (2.29)$$

with

$$\delta \mathbf{n} = \hat{\boldsymbol{\theta}} \delta n_\theta + \hat{\boldsymbol{\phi}} \delta n_\varphi, \quad (2.30)$$

where $|\delta n_\theta| \ll 1$ and $|\delta n_\varphi| \ll 1$ are the fluctuation amplitudes. We define the magnon wave function $\psi \equiv \delta n_\theta + i \delta n_\varphi$. Next, we insert this in the Landau Lifschitz Gilbert equation (2.25) and linearize in the fluctuations to obtain a Schroedinger-like equation for ψ :

$$\frac{M_S}{\gamma K_z} \partial_t \psi = -\lambda^2 \nabla^2 + \cos 2\theta. \quad (2.31)$$

The potential part $\cos 2\theta$ has a local minimum at the DW position as illustrated in figure 2.7b. This equation admits two type of solutions:

$$\psi_b = \sin \theta e^{i(q_x x)}, \quad \psi_p = (-i\lambda q_y + \cos \theta) e^{i(q_x x + q_y y)} \quad (2.32)$$

The last one is a spinwave propagating through the bulk, which picks up a phase difference of p when moving past the DW. The first one is a spinwave bound to the DW since $\sin \theta$ is strongly localized around x_{DW} (see figure 2.7b). Whereas the bulk modes are gapped, the bound magnons are gapless, so by control of the frequency the type of magnons in the system is controlled.

2.6 SPINTRONICS

My research aims to advance the field of spintronics where the exchange interaction and other micro magnetic mechanisms are employed to create devices with great benefits over their electronic counterparts (for a recent review, see [27]). In spintronic devices, instead of the charge of the electron, its spin is what will store, transfer and configure information.

The field of spintronics emerged with the discovery of giant magneto resistance, which arises due to the spin of the moving electrons. We consider the current being composed of two components: an electric current and a spin

Figure 2.9: (a) The racetrack memory lets magnetic domains move past writing and reading heads to store and access information [1]. (b) A magnonic majority gate [31].

current. The resistance of the spin current depends on the alignment between the spin and the magnet. As a consequence, the resistance experienced by a current moving through two ferromagnets depends strongly on the alignment of the magnetization of these two magnets [28, 29]. The mechanism is used for example to sense magnetic fields in hard disk drives to read data, in biosensors, and in micro-electro-mechanical devices [30].

Since then, many more spintronic devices were thought up. Perhaps the most famous is the racetrack memory by Parkin [1], illustrated in figure 2.9a. A ferromagnetic track accommodates a number of domains with a homogeneous magnetization that represent bits corresponding to the two ground states. These domains can move around the track, by means of a spin current or an external magnetic field, to pass by a reading and writing point. Such a device has no mechanically moving parts which is a great advantage over magnetic hard disk memory devices.

Magnonic devices go even further by eliminating the electron current. In such devices information is carried by spin waves [32–37], leaving electrons in their place. This has the benefit of a low power usage and thereby a low generation

Figure 2.8: The torques from the Landau Lifshitz Gilbert equation.

of heat. Figure 2.9b depicts a magnonic majority gate.

Spintronics does come with some extra challenges because spin is generally not conserved, whereas charge is. This is due to the fact that electrons can exchange their angular momentum with the lattice. These interactions with phonons cause damping of spin waves which for example limits the distance of non-local magnon transport.

2.7 ENTANGLEMENT

As we have seen, quantum mechanics play a crucial role in our understanding of magnetism. Beyond that, the quantum nature can also be employed for computational and information processing purposes. This field of research is called quantum information and has a branch that is focused on entanglement.

In 1932 Von Neumann established non relativistic quantum mechanics as a mathematical theory [38]. Shortly after that, in 1935, Einstein, Podolsky and Rosen uncovered a troublesome paradox about quantum entanglement [39]. The idea of quantum entanglement is that the state of a quantum system is not necessarily a direct product of the states in the individual subsystems [39]. *“The best possible knowledge of a whole does not include the best possible knowledge of its parts”*, as Schroedinger put it [40]. I refer the reader to section 1.4 for an illustrative thought experiment to clarify the concept of entanglement.

Apart from theoretical and philosophical interest, the study of entanglement is promising for applications in cryptography [41, 42], quantum teleportation [43], dense coding [44, 45], and quantum computation [46]. Furthermore, macroscopic properties of solids may be interpreted by quantum entanglement [47] and the entanglement can indicate quantum phase transitions [48–50].

The degree of entanglement is given by the von Neumann entropy [38, 51]: A system consisting of subsystem A and its complement, has a product Hilbert space $H = H_A \otimes H_{A^c}$. A general state $|\psi\rangle$ of the system however cannot be written as a direct product of states in subsystems. The reduced density matrix of subsystem A is defined from the total density matrix $\rho = |\psi\rangle\langle\psi|$, by tracing out the the eigenstates $|\varphi_i\rangle$ in H_{A^c} :

$$\rho_A = \text{Tr}_{A^c}(\rho) = \sum_i \langle\varphi_i|\rho|\varphi_i\rangle. \quad (2.33)$$

Given the reduced density matrix ρ_A , the entanglement entropy is defined by the formula

$$S_A = -\text{Tr}[\rho_A \log \rho_A], \quad (2.34)$$

where the trace is over the remainder of the Hilbert space H_A .

The area law is a well established result on the entropy of entanglement, stating that the entanglement entropy scales with the area between the considered

subsystems in a theory with local interactions [52]. The area law seen to apply in conformal field theory as well [53]. This invites the use of the AdS/CFT correspondence which relates the entanglement entropy to black hole physics. Particularly, Einsteins equation on gravity [54, 55] follow from the area law in this correspondence.

Of special interest also is a class of conformal field theories which respect the so called Lifshitz scaling symmetry [56, 57]. Such fields obey a scaling symmetry that the system is unchanged when we transform space and time according to

$$x \rightarrow \lambda x, \quad t \rightarrow \lambda^z t. \quad (2.35)$$

Here λ is the scaling value, and z is the Lifshitz scaling exponent. For fermions the Dirac Lagrangian is adjusted by replacing taking all spatial derivatives to the z -th power (and preserving hermiticity). The scaling of the fermion wave function is then $\psi \rightarrow \lambda^{-1/2}\psi$, independent of z . Such theories introduce an anisotropy between space and time, which provides an alternative to the ultraviolet completion of Einsteins theory of gravity [58]. Furthermore, Lifshitz symmetry arises at quantum critical points of several condensed matter systems modeled by a Lifshitz scalar field [59–61]. Our interest restricts itself to the entanglement of fermionic systems with this scaling anisotropy.

2.8 THIS THESIS

The introduction in this chapter serves as a foundation upon which to build the theory from chapters 3 to 7. In chapter 3 a detailed model of the creep motion of DWs is established that accounts for the effect of its local chirality on the stiffness of the DW manifold. We find the model to agree well with our experiments, explaining previously enigmatic features of DW creep dynamics.

Shifting our focus to the setup where angular momentum is injected from a spin biased metallic lead into a ferromagnet, in chapter 4 we unearth a static magnetic texture, resembling that of a virtual DW. We find the magnetic DW texture to be rich enough to statically accommodate the boundary condition from the spin bias, but in doing so, it demonstrates threshold behavior as a function of that bias. This leads to the proposal of two types of spin-bias switched gate devices.

In chapter 5 the DW is employed as a waveguide to accomodate non-local magnon transport. To facilitate technological development we derive a theoretical model that captures the bound magnons in the collective coordinates of the DW. One great advantage of our model is that it does not require a $U(1)$ symmetry, only a twofold degenerate ground state.

To bridge between magnetism and entanglement, in chapter 6 we study the magnon entanglement between the two sublattices of an antiferromagnet. We find an extensive entanglement entropy that is universal with respect to

details of the material and unaffected by weak external magnetic fields. Our analytical results agree well with numerical calculations and may facilitate the development of quantum information protocols.

In chapter 7 we study the entanglement in fermion chains with the Lifshitz scaling symmetry. Both a direct analytical approach as well as holographic methods confirm the remarkable feature that for systems with only next-to-nearest neighbor interactions there is no entanglement. The thermal state does yield a finite entanglement entropy and for high temperatures, the range of interaction no longer plays a distinguishing role.

CREEP OF CHIRAL DOMAIN WALLS

Recent experimental studies of magnetic domain expansion under easy-axis drive fields in materials with a perpendicular magnetic anisotropy have shown that the domain wall velocity is asymmetric as a function of an external in plane magnetic field. This is understood as a consequence of the inversion asymmetry of the system, yielding a finite chiral Dzyaloshinskii-Moriya interaction. Numerous attempts have been made to explain these observations using creep theory, but, in doing so, these have not included all contributions to the domain wall energy or have introduced additional free parameters. In this article we present a theory for creep motion of chiral domain walls in the creep regime that includes the most important contributions to the domain-wall energy and does not introduce new free parameters beyond the usual parameters that are included in the micromagnetic energy. Furthermore, we present experimental measurements of domain wall velocities as a function of in-plane field that are well described by our model, and from which material properties such as the strength of the Dzyaloshinskii-Moriya interaction and the demagnetization field are extracted.

3.1 INTRODUCTION

The interest in nanomagnetic materials has grown steadily since magnetic storage devices, such as the racetrack memory, were proposed as a new tool to meet the ever increasing demand for computer storage capacity [1, 62–64]. For such applications the DW chirality is an important parameter as it affects the speed and direction of DW motion. The interfacial Dzyaloshinskii-Moriya-interaction (DMI) [15, 16] arises from perpendicular inversion asymmetry in the system and affects the DW chirality. Hence it is of paramount importance to be able to measure the magnitude of the DMI using a simple experimental method. The interfacial DMI is modeled as an effective field that lies in-plane (IP) and is always perpendicular to the DW normal, hence preferring a Néel wall [65]. Superpositioning the DMI field with an externally applied IP magnetic field could provide means of measuring it. This has led to a boom of experimental studies on DW dynamics under the influence of an IP magnetic field [23, 66–74].

There are several regimes of DW dynamics, determined by the strength of the DW driving force compared to the pinning force. In the flow regime the driving force is significantly higher than the pinning force and in this regime IP magnetic fields and DMI is successfully modeled by means of the Landau-Lifschitz-Gilbert equation [75–77]. In the creep-regime however, the DW is considered to be mostly pinned and in local equilibrium and has a net displacement because the bias is assisted by thermal fluctuations.

The creep model was successfully implemented to interpret magnetic domain growth driven by an external magnetic field H_z in the direction of the magnetization of one of the domains, resulting in the famous universal creep law for the DW velocity v : $\ln(v) \propto H_z^{-1/4}$ [20]. When introducing a magnetic field perpendicular to the magnetization direction of the domains, a modification to this creep law was proposed: $\ln(v) \propto (E_{el}/H_z)^{1/4}$, where E_{el} is the elasticity of the DW [66]. This modification turned out to describe experimental findings well for small IP magnetic fields, but is not able to describe the high-field region [68]. Recent attempts to improve the theoretical model exposed the dispersive nature of the elasticity but compromised on universality as extra free parameters were introduced. [23] Chiral damping was proposed to explain the asymmetric component of the velocity profiles [69, 72, 73, 78]. We contend however that in the quasi-static creep regime dynamic effects such as chiral damping should not play a significant role.

In this chapter we construct a theory for motion of chiral DWs in the creep regime which does not involve the free parameters introduced in Ref. [23]. We use it to interpret our experimental data on the DW velocity as a function of the IP magnetic field. We show that our model allows for quantitative determination of the interfacial DMI strength from field-driven DW creep measurements.

3.2 MODEL

In figure 3.1a the deformation of a DW due to a thermal fluctuation in the presence of an easy axis driving field H_z is illustrated. The deformation size L determines the balance between the gained Zeeman energy from the driving field and the elastic energy cost. The deformations can be seen as nucleations whose chance of survival is determined by L . For such a nucleation process, Arrhenius' law tells us that the rate at which these surviving deformations will occur is determined by the height of the energy barrier F_b (i.e. the free energy at the tipping point): $\ln(v) \propto -F_b/(k_B T)$ [18, 19]. Jeudy et al. have shown that defining $F_b = T_d \left[(H_d/H_z)^{1/4} - 1 \right]$, in terms of the depinning field H_d and an effective disorder temperature T_d can describe the DW motion accurately in both the creep and depinning regime [79, 80]. A recent study also used this form to capture the in-plane magnetic field effects into the depinning field [81]. Instead of postulating a form of F_b we will determine it from micromagnetics

Figure 3.1: (a): Top view of a DW (blue lines) that gets deformed over a length L and displaced over a distance u due to a thermal fluctuation. The DW can be tilted over an angle α . The magnetization is indicated by the red vectors, which at the DW are characterized by the IP angle φ . Note that the IP magnetization changes due to the displacement, affecting the elasticity. The IP magnetic field H_x (green) as well as the effective DMI field H_D (yellow) and effective Bloch field H_B (purple) are indicated locally. (b): Model to describe the deformation. (c): When an IP magnetic field is applied to a sample with PMA, the magnetization inside a domain tilts towards the IP magnetic field by an angle θ_t determined by the balance of PMA and IP magnetic field $\beta = M_S H_x \cos(\varphi) / K_P = \sin(\theta_t)$ (orange) compared to the $\beta = 0$ case (red).

well inside the creep regime; $F_b = \max_L F(L)$. We capture the complexity of the asymmetric DW dynamics in $F(L)$. As a consequence the optimization required to determine F_b is semi-analytical. Here, we introduce an insightful numerical procedure as opposed to a full analytical treatment as has been done extensively in literature [20, 23, 66, 79]. This numerical approach allows to address a plethora of effects in the underlying physics of DW dynamics in the creep regime.

$F(L)$ is composed of the elastic energy cost and the Zeeman energy gain, which depend not only on L , but also on the DW displacement u : $F(u, L) = E_{\text{el}}(u, L) + E_{\text{Zeeman}}(u, L)$. To express u in terms of L we use $u(L) = u_c(L/L_c)^{2/3}$ [20–22], where L_c is the Larkin length scale determined by minimizing the sum

of the elastic and pinning energy density for $u = \xi$, and u_c is a proportionality constant. Hence, the next step is to determine the elastic energy to be able to compute L_c and express u , and thereby F , in terms of L .

The elastic energy is defined as the difference in internal, i.e. excluding pinning and driving, energy between the domain wall before and after the deformation. Due to the application of the external IP magnetic field the DW energy density itself depends on the orientation of the DW with respect to this applied field. Furthermore, the IP magnetization of the sample at the DW is affected by the exchange interaction.

Following Blatter et al. we model the deformation as an angular shape for simplicity, see figure 3.1b. [17] Other shapes are possible, but this is the lowest order approximation. Note that Pellegren et al. chose an arc shape [23], but did not implement the exchange energy cost due to the kink in the connection with the straight DW segments, resulting in unphysical divergences (as demonstrated in appendix 3.B) that do not occur in our theory.

We have approximated the IP magnetization of each segment to be constant and implement a nearest neighbor exchange interaction at the bending points. The energy of the system is then minimized (numerically) over the IP magnetization angle of the two segments.

We compute the energy density of the domain wall by inserting the domain-wall solution into the micromagnetic energy functional. For more details see appendices 3.A and 3.B.

$$\mathcal{E}(\alpha, \varphi) = 2\sqrt{1 - \beta^2} \frac{J}{\lambda} + M_S \pi \lambda \left(g(\beta) H_B \cos^2(\varphi - \alpha) - f(\beta) (H_x \cos(\varphi) + H_D \cos(\varphi - \alpha)) \right). \quad (3.1)$$

The first term is the exchange interaction J over λ , the DW thickness. The second term is the demagnetization energy, expressed in terms of the effective Bloch field H_B (this energy favors a Bloch DW, hence the nomenclature), the angle α between the DW normal and the x -axis and the angle φ the IP magnetization at the DW with the direction of the IP magnetic field, see figure 3.1a. The third term is the Zeeman energy due to the applied IP magnetic field H_x and the fourth is the DMI expressed in terms of an effective field H_D favoring a Néel type DW. The prefactors involving β incorporate the tilting in the x -direction of the magnetic domains due to the external IP magnetic field (see figure 3.1c). The functions f and g are given in appendix 3.A.

Similarly, we obtain the Zeeman energy from the driving field H_z

$$E_{\text{Zeeman}}(u, L) = M_S H_z t u L \sqrt{1 - \beta^2}. \quad (3.2)$$

Again, the factor $\sqrt{1 - \beta^2}$ comes from the tilted domains as illustrated in figure 3.1c. By dividing out $D(= M_S \lambda H_D)$ in equations (3.1) and (3.2), the relevant

Figure 3.2: (a) α_0 as a function of the applied IP magnetic field (i) and the corresponding minimized azimuthal angle of the internal magnetization ϕ_0 (ii). The green curve shows the solution for ϕ_0 when α is fixed at 8° , which switches sign at $H_x = 0$. The corresponding energy density however, remains continuous and smooth. Note that the green curve does not saturate in but converges to the Néel wall. (b) Dependence of the IP magnetic field $\tilde{H}_x$ of the DW velocity (iii) and the assymetric component of the velocity A for $\tilde{H}_B = 0.5$ (iv). The profiles in (iii) are given a vertical offset for clarity. The dashed lines represent the result for fixing $\alpha = 8^\circ$. For this calculation $H_z = 10\text{mT}$.

dimensionless parameters become $\tilde{J} \equiv J\lambda^{-1}D^{-1}$, $\tilde{H}_B \equiv 2H_B/H_D$, $\tilde{H}_x \equiv H_x/H_D$ and $\tilde{H}_z \equiv H_z/H_D$.

Using equation (3.1), we compute the optimal orientation angle α_0 of the undeformed DW and the corresponding internal magnetization IP angle ϕ_0 by minimizing $\mathcal{E}(\alpha, \varphi)/\cos(\alpha)$. The factor $1/\cos(\alpha)$ arises because we allow the DW to orient itself with respect to the IP magnetic field at the cost of elongating. For example, a mixed Bloch-Néel DW tilts its normal to better align with the external IP magnetic field. This tilting however would induce a stretching factor of $1/\cos\alpha$, increasing the energy cost. This effect is illustrated in figure 3.1a and the optimal angle α_0 and corresponding minimized angle ϕ_0 are shown as a function of H_x in figure 3.2a. The energy of the unperturbed DW is then given by $Lt\mathcal{E}(\alpha_0, \phi_0)$.

The profile of ϕ_0 shown in figure 3.2ab exhibits sharp kinks for both $\alpha(H_x) = \alpha_0(H_x)$ and $\alpha(H_x) = 0$. This feature arises because in the energy density of equation (3.1) we neglected higher order anisotropy terms proportional to $\cos^n(\varphi - \alpha)$ for $n > 2$ which are allowed by symmetry. As a consequence, this simplified energy density yields a sharp transition in DW type, from mixed Bloch-Néel to pure Néel at $f(\beta)(\tilde{H}_x - 1) = g(\beta)\tilde{H}_B$ as demonstrated in figure 3.2a where ϕ_0 saturates to 0 or π . To effectively include for the higher

order terms in the energy density, we adjust the value of α to some non-zero value, e.g. $\alpha = 8^\circ$ as done by Pellegren et al. [23]. This removes the symmetry between the two deformed segments and prevents the saturation of φ . With this modification, φ is smooth around $\varphi = 0$ or $\varphi = \pi$ as demonstrated by the green curve in figure 3.2ab. We will consider $D > 0$ here, the results for $D < 0$ are obtained by $\tilde{H}_x \rightarrow -\tilde{H}_x$.

A kink between two DW segments, as illustrated in figure 3.1b, gives an energy cost

$$\mathcal{E}_{\text{ben}}(\varphi_1, \varphi_2) = \frac{J\lambda}{a}(1 - \cos(\varphi_1 - \varphi_2)), \quad (3.3)$$

with φ_1 and φ_2 the IP angles of the internal magnetization of the segments. Here, a is the distance between neighboring atoms in the magnetic layer. Due to variations in the lattice structure and to account for non-nearest neighbor interactions, an effective value of $a \sim 1$ nm is used. The effect of a on the DW dynamics is investigated in appendix 3.B.

The elastic energy is computed by minimizing over φ_1 and φ_2 :

$$\begin{aligned} \frac{E_{\text{el}}}{t} = \min_{\varphi_1, \varphi_2} & \left[\frac{L}{2} \sqrt{1 + \left(\frac{2u}{L}\right)^2} (\mathcal{E}(\alpha_1, \varphi_1) + \mathcal{E}(\alpha_2, \varphi_2)) \right. \\ & + \frac{J\lambda}{a} (3 - \cos(\varphi_0 - \varphi_1) - \cos(\varphi_0 - \varphi_2) - \cos(\varphi_1 - \varphi_2)) \\ & \left. - L\mathcal{E}(\alpha_0, \varphi_0) \right] \end{aligned} \quad (3.4)$$

The first term is the length of each of the two segments of the deformed DW multiplied by their respective energy densities. α_1 and α_2 are the orientations of the respective segments. The second term is the bending energy for the three corners, see figure 3.1b. The third term is the energy of the unperturbed DW.

With this expression we compute L_c , express u in terms of L and thereby obtain $F(L) = F(u(L), L)$ from which the DW velocity is found as $\ln(v) \propto -F_b/(k_B T)$. For more detail, see appendix 3.B. In summary, the derivation of the DW velocity involves multiple optimization steps to determine α_0 , φ_0 , φ_1 , φ_2 , L_c and finally F_b . Due to the complexity of the elastic energy, our results are obtained numerically.

Approximating the elasticity to be proportional to u^2/L does allow for analytic solutions, but these are not able to fully explain recent experimental observations. For example, Je et al. approximated equation (3.4) by setting $\alpha_0 = 0$, $\varphi_0 = \varphi_1 = \varphi_2$ and neglecting the $\pm \arctan(2u/L)$ in the first two terms [66]. Because Pellegren et al. have chosen a different DW profile, we cannot directly compare the expression in equation (3.1) with their results [23]. They do, however, treat L as a free parameter and do not find it by optimization. Moreover, they do not account for the bending costs.

3.3 RESULTS

In figure 3.2ba the modeled DW velocity as a function of the applied IP magnetic field H_x is shown for different values of $\tilde{H}_B$. Figure 3.2bb shows the asymmetric component $A = \ln(v(\uparrow\downarrow)/v(\downarrow\uparrow))$ for $\tilde{H}_B = 0.5$. The kinks in the solid lines at $\tilde{H}_x = 1 \pm \tilde{H}_B$ mark the saturation of internal DW magnetization angle into a Néel wall perpendicular to the IP magnetic field. These are expected from the form of equation (3.1) where we neglected terms $\mathcal{O}(\cos^4(\varphi))$. The dashed curves are the result of setting $\alpha = 8^\circ$ fixed to compensate for the simplified energy density.

In the high IP magnetic field regime, i.e. $|\tilde{H}_x| > \tilde{H}_B$, the profile straightens out. In this regime the azimuthal angle of the internal magnetization is saturated to align with the IP magnetic field, yielding a Néel DW. Due to this saturation, the orientation dependence of the elasticity no longer varies with further increasing $|H_x|$. As a result, the elasticity becomes isotropic and the logarithmic increase in velocity is solely due to the gained Zeeman energy.

Note that the demonstrated asymmetry of the profile compares well with experiments [23, 68, 69, 72–74, 82]. Furthermore, the minimal velocity is not attained at $\tilde{H}_x = 1$ as in the model of Je et al. [66].

Note moreover that the asymmetric velocity component switches sign as $|\tilde{H}_x|$ increases. This feature has been observed experimentally and explained by chiral damping [69, 71, 72]. In our model there are no chiral damping effects, showing that this feature need not be an indication for chiral damping.

Finally, we compared and fitted our model to experimental data. The results are shown in figure 3.3 showing good quantitative agreement in a broad variety of samples over a wide range of IP magnetic fields. The asymmetric behavior is clearly demonstrated in the experiment.

We performed measurements on two different samples stacks, see figure 3.3 (b) and (c). The samples are grown via Ar DC magnetron sputter deposition in a sputter chamber with a base pressure of $\sim 3 \times 10^{-9}$ mbar. The detailed composition of the samples is:

Sample a	SiO ₂ /Ta(4)/Pt(4)/Co(0.6)/Pt(4);
Sample b	SiO ₂ /Ta(4)/Pt(4)/Co(0.8)/Gd(3)/Pt(2);
Sample c	SiO ₂ /Ta(4)/Pt(4)/Co(0.9)/Ir(4).

The number in parentheses indicates the thickness of the layer in nanometers. These samples are representatives of the variety of the velocity profiles observed in the literature of asymmetric domain expansion experiments.[66, 68, 70, 71]. We image the magnetic domains and the expansion of those domains with a Kerr microscope setup. We use an OOP pulse magnet with a pulse length of 0.8-400 ms and strength up to ± 33 mT, and an IP magnet with a strength up to ± 300 mT. More details of the experimental method are given in appendix 3.C. Furthermore, we also interpret data from previous research of Ref.[68] in figure 3.3a. In figure 3.3d the obtained values for H_D are plotted as a function

Figure 3.3: Fitted DW velocity curve (dashed line) to experimental data (dots) of three different samples. The data shown in (b) and (c) is obtained for this paper. The data in (a) is from Ref.[68]. The obtained fit parameters are shown in the table. (d) shows the $H_D(1/t_{film})$ trend for a Cobalt film thickness sample study in Pt/Co(t_{film})/Gd (orange) and Pt/Co(t_{film})/Ir (blue) stacks. The corresponding data and fits can be found in appendix 3.D.

of the film thickness t and confirm our expectation that H_D should decrease as a function of t [77, 83].

3.4 CONCLUSION

The DMI and IP magnetic field complexify DW dynamics significantly due to the orientation dependence of elasticity. To grasp and expose this complexity, we defined a model following creep theory and solving the dynamics semi-analytically. The model has a profound sensitivity to DMI and demagnetization. As a result, the model provides a quantitative interpretation of experimental data of DWs that demonstrate asymmetric velocity profiles as a function of H_x .

Experimental studies that do not exhibit a kink at $H_x/H_D = \pm 2H_B$ are often fitted with the constant elasticity model proposed by Je et al. [66]. In these studies the measurement range of H_x might not be large enough to expose these kinks. figure 3.3b demonstrates that our model resembles results from the constant elasticity model of Je et al. [66], but yields a different value of the DMI: at $H_x = H_D$, the velocity is not minimized.

The parameter α has been set to a fixed value to account for the omission of higher order anisotropy terms in the energy density. As a result the angle φ will not saturate for large H_x . Previous research used α as a fitting parameter as to account for roughness [23]. If roughness forces the DW to tilt, the tilting angle is not fixed to one value. Hence a fixed value of α should not be interpreted as a physical tilting of the DW. Thus a fixed value of α should not be interpreted as a physical tilting of the DW.

We remark that assuming φ to be constant along an axis normal to the DW is only a first approximation. For a Mixed Bloch-Néel DW, φ will adjust so that the magnetization aligns with the IP magnetic field well inside the domains, but does not at the DW. As φ plays a key role in the DW dynamics, future research could focus on the exact behavior of φ .

In recent publications the asymmetric shape of the DW velocity profile as a function of H_x is used as an argument for significant effect of chiral damping on the DW dynamics [69, 72, 73]. However, our model demonstrates a similar asymmetry without chiral damping. Furthermore, in the quasi-static creep regime dynamic effects such as chiral damping should not affect creep motion.

The comparison experimental data demonstrates the broad applicability of our model. Future research could apply our model to an extensive sample study to investigate the effects of sample growth parameters and layer thickness on parameters of the model such as the effective lattice spacing a . Furthermore, measurements over a broader range in H_z could be performed to test the universality.

APPENDIX

3.A MAGNETIZATION PROFILE AND ENERGY DENSITY

DW profile for tilted domains

An IP magnetic field also changes the orientation of the magnetic moment inside the domains. The equilibrium orientation has to balance PMA and the IP magnetic field, yielding a different internal magnetization profile. The energy density well inside the domain is given by

$$\mathcal{E}_{\text{in}} = -M_S \left(\mathbf{H}_{\text{IP}} \cdot \hat{m} \sin(\theta) + \frac{H_{\text{PMA}}}{2} \cos^2(\theta) \right). \quad (3.5)$$

Here $H_P = K_P/M_S$ is the effective magnetic field of the PMA, $\mathbf{H}_{IP}$ is the IP vector of the applied magnetic field and $\hat{m}(\varphi)$ the IP vector of the internal magnetization. We minimize this energy, where we assume $H_P \gg |\mathbf{H}_{IP} \cdot \hat{m}|$, because else there will be no perpendicularly magnetized domains anymore. The two solutions are

$$\theta = \arcsin(\beta), \quad \text{or} \quad \theta = \pi - \arcsin(\beta), \quad (3.6)$$

$$\text{with} \quad \beta = \frac{\mathbf{H}_{IP} \cdot \hat{m}}{H_{PMA}}. \quad (3.7)$$

Which precisely characterizes the magnetization in the the up and down domains respectively. We have the bounds $|\beta| \ll 1$, thus our model has a bound on the strength of $\mathbf{H}_{IP}$. This makes sense because when the IP magnetic field is too large, the sample completely loses its different domains as they all align with $\mathbf{H}_{IP}$.

This immediately yields an effect on the driving force as the Zeeman energy difference now is proportional to

$$\begin{aligned} \Delta E_{\text{Zeeman}} &\propto \frac{1}{2} H_z (\cos(\pi - \theta) - \cos(\theta)) \\ &= H_z \cos(\theta) \\ &= H_z \sqrt{1 - \beta^2}. \end{aligned} \quad (3.8)$$

We investigate how these new boundary conditions affect the energy density. Accounting for IP magnetic fields and DMI, the energy density reads

$$\begin{aligned} \mathcal{E} = \frac{J}{2} (\nabla \theta)^2 + M_S \left(-\frac{H_P}{2} \cos(\theta)^2 - \mathbf{H}_{IP} \cdot \hat{m} \sin(\theta) \right. \\ \left. + H_D \lambda \cos(\varphi - \alpha) \nabla \theta \right). \end{aligned} \quad (3.9)$$

The Euler-Lagrange formalism yields

$$J \partial_x^2 \theta = M_S (H_P \cos(\theta) \sin(\theta) - \mathbf{H}_{IP} \cdot \hat{m} \cos(\theta)). \quad (3.10)$$

We divide out $M_S H_P$ and use the definitions of λ and β to rewrite the above as

$$\lambda^2 \partial_x^2 \theta = \cos(\theta) (\sin(\theta) - \beta). \quad (3.11)$$

As we can see, the DMI has no effect on the DW profile because it depends only linearly on $\nabla \theta$. Using $2(\partial_x \theta)(\partial_x^2 \theta) = \partial_x ((\partial_x \theta)^2)$ and $2(\partial_x \theta) \cos(\theta) \sin(\theta) = \partial_x \sin(\theta)^2$, the above yields

$$(\lambda \partial_x \theta)^2 = (\sin(\theta)^2 - 2\beta \sin(\theta)) + C, \quad (3.12)$$

Figure 3.4: Effect of IP magnetic field on the DW profile parameter . A side view of the corresponding magnetization is illustrated in figure 3.1c.

where C is a constant which we now determine from the boundary conditions. As $x \rightarrow \infty$, $\partial_x \theta \rightarrow 0$ and $\sin(\theta) \rightarrow \beta$, we obtain:

$$C = \beta^2. \quad (3.13)$$

Thus

$$(\lambda \partial_x \theta)^2 = (\sin(\theta) - \beta)^2, \text{ that is } \lambda \frac{\partial \theta}{\partial x} = \sin(\theta) - \beta. \quad (3.14)$$

With boundary condition $\theta(x_{\text{dw}}) = \pi/2$, we solve this equation and obtain

$$\theta(x) = 2 \arctan \left(e^{\frac{x-x_{\text{dw}}}{\lambda}} \right) - \beta \tanh \left(\frac{x-x_{\text{dw}}}{\lambda} \right) + \mathcal{O}(\beta^2). \quad (3.15)$$

This DW shape is shown in figure 3.4 for $\beta = 1/4$ along with the $\beta = 0$ case. The result leaves the DW thickness λ intact, as can also be seen in the figure.

Energy density

Phenomenologically, we introduce an anisotropy energy for an easy z -axis, by

$$E_{\text{PMA}}(\theta, \varphi) = -\frac{K_P}{2} \int d^3\mathbf{x} \cos^2(\theta). \quad (3.16)$$

The subscript P stands for PMA. From symmetry arguments higher order terms in φ and θ could also be included, but we omit them for simplicity. It will be convenient to define an effective anisotropy magnetic field strength $H_P = K_P/M_S$ with M_S the saturation magnetization. It should be noted that

H_P is only introduced for notational convenience and cannot be physically interpreted as an effective magnetic field. H_P has typical values around 0.1 T and M_S around $5 \times 10^5 \text{ Am}^{-1}$.

Dipole-dipole interactions also contribute to the energy and prefer Bloch DWs. The dipole-dipole potential energy is written as a Zeeman energy from the demagnetization field $\mathbf{H}_d$

$$E_d = -\frac{1}{2} \int d^3\mathbf{x} \mathbf{M}(\mathbf{x}) \cdot \mathbf{H}_d(\mathbf{x}). \quad (3.17)$$

Following Tarasenko et al. [84], we perform a Fourier transform of the demagnetisation field, its corresponding magnetic potential Φ (such that $\mathbf{H}_d = \nabla\Phi$) and the magnetization $\mathbf{M} = M_S \mathbf{n}$ to obtain an expression for the above energy in terms of the fourier modes using Maxwell's equation

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\mathbf{H} + 4\pi\mu_0\mathbf{M}) = 0; \quad (3.18)$$

we thus find

$$E_d = \frac{\mu_0}{4\pi^2} \int d^3\mathbf{k} \frac{1}{k^2} (\mathbf{M}_{\mathbf{k}} \cdot \mathbf{k})(\mathbf{M}_{-\mathbf{k}} \cdot \mathbf{k}). \quad (3.19)$$

Here, $k = |\mathbf{k}|$. Now we perform the inverse Fourier transform of the z component.

$$\mathbf{M}_{\mathbf{k}} = \int dz \mathbf{M}_{\kappa}(z) e^{-ik_z z}, \quad (3.20)$$

where $\kappa = (k_x, k_y)^T$. This results in

$$E_d = \frac{\mu_0}{4\pi^2} \int d^2\kappa \int dz \int dz' \int dk_z \frac{1}{\kappa^2 + k_z^2} e^{-ik_z(z-z')} (\mathbf{M}_{\kappa}(z) \cdot \kappa + M_{\kappa}^z(z)k_z)(\mathbf{M}_{-\kappa}(z') \cdot \kappa + M_{-\kappa}^z(z')k_z). \quad (3.21)$$

We take out an effective contribution to the easy z -axis anisotropy by using:

$$\begin{aligned} \int d^3\mathbf{x} M_z^2 &= \int dz \int \frac{d^2\kappa d^2\kappa'}{(2\pi)^4} M_{\kappa}^z M_{\kappa'}^z e^{i\mathbf{x} \cdot (\kappa + \kappa')} \\ &= \int dz \int \frac{d^2\kappa}{(2\pi)^2} M_{\kappa}^z M_{-\kappa}^z. \end{aligned} \quad (3.22)$$

Then we evaluate the integral over k_z and are left with

$$\begin{aligned} E_d &= \frac{\mu_0}{4\pi} \int d^2\kappa \int dz \int dz' e^{-\kappa|z-z'|} \\ &\quad \left(\frac{1}{\kappa} (\mathbf{M}_{\kappa}(z) \cdot \kappa)(\mathbf{M}_{-\kappa}(z') \cdot \kappa) - \kappa M_{\kappa}^z(z) M_{-\kappa}^z(z') \right. \\ &\quad \left. - 2i \text{sign}(z - z') (\mathbf{M}_{\kappa}(z) \cdot \kappa) M_{-\kappa}^z(z') \right). \end{aligned} \quad (3.23)$$

The first term is the energy of the magnetic field declination away from the DW. The second term is the energy associated with the change in magnetization at the boundaries. The third term describes the twisting of the DW. [85, 86]

Next, under the assumption that the film thickness t is small (i.e. $t < \lambda$), the magnetization is homogeneous along the z direction and we evaluate the integrals over z and z' . We use

$$\int_{-t/2}^{t/2} dz \int_{-t/2}^{t/2} dz' e^{-|z-z'|\kappa} = \frac{2t}{\kappa} \left(1 - \frac{1 - e^{-\kappa t}}{\kappa t} \right) \quad (3.24)$$

$$\approx t^2, \text{ for } \kappa t \ll 1,$$

and

$$\int_{-t/2}^{t/2} dz \int_{-t/2}^{t/2} dz' e^{-|z-z'|\kappa} \text{sign}(z - z') = 0, \quad (3.25)$$

by symmetry. Furthermore, we rotate the IP components of $\mathbf{M}$ such that the x component aligns with the DW normal. This means $\mathbf{n}(\varphi, \theta) \rightarrow \mathbf{n}(\varphi - \alpha, \theta)$, where α is the angle (w.r.t. the x -axis) of the DW normal. And by translation symmetry the field is also homogeneous in the y direction. That means

$$\mathbf{M}_\kappa \cdot \boldsymbol{\kappa} = M_\kappa^x, \text{ and} \quad (3.26)$$

$$M_\kappa^x = 2\pi\delta(k_y)M_{k_x}^x. \quad (3.27)$$

For the DW dynamics only the first term in equation (3.23) is relevant. As we shall see from the DW profile in the following, M_{k_x} converges rapidly to 0 for increasing k_x , hence for small t , we assume $\kappa t \ll 1$. Then the demagnetization energy is reduced to

$$E_d \approx \frac{t^2\mu_0}{2} \int dk |k| M_k^x M_{-k}^x \int dy. \quad (3.28)$$

The integral over y indicates that the demagnetization energy increases linearly with the DW length. Once we know the DW profile (i.e. θ and φ as a function of x), we then explicitly compute the inverse Fourier transform

$$\begin{aligned} M_k^x &= \int_{-\infty}^{\infty} dx M_x e^{ikx} \\ &= M_S \int_{-\infty}^{\infty} dx \sin(\theta) \cos(\varphi - \alpha) e^{ikx}, \end{aligned}$$

to obtain the final expression for the demagnetization energy by working out all the integrals.

The energy contribution of the external magnetic field $\mathbf{H}$ is the Zeeman energy, given by

$$\begin{aligned} E_{\text{external}}(\theta, \varphi) &= M_S \int d^3\mathbf{x} (-\mathbf{n}(\mathbf{x}) \cdot \mathbf{H}) \\ &= -M_S \int d^3\mathbf{x} (H_x \sin(\theta) \cos(\varphi) \\ &\quad + H_y \sin(\theta) \sin(\varphi) + H_z \cos(\theta)). \end{aligned} \quad (3.29)$$

For convenience we rotate our axes such that the gradient of the magnetization $\mathbf{n}$ points along the x axis and the interface between the ferromagnet and (heavy spin orbit coupling) normal metal is perpendicular to the z axis. For two neighboring spins along the x direction, the energy contribution due to the interfacial Dzyaloshinskii-Moriya interaction (DMI) is given by

$$E_{\text{DMI}} = \mathbf{D}_s \cdot (\mathbf{n}(\mathbf{x}) \times \mathbf{n}(\mathbf{x} + a\hat{x})), \quad (3.30)$$

where $\mathbf{D}_s = D_s \hat{y}$ is the DMI. From Ref.[65] we compute D_s to be in the order of 10^{-23} J. We approximate the magnetization direction of the neighboring particle ($\mathbf{n}(\mathbf{x} + a\hat{x})$) in terms of $\mathbf{n}(\mathbf{x})$ by $\mathbf{n}(\mathbf{x} + a\hat{x}) \approx \mathbf{n}(\mathbf{x}) + a \frac{\partial \mathbf{n}(\mathbf{x})}{\partial x}$. Then, summing over all sites and taking the continuum limit, we get

$$E_{\text{DMI}} = D_s \int \frac{dy}{a} \int dx (\hat{y} \cdot (\mathbf{n}(x) \times \frac{\partial \mathbf{n}}{\partial x})). \quad (3.31)$$

Where we used that the system is homogeneous along the y direction. We then define $D = D_s a^{-1} t^{-1}$ to obtain an interfacial DMI energy per DW area. D has typical values around 10^{-4} J m^{-2} . We also define an effective DMI field $H_D = D / (M_S \lambda)$. In appendix 3.D we apply our model to DW velocity measurements of two series of samples with varying thickness to extract the value of H_D of these samples and verify the proportionality $H_D \propto t^{-1}$.

Inserting the definition of n and working out the cross and inner product, we obtain in terms of the spherical coordinates θ and ϕ .

$$E_{\text{DMI}}(\theta, \varphi) = t M_S \lambda H_D \int dx dy \cos(\varphi) \frac{\partial \theta}{\partial x}. \quad (3.32)$$

In this case H_D is not only introduced for notational convenience, but also has the physical interpretation of a local IP magnetic field directed along $-\nabla\theta$. Let α be the angle between the DW normal and the applied external IP magnetic field. Then we rotate our axes back such that the x axis lies parallel to the IP magnetic field. This effectively shifts $\varphi \rightarrow \varphi - \alpha$.

We now assume that along the x -axis φ is constant. Then we integrate our energy density over x using equation (3.15) (or equation (2.12) for the untilted case), subtracting the energy density of a system with a DW at $x_{\text{dw}} = 0$, and we are left with an equilibrium (i.e. $H_z = 0$) energy density. We work out the

case for the demagnetization energy explicitly based on the work of Tarasenko et al.[84]. We start out by reproducing the known results for $\beta = 0$.

For convenience we define $X = (x - x_{\text{dw}})/\lambda$, so $dx = \lambda dX$. Note that $\sin(\theta) = 1/\cosh(X)$, which has poles at $X_n = \frac{\pi i}{2}(2n+1)$ for $n \in \mathbb{Z}$. We now evaluate equation (3.29) by closing the contour in the upperhalf of the complex plane along an infinity modulus semi circle. Due to the exponent, this contribution vanishes, and we use the residue theorem. The residue of the integrand at X_n is given by $-ie^{\frac{1}{2}\pi k\lambda(2n+1)}/\cos(n\pi)$. Now we sum over all poles in within the contour (i.e. $n > 0$) to find

$$M_k^x = M_{-k}^x = M_S \cos(\varphi - \alpha) \frac{\pi\lambda}{\cosh\left(\frac{k\pi\lambda}{2}\right)}. \quad (3.33)$$

Which indeed vanishes rapidly for increasing k . Now we evaluate the integral over k in equation (3.28) to obtain the energy density (per sample thickness per DW length)

$$\mathcal{E}_d = 4 \ln(2) t \mu_0 M_S^2 \cos^2(\varphi - \alpha). \quad (3.34)$$

Hence we define an effective Bloch anisotropy field (referring to the fact that this energy contribution favors a Bloch type DW) $H_B = 4 \ln(2) \frac{t \mu_0 M_S}{\pi \lambda}$.

In total, the energy density is given by

$$\begin{aligned} \mathcal{E}(\alpha, \varphi) = & 2 \frac{J}{\lambda} + M_S \pi \lambda \left[H_B \cos^2(\varphi - \alpha) - (H_x \cos \varphi + H_y \sin \varphi \right. \\ & \left. + H_D \cos(\varphi - \alpha)) \right] \end{aligned} \quad (3.35)$$

When φ varies along the DW, the energy density obtains a bending contribution following from the exchange interaction:

$$\mathcal{E}_{\text{bend}} = 2J\lambda \left(\frac{\partial \varphi}{\partial s} \right)^2. \quad (3.36)$$

Now we calculate the same energy density for $\beta \neq 0$. The more elegant equation (3.14) is used to simplify these integrals. For convenience we first define

$$f(\beta) = \frac{4}{\pi} \left(\arcsin\left(\sqrt{\frac{1+\beta}{2}}\right) - \arcsin(\beta) \right) \quad (3.37)$$

$$\approx 1 - \frac{2}{\pi}\beta + \mathcal{O}(\beta^3).$$

$$g(\beta) = \frac{(1-\beta^2)}{4 \ln(2)} \left(-H_{-b} - H_b + b(\psi^{(1)}(1-b) - \psi^{(1)}(1+b)) \right) \quad (3.38)$$

The results are:

Figure 3.5: (a) The effect of the β parameter on H_B (blue), H_D and H_x (orange), J , K_P and H_z (green), normalized to their respective values for $\beta = 0$. (b) The DW velocity corrected for tilting of the magnetic domains (blue) compared with the model where this effect is neglected and β is set to 0 (light blue). The effect is significant and therefore implemented in our model.

- Exchange interaction

$$E_{\text{Exchange}}(\varphi) = \frac{J}{\lambda} \left(\sqrt{1 - \beta^2} - \frac{\pi\beta}{2} f(\beta) \right). \quad (3.39)$$

- PMA

$$E_{\text{PMA}}(\varphi) = K_P \lambda \left(\sqrt{1 - \beta^2} + \frac{\pi\beta}{2} f(\beta) \right). \quad (3.40)$$

Using the definition of λ we combine the two above expressions, cancelling the inverse cosine functions. The result is $2 \frac{J}{\lambda} \sqrt{1 - \beta^2}$.

- Bloch anisotropy

Using the residue theorem we compute M_k^x so that the Bloch anisotropy energy is now given by

$$\begin{aligned} E_{\text{Bloch}}(\varphi) &= 4\mu_0 M_S^2 \pi^2 \lambda^2 \cos^2(\varphi - \alpha) \times \\ &\quad \int_{-\infty}^{\infty} dk \frac{\sinh^2 \left(\frac{k\lambda \arccos(\beta)}{\sqrt{1 - \beta^2}} \right)}{\sinh^2 \left(\frac{k\lambda \pi}{\sqrt{1 - \beta^2}} \right)} |k| \\ &= 8\mu_0 M_S^2 t \cos^2(\varphi - \alpha) (1 - \beta^2) \times \\ &\quad \int_0^{\infty} dK \frac{\sinh^2(Kb)}{\sinh^2(K)} |K|. \end{aligned} \quad (3.41)$$

We have changed variables $K = \frac{k\pi\lambda}{\sqrt{1-\beta^2}}$ and defined $b = \arccos(\beta)/\pi$, $0 \leq b \leq 1$. Now we work out the integral using the geometric series

$$\sum_{n=0}^{\infty} (n+1)e^{-2nK} = \frac{1}{(1-e^{-2K})^2}. \quad (3.42)$$

We write out the definition of hyperbolic sine in terms of exponentials and use

$$\int_0^{\infty} dK K e^{-aK} = \frac{1}{a^2}, \quad (3.43)$$

for $a > 0$, to evaluate the integral. Finally, we evaluate the sum over n to find

$$E_{\text{Bloch}}(\varphi) = 8\mu_0 M_S^2 t \cos^2(\varphi - \alpha) \ln(2) \frac{(1-\beta^2)}{4 \ln(2)} \times \left(-H_{-b} - H_b + b(\psi^{(1)}(1-b) - \psi^{(1)}(1+b)) \right). \quad (3.44)$$

H_z is the Harmonic number of z and $\psi^{(1)}$ is the first derivative of the digamma function. When $\beta = 0$, $b = 1/2$ and $2 + H_{-1/2} = H_{1/2} = 2 - 2 \ln(2)$ and $\psi^{(1)}(3/2) = \psi^{(1)}(1/2) - 4 = \pi^2/2$, so we indeed obtain the known limit. This result is shown in figure 3.5a as a function of β .

- Zeeman energy

$$E_{\text{Zeeman}}(\varphi) = -M_S \lambda \pi (H_x \cos(\varphi) + H_y \sin(\varphi)) f(\beta) \quad (3.45)$$

- DMI

$$E_{\text{DMI}}(\varphi) = 4\pi D \cos(\varphi) f(\beta) \quad (3.46)$$

We absorb this effect by a redefinition of the effective fields and parameters as follows

- $J \rightarrow J\sqrt{1-\beta^2}$;
- $K_P \rightarrow K_P\sqrt{1-\beta^2}$, which keeps λ fixed;
- $H_{\text{IP}} \rightarrow H_{\text{IP}}f(\beta)$;
- $H_z \rightarrow H_z\sqrt{1-\beta^2}$;
- $H_D \rightarrow H_D f(\beta)$;
- $H_B \rightarrow H_B g(\beta)$.

In figure 3.5a the different effects of β are plotted, normalized with respect to the $\beta = 0$ value. The effect of this domain tilting on the final DW velocity as a function of H_x is shown in figure 3.5b.

3.B DISPERSIVE CREEP THEORY

In our theory of creep there are three components that make up elasticity. Most familiar is stretching: There is an DW energy cost per length, and a deformation elongates the DW. To minimize this energy cost, the DW is kept short and thus straight. Some systems (including ours) need to distinguish between stretching and bending; although both try to keep the manifold straight the underlying mechanisms are different. In our system both stretching and bending energy costs are caused by the exchange interaction. A third elasticity component is formed by orientation. In our system this component starts playing a role when applying an external field. For example a Bloch DW becomes more flexible when applying an IP magnetic field perpendicular to the DW normal as the magnetic moment at the DW will then be more aligned with the IP magnetic field upon any deformation. Similarly a Néel wall will become stiffer in the same situation.

So suppose in general we have some DW energy density $\mathcal{E}(\varphi, \alpha)$ which is a function of the azimuthal angle of the magnetization of the sample at the DW φ , and the angle α of the DW normal w.r.t. the applied in-plane magnetic field. After minimizing it for φ this energy $\mathcal{E}_{\min}(\alpha)$ is only a function of α . We assume a DW deformation that has some profile $\alpha(s)$ parametrized by s , which runs along the length of the DW, such that $\alpha(s) - \alpha(0)$ is an odd function of s (i.e. the deformation has a reflection symmetry in a line along $\alpha(0)$). So in general, the energy difference is given by

$$E_{\text{el}} = \int ds \mathcal{E}_{\min}(\alpha(s)) - L \mathcal{E}_{\min}(0). \quad (3.47)$$

For a situation such as studied above, with $u/L \ll 1$, we do a second order approximation of the right hand side in u/L . The result is

$$E_{\text{el}} \approx \frac{2u^2}{L} \left(\mathcal{E}_{\min}(0) + \frac{\partial^2 \mathcal{E}_{\min}}{\partial \alpha^2}(0) \right). \quad (3.48)$$

On the right hand side of this equation the two elasticity components are made explicit. The first term is the stretching energy density and the second term the orientation energy density.

When considering bending, it becomes hard to determine $\mathcal{E}_{\min}$ analytically. In the previous section we have computed the bending contribution from the exchange interaction to the energy density given by equation (3.36). To solve such a system one could expand the energy density around $\alpha_0 = \alpha(0)$ and

$\varphi_0 = \varphi(0)$, which is the equilibrium angle corresponding to a DW orientation $\alpha(0)$ (so $\frac{\partial \mathcal{E}}{\partial \varphi}(\alpha_0, \varphi_0) = 0$):

$$\begin{aligned} \mathcal{E}(\alpha, \varphi) \approx & \mathcal{E}(\alpha_0, \varphi_0) + (\alpha - \alpha_0)\mathcal{E}_\alpha + \frac{1}{2}(\alpha - \alpha_0)^2\mathcal{E}_{\alpha\alpha} \\ & + \frac{1}{2}(\varphi - \varphi_0)^2\mathcal{E}_{\varphi\varphi} + (\alpha - \alpha_0)(\varphi - \varphi_0)\mathcal{E}_{\alpha\varphi}, \end{aligned} \quad (3.49)$$

with $\mathcal{E}_\alpha = \frac{\partial \mathcal{E}}{\partial \alpha}(\alpha_0, \varphi_0)$ and $\mathcal{E}_{ab} = \frac{\partial^2 \mathcal{E}}{\partial a \partial b}(\alpha_0, \varphi_0)$. If we neglect bending and solve the Euler Lagrange equation for φ , we find

$$\varphi = \varphi_0 - (\alpha - \alpha_0) \frac{\mathcal{E}_{\alpha\varphi}}{\mathcal{E}_{\varphi\varphi}}. \quad (3.50)$$

Inserting this in equation (3.49) we get

$$\begin{aligned} \mathcal{E}(\alpha) \approx & \mathcal{E}(\alpha_0, \varphi_0) + (\alpha - \alpha_0)\mathcal{E}_\alpha + \frac{1}{2}(\alpha - \alpha_0)^2\mathcal{E}_{\alpha\alpha} \\ & - \frac{1}{2}(\alpha - \alpha_0)^2 \frac{\mathcal{E}_{\alpha\varphi}^2}{\mathcal{E}_{\varphi\varphi}}. \end{aligned} \quad (3.51)$$

To find the energy of the deformed wall we just have to integrate over s . As $\alpha(s) - \alpha(0)$ is odd, the second term vanishes upon integration. After subtracting the unperturbed DW energy we obtain the energy difference

$$E_{\text{el}} \approx \Delta L \mathcal{E}(\alpha_0, \varphi_0) + S \left(\varepsilon_{\alpha\alpha} - \frac{1}{2} \frac{\varepsilon_{\alpha\varphi}^2}{\varepsilon_{\varphi\varphi}} \right), \quad (3.52)$$

where $\Delta L = \int ds - L$ and $S = \int (\alpha(s) - \alpha_0)^2 ds$. Note that although this result is more general than the result of equation (3.49), it is less accurate, because of the assumption that not only $\alpha(s) - \alpha_0$, but also $\varphi(s) - \varphi_0$ has to be small.

When bending cannot be neglected the Euler Lagrange formalism gets an additional term which makes it a differential equation.

$$\varphi - 4 \frac{J\lambda}{\mathcal{E}_{\varphi\varphi}} \frac{\partial^2 s}{\partial \varphi^2} = \varphi_0 - (\alpha - \alpha_0) \frac{\mathcal{E}_{\alpha\varphi}}{\mathcal{E}_{\varphi\varphi}}. \quad (3.53)$$

The solution depends on the profile $\alpha(s)$ of the DW. In [23] a circle segment shape is chosen and the equation is solved when there is no boundary condition for φ . At the end of this section we reproduce their model and extend it by optimizing for the segment length L to find the free energy barrier.

However, as this solution does not account for bending at the boundaries, we first work out the model of this paper, which implements the bending energy in a rough and simplified manner: We choose the deformation to be triangularly shaped, determined by u and L , following Blatter et al.[17] The segments will

have a constant value for φ each and at the bending points a nearest neighbor exchange interaction is implemented. The elasticity is then enriched by the following bending energy contribution:

$$E_{\text{bend}} = \frac{J\lambda}{a}(3 - \cos(\varphi_0 - \varphi_1) - \cos(\varphi_0 - \varphi_2) - \cos(\varphi_1 - \varphi_2)). \quad (3.54)$$

Here φ_0 is the azimuthal angle of the magnetization of the unperturbed DW, and φ_1 and φ_2 are the optimized angles of the upper and lower segment respectively. The factor 3 comes from the subtraction with the straight DW. Due to variations in the lattice structure and to account for non-nearest neighbor interactions, an effective value of a should be used, but as demonstrated in figure 3.6a the precise value of a has no significant effect on the DW dynamics. The elasticity is now given by

$$\begin{aligned} \frac{E_{\text{el}}(u, L)}{t} = \min_{\varphi_1, \varphi_2} & \left[\frac{L}{2} \sqrt{1 + \left(\frac{2u}{L}\right)^2} \left\{ \mathcal{E}(\alpha_0 + \arctan(2u/L), \varphi_1) \right. \right. \\ & \left. \left. + \mathcal{E}(\alpha_0 - \arctan(2u/L), \varphi_2) \right\} \right. \\ & \left. + \frac{J\lambda}{a} \left\{ 3 - \sum_{i>j} \cos(\varphi_i - \varphi_j) \right\} \right] - L\mathcal{E}(\alpha_0, \varphi_0). \end{aligned} \quad (3.55)$$

After minimizing the elasticity in equation (3.55) over φ_1 and φ_2 , one obtains the elasticity as a function of u and L . This dispersive elasticity is used in the main article as a starting point for our model and is made dimensionless by dividing out D :

$$\begin{aligned} \tilde{\mathcal{E}}(\alpha, \varphi) \equiv \frac{\mathcal{E}(\alpha, \varphi)}{D} = 2\tilde{J}\sqrt{1 - \beta^2} + \pi \left(g(\beta) \frac{\tilde{H}_B}{2} \cos^2(\varphi - \alpha) \right. \\ \left. - f(\beta) \tilde{H}_x \cos(\varphi) + f(\beta) \cos(\varphi - \alpha) \right), \end{aligned} \quad (3.56)$$

with $\tilde{J} \equiv J\lambda^{-1}D^{-1}$, $\tilde{H}_B \equiv 2H_B/H_D$ and $\tilde{H}_x \equiv H_x/H_D$. By assuming $u \ll L$, setting $\varphi_1 = \varphi_2 = \varphi_0$ and $\alpha_0 = 0$, the elasticity is approximated to be proportional to u^2/L : $E_{\text{el}} \approx \frac{u^2}{L} \mathcal{E}_{\text{el},0}$ with

$$\mathcal{E}_{\text{el},0} = 4\tilde{J}\sqrt{1 - \beta^2} + \begin{cases} +2\pi f(\beta)|\tilde{H}_x| - \pi g(\beta)\tilde{H}_B & , \text{ if } |\tilde{H}_x - 1| > \tilde{H}_B; \\ -\pi \frac{f(\beta)^2}{\tilde{H}_B g(\beta)} (\tilde{H}_x - 1)(\tilde{H}_x - 3) & , \text{ otherwise.} \end{cases} \quad (3.57)$$

This formula is used to fit data using $\ln(v) \propto -(\mathcal{E}_{\text{el},0}/H_z)^{1/4}$. However, due to the severe assumptions made to obtain it, the fit will not be good and should be

Figure 3.6: (a) The effective lattice spacing affects the DW dynamics. (b) The exact creep velocity as a function $H_z^{-1/4}$ for $\tilde{H}_B = 1.3$, $\tilde{H}_x = 0.5$ (blue) and $\tilde{H}_x = 2$ (orange). The good linear fit shows that even for large $|\tilde{H}_x|$ the universal creep law is still obeyed.

used to obtain good guesses for the relevant parameters. To determine the exact velocity we continue with the exact elasticity of equation (3.55). The optimization over φ_1 and φ_2 is done numerically.

The Larkin length now is found as usual inserting $u = \xi$ as described above. However, this has to be done numerically as well as the computation of the free energy barrier. The result is shown and discussed in the main paper. We wish to remark here that for $|\tilde{H}_x| \leq \tilde{H}_B$ the DW is mixed Bloch-Néel and the DW velocity provides a distinguishing feature regarding where the steepest slope of the velocity profile with respect to $\tilde{H}_x = 0$ and $\tilde{H}_x = 1$ is found: When the steepest slope is attained at $|\tilde{H}_x| > 1$, then $\tilde{H}_B < 1$ (that is, at $\tilde{H}_x = 0$ the DW is purely Néel). Otherwise, there is a steep slope around $H_x = 0$ (where the DW now is mixed Bloch-Néel). This distinction thus indicates the strength of demagnetization relative to DMI.

In figure 3.6b we plot $\ln(v)$ as a function of $-H_z^{-1/4}$ for two values of $\tilde{H}_x$. Our exact theoretical creep model shows good agreement with the linear fit, demonstrating the universality of the creep law $\ln(v) \propto -H_z^{-1/4}$.

Instead of considering a DW normal to the applied IP magnetic field, one could tilt the DW itself over a fixed angle α as suggested by Pellegren et. al. [23]. From equation (3.56) it becomes clear that the pure Néel DW then no longer minimizes the energy density for $|H_x - H_D| > 2H_B$; the Néel DW is the asymptotic limit as $H_x \rightarrow \infty$. Because there will be no saturation of the azimuthal angle φ , the velocity profile becomes more smooth. As described in

the main text, we do not need to interpret a fixed value of α as a physical tilting of the DW, but use it solely to account for higher order anisotropy terms.

Figure 3.7: An illustration of the deformation model considered by Pellegren et al. [23].

To conclude this section, we reproduce and expand on the results from Pellegren et al., who considered an arc shape deformation parametrized not by the displacement u , but by the radius of the circle segment illustrated in figure 3.7. The relation between u and R is given by

$$R = \frac{L^2 + 4u^2}{8u}. \quad (3.58)$$

Using equation (3.49) enriched with equation (3.36), we find φ by the Euler Lagrange formalism. The boundary conditions are not given by a restriction on the value of φ at the boundary of the segment, but by minimizing the energy. Assuming $R \ll L$ the solution for φ can be approximated by:

$$\varphi(s) = \varphi_0 + \frac{\mathcal{E}_{\alpha\varphi}}{\mathcal{E}_{\varphi\varphi}} \frac{1}{R} (s - \Lambda \sinh(s/\Lambda)). \quad (3.59)$$

Now we compute the elasticity as a function of R and L . Using the Larkin length and the wandering exponential relation combined with equation (3.58), the elasticity is only a function of L . Then the optimal length L_{opt} is computed as a function of H_x as well as the corresponding free energy barrier.

In figure 3.8 this optimal length is plotted along with the DW velocity computed from the corresponding energy barrier. In the same figure the elasticity for a fixed value of L is also shown to emphasize the effect of optimizing for L . These results are generated for $\alpha_0 = 0$ (left column) and $\alpha_0 = 8^\circ$ (right column). Notice that for $\alpha_0 = 0$ the same kinks occur, which has the same cause: the saturation of φ_0 .

Figure 3.8: The exact results for the arc deformation model proposed by Pellegrin et al. [23] for $\alpha_0 = 0$ (blue) and $\alpha_0 = 8^\circ$ (orange). The two columns correspond to $\tilde{H}_B = 0.5$ (left) and $\tilde{H}_B = 2$ (right). The optimized deformation length L_{opt} is not constant as a function of H_x (a) and (b), as a result the velocity profiles (c) and (d) differ from the elasticity with $L(H_x) = L$ fixed shown in (e) and (f) respectively. Furthermore, due to the saturation of φ_0 at $\alpha_0 = 0$, the results again show kinks.

3.C EXPERIMENTAL METHOD

The samples are grown via Ar DC magnetron sputter deposition in a sputter chamber with a base pressure of $\sim 3 \times 10^{-9}$ mbar. The detailed composition of the samples introduced in the main article is:

Sample a	$\text{SiO}_2/\text{Ta}(4)/\text{Pt}(4)/\text{Co}(0.6)/\text{Pt}(4)$
Sample b	$\text{SiO}_2/\text{Ta}(4)/\text{Pt}(4)/\text{Co}(0.8)/\text{Gd}(3)/\text{Pt}(2)$
Sample c	$\text{SiO}_2/\text{Ta}(4)/\text{Pt}(4)/\text{Co}(0.9)/\text{Ir}(4)$

The number in parentheses indicates the thickness of the layer in nanometers. These samples are representatives of the variety of the velocity profiles observed in the literature of asymmetric domain expansion experiments.[66, 68, 70, 71].

Figure 3.9: Kerr microscope images of a magnetic domain. The magnetization in the light areas is pointing out of the plane (up direction) and the magnetization in the black areas is pointing into the plane (down direction). In (a) an unprocessed Kerr image is shown. A binary image is obtained via thresholding. In (b) the nucleation of the domain is shown in white and the three expansion steps with a darkening gray scale. Only out-of-plane magnetic field pulses are applied and this results in a symmetric expansion of the magnetic domain to all sides. In (c) an additional in-plane field is applied in the x-direction and this results in an asymmetric expansion along that direction.

The data of sample (a) was obtained from Ref. [[68]]. The sample is a symmetric stack where in an ideal case no DMI should be present. A finite DMI is obtained when the difference of the interfacial quality of the sputtered layers is considered. The authors studied the interfacial quality by changing the growth pressure of the top platinum layer. A growth pressure of 1.12 Pa is used for the top layer in sample (a).

The data of samples (b) and (c) are part of a larger sample study performed for this paper on the effect of the Cobalt layer thickness. The results of the sample study are shown in the following section.

For all samples a perpendicular magnetic anisotropy is determined via angle-dependent anomalous Hall effect measurements. The saturation magnetization M_S is obtained via SQUID-VSM measurements. We image the magnetic domains and the expansion of those domains with a Kerr microscope setup. In this setup

a difference in contrast in the image corresponds to a different magnetization in the z -direction. A dark contrast in the images indicates that the magnetization is pointing into the plane, referred to as a down domain. A light contrast in the image corresponds to an up domain. These domains are nucleated around intrinsic impurities and defects in the sample, by applying out-of-plane magnetic field pulses. The pulses can vary in length between 0.8 – 400 ms and can reach a field strength up to ± 33 mT.

In figure 3.9a the nucleation of an up domain is shown. The image is processed to a binary image by thresholding. After the nucleation the domain is expanded in three steps via additional out-of-plane field pulses. In figure figure 3.9b the nucleation of an up domain is shown in white and the three expansion steps are indicated by a darkening gray scale. We find that the expansion of the magnetic domain is symmetric to all sides. In figure 3.9c an in-plane magnetic field along the x -direction is added during the expansion of the domain. In this case, the right side of the domain expands much faster than the left side, resulting in an asymmetric magnetic domain expansion. The velocity of the domain boundary is obtained by measuring its displacement for a known pulse length. Repeating this as a function of the in-plane field H_x results in a velocity profile.

3.D SAMPLE STUDY: EFFECT OF FERROMAGNETIC LAYER

We applied our model to determine the effect of the layer thickness of the ferromagnet on the strength of the DMI. Since the DMI is an interfacial effect, we would expect that the DMI decreases for an increase in the cobalt layer thickness. We studied two types of stacks: $\text{SiO}_2/\text{Ta}(4)/\text{Pt}(4)/\text{Co}(x)/\text{Gd}(3)/\text{Pt}(2)$ and $\text{SiO}_2/\text{Ta}(4)/\text{Pt}(4)/\text{Co}(x)/\text{Ir}(4)$. The result is shown in figure 3.10.

The obtained trends are in line with our expectations described in appendix 3.A (i.e. that $H_D \propto t^{-1}$) and compare well to previous experiments. [77, 87] Future research could focus on a larger sample study with a thorough investigation of uncertainties in the measurement and determination of the sample parameters (such as the thickness of the saturation magnetization) to establish a proportionality $D \propto t^{-1}$.

Figure 3.10: Effect of ferromagnetic layer thickness on the velocity profile as a function of H_x . The data (dots) are fitted (dashed lines) with our model to obtain a value for the effective DMI field H_D , summarized in (h). (a) - (d) show the results for the Pt/Co(x)/Ir stacks and (e) - (f) show the results for the Pt/Co(x)/Gd stacks with varying thickness of the Co layer.

SPIN-BIAS INDUCED MAGNETIC TEXTURE

We consider the influence of a spin accumulation in a normal metal on the magnetic statics and dynamics in an adjacent magnetic insulator. In particular, we focus on arbitrary angles between the spin accumulation and the easy-axis of the magnetic insulator. Based on Landau-Lifshitz-Gilbert phenomenology supplemented with magnetoelectronic circuit theory, we find that the magnetic texture twists into a stable configuration that turns out to be described by a virtual, or image, domain wall configuration, i.e., a domain wall outside the ferromagnet. We show that even when the spin accumulation is perpendicular to the anisotropy axis, the magnetic texture develops a component parallel to the spin accumulation for sufficiently large spin bias. The emergence of this parallel component gives rise to threshold behavior in the spin Hall magnetoresistance and nonlocal magnon transport. This threshold can be used to design novel spintronic and magnonic devices that can be operated without external magnetic fields.

4.1 INTRODUCTION

The use of propagating spin waves, or magnons, to transmit and process information has the potential advantage of lower energy consumption over electronic currents. Especially insulating ferromagnets (IFM), such as yttrium-iron garnet (YIG), are able to accommodate a spin current efficiently as the damping of the magnetic dynamics is relatively low [88]. This has raised an increased interest in the possibilities of magnonic devices and how these could replace current electronic devices [24, 89]. Specifically, the behavior of magnons in magnetic domain wall textures can have promising applications [90, 91].

A typical experiment achieves transfer of angular momentum into an IFM through a spin current from a normal metal (NM) lead, usually platinum, by generating a spin accumulation at the interface by the spin-Hall effect [88, 92–94]. The angle of this spin accumulation with respect to the magnetization at the interface determines the efficiency of spin current injection. In this chapter we consider the effect of a sufficiently large spin bias which locally affects the magnetic texture and thereby the transfer of angular momentum. We propose an analytical solution for the magnetization texture of the IFM for

a general orientation of the spin accumulation. Results for nonlocal magnon transport [95] and the spin Hall magnetoresistance [96, 97] are derived. We find threshold behavior in both local and nonlocal setups for a critical magnitude of the spin accumulation. This threshold behavior may be employed as a useful functionality in novel spintronic and magnonic devices that, as a result, do not require a cumbersome external magnetic field to access their different states. While threshold behavior is commonly associated with spin superfluidity [98, 99], our results show a threshold that is related to a change in the stable magnetic texture, and not to a spin superfluid state.

Figure 4.1: (a): The magnetic texture $\mathbf{n}(x)$ (blue arrows) of the semi-infinite IFM nanowire (green region) with an easy-axis anisotropy in the z direction. In the NM (orange region) an electric current generates a spin accumulation μ with polar angle θ_μ at the interface (red arrow) that deforms the magnetic texture. (b): The opaque arrows in the NM region are virtual and illustrate that the magnetic texture is that of two oppositely oriented domains with the center of the virtual domain wall, x_{DW} , outside the IFM. Such a virtual domain wall solution is found analytically for any magnitude and orientation of the spin accumulation.

4.2 EQUATIONS OF MOTION

A one dimensional semi-infinite IFM nanowire with an interface with a non-magnetic metal at $x = 0$ is studied. At the interface a spin accumulation μ is generated, e.g. by means of the spin-Hall effect, which results in a boundary condition on the spin current in the ferromagnet. A possible configuration of the

system is illustrated in figure 4.1a. Our aim is to determine the magnetic texture of the ferromagnet and its stability as a function of μ . We define $\mathbf{n} = \mathbf{M}/M_s$ as the unit vector in the direction of the magnetization, where M_s is the saturation magnetization. The energy of our system is given by

$$E = \int_V dV \frac{1}{2} \left(A |\partial_x \mathbf{n}|^2 - K n_z^2 \right), \quad (4.1)$$

with V the volume of the IFM, A the spin stiffness, $K > 0$ the easy-axis anisotropy and $n_z = \hat{\mathbf{z}} \cdot \mathbf{n}$. We consider an easy z axis anisotropy, but the results apply to other easy-axis directions similarly. The Landau-Lifschitz-Gilbert (LLG) equation supplemented with spin-transfer torques and spin-pumping terms that follows from magnetoelectronic circuit theory reads [100, 101]

$$(1 + \alpha_G \mathbf{n} \times) \dot{\mathbf{n}} = -\gamma \mathbf{n} \times \mathbf{H}_{\text{eff}} - \delta(x) \frac{g^{\uparrow\downarrow}}{4\pi s} \left[\mathbf{n} \times (\mathbf{n} \times \frac{\boldsymbol{\mu}}{\hbar} + \dot{\mathbf{n}}) \right]. \quad (4.2)$$

The left hand side describes the damped time evolution of $\mathbf{n}$, where α_G is the dimensionless phenomenological Gilbert damping constant. The first term on the right hand side is the torque due to effective magnetic field $M_s \mathbf{H}_{\text{eff}} = -\delta E / \delta \mathbf{n}$, where $\gamma > 0$ is the gyromagnetic ratio. The second is the interfacial spin transfer torque and spin pumping respectively, where $g^{\uparrow\downarrow}$ is the interface spin flip scattering per surface area, i.e., the spin-mixing conductance, and s the spin density.

The characteristic length scale of the ferromagnet is the exchange length $\lambda = \sqrt{A/K}$ and the ferromagnetic resonance frequency $\omega_F = \gamma K / M_s$ sets the timescale. Finally, we define $\alpha(x) = \alpha_G + \lambda \delta(x) \alpha'$, with $\alpha' = \frac{g^{\uparrow\downarrow}}{4\pi \lambda s}$. We integrate the LLG equation around an infinitesimal interval around the interface to obtain the boundary condition on the spin current density:

$$\mathbf{j}^s|_{x=0} = \frac{\hbar s}{2} \frac{\gamma A}{M_s} \mathbf{n} \times \partial_x \mathbf{n} \Big|_{x=0} = -\lambda \alpha' \frac{\hbar s}{2} \mathbf{n} \times \left(\mathbf{n} \times \frac{\boldsymbol{\mu}}{\hbar \omega_F} + \dot{\mathbf{n}} \right) \Big|_{x=0}. \quad (4.3)$$

Furthermore, we have the boundary condition $\mathbf{n} \rightarrow \hat{\mathbf{z}}$ as $x \rightarrow \infty$.

4.3 VIRTUAL DOMAIN WALL SOLUTION

It turns out that the stationary magnetization profile that obeys equation (4.2) and the boundary condition equation (4.3) is similar to a domain wall (DW) texture, but with the DW position outside of the ferromagnet: the DW is a stationary solution to the bulk part of the LLG equation, and the freedom of the DW position allows us to satisfy the boundary conditions. We refer to this situation as a virtual DW. For convenience we switch to a local spherical basis whose radial unit vector is given by $\mathbf{n}_{DW}$ as given in equations (2.26) to (2.28) and (4.31).

Figure 4.2: The component of the spin accumulation parallel to the magnetization at the interface ((a), (b)) and the virtual DW position x_{DW} ((c), (d)) as a function of the value of the total spin accumulation μ ((a), (c)) and its polar angle w.r.t. the z-axis ((b), (d)). These plots indicate the effect of the spin accumulation on the magnetic texture: As the spin accumulation increases, the virtual DW position approaches the interface, which will only be reached when $\theta_\mu = \pi/2$, i.e., when μ is perpendicular to the anisotropy axis. For $|\mu| \leq \hbar \omega_F / \alpha'$, μ is also perpendicular the magnetization at the interface. But in the regime $|\mu| > \hbar \omega_F / \alpha'$ there will be a finite parallel component of the spin accumulation.

Next, we study the boundary condition equation (4.3) of the spin current. It follows that $\lambda \partial_x \mathbf{n}_0 = -\sin \theta \hat{\theta}$ (see appendix 4.B for useful properties of the polar angle θ). Hence,

$$\frac{2\mathbf{j}^s|_{x=0}}{s\lambda\alpha'} = -\frac{\hbar\omega_F \sin \theta_0}{\alpha'} \hat{\phi} = (\mu \cdot \hat{\theta}) \hat{\theta} + (\mu \cdot \hat{\phi}) \hat{\phi}|_{x=0}. \quad (4.4)$$

where $\theta_0 = \theta(x=0)$. This gives us two equations:

$$\mu \cdot \hat{\theta}|_{x=0} = 0, \text{ and } \mu \cdot \hat{\phi}|_{x=0} = -\frac{\hbar\omega_F \sin \theta_0}{\alpha'}, \quad (4.5)$$

The virtual DW position x_{DW} and its azimuthal angle ϕ are determined by solving these equations for the boundary condition equation (4.3). We write

$$\mu \cdot \hat{\theta} = \mu_R \cos(\varphi - \varphi_\mu) \cos \theta - \mu_z \sin \theta; \quad (4.6)$$

$$\mu \cdot \hat{\phi} = -\mu_R \sin(\varphi - \varphi_\mu), \quad (4.7)$$

where we express μ in rescaled cylindrical coordinates: $\mu_z = \mu \cdot \hat{z} / \hbar \omega_F$; $\mu_R = \sqrt{(\mu \cdot \hat{x})^2 + (\mu \cdot \hat{y})^2} / \hbar \omega_F$; $\varphi_\mu = \arctan(\mu \cdot \hat{y}) / (\mu \cdot \hat{x})$, and define $\mu = |\mu| / \hbar \omega_F$. From equation (4.7), we obtain an expression for the azimuthal angle φ of the

virtual DW in terms of φ_μ and the polar angle θ_0 of the virtual DW at the interface:

$$\varphi - \varphi_\mu = \begin{cases} \arcsin\left(\frac{\sin\theta_0}{\alpha'\mu_R}\right), & \text{for } \mu_z \geq 0; \\ \pi - \arcsin\left(\frac{\sin\theta_0}{\alpha'\mu_R}\right), & \text{otherwise.} \end{cases} \quad (4.8)$$

Note that φ is only properly defined when $\mu_R \neq 0$. Indeed, if $\mu_R = 0$ the boundary conditions fix $\sin\theta = 0$, i.e., the magnetization is homogeneous along the z direction and an azimuthal angle is ill-defined. By inserting equation (4.8) into equation (4.6), we rewrite equation (4.5) and take the square to obtain $(\alpha'^2\mu_R^2 - u)(1 - u) = \alpha'^2\mu_z^2u$, with $u = \sin^2\theta_0$. This is solved for $0 \geq u \geq 1$ to obtain the expression for x_{DW} :

$$x_{\text{DW}} = -\text{arcsech}\left(\sqrt{\frac{1 + \alpha'^2\mu^2 - \sqrt{(1 - \alpha'^2\mu^2)^2 + 4\alpha'^2\mu_z^2}}{2}}\right). \quad (4.9)$$

Note that although the semi-infinite ferromagnet lies on the $x \geq 0$ axis, a virtual DW texture, i.e., $x_{\text{DW}} \leq 0$, is the only physical solution, as this will minimize the energy of the system. This is seen directly from equation (4.1) as the gradient in the first term is maximal around the virtual DW position. The role of x_{DW} is merely to configure the virtual DW profile in such a way that the boundary conditions are met. The behavior of the magnetic texture as a function of μ is plotted in figure 4.2. The figure demonstrates the effect of the spin bias on the magnetic texture in terms of the virtual DW position and the component of the spin accumulation that is parallel to the magnetization at the interface.

A remarkable feature is that for increasing $|\mu|$ the virtual DW position approaches the interface. Precisely when $\theta_\mu = \pi/2$, the virtual DW position will reach the interface when $|\mu| = \hbar\omega_F/\alpha'$. When $|\mu|$ increases further, the virtual DW position remains at the interface, but the azimuthal angle of the virtual DW now starts changing to pull the magnetization more parallel to the spin accumulation, resulting in the threshold behavior in the parallel component $\mu_{||} = \mu \cdot \mathbf{n}_0|_{x=0}$ of the spin accumulation.

4.4 SPIN HALL MAGNETORESISTANCE

When applying an electric current through a NM|IFM system, the electrical resistance depends on the orientation of the magnetization of the IFM with respect to the current direction. The electric current $\mathbf{j}^e$ will generate a spin current $\mathbf{j}_x^s$ through the interface by the spin Hall effect. The magnitude of this current depends on the relative orientation of the magnetization of the IFM to the spin accumulation μ at the interface [96, 97]: The spin current is maximized (minimized) when the spin accumulation and magnetization at the interface are

perpendicular (parallel) as then the most (no) angular momentum is transferred. As a result the resistivity in the NM is maximal (minimal) due to the inverse spin Hall effect.

Considering figure 4.2a, we expect a threshold effect in this spin Hall magnetoresistance of the normal metal when the angle θ_j between the electrical current through the NM and the anisotropy axis vanishes. The applied electric field thus has a threshold value E_c , such that $\mu > 1/\alpha'$, where the spin accumulation deforms the magnetic texture such that the transfer of angular momentum is reduced.

Following [97] we solve the coupled charge and spin current drift-diffusion equations as a function of the angle θ_j between the electrical current and the anisotropy axis by inserting the boundary conditions for the spin current from equation (4.3), assuming that μ obeys a diffusion equation. Details on the derivation are in appendix 4.C. In the large thickness (along the x direction) limit for the NM and parallel current $\theta_j = 0$, the critical electric field for which the magnetic texture develops a component parallel to the spin accumulation, i.e., $\mu = 1/\alpha'$, is given by

$$E_c = \frac{\lambda s \gamma K}{\theta_{SH} M_s} \left(\frac{2\pi\hbar}{l_s g^{\uparrow\downarrow} e} + \frac{e}{\sigma} \right), \quad (4.10)$$

with θ_{SH} the spin-Hall angle of the NM, l_s the spin diffusion length, $e > 0$ the elementary charge and σ the electrical conductivity. To estimate this effect we consider a Pt|YIG interface where the critical electric field has a value of approximately 21 V/ μm [88, 102–104].

In figure 4.3 we plot the normalized difference in resistance in the NM as a function of the applied electric field for a Pt|YIG interface. One clearly sees the threshold behavior of the resistance due to the change in magnetic texture as a function of the spin accumulation.

4.5 MAGNON TRANSPORT

As we have seen, there is no transfer of spin when the spin accumulation and magnetization are parallel. Despite this, the IFM can accommodate the transfer of angular momentum by means of fluctuations (either thermal or quantum) in the form of spin waves, i.e., magnons. The magnons are injected and detected through spin-flip scattering at the interface with NM leads. The efficiency of the transfer of angular momentum is optimal when the spin accumulation is parallel to the magnetization at the interface. As a consequence, threshold behavior is expected in the nonlocal magnon transport signal.

A typical experiment that quantifies the magnon transport attaches a lead at some position $x = d \gg \lambda$ and measures the electric current generated by the

Figure 4.3: Relative decrease in resistivity as a function of the normalized electric field E/E_c for different angles θ_j between the electric field and the anisotropy axis, where E_c is the electric field that generates a spin accumulation $|\mu| = \hbar\omega_F/\alpha'$ at the interface. For $E \leq E_c$ and $\theta_j = 0$ the conductivity is not affected by the change in magnetization of the IFM as the magnetization remains perpendicular to μ . For $E > E_c$ the magnetization at the interface aligns more with μ and the spin Hall magnetoresistance decreases.

inverse spin-Hall effect [24]. To consider magnons, we add a perturbation to our stationary solution:

$$\mathbf{n} = \mathbf{n}_0 + \delta\mathbf{n}, \text{ with } \delta\mathbf{n} = \hat{\boldsymbol{\theta}}\delta n_\theta + \hat{\boldsymbol{\phi}}\delta n_\varphi, \quad (4.11)$$

where we make the ansatz $|\delta n_\theta(x, t)| \ll 1$ and $|\delta n_\varphi(x, t)| \ll 1$ are homogeneous along the y and z direction as we assume translation symmetry along the interface. The magnon field is defined as $\psi = \delta n_\theta + i\delta n_\varphi$, and thermal fluctuations are modeled by adding a stochastic field $\mathbf{h}$ to the LLG equation (4.2) [95, 105]. Fourier transforming ψ and $\mathbf{h}$, we obtain a Schrödinger-like equation from the linearized LLG equation:

$$(1 + i\alpha)\omega\psi_\omega = \left(-\frac{\gamma A}{M_s}\partial_x^2 + \omega_F \cos 2\theta + i\delta(x)\lambda\alpha'\frac{\mu_{||}}{\hbar} \right)\psi_\omega - \gamma h_\omega. \quad (4.12)$$

where $h = h_\theta + ih_\varphi$. The second term on the right hand side plays the role of a local potential with a minimum at the virtual DW position. The stochastic fields

at the interfaces and in the bulk are combined into $\mathbf{h} = d\mathbf{h}^l\delta(x) + \mathbf{h}^b + d\mathbf{h}^r\delta(x-d)$, where each stochastic field obeys the fluctuation dissipation theorem [95]:

$$\langle h_\omega^*(x)h_{\omega'}(x') \rangle = 4\pi \frac{\alpha_G \hbar \omega d}{\gamma M_s V} \frac{\delta(x-x')\delta(\omega-\omega')}{\tanh(\hbar\omega/2k_B T)}; \quad (4.13)$$

$$\langle h_\omega^{l*}h_{\omega'}^l \rangle = \frac{g^{\uparrow\downarrow}\hbar}{M_s^2 V d} \frac{(\hbar\omega - \mu_{||})\delta(\omega-\omega')}{\tanh\left(\frac{\hbar\omega - \mu_{||}}{2k_B T}\right)}; \quad (4.14)$$

$$\langle h_\omega^{r*}h_{\omega'}^r \rangle = \frac{g^{\uparrow\downarrow}\hbar}{M_s^2 V d} \frac{\hbar\omega\delta(\omega-\omega')}{\tanh(\hbar\omega/2k_B T)}, \quad (4.15)$$

and the temperature T is assumed constant and equal in the bulk and at the leads as we are only interested in the nonlocal transport due to the spin bias. In this way, magnon dissipation at the boundaries and in the bulk is considered.

The observable we are interested in is the average spin current injected into the right lead at $x = d$. We have

$$\langle j^s \rangle = \frac{M_s}{2\gamma} \text{Im} \langle \psi^* (\lambda\alpha'\dot{\psi} + d\gamma h^r) \rangle \Big|_{x=d}, \quad (4.16)$$

where we defined $\langle j^s \rangle = \langle \mathbf{j}^s \rangle \cdot \mathbf{n}_0|_{x=d}$. We use Green's functions to express ψ in terms of the stochastic field and find an analytical solution using two types of solutions for the bulk part of equation (4.12) [90, 95]:

$$\psi_{\omega,\pm} = (\mp i\lambda k(\omega) + \cos\theta)e^{\pm ikx}; \quad (4.17)$$

with $k(\omega) = \lambda^{-1}\sqrt{(1+i\alpha_G)\omega/\omega_F - 1}$. Remarkably, these magnon modes are stable regardless of the orientation and magnitude of the spin accumulation.

The result for the spin current at the right interface is written in the familiar Landauer form:

$$\langle j^s \rangle = \int \frac{d\omega}{2\pi} \mathcal{T}(\omega) \left[N_B\left(\frac{\hbar\omega - \mu_{||}}{k_B T}\right) - N_B\left(\frac{\hbar\omega}{k_B T}\right) \right], \quad (4.18)$$

where $\mathcal{T}(\omega) = \alpha'^2 \hbar d \omega (\omega - \mu_{||}/\hbar) |G_\omega(0,d)|^2 / V \omega_F^2$ is the transmission coefficient, $N_B(\epsilon) = (e^\epsilon - 1)^{-1}$ is the Bose distribution function and $G_\omega(x, x')$ is the retarded magnon Green's function that solves equation (4.12). Note that this spin current vanishes when $\mu_{||} = 0$.

In figure 4.4 the spin current injected into the right lead $x = d$ is plotted as a function of the spin accumulation at the left lead $x = 0$, where the polar angle θ_μ between the spin accumulation and the anisotropy axis is $\pi/2$. Our results show that for large bias, the spin accumulation affects the magnetic texture significantly. In particular, for $|\mu| > \hbar\omega_F/\alpha'$ there is a non-zero current even though the spin accumulation is perpendicular to the anisotropy axis. Such threshold behavior is also seen in experiments [98].

Figure 4.4: Threshold behavior in the spin current as a function of the bias. For $|\mu| > \hbar\omega_F/\alpha'$ there is a finite spin current when the spin accumulation is perpendicular to the anisotropy axis. This threshold behavior is caused by the deformation of the magnetic texture, generating a parallel component of the magnetization as is seen in figure 4.2a.

4.6 CONCLUSION

We have shown that a spin accumulation at the interface of a NM with an IFM affects the magnetic texture and thereby moderates the transfer of angular momentum across the interface. The magnetic texture is found analytically and is interpreted as a virtual DW, where the DW position always lies outside or at the boundary of the ferromagnet.

Note that we do not fix the magnetization of the IFM at the interface as was done by Sitte et al. [106] where a conducting ferromagnet is considered. There the authors demonstrate that by fixing the magnetization at the interface there is a critical current above which DWs are injected into the nanowire. Similarly, DWs are injected into our IFM system for sufficiently large spin accumulation and when the magnetization is fixed, which would physically correspond to a very large interface anisotropy.

Furthermore, we have shown that this interaction between the spin accumulation and the magnetization at the interface results in threshold behavior in spin Hall magnetoresistance and nonlocal magnon transport: When the spin accumulation exceeds the critical value $\hbar\omega_F/\alpha'$ the spin Hall magnetoresistance drops suddenly when the electric current is parallel to the anisotropy axis. Moreover, above the critical value a finite nonlocal magnon current can be measured even when the spin accumulation is oriented perpendicular to the anisotropy axis. These results provide a novel route to control both local and nonlocal spin transport signals via the electric current, without the need for an external

magnetic field. We provide a possible geometry for an experimental setup in appendix 4.D.

We have assumed that the system size is large relative to the exchange length λ . For a smaller system, the exchange energy of the magnet cannot compensate the spin transfer torque, which leads to spin-torque oscillator instabilities [107] that prevent the formation of the virtual DW texture. Furthermore, we assume that the contact size of the biased lead is small compared to the distance between the leads to ensure that the magnons do not form a Bose-Einstein condensate when $\mu_{||} > \hbar\omega_F/\alpha'$ [108].

The electric field required to arrive at the threshold for a Pt|YIG bilayer is still two orders of magnitude higher than electric fields that have recently been applied in this kind of system [109], but the expression for the critical electric field (4.10) holds for any material, hence the threshold is more accessible for materials with a lower spin density, for example.

Remarkably, it is often argued that threshold behavior in nonlocal magnon transport indicates a metastable spin superfluid state [98, 99, 110]. However, we have demonstrated that even a stable magnetic texture may also lead to threshold behavior in the nonlocal magnon transport. We expect that an external magnetic field or a non-zero Dzyaloshinskii-Moriya interaction (DMI) might smoothen the threshold behavior as this will affect the azimuthal angle of the virtual DW.

In future research our theory can be applied to interpret experimental results on such threshold behavior. Moreover, the model can be extended to antiferromagnets. Furthermore, the model can be enriched by considering the effects of a weak magnetic field or DMI.

APPENDIX

4.A DOMAIN WALL INJECTION FOR INSULATING FERROMAGNETS

Recently it was shown by Sitte et al. [106] that passing an electrical current through a conducting ferromagnetic nanowire injects domain walls (DWs) into the magnet when the magnetization at the interface is fixed. Physically, the situation corresponds to an interface anisotropy that is so large that it dominates all other terms and fixes the magnetization direction at the interface. Here we show that the same result can be achieved for an insulating ferromagnet (IFM) by means of a spin accumulation at the interface with a normal metal. We will attempt to follow the derivation by Sitte et al. as closely as possible.

The system of interest is a semi-infinite ferromagnetic nanowire with an easy axis along the wire. At the left end of the wire ($x = 0$) we fix the magnetization orientation $\mathbf{n}(0) = \hat{z}$. The free energy of this system is given by

$$F = \int_0^\infty \frac{1}{2} \left(A(\partial_x \mathbf{n})^2 + K\Pi(n_x) \right) dx \quad (4.19)$$

where A is the exchange interaction, and K the anisotropy for the one dimensional system (note that in the main text we consider a three dimensional system with translation invariance). Π is a general form for the anisotropy, from which we only require $\Pi(0) = 1$, $\Pi(1) = 0$ and is monotonic and differentiable with $\Pi'(0) = 0$.

At $x = 0$ there is a spin accumulation $\boldsymbol{\mu} = \mu_x \hat{x} + \mu_y \hat{y}$ (an accumulation in the z -direction does not contribute). The LLG equation is given in equation (2.25).

We aim to determine the critical spin accumulation energy below which there is a stable solution $\mathbf{n}$. Above this energy the dynamics will be slow and considered adiabatic, hence we will ignore dissipation. We thus reduce the LLG to

$$\partial_t \mathbf{n} = -\frac{\gamma}{M_s} \mathbf{n} \times \frac{\delta F_{\text{eff}}}{\delta \mathbf{n}}, \quad (4.20)$$

where we defined

$$F_{\text{eff}} = F + \int_0^\infty \Omega_\mu dx, \quad (4.21)$$

with the Berry phase like term Ω_μ satisfying

$$\frac{\delta}{\delta \mathbf{n}} \left(\int_0^\infty \Omega_\mu dx \right) = \delta(x) \frac{M_s}{\gamma \hbar} \frac{g^{\uparrow\downarrow}}{4\pi s} (\mathbf{n} \times \boldsymbol{\mu}). \quad (4.22)$$

Now we consider F_{eff} as an action with corresponding Lagrangian

$$\mathcal{L} = \frac{A}{2} (\partial_x \mathbf{n})^2 + \frac{K}{2} \Pi(n_x) + \Omega_\mu. \quad (4.23)$$

We may also define a Hamiltonian density

$$\mathcal{H} = \frac{A}{2} (\partial_x \mathbf{n})^2 - \frac{K}{2} \Pi(n_x), \quad (4.24)$$

which should be conserved (w.r.t. x , i.e. translationally invariant). So evaluating at $x \rightarrow \infty$ we have that

$$\frac{A}{2} (\partial_x \mathbf{n})^2 - \frac{K}{2} \Pi(n_x) = 0. \quad (4.25)$$

Next, we consider the x component of the LLG and integrate to obtain

$$\int_0^\infty \left(\dot{n}_x + \frac{\gamma}{M_s} A (n_y \partial_x^2 n_z - n_z \partial_x^2 n_y) \right) dx = -\frac{g^{\uparrow\downarrow}}{4\pi s \hbar} \mu_x. \quad (4.26)$$

For a static solution $\dot{n}_x = 0$, and with the boundary conditions $n_y(0) = n_z(\infty) = \partial_x n_z(\infty) = 0$ and $n_z(0) = 1$ we use partial integration

$$\int_0^\infty n_y \partial_x^2 n_z dx = \int_0^\infty n_z \partial_x^2 n_y dx + \partial_x n_y|_{x=0}, \quad (4.27)$$

to obtain

$$\partial_x n_y|_{x=0} = -\frac{g^{\uparrow\downarrow}}{4\pi s} \frac{M_s \mu_x}{\gamma A \hbar}. \quad (4.28)$$

If we evaluate equation (4.25) at $x = 0$, where we have $\partial_x \mathbf{n} \perp \hat{z}$, and insert the above result, we obtain the condition

$$\mu_x \leq \frac{4\pi s \gamma \hbar}{g^{\uparrow\downarrow} M_s} \sqrt{AK} = \frac{\hbar \omega_F}{\alpha'}, \quad (4.29)$$

for a stationary solution. For $\mu_x > \hbar \omega_F / \alpha'$ the texture thus becomes unstable, and a numerical calculation demonstrates that domain walls are injected into the insulating ferromagnet.

4.B PROPERTIES OF THE DOMAIN WALL PROFILE

As stated in the main article a stationary solution to the bulk part of the LLG equation is the domain wall, written in spherical coordinates as

$$\mathbf{n}_0 = \hat{x} \sin \theta \cos \varphi + \hat{y} \sin \theta \sin \varphi + \hat{z} \cos \theta, \quad (4.30)$$

with φ a constant azimuthal angle throughout the nanowire and θ the polar angle given by

$$\theta = 2 \arctan \left(e^{(x_{DW} - x)/\lambda} \right). \quad (4.31)$$

Here x_{DW} is the position of the domain wall (DW) and λ the DW width. To show that this indeed is a solution to the LLG equation, we first prove some usefull identities. For clarity we define $X = (x_{DW} - x)/\lambda$

- We have the following identities

$$\sin(\theta) = \operatorname{sech}(-X); \quad (4.32)$$

$$\cos(\theta) = \tanh(-X). \quad (4.33)$$

Indeed $\operatorname{sech}^2(-X) + \tanh^2(-X) = 1$, so there must be some angle θ such that $\sin(\theta) = \operatorname{sech}(-X)$ and $\cos(\theta) = \tanh(-X)$. Now we will show that this is satisfied by equation (4.31). This angle θ must satisfy

$$\begin{aligned} \tan\left(\frac{\pi}{2} - \theta\right) &= \cot(\theta) = \frac{\tanh(-X)}{\operatorname{sech}(-X)} \\ &= \sinh(-X) = \frac{e^{-X} - e^X}{2}. \end{aligned} \quad (4.34)$$

Now we can find a and b such that

$$\begin{aligned}\tan(a) &= e^{-X}, \quad \text{and} \\ \tan(b) &= e^X = \cot(a) = \tan\left(\frac{\pi}{2} - a\right).\end{aligned}\tag{4.35}$$

That is

$$\begin{aligned}a &= \frac{\pi}{2} - \arctan(e^X), \quad \text{and} \\ b &= \arctan(e^X).\end{aligned}\tag{4.36}$$

Now continuing equation (4.34)

$$\begin{aligned}\frac{e^{-X} - e^X}{2} &= \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)} \\ &= \tan(a - b) \\ &= \tan\left(\frac{\pi}{2} - 2\arctan(e^X)\right),\end{aligned}\tag{4.37}$$

where we used the difference formula of the tangent for the last identity of the first line. Thus we obtain that indeed $\theta = 2\arctan(e^X)$.

- The following identities hold

$$\lambda \partial_x \theta = -\sin \theta;\tag{4.38}$$

$$\lambda \partial_x \sin \theta = -\cos \theta \sin \theta;\tag{4.39}$$

$$\lambda^2 \partial_x^2 \sin \theta = \cos 2\theta \sin \theta;\tag{4.40}$$

$$\lambda \partial_x \cos \theta = \sin^2 \theta;\tag{4.41}$$

$$\begin{aligned}\lambda^2 \partial_x^2 \cos \theta &= -2 \sin^2 \theta \cos \theta; \\ &= (\cos 2\theta - 1) \cos \theta.\end{aligned}\tag{4.42}$$

We only need to show the first identity and the rest will follow trivially. Note that from equations (4.32) and (4.33) we have $e^X = \tan \theta/2$, so

$$\lambda \partial_x \tan \theta/2 = -e^X = -\tan \theta/2\tag{4.43}$$

$$= \lambda \frac{\partial_x \theta}{2} \cos^{-2} \theta/2.\tag{4.44}$$

Hence, rewriting the above gives

$$\lambda \partial_x \theta = -2 \sin \theta/2 \cos \theta/2 = -\sin \theta.\tag{4.45}$$

LLG equation is written as

$$(1 + \alpha \mathbf{n} \times) \partial_t \mathbf{n} = -\frac{\gamma}{M_s} \mathbf{n} \times \left[A \partial_x^2 \mathbf{n} + K n_z \hat{\mathbf{z}} + \delta(x) \lambda \alpha' \left(\mathbf{n} \times \frac{\boldsymbol{\mu}}{\hbar \omega_F} \right) \right].\tag{4.46}$$

Now we set out to obtain the DW texture solution to the bulk part of equation (4.46) and use that to satisfy the boundary condition given in the main text. To show that $\mathbf{n}_0$ in equation (4.30) indeed satisfies the LLG equation we derive

$$\lambda^2 \partial_x^2 \mathbf{n}_0 + (\mathbf{n}_0 \cdot \hat{\mathbf{z}}) \hat{\mathbf{z}} = \mathbf{n}_0 \cos 2\theta, \quad (4.47)$$

which is obtained readily with the help of equations (4.40) and (4.42). As this is clearly parallel to $\mathbf{n}_0$, all terms in the LLG equation vanish.

4.C SPIN HALL MAGNETORESISTANCE

In this section we compute the resistance of a normal metal wire connected to a ferromagnetic insulator whose magnetization is parallel to the applied electric current. As the electric current will generate a spin accumulation at the interface, perpendicular to the magnetization, we expect an increase in the resistance once the applied voltage is above a threshold value, such that $|\mu| > \hbar\omega_F/\alpha'$ where the spin accumulation deforms the magnetic texture to allow for a nonzero spin current out of the normal metal.

Considering the spin current $\mathbf{j}_x^s$ flowing perpendicular to the interface (the vector part describes the orientation of the spin), the relevant spin diffusion equations are [111]:

$$j_z^e = \frac{\sigma}{e} \partial_z \mu_e - \frac{\sigma \theta_{SH}}{2e} \partial_x \mu_y; \quad (4.48)$$

$$\frac{\hbar}{2e} \mathbf{j}_x^s = -\frac{\sigma}{2e} \partial_x \mu - \frac{\sigma \theta_{SH}}{e} \partial_z \mu_e \hat{\mathbf{y}}. \quad (4.49)$$

Here, σ is the electric conductivity of the normal metal in units $\Omega^{-1}\text{m}^{-1}$, $e > 0$ the elementary charge and θ_{SH} the spin Hall angle. μ_e is the electric potential, so $\partial_z \mu_e = F$ is the applied electric force. We rescale to make these equations dimensionless, defining $\tilde{x} = x/\lambda$, $\tilde{t} = t\omega_F$, $\tilde{\mu}_e = \mu_e/\hbar\omega_F$ and $\tilde{F} = F\lambda/\hbar\omega_F$. The currents will be normalized as $\tilde{\mathbf{j}}^e = \mathbf{j}^e/j_c^e$, with $j_c^e = \hbar\omega_F\sigma/\lambda e$, and $\tilde{\mathbf{j}}^s = \mathbf{j}^s/j_c^s$, with $j_c^s = \hbar\omega_F\sigma\lambda/2$. Furthermore, we introduce the dimensionless constant $c_j = 2ej_c^s/\hbar j_c^e$ such that, omitting the tildes for clarity, equations (4.48) and (4.49) reduce to

$$j_z^e = F - \frac{\theta_{SH}}{2} \partial_x \mu_y; \quad (4.50)$$

$$c_j \mathbf{j}_x^s = -\frac{1}{2} \partial_x \mu - \theta_{SH} F \hat{\mathbf{y}}. \quad (4.51)$$

The spin accumulation obeys the diffusion equation $\mu = \partial_x^2 \mu / l_s^2$, with l_s the rescaled (w.r.t. λ) spin diffusion length. The solution is of the form

$$\mu(x) = \mathbf{a}_- e^{-x/l_s} + \mathbf{a}_+ e^{(x+t)/l_s}, \quad (4.52)$$

with t the rescaled thickness of the normal metal (along the x axis). We have the following boundary conditions for equation (4.51):

$$\mathbf{j}_x^s(x = -t) = 0; \quad (4.53)$$

$$\mathbf{j}_x^s(x = 0) = \sin \theta_0 \hat{\boldsymbol{\phi}}, \quad (4.54)$$

where $\theta_0 = \theta(x = 0)$. The latter condition is obtained from the LLG equation in the ferromagnet, as discussed in the main text. For this particular orientation of the spin accumulation (that is $\mu_z = 0$), the solution for the DW position reduces to

$$x_{DW} = \begin{cases} 0, & \text{for } |\alpha' \mu_y| \geq 1; \\ -\text{arcsech} \sqrt{|\alpha' \mu_y|}, & \text{otherwise.} \end{cases} \quad (4.55)$$

Using equation (4.32) we obtain $\sin \theta_0$ and thereby $\varphi = \varphi_\mu + \arcsin(\sin \theta_0 / \alpha' \mu_y)$. Thus the boundary conditions at $x = 0$ become

$$\mathbf{j}_x^s(x = 0) = -\alpha' \boldsymbol{\mu} \quad (4.56)$$

for $|\alpha' \mu_y| \leq 1$. Otherwise

$$j_{x,x}^s(x = 0) = -\frac{\sqrt{(\alpha' \mu_R)^2 - 1}}{\alpha' \mu_R} \approx -\frac{\sqrt{(\alpha' \mu_y)^2 - 1}}{\alpha' \mu_y}; \quad (4.57)$$

$$j_{x,y}^s(x = 0) = -\frac{1}{\alpha' \mu_R} \approx -\frac{1}{\alpha' \mu_y}, \quad (4.58)$$

where we approximated around $\mu_x = 0$. We can solve the system exactly using numerics, or, with this approximation, obtain an analytical expression for a_- and a_+ , which allows us to determine the electrical current as a function of the applied electric force and study their non-linear dependence.

Furthermore, we generalize the result to the setup where the polar angle θ_j between the electric current and the anisotropy axis is varied. Effectively, the boundary condition in equation (4.54) gets rotated over the angle $-\theta_j$. The results are obtained numerically and discussed and illustrated in figure 4.3.

4.D FLUCTUATION ASSISTED TRANSPORT

The observable we are interested in is the average spin current into the right lead at $x = d \gg \lambda$. We use $\delta n_\theta = (\psi + \psi^*)/2$ and $\delta n_\varphi = (\psi - \psi^*)/2i$. Similar for the stochastic field $h_\theta^r = (h^r + h^{r*})/2$ and $h_\varphi^r = (h^r - h^{r*})/2i$. When taking the average, terms linear in any of the independent components of the stochastic field

$\mathbf{h}$ vanish. Note furthermore that ψ does not depend on the radial component $\mathbf{n}_0 \cdot \mathbf{h}$. Hence we find

$$\langle \delta \mathbf{n} \times \delta \mathbf{n} \rangle = n_0 \text{Im} \langle \psi^* \psi \rangle; \quad (4.59)$$

$$\langle \delta \mathbf{n} \times \mathbf{h}^r \rangle = n_0 \text{Im} \langle \psi^* h^r \rangle. \quad (4.60)$$

The superscript indicates that we are considering the stochastic field at the right lead. Thus, averaging over the spin current at the right lead, we will determine

$$\begin{aligned} \langle \mathbf{j}_s \rangle|_{x=d} \cdot \mathbf{n}_0 &= \frac{\hbar}{2} s d \left\langle \delta \mathbf{n} \times \left(\frac{\lambda}{d} \alpha' \delta \mathbf{n} + \mathbf{h}^r \right) \right\rangle \Big|_{x=d} \cdot \mathbf{n}_0 \\ &= \frac{\hbar}{2} s d \text{Im} \left\langle \psi^* \left(\frac{\lambda}{d} \alpha' \psi + h^r \right) \right\rangle \Big|_{x=d}, \end{aligned} \quad (4.61)$$

by working out to two terms on the right hand side separately. At the right interface we also have spin flip scattering, so now $\alpha(x) = \alpha_G + \lambda \alpha' [\delta(x) + \delta(x - d)]$. Starting from equation (4.12), we will use the magnon's Greens function, defined by the equation

$$\left[(1 + i\alpha) \frac{\omega}{\omega_F} + \lambda^2 \partial_x^2 - \cos 2\theta - i\delta(x) \lambda \alpha' \frac{\mu_{||}}{\mu_c} \right] G_\omega(x, x') = \lambda \delta(x - x'), \quad (4.62)$$

to express ψ in terms of the stochastic field:

$$\omega_F \psi_\omega(x) = -\gamma \int \frac{dx'}{\lambda} G_\omega(x, x') h_\omega(x'), \quad (4.63)$$

with $h = d\delta(x)h^l + h^b + d\delta(x - d)h^r$, and

$$\psi = \int \frac{d\omega}{2\pi} \psi_\omega e^{i\omega t}. \quad (4.64)$$

For our purposes, we only need to consider the system of equations

$$\left[(1 + i\alpha_G) \frac{\omega}{\omega_F} + \lambda^2 \partial_x^2 - \cos 2\theta \right] G_\omega(x, d) = 0; \quad (4.65)$$

$$\left[i\alpha' \left(\frac{\omega}{\omega_F} - \frac{\mu_{||}}{\hbar\omega_F} \right) + \lambda \partial_x \right] G_\omega(0, d) = 0; \quad (4.66)$$

$$\left[i\alpha' \frac{\omega}{\omega_F} - \lambda \partial_x \right] G_\omega(d, d) = 1. \quad (4.67)$$

To find a solution for $G_\omega(x, d)$ we attempt

$$G_\omega(x, d) = c_+ (-i\lambda k + \cos \theta) e^{ikx} + c_- (i\lambda k + \cos \theta) e^{-ik(x-d)}, \quad (4.68)$$

with $k(\omega) = \lambda^{-1} \sqrt{(1 + i\alpha_G)\omega/\omega_F - 1}$ so that equation (4.65) is satisfied. The remaining two equation are used to determine c_+ and c_- analytically.

We define the self energies

$$\Sigma_{\omega}(x, x') = i\alpha_G \frac{\omega}{\omega_F} \frac{1}{\lambda} \delta(x - x'); \quad (4.69)$$

$$\Sigma_{\omega}^l(x, x') = i\alpha' \left(\frac{\omega}{\omega_F} - \frac{\mu_{||}}{\mu_c} \right) \delta(x) \delta(x - x'); \quad (4.70)$$

$$\Sigma_{\omega}^r(x, x') = i\alpha' \frac{\omega}{\omega_F} \delta(x - d) \delta(x - x'). \quad (4.71)$$

We can Fourier transform the time coordinate and express the wave function in terms of the Greens function and the total stochastic field to write

$$\begin{aligned} \frac{\lambda}{d} \alpha' \langle \delta \mathbf{n} \times \delta \mathbf{n} \rangle \cdot \mathbf{n}_0 = & -2 \int \frac{d\omega}{2\pi} \int dx dx' dx'' dx''' \Sigma_{\omega}^r(x, x') G_{\omega}^*(x', x'') \\ & \left(\Sigma_{\omega}(x'', x''') F_{\omega} + \Sigma_{\omega}^l(x'', x''') F_{\omega}^l + \Sigma_{\omega}^r(x'', x''') F_{\omega}^r \right) G_{\omega}(x, x'''); \end{aligned} \quad (4.72)$$

$$= -2 \int \frac{d\omega}{2\pi} \text{Tr} \left[\hat{\Sigma}_{\omega}^r \hat{G}_{\omega}^{\dagger} \left(\hat{\Sigma}_{\omega} F_{\omega} + \hat{\Sigma}_{\omega}^l F_{\omega}^l + \hat{\Sigma}_{\omega}^r F_{\omega}^r \right) \hat{G}_{\omega} \right]. \quad (4.73)$$

Where we introduced a matrix (of infinite dimension) notation, where we interpret a function $f(x, x')$ as a matrix $\hat{f}$ with components labeled by x and x' . The matrix product is defined by integration $(\hat{f} \cdot \hat{g})_{xx'} = \int dx'' f(x, x'') g(x'', x)$. The trace is defined by integrating over the diagonal components. Similarly, we express

$$\langle \delta \mathbf{n} \times \mathbf{h}^r \rangle \cdot \mathbf{n}_0 = 2 \int \frac{d\omega}{2\pi} \text{Re} \left(\text{Tr} \left[\hat{\Sigma}_{\omega}^r \hat{G}_{\omega}^{\dagger} (\hat{G}_{\omega})^{-1} \hat{G}_{\omega} \right] F_{\omega}^r \right); \quad (4.74)$$

$$= 2 \int \frac{d\omega}{2\pi} \text{Re} \left(\text{Tr} \left[\hat{\Sigma}_{\omega}^r \hat{G}_{\omega}^{\dagger} \left(\hat{\omega} + \hat{H} + \hat{\Sigma}_{\omega} + \hat{\Sigma}_{\omega}^l + \hat{\Sigma}_{\omega}^r \right) \hat{G}_{\omega} \right] F_{\omega}^r \right); \quad (4.75)$$

$$= 2 \int \frac{d\omega}{2\pi} \text{Tr} \left[\hat{\Sigma}_{\omega}^r \hat{G}_{\omega}^{\dagger} \left(\hat{\Sigma}_{\omega} + \hat{\Sigma}_{\omega}^l + \hat{\Sigma}_{\omega}^r \right) \hat{G}_{\omega} \right] F_{\omega}^r. \quad (4.76)$$

Where we inserted an identity in the first line and read off the inverse matrix equation (4.62), to be inserted in the second line, with $H(x, x') = \lambda^{-1} \delta(x - x') (\lambda^2 \partial_x^2 - \cos 2\theta)$. Note that the term proportional to $\hat{\omega} + \hat{H}$ drops out, as it is purely imaginary because H is hermitian. Combining the two results we have

$$\begin{aligned} \langle \mathbf{j}^s \rangle|_{x=L} \cdot \mathbf{n}_0 = & -\hbar s d \int \frac{d\omega}{2\pi} \text{Tr} \left[\hat{\Sigma}_{\omega}^r \hat{G}_{\omega}^{\dagger} \left(\hat{\Sigma}_{\omega} (F_{\omega} - F_{\omega}^r) + \hat{\Sigma}_{\omega}^l (F_{\omega}^l - F_{\omega}^r) \right) \hat{G}_{\omega} \right]; \\ = & \hbar s d \int \frac{d\omega}{2\pi} \alpha' \frac{\omega}{\omega_F} \left\{ \alpha_G \frac{\omega}{\omega_F} \int \frac{dx}{\lambda} |G_{\omega}(d, x)|^2 \left[N_B \left(\frac{\omega}{T(x)} \right) - N_B \left(\frac{\omega}{T_r} \right) \right] \right. \\ & \left. + \alpha' \left(\frac{\omega}{\omega_F} - \frac{\mu_{||}}{\mu_c} \right) |G_{\omega}(d, 0)|^2 \left[N_B \left(\frac{\omega - \mu_{||}}{T_l} \right) - N_B \left(\frac{\omega}{T_r} \right) \right] \right\}, \end{aligned} \quad (4.77)$$

setting all temperatures equal yields the spin current as given in the main article.

We propose a geometry for an experimental setup to measure the threshold behavior in the non-local magnon transport. A top view is illustrated in figure 4.5. The NM leads are attached perpendicular to each other and the magnetization anisotropy is parallel to the length of the injecting lead, so that the spin accumulation is perpendicular to the anisotropy axis for the injecting lead. At the detecting lead the magnetization will be perpendicular to the length of the lead.

Figure 4.5: A geometry for an experiment to measure the threshold behavior in the non-local magnon transport. The NM leads (orange) are attached perpendicular to each other in such a way that the injecting lead is parallel to the anisotropy axis (blue arrow) of the FMI (green). The theoretical model described in this paper is a good approximation if the distance between the leads d is much larger than the length of the leads l .

GUIDING MAGNONS BY DOMAIN WALLS

Magnetic domain walls function as a wave guide for low energy magnons. In this chapter we develop the theory for the non-local transport of these bound magnons through a ferromagnetic insulator that are injected and detected electrically in adjacent normal metal leads by spin-flip scattering processes and the (inverse) spin-Hall effect. Our set-up requires a twofold degeneracy of the magnetic ground state, which we realize by an easy axis and hard axis anisotropy, in the ferromagnetic insulator. This is readily provided by a broad range of materials. The domain wall is a topologically protected feature of the system and we obtain the non-local spin transport through it. Thereby we provide a framework for reconfigurable magnonic devices.

5.1 INTRODUCTION

Magnonic devices, where information is processed and transmitted by the manipulation and control of spin waves, have multiple benefits over electronic devices, ranging from lower power usage to higher processing speeds [34, 35, 112, 113]. To realize the control of spin waves, the magnetic domain wall (DW) has been proposed as a wave guide, because it hosts a soft spin wave mode that propagates along the DW [90, 91, 114–116]. The added benefit of using DWs is that such a magnonic device is easily reconfigured for instance by applying an external magnetic field or by bulk spin waves [117, 118]. Furthermore, the DW is a solitary wave which enjoys topological protection from the boundary conditions [13], i.e. between two different magnetic domains there is always a DW along the entire interface between these domains. The DW can be moved, bent and contain Bloch points, but still house the bound spin wave modes after such reconfigurations [90, 91].

Unlike bulk magnons that usually have an energy gap due to anisotropies and external magnetic fields, the bound magnons on the DW are always gapless, irrespective of microscopic details. As such, they are easily excited at low energies. The gapless nature of the bound magnons as well as their robustness to DW deformations also has an intuitive interpretation as a topological property of the magnetic texture: The DW is a boundary between two topologically

distinct phases in which the bulk magnons have opposite polarizations, thus there must be a gapless mode at the interface of the two domains, analogous to the edge states of topological insulators [72]. Physically, this mode corresponds to oscillations of the DW position. Recent simulations have shown that the bound DW modes are also robust against impurities and can be excited well below the bulk energy gap [119].

This work presents a comprehensive theory of the non-local transport of spin angular momentum between two normal metal (NM) leads that is facilitated by the spin waves bound to the DW. The system is illustrated in figure 5.1. From the well-established notion of the bound magnon [90, 115, 120], we go beyond the single mode studies to a model that captures the entire spectrum of the spin waves that are localized at the DW, excluding the bulk magnons. We explicitly include both the coupling to the lead electrons as well as the Gilbert damping of the magnetization in the ferromagnetic insulator (FMI), both of which are crucial for the description of realistic experiments. This allows for an experimental verification in a setup similar to recent setups that measure the non-local magnon transport through homogeneously magnetized textures [24, 104]. Our model is further enriched by an additional anisotropy perpendicular to the dominant anisotropy. We analytically obtain a Landauer formula for the transport and numerically demonstrate promising properties of the DW channel. Namely, the DW does not only serve as a wave guide, but also as a wave focus, yielding long magnon relaxation length scales.

Recent theoretical works also show that DWs support superfluid spin transport, but this requires a global $U(1)$ symmetry with respect to rotations of the spin around an easy axis [72]. However, it remains a challenge to confirm the superfluid nature of the transport experimentally [121], mainly because the physical systems do not respect the symmetry. For our model, only a $\mathbb{Z}(2)$ symmetry of the magnetic ground state is required, i.e. having a twofold degenerate ground state composing the two magnetic domains separated by the DW. Such a symmetry is easily realized in physical systems with an easy axis and hard axis anisotropy.

In the following we will first set out to discuss the bulk dynamics of the bound magnon in detail. Then we introduce the setup and model for the non-local magnon transport between two leads that we consider and obtain the main results. After demonstrating the relevant results on the relaxation length, we conclude by proposing two potential applications. The appendix to this chapter contains more details on the derivation of our theoretical model.

Figure 5.1: (a) A FMI (light red) with a DW texture has an easy y - and z -axis anisotropies $K_y \ll K_z$. NM leads (light yellow) can inject and absorb magnons due to spin-flip scattering. By applying a bias in the spin population in the left lead, e.g. using the spin Hall effect by applying an electric current in the y direction, there is a net transport of spin angular momentum between the leads. This transport is facilitated by spin waves bound to the DW (green). (b) Magnons that are bound to the DW are modeled by oscillations in the azimuthal angle $\varphi(x, t)$ and the DW position $y_{\text{DW}}(x, t)$ (dashed). Notice that the spin wave displacement (green) along the y -axis is proportional to $-y_{\text{DW}}$.

5.2 BOUND MAGNONS

We describe the magnetic texture in spherical coordinates $\mathbf{n} = \mathbf{M}/M_s = \hat{\mathbf{x}} \sin \theta \sin \varphi + \hat{\mathbf{y}} \cos \theta + \hat{\mathbf{z}} \sin \theta \cos \varphi$, with M_s the saturation magnetization. We start out by considering the classical Lagrangian for the uniaxial ferromagnet

$$L_{\text{FM}}[\mathbf{n}] = \int dV (\phi \chi - \mathcal{H}[\mathbf{n}]); \quad (5.1)$$

$$\mathcal{H}[\mathbf{n}] = \frac{1}{2} A (\nabla \mathbf{n})^2 - \frac{1}{2} K_y (n_y^2 + \kappa n_z^2),$$

where $\chi = -\hbar s \cos \theta$ is the conjugate momentum field to the azimuthal angle field φ , with s the spin density, A is the exchange stiffness and $\kappa = K_z/K_y \ll 1$ is the ratio of the two easy-axis anisotropy constants. The easy y -axis anisotropy constant K_y together with the boundary conditions $\mathbf{n}|_{y=\pm\infty} = \pm \hat{\mathbf{y}}$ ensure that the ground state for the magnetic texture is that of a DW. Indeed, the equations of motion give a static solution $\mathbf{n}_0(\varphi_0, \theta_0)$ with

$$\theta_0 = 2 \arctan \exp \left(\frac{y - y_{\text{DW}}}{\tilde{\lambda}} \right), \quad (5.2)$$

where $\tilde{\lambda} \equiv \sqrt{A/K_y(1-\kappa)}$ is the DW width, which is larger than the exchange length $\lambda = \sqrt{A/K_y}$, due to the off-axis anisotropy K_z . This weaker easy z -

axis anisotropy enforces a Néel DW, $\varphi_0 = 0$, and we assume the system is thin and homogeneous along this axis. Therefore all densities presented here are quantities per area, which are obtained from the volumetric densities by multiplication with the film thickness.

The DW solution is parametrized by its position y_{DW} and azimuthal angle φ . However, its energy is independent of the DW position. Therefore the DW position is a zero mode of the system, and there is a soft mode associated with its long-wavelength fluctuations [13]. To capture the dynamics of this mode, we promote the domain wall parameters y_{DW} and φ to slowly varying functions of the spatial coordinate x and time t . These collective coordinates $y_{\text{DW}}(x, t)$ and $\varphi(x, t)$ that describe the low-energy DW dynamics are illustrated in figure 5.1b. Fluctuations in y_{DW} and φ perturb the magnetization as $\delta \mathbf{n} \approx \sin \theta_0 [\hat{\varphi} \delta \varphi - \hat{\theta} \delta y_{\text{DW}} / \tilde{\lambda}]$. It is clear, from the prefactor $\sin \theta_0$ which vanishes when $|y/y_{\text{DW}}| \gg 1$, that these are indeed the bound modes. By aligning the spin accumulation in the normal metal parallel to the magnetization at the center of the DW, we directly excite only thermal magnons on the DW.

We expand the FMI Lagrangian in the collective coordinates and evaluate the integral over y , yielding

$$L_{\text{FM}} = \int_{\text{FMI}} dx \left[-2\hbar s \dot{\varphi} y_{\text{DW}} - 2K_y \tilde{\lambda} \left(\frac{\lambda^2}{2} (\partial_x \varphi)^2 + \frac{1-\kappa}{2} (\partial_x y_{\text{DW}})^2 + \kappa \varphi^2 \right) \right]. \quad (5.3)$$

The canonical momentum conjugate to φ is $p = -2\hbar s y_{\text{DW}}$, so we quantize by promoting our collective coordinates to operators with canonical commutation relation $[\varphi(x), p(x')] = i\hbar \delta(x - x')$. The bound magnon is then described by the boson field

$$\zeta(x, t) = \begin{pmatrix} \eta(x, t) \\ \eta^*(x, t) \end{pmatrix}; \quad \eta = \sqrt{s\tilde{\lambda}} \left(\varphi - i \frac{y_{\text{DW}}}{\tilde{\lambda}} \right). \quad (5.4)$$

The minus sign in the definition of η is due to the fact that the spin wave oscillation transverse to the DW, i.e. along the y -axis, is proportional to the oscillation of $-y_{\text{DW}}$ as illustrated in figure 5.1b.

Next, we add Gilbert damping and obtain the equations of motion for our bound magnon field ζ ,

$$\left\{ (i\sigma_3 - \alpha_G) \hbar \partial_t - \frac{K_y}{s} \left[\lambda^2 \partial_x^2 + \kappa(1 + \sigma_1) \right] \right\} \cdot \zeta = 0, \quad (5.5)$$

where α_G is the Gilbert damping parameter and σ_i are the Pauli matrices. From the plane wave ansatz $\zeta \sim e^{i(qx - \omega t)} \zeta_\omega$ the dispersion relation follows and we obtain four branches determined by solving for ω from

$$\lambda^2 q^2(\omega) = -\kappa + i\alpha_G \frac{\omega}{\omega_K} \pm \sqrt{\kappa^2 + \frac{\omega^2}{\omega_K^2}}, \quad (5.6)$$

with $\omega_K = K_y/\hbar s$ the anisotropic resonance frequency. Here the two branches with the minus sign in front of the square root are exponentially decaying surface states which do not contribute significantly to the transport as long as the decay length ($\sim \lambda/\sqrt{2\kappa}$) is shorter than the distance between the leads. The two plus sign branches correspond to bound magnons that propagate forward and backward along the DW. Note also that these propagating modes are indeed gapless, which is due to the fact that the domain wall position is a zero mode. The effect of the localized modes on the DW profile at the interface have been taken into account by the interfacial interaction Lagrangian (see below). Because of the additional anisotropy κ , the propagating modes furthermore exhibit a linear dispersion $\omega = v|q|$ at long wavelengths, with a characteristic velocity $v = \sqrt{2\kappa}\lambda\omega_K$. This velocity v is in the order of 50m/s for Yttrium-Iron-Garnet (YIG) with $K_y \approx 600\text{Jm}^{-3}$ and $\kappa = 0.1$. For YIG anisotropies are small, so other materials with larger anisotropy have larger velocities and frequency gaps.

5.3 NON-LOCAL MAGNON TRANSPORT

The set-up we study is illustrated in figure 5.1a. At the left NM lead there is a net spin bias $\mu \equiv \mu_\uparrow - \mu_\downarrow$ which can be generated by the spin Hall effect [92]. At the interface with the FMI the exchange interaction between itinerant spins in the NM and spins on the FMI lattice accommodates spin-flip scattering and thereby transfer of spin angular momentum, which excites the bound magnon states of the DW. By aligning the spin accumulation in the normal metal parallel to the magnetization at the center of the domain wall, we make sure that we only directly excite thermal magnons in the domain wall. Gapless magnons then travel to the right interface and can be detected by the reciprocal processes.

The Lagrangian of the FMI-NM interaction is given by

$$L_{int} = - \int_{FM} dV \int_{NM} dV' J_{int}(\mathbf{r}, \mathbf{r}') \mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}'), \quad (5.7)$$

with $\mathbf{s}$ the electron spin density of the NM, which we assume to be in a biased steady state, and $J_{int}(\mathbf{r}, \mathbf{r}')$ the interfacial exchange interaction. Furthermore, we assume the electrons to be non-interacting such that we are able to integrate them out. The Keldysh partition function of the coupled electron-DW magnon system is given by

$$Z = \int \mathcal{D}[\eta, c] e^{\frac{i}{\hbar}(S_{FMI}[\eta] + S_{int}[\eta, c] + S_{NM}[c])} = \int \mathcal{D}[\eta] e^{\frac{i}{\hbar}S[\eta]}, \quad (5.8)$$

Figure 5.2: As the distance d between the leads increases, the signal strength decays exponentially, $I_r \sim e^{-d/d_r}$, over a relaxation length scale d_r . In (a) the temperature dependence of this length scale is investigated and fitted with $d_r^{-1} = a + bT + cT^2$ for different values of the Gilbert damping parameter α_G and setting $\kappa = 10^{-2}$. (b) The relaxation length as a function of the ratio between the two anisotropies κ for $k_B Ts/K_y = 10^{-4}$.

where S is the effective action for the magnons on the DW that is obtained by integrating out the leads:

$$S = S_{FMI} + \langle S_{\text{int}} \rangle_c + \frac{i}{2\hbar} \left(\langle S_{\text{int}}^2 \rangle_c - \langle S_{\text{int}} \rangle_c^2 \right) + \mathcal{O}(S_{\text{int}}^3), \quad (5.9)$$

with

$$\langle \dots \rangle_c = \int \mathcal{D}[c] \dots e^{\frac{i}{\hbar} S_{NM}[c]}. \quad (5.10)$$

Then, after applying a Hubbard-Stratonovich transformation to the Keldysh partition function we obtain a stochastic equation of motion for the Fourier transformed bound magnon field $\zeta(x, t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \zeta_\omega(x)$ (see the appendix for more details),

$$\left[\hbar\omega\sigma_3 - H_{DW} - \frac{1}{2\hbar}\sigma_1\Sigma_\omega \right] \cdot \zeta_\omega = h_\omega, \quad (5.11)$$

where the Hamiltonian operator is given by $H_{DW} = K_y[\lambda^2\partial_x^2 - \kappa(1 + \sigma_1)]/s$ and the self-energies $\Sigma_\omega = \Sigma_\omega^b + \Sigma_\sigma \Sigma_{\sigma,\omega}^l + \Sigma_\omega^r$ are given by

$$i\Sigma_{\sigma,\omega}^l = \frac{\alpha'\hbar}{8}\delta(x-x')\delta(x)(\hbar\omega + \sigma\mu) \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix}; \quad (5.12)$$

$$i\Sigma_\omega^r = \frac{\alpha'}{4}\delta(x-x')\delta(x-d)\hbar^2\omega \begin{pmatrix} \pi^2 - 4 & \pi^2 + 4 \\ \pi^2 + 4 & \pi^2 - 4 \end{pmatrix}; \quad (5.13)$$

$$i\Sigma_\omega^b = 2\alpha_G\delta(x-x')\hbar^2\omega\sigma_1, \quad (5.14)$$

with $\sigma \in \{\uparrow, \downarrow\}$ and the effective spin-flip length

$$\alpha' = \frac{\tilde{\lambda} g^{\uparrow\downarrow}}{2\pi s}. \quad (5.15)$$

Here, $g^{\uparrow\downarrow}$ is the spin-mixing conductance [122, 123], which is related to the interfacial exchange coupling J_{int} [95]. The stochastic field $h_\omega = h_\omega^l + h_\omega^b + h_\omega^r$ satisfies the following fluctuation dissipation rules

$$\langle h_\omega^l h_{\omega'}^{l\dagger} \rangle = 2\pi i \delta(\omega + \omega') \sigma_1 \sum_\sigma \Sigma_{\sigma,\omega}^l F_{\sigma,\omega}^l; \quad (5.16)$$

$$\langle h_\omega^{r\backslash b} h_{\omega'}^{r\backslash b\dagger} \rangle = 2\pi i \delta(\omega + \omega') \sigma_1 \Sigma_\omega^{r\backslash b} F_\omega^{r\backslash b}, \quad (5.17)$$

with $F_{\sigma,\omega}^l = n_B[(\hbar\omega + \sigma\mu)/k_B T] + 1/2$ expressed in terms of the Bose-Einstein distribution function n_B , and $F_\omega^{r\backslash b} = n_B[\hbar\omega/k_B T] + 1/2$, assuming homogeneous temperature. Notice the differences with the regular fluctuation dissipation theorem [95]: We obtain expressions that resemble a fluctuation dissipation theorem for each of the two polarizations of the electronic spin as long as there is a finite spin accumulation μ . The reason for this is that the leads are in a biased steady state and not in equilibrium.

The stochastic equation of motion 5.11 can be solved formally in terms of a Green's function $g_\omega(x)$ for the bound magnon,

$$\zeta_\omega(d) = \int dx g_\omega(x) \cdot h_\omega(x). \quad (5.18)$$

While it is possible to obtain this Green's function analytically for our model, it is however rather involved so that we do not show it here. Finally, the spin current into the right lead is obtained in a Landauer form as

$$I_r = \int d\omega \sum_\sigma \mathcal{T}_\sigma(\omega) \left(F_\sigma^l(\omega) - F^r(\omega) \right); \quad (5.19)$$

$$\mathcal{T}_\sigma = \frac{1}{16s\hbar^2} \text{Re} \left\{ \text{Tr} \left[g_{-\omega}^\dagger(0) \cdot \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \Sigma_\omega^r \cdot g_\omega(0) \cdot \sigma_1 \cdot \Sigma_{\sigma,\omega}^l \right] \right\}. \quad (5.20)$$

See the appendix for more details on the derivation of this result.

To study the long-range spin transport, we investigate the decay of the signal strength as a function of the distance d between the leads for different values of the bulk Gilbert damping α_G . In figure 5.2a we plot the extracted relaxation length scale as a function of temperature and in figure 5.2b as a function of the anisotropy ratio κ . In Yttrium-Iron-Garnet the DW width λ is estimated to be around 78nm for typical values of the exchange A and anisotropy energy K_y [102, 103]. Hence, we obtain relaxations lengths in the order of micrometers. We fit the temperature dependent curves to the function $d_r^{-1} = a + bT + cT^2$, where

b is a term we expect from the linear dispersion relation [24, 124] and a and c are corrections due to the spin current contact resistance at the NM|FMI interfaces, and the deviation from the linear dispersion for small κ or large ω . This form is motivated by the general form $d_r \sim v\tau_t$ where v is the spin wave speed and τ_t is the total relaxation time obtained from the combination of different relaxation processes.

Remarkably, far from being detrimental to the transport, larger anisotropies κ actually enhance the relaxation length, as shown in figure 5.2b. This can be explained from the spectrum in equation (5.6) of the bound magnons: For larger κ , the propagating modes have a higher velocity $v \propto \sqrt{\kappa}$, and the linear regime where all modes propagate with this velocity becomes more dominant.

An important note regards how the remaining bulk of the magnet responds to the biased leads. For large enough bias, the texture will be distorted [125], so we have the restriction $\mu \ll \hbar\omega_K/\alpha'$. Furthermore, the interfacial torques on the texture do not lead to significant a distortion or displacement of the DW, but in fact counteract each other on each side of the DW.

5.4 APPLICATION

With the theory at hand of non-local transport of ungapped magnons bound to the DW, we propose two types of devices that employ this specific mechanism. The state of the FMI determines the transport and thereby determines the spin angular momentum injected into the right lead. Due to the inverse spin-Hall effect [92, 97], this will generate an electric current which can be detected as is done in typical non-local magnon transport experiments [24, 93, 94]. The gapless magnons are easily excited and thus more efficiently generated compared to bulk magnons in homogeneously magnetized systems. However, the width $\tilde{\lambda}$ of the DW channel often is narrow compared to the total length of the interface available for the bulk magnons. The efficiency of the injection and detection of a signal thus depends on system and material parameters such as the energy gap, magnon conductivity, anisotropy and the DW width. This complexity is captured in the transmission function in equation (5.20). Future research could investigate in what materials and dimensions the bound DW magnon transport dominates over bulk homogeneous magnons.

The first type that we propose is a configurable switching device, which is illustrated in figure 5.3. In the left lead an input spin bias, e.g. by means of an electric current employing the spin-Hall effect, sends a signal to one of the two detectors on the right lead. The position of the DW determines which and can be configured by an external magnetic field aligned with one of the magnetic domains.

The second application we propose is a device that counts magnetic domains by counting the DWs. Data stored as magnetic domains, for instance on the

Figure 5.3: The output signals (V_1 and V_2) are switched on by configuring the DW (blue): When the DW position is in the top half, as illustrated, there will be a finite voltage measured by V_1 due to the inverse spin-Hall effect, whereas there will be no signal measured by V_2 . The proposed materials used are platinum (Pt) for the normal metal leads, and Yttrium-Iron-Garnet (YIG) for the FMI as these are commonly used in non-local magnon transport experiments [93].

Figure 5.4: Instead of reading out the orientation of the magnetic domains, this device measures whether a DW (blue) passes and from there infers the data stored on the magnetic track.

racetrack memory [1], is read out by measuring the orientation of the magnetization. But by attaching two leads to such a strip of magnetic domain data as illustrated in figure 5.4, one can read out when a DW passes the leads. Then, the counting of 0's and 1's is just a matter of keeping track of when a DW passes. A possible restriction to the DW speed might come from the velocity of the bound spin waves, which has to be significantly larger than the DW velocity.

Magnetic storage devices often use ferromagnetic materials, because the data can be easily read out using the local magnetic field generated by the magnetic domains. Although antiferromagnets have many benefits over ferromagnetic devices - for instance DWs move much faster - they are so far not employed as materials for storage devices, because it is hard to distinguish and read out the domains. Measuring the antiferromagnetic DWs instead circumvents this problem and might pave a way for the introduction of antiferromagnetic materials to the playfield of magnetic storage devices. An extension of our theoretical model to an antiferromagnetic texture will benefit this path.

The energy gap, proportional to ω_K for bulk magnons that is opened by the anisotropy makes the gapless bound magnons more easily accessible. This is also the case for antiferromagnets [126]. Hence, the suitability of materials for applications exploiting the DW channel for non-local magnon transport is characterized mainly by the strength of their magnetic anisotropy.

5.5 CONCLUSION

The ferromagnetic texture with a DW allows two types of spin waves: gapped magnons that propagate through the bulk and ungapped bound magnons that reside on the DW. We have developed a theoretical model for the non-local transport of magnons that is accommodated on DWs. We extracted the transport of spin angular momentum between two normal metal leads by treating the bound spin waves as oscillations of the collective coordinates of the DW solution (see figure 5.1b), which explicitly exclude the bulk magnons.

The resulting framework leads to an analytical expression of the transport that takes the familiar Landauer form given in equation (5.19), which leads to the intuitive interpretation: The DW connects two baths which have a relative difference in spin population. This bias thus results in a net flow of spin angular momentum. Moreover, there are no significant requirements on our considered system, such as $U(1)$ symmetry, that would make the setup hard to realize experimentally. On the contrary, our results show that the breaking of $U(1)$ symmetry is actually beneficial to the transport, because it leads to longer relaxation lengths. Our numerical results show furthermore that the obtained transport enjoys benefits from the DW guiding in the form of a large relaxation length. We recognize these features as benefits from the linear and gapless

dispersion relation. However, an analytic understanding of the relaxation length is still an open issue.

We expect these DW wave guides to be very versatile and (topologically) robust [90]. For example, we expect that our theory is also applicable to curved DWs [90, 116], as long as the radius of curvature is large compared to the domain wall thickness. The presence of a pinning potential, caused by impurities in real materials, can cause the bound magnons to have a gap in their energy spectrum [127]. However, they can still be excited well below the bulk magnon energy gap [119]. Based on our findings, we proposed two types of devices that employ the DW bound magnons to switch a signal by configuring the DW and to read magnetic memory data. Future research should first focus on the experimental verification of our theoretical results before investigating the realizability of these devices. Theoretical progress is yet to be made in extending our model to antiferromagnets.

APPENDIX

5.A SPIN TRANSPORT FORMALISM

In this appendix we give some more details on the derivation of the spin transport from the action of the total system. The full action for our system consists of three parts: the ferromagnetic insulator (FMI), the interaction (int) and the normal metal (NM) leads. The Lagrangian part of the FMI is given in the main text. We assume the action of the electrons in the NM leads to be quadratic in the fermionic operators and diagonal in the spin label. The Lagrangian of the interaction part is given by the exchange interaction between spins from the FMI and the NM,

$$L_{int} \approx -\tilde{\lambda} s \pi \int_{FMI} dx \delta(x) \left(s_x(0) \varphi - \frac{1}{2} s_z(0) \varphi^2 + 2s_y(0) \frac{y_{DW}}{\pi \tilde{\lambda}} \right). \quad (5.21)$$

Here, we assumed a coarse grained, long wavelength coupling $J_{int}(\mathbf{r}, \mathbf{r}') = J_{int} \delta(\mathbf{r} - \mathbf{r}') \delta(x)$, expanded in the domain wall collective coordinates φ and y_{DW} , and dropped constant terms. Now we quantize by promoting our coordinates to operators with:

$$\mathbf{s} = \frac{1}{2} \left[(c_{\uparrow}^{\dagger} c_{\downarrow} + c_{\downarrow}^{\dagger} c_{\uparrow}) \hat{\mathbf{x}} + i(c_{\uparrow}^{\dagger} c_{\downarrow} - c_{\downarrow}^{\dagger} c_{\uparrow}) \hat{\mathbf{y}} + (c_{\uparrow}^{\dagger} c_{\uparrow} - c_{\downarrow}^{\dagger} c_{\downarrow}) \hat{\mathbf{z}} \right], \quad (5.22)$$

where $c_\sigma^\dagger$ creates an electron with spin σ . We express the Hamiltonian $H = H_{int} + H_{FMI}$ in terms of the boson field η as

$$H = \int_{FMI} dx \left[\frac{K_y}{s} \left(\lambda^2 (\partial_x \eta) (\partial_x \eta^\dagger) + \frac{1}{2} \kappa (\eta + \eta^\dagger)^2 \right) + \pi \delta(x) \left(\sqrt{s\tilde{\lambda}} s_x \frac{1}{2} (\eta + \eta^\dagger) - \frac{1}{8} s_z (\eta + \eta^\dagger)^2 + \frac{i}{\pi} s_y \sqrt{s\tilde{\lambda}} (\eta - \eta^\dagger) \right) \right]. \quad (5.23)$$

Now we insert the DW solution into $\mathbf{S} = -s\hat{\mathbf{n}}$ and introduce $n_{\uparrow\downarrow} = n_x \pm in_y$. Then we write

$$S_{int} = -\frac{s\hbar}{2} \sum_{p \in \pm} p \int dt \int_{FMI} dV \int_{NM} dV' J_{int}(\mathbf{r}, \mathbf{r}') \times \left[c_\uparrow^{p\dagger} c_\downarrow^p n_\uparrow^p + c_\downarrow^{p\dagger} c_\uparrow^p n_\downarrow^p + \left(c_\uparrow^{p\dagger} c_\uparrow^p - c_\downarrow^{p\dagger} c_\downarrow^p \right) n_z^p \right]. \quad (5.24)$$

Where p indicates which part of the Keldysh contour we evaluate. The electronic Green's functions are defined as

$$iG_{\sigma\sigma'}^{pp'} = \langle c_\sigma^{p\dagger} c_{\sigma'}^{p'} \rangle_c. \quad (5.25)$$

We assume that the leads are in a biased stationary state and neglect the influence of the interfacial coupling J on the electrons, so the Green's functions depend only on the difference in coordinates. Because the lead action is diagonal in spin label, we have that $G_{\sigma\sigma'}^{pp'} = 0$ when $\sigma \neq \sigma'$. So

$$\langle S_{int} \rangle_c = i \frac{\pi\hbar}{16} \sum_{p,p' \in \pm} pp' \int dt dt' \int dx dx' J_{int} \delta(x) \delta(x') \quad (5.26)$$

$$\times p' \delta_{p,p'} \sum_{\sigma \in \uparrow\downarrow} \sigma G_\sigma^{pp} [\zeta^p]^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \zeta^{p'}.$$

$$\langle S_{int} \rangle_c^2 = -\frac{s^2 \pi^2 \tilde{\lambda}^2 \hbar^2}{4} \sum_{p,p' \in \pm} pp' \int dt dt' \int dx dx' J_{int}^2 \delta(x) \delta(x') \quad (5.27)$$

$$\sum_{\sigma \in \uparrow\downarrow} \left(G_\sigma^{pp} G_\sigma^{p'p'} - G_\sigma^{pp} G_\sigma^{p'p'} \right) \cos \varphi^p \cos \varphi^{p'}.$$

And using Wick's theorem

$$\langle c_{\sigma_1}^{p_1\dagger} c_{\sigma_2}^{p_2\dagger} c_{\sigma_3}^{p_3\dagger} c_{\sigma_4}^{p_4} \rangle = G_{\sigma_4\sigma_1}^{p_4p_1} G_{\sigma_2\sigma_3}^{p_2p_3} - G_{\sigma_2\sigma_1}^{p_2p_1} G_{\sigma_4\sigma_3}^{p_4p_3} \quad (5.28)$$

we find

$$\begin{aligned}
\langle S_{int}^2 \rangle_c = & \frac{s^2 \hbar^2}{4} \sum_{p,p' \in \pm} pp' \int dt dt' \int dx dx' J_{int}^2 \delta(x) \delta(x') \\
& \times \sum_{\sigma \in \uparrow \downarrow} \left[G_{\sigma}^{p'p} G_{\bar{\sigma}}^{pp'} (\tilde{\lambda} \pi \phi^p + 2i\sigma y_{DW}^p) (\tilde{\lambda} \pi \phi^{p'} - 2i\sigma y_{DW}^{p'}) \right. \\
& \left. + \left(-G_{\sigma}^{pp} G_{\sigma}^{p'p'} + G_{\sigma}^{pp} G_{\bar{\sigma}}^{p'p'} + G_{\sigma}^{p'p} G_{\sigma}^{pp'} \right) \pi^2 \tilde{\lambda}^2 \cos \phi^p \cos \phi^{p'} \right], \quad (5.29)
\end{aligned}$$

such that

$$\begin{aligned}
\langle S_{int}^2 \rangle_c - \langle S_{int} \rangle_c^2 = & \frac{s \tilde{\lambda} \hbar^2}{16} \sum_{p,p' \in \pm} pp' \int dt dt' \int dx dx' J_{int}^2 \delta(x) \delta(x') \\
& \times \sum_{\sigma \in \uparrow \downarrow} [\zeta^p]^T \left[G_{\sigma}^{p'p} G_{\bar{\sigma}}^{pp'} \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \right. \\
& \left. + (G_{\sigma}^{p\bar{p}} G_{\sigma}^{\bar{p}p} - G_{\sigma}^{pp} G_{\sigma}^{pp'}) \delta_{p,p'} \pi^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] \zeta^{p'}. \quad (5.30)
\end{aligned}$$

Therefore the correction to the DW magnon action due to the interaction with the electrons is given by

$$\begin{aligned}
\Delta S = & S - S_{FMI} = \langle S_{int} \rangle_c + \frac{i}{2\hbar} \left(\langle S_{int}^2 \rangle_c - \langle S_{int} \rangle_c^2 \right) \\
& \approx \frac{i\hbar}{16} \sum_{p,p' \in \pm} pp' \sum_{\sigma \in \uparrow \downarrow} \int dt dt' \int dx dx' J_{int}^2 \delta(x) \delta(x') [\zeta^p]^T \\
& \left\{ \pi p' \sigma G_{\sigma}^{pp} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{s \tilde{\lambda}}{2} J_{int} \left[G_{\sigma}^{p'p} G_{\bar{\sigma}}^{pp'} \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \right. \right. \\
& \left. \left. + (G_{\sigma}^{p\bar{p}} G_{\sigma}^{\bar{p}p} - G_{\sigma}^{pp} G_{\sigma}^{pp'}) \delta_{p,p'} \pi^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] \right\} \zeta^{p'}. \quad (5.31)
\end{aligned}$$

Next, we perform the Keldysh rotation

$$\zeta^p = \frac{1}{\sqrt{2}} (\zeta^c + p \zeta^q); \quad G^{pp'} = \frac{1}{2} (p' G^R + p G^A + G^K), \quad (5.32)$$

where ζ^c is the classical field, ζ^q is the quantum field, G^R is the retarded, G^A the advanced and G^K the Keldysh Green's function. Then we have

$$\Delta S = -\frac{1}{2} \int dt dt' \int dx dx' \left\{ [\zeta^c]^T \Sigma^A \zeta^q + [\zeta^q]^T \Sigma^R \zeta^c + [\zeta^q]^T \Sigma^K \zeta^q \right\}, \quad (5.33)$$

with

$$\Sigma^{R \setminus A} = -\frac{i}{16} \hbar J_{int} \delta(x - x') \delta(x) \sum_{\sigma \in \uparrow \setminus \downarrow} \left\{ \delta(t - t') \pi \sigma G_{\sigma}^K(0, 0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right. \quad (5.34)$$

$$+ \frac{1}{2} \tilde{\lambda} s J_{int} \left[\left(G_{\tilde{\sigma}}^{R \setminus A}(\tilde{t}) G_{\sigma}^K(-\tilde{t}) + G_{\tilde{\sigma}}^K(\tilde{t}) G_{\sigma}^{A \setminus R}(-\tilde{t}) \right) \right. \\ \left. \times \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \right. \quad (5.35)$$

$$\left. - \pi^2 \delta(\tilde{t}) \left(G_{\sigma}^R(0) G_{\sigma}^K(0) + G_{\sigma}^K(0) G_{\sigma}^A(0) \right) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] \Big\},$$

$$\Sigma^K = -\frac{i}{32} \tilde{\lambda} s \hbar J_{int}^2 \delta(x - x') \delta(x) \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \quad (5.36)$$

$$\sum_{\sigma \in \uparrow \setminus \downarrow} \left(G_{\tilde{\sigma}}^A(\tilde{t}) G_{\sigma}^R(-\tilde{t}) + G_{\tilde{\sigma}}^R(\tilde{t}) G_{\sigma}^A(-\tilde{t}) + G_{\tilde{\sigma}}^K(\tilde{t}) G_{\sigma}^K(-\tilde{t}) \right),$$

with $\tilde{t} = t - t'$ and the spatial dependence of the Green's functions is omitted due to the delta functions. Now we Fourier transform

$$\Sigma(\omega) = \int dt e^{i\omega t} \Sigma(t, 0), \quad G(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} G(\omega), \quad (5.37)$$

to obtain

$$\Sigma^{R \setminus A}(\omega) = -\frac{i\hbar}{16} J_{int} \delta(x - x') \delta(x) \sum_{\sigma \in \uparrow \setminus \downarrow} \left\{ \pi \sigma \int \frac{d\omega'}{2\pi} G_{\sigma}^K(\omega') \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right. \\ + \frac{1}{2} \tilde{\lambda} s J_{int} \left[\int \frac{d\omega'}{2\pi} \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \right. \\ \times \left(G_{\tilde{\sigma}}^{R \setminus A}(\omega') G_{\sigma}^K(\omega' - \omega) + G_{\tilde{\sigma}}^K(\omega') G_{\sigma}^{A \setminus R}(\omega' - \omega) \right) \quad (5.38) \\ \left. - \pi^2 \int \frac{d\omega'}{2\pi} \frac{d\omega''}{2\pi} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right. \\ \left. \times \left(G_{\sigma}^R(\omega') G_{\sigma}^K(\omega'') + G_{\sigma}^K(\omega') G_{\sigma}^A(\omega'') \right) \right] \Big\},$$

$$\Sigma^K(x, x'; \omega) = -\frac{i}{32} \tilde{\lambda} s \hbar J_{int}^2 \delta(x - x') \delta(x) \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \\ \sum_{\sigma \in \uparrow \setminus \downarrow} \int \frac{d\omega'}{2\pi} \left(G_{\tilde{\sigma}}^A(0, \omega') G_{\sigma}^R(0, \omega' - \omega) \right. \quad (5.39) \\ \left. + G_{\tilde{\sigma}}^R(0, \omega') G_{\sigma}^A(0, \omega' - \omega) + G_{\tilde{\sigma}}^K(0, \omega') G_{\sigma}^K(0, \omega' - \omega) \right).$$

In the next step, we write

$$G_{\sigma}^{R \setminus A}(0, \omega') = \lim_{\epsilon \downarrow 0} \int \frac{d\omega''}{2\pi} \frac{A(\omega'')}{\omega' - \omega'' \pm i\epsilon'} \quad (5.40)$$

where $A(\omega)$ is the electronic spectral function at the interface. It is independent of the spin label σ because we are considering normal metal leads. Then, using the following identity

$$\delta(\omega - \omega') = \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\pi} \frac{1}{(\omega - \omega')^2 + \epsilon^2}, \quad (5.41)$$

we write

$$\begin{aligned} G_\sigma^K(\omega') &= \tanh\left(\frac{\hbar\omega' - \mu_\sigma}{2k_B T}\right) \left[G_\sigma^R(\omega') - G_\sigma^A(\omega')\right] \\ &= -iA(\omega') \tanh\left(\frac{\hbar\omega' - \mu_\sigma}{2k_B T}\right). \end{aligned} \quad (5.42)$$

Now, assuming $|\mu_\sigma|, |\omega|, k_B T \ll \varepsilon_F$, we expand

$$\tanh\left(\frac{\hbar\omega' - \mu_\sigma}{k_B T}\right) - \tanh\left(\frac{\hbar(\omega' - \omega) - \mu_\sigma}{k_B T}\right) \approx 2\delta(\hbar\omega' - \varepsilon_F)(\hbar\omega + \sigma\mu), \quad (5.43)$$

with $\mu = \mu_\uparrow - \mu_\downarrow$. Then we obtain

$$\begin{aligned} \Sigma^R - \Sigma^A &= -i\delta(x - x')\delta(x) \frac{\tilde{\lambda}s\hbar}{32\pi} J_{int}^2 A^2 \\ &\times \sum_\sigma (\hbar\omega + \sigma\mu) \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix}. \end{aligned} \quad (5.44)$$

Moreover, the identity

$$1 - \tanh(x) \tanh(y) = (\tanh(x) - \tanh(y)) \coth(x - y), \quad (5.45)$$

yields

$$\begin{aligned} \Sigma^K &= -i\delta(x - x')\delta(x) \frac{\tilde{\lambda}s\hbar}{32\pi} J_{int}^2 A^2 \\ &\times \sum_\sigma (\hbar\omega + \sigma\mu) \begin{pmatrix} \pi^2 - 4 & (\pi - 2\sigma)^2 \\ (\pi + 2\sigma)^2 & \pi^2 - 4 \end{pmatrix} \coth\left(\frac{\hbar\omega + \sigma\mu}{2k_B T}\right). \end{aligned} \quad (5.46)$$

We obtain the results for the right interface by replacing $\delta(x) \rightarrow \delta(x - d)$ and recognizing the interface damping length $\alpha' = \tilde{\lambda}sJ_{int}^2 A(\varepsilon_F/\hbar)^2/4\pi$. Then, using a Hubbard-Stratonovich transformation we obtain the equation of motion for the classical field,

$$\left[\hbar\omega\sigma_3 - H_{DW} - \frac{1}{2\hbar}\sigma_1 \left(\sum_\sigma \Sigma_\sigma^l + \Sigma^b + \Sigma^r \right) \right] \cdot \zeta^c(\omega) = h(\omega), \quad (5.47)$$

Between ζ^c and the conventional classical field $\zeta = (\zeta^+ + \zeta^-)/2$ is a factor of $\sqrt{2}$ difference, which is resolved by a redefinition of the stochastic field, resulting in

a factor of 1/2 in the fluctuation dissipation theorem equations (5.16) and (5.17). To obtain a Landauer formula we insert the stochastic equation of motion to write

$$\begin{aligned}
 \int dy \langle \partial_t n_z \rangle|_{x=d, t=0} &\approx \frac{-\pi}{8s} \text{Tr} \left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \langle \zeta^c(d, 0) (\zeta^c)^\dagger(d, 0) + \zeta^c(d, 0) (\zeta^c)^\dagger(d, 0) \rangle \right] \\
 &= \frac{-1}{16\pi s \hbar} \int d\omega \int d\omega' \text{Im} \left[\text{Tr} \left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \sigma_3 \cdot \left\{ \underbrace{\langle h(x, \omega) (\zeta^c)^\dagger(d, \omega') \rangle}_{\text{Fluctuations}} \right. \right. \right. \\
 &\quad \left. \left. \left. + \left(\underbrace{-i\alpha_G \hbar \omega + H_{DW}}_{\text{Transport}} - \underbrace{\frac{1}{2\hbar} \sigma_1 \cdot \Sigma^r}_{\text{Lead damping}} \right) \cdot \langle \zeta^c(d, \omega) (\zeta^c)^\dagger(d, \omega') \rangle \right\} \right] \right].
 \end{aligned} \tag{5.48}$$

This is the averaged change in magnetization at the right interface. We identify the total transport through the right lead as the combination of the lead damping and fluctuations. Neglecting the bulk to lead transport, i.e. at homogeneous temperature, we obtain the transport between the leads as

$$\begin{aligned}
 I_{l \rightarrow r} &= \frac{1}{8s\hbar^2} \int d\omega \text{Re} \left\{ \text{Tr} \left[g_{-\omega}^\dagger(0) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \sigma_3 \right. \right. \\
 &\quad \left. \left. \cdot \left(\frac{1}{2} \sigma_1 \Sigma^r(\omega) \cdot g_\omega(0) \cdot \sigma_1 \cdot \sum_\sigma \Sigma_\sigma^l(\omega) \left[F_\sigma^l(\omega) - F^r(\omega) \right] \right) \right] \right\}.
 \end{aligned} \tag{5.49}$$

INTERSUBLATTICE ENTANGLEMENT IN ANTI-FERROMAGNETS

Recent advancements in our understanding of ordered magnets call for a quantification of their entanglement content on an equal footing with classical thermodynamic quantities, such as the total magnetic moment. We evaluate the entanglement entropy (EE) between the two sublattices of a bipartite ordered antiferromagnet finding it to scale with volume. Thus, the EE density becomes an intensive property and is evaluated to be a universal dimensionality-dependent constant when exchange is the dominant interaction. Our analytic results are validated against the DMRG-based analysis of a one-dimensional (1D) system, finding good agreement. Further, our evaluated EE per bond provides a useful shortcut towards obtaining the central-cut EE in 1D, and the area law in higher-dimensional magnets.

6.1 INTRODUCTION

Antiferromagnets (AFMs) and their different phases pervade condensed matter physics. They underlie research fields, such as spin fluctuations mechanism of high- T_c superconductivity [128–130] and quantum spin liquids [131–135], as well as applications, such as exchange-biasing in magnetic read heads [136–138]. There exists a sharp contrast between the two widespread approaches towards understanding AFMs. In the first “quantum” approach [133–135], determining the ground state wavefunction for various model AFMs is a major goal. The true ground state, comprising distant entangled spins, is often not known and is complicated. Various numerical methods are employed in approximating the ground state and the excitations. The investigated system size is often limited by computational power. In the second “semiclassical” approach [139–144], a mean-field approximation is made and spatially resolved spin densities or magnetizations become classical fields within the Landau-Lifshitz description [145, 146]. A Néel ordered ground state is assumed and yields results consistent with many experiments [140, 142, 144, 147]. Macroscopic averaging is one of the reasons why nonlocal spin correlations and entanglement, fundamental in the quantum approach, appear to not affect several experiments consistent with the semiclassical approach.

Entanglement is an important resource in quantum information and computing protocols [148–150]. Two subsystems are said to be entangled if the wavefunction describing the total system cannot be factored into a product of two wavefunctions, one for each subsystem. Further, in the quantum approach discussed above, entanglement offers a powerful metric for characterizing ground state and excitations, mapping complicated wavefunctions existing in very high-dimensional spaces to a scalar quantity [134, 151–153]. In a widely employed technique, a 3-dimensional AFM is partitioned via a closed surface and the entanglement entropy (EE) between the two partitions is evaluated. The EE then bears a contribution proportional to the partition surface area, known as the area law [134, 153–155]. An additional, and sometimes universal, offset in the area law probes and characterizes topology and long-range entanglement in ground and excited states of the AFM [153, 156].

Recent works relate magnons with squeezed states studied in quantum optics [157–160] to demonstrate a nonzero entanglement in magnets [158–161], even in the mean-field approximation employed in the magnon-based semiclassical theory. This calls for a systematic quantification of EE as a quantum property describing such ordered magnets. We note two key motivators for this. First, the EE offers a simple scalar metric from which the proximity of a numerically evaluated ground state wavefunction of an ordered state can be measured. This facilitates analysis and approximations for quantum ground states. Second, the recent breakthroughs in robust experimental control of AFMs [140, 144, 147] obeying the semiclassical approach outlined above pave the way for using them as a resource or battery for entanglement [153, 159, 162]. Such efforts benefit from adding EE to the (quantum) thermodynamic description of magnets.

In this chapter, partitioning the AFM into two sublattices (figure 6.1), we establish the EE (density) as an extensive (intensive) quantum property characterizing ordered AFMs. Working within the mean-field approximation and magnon picture, we analytically evaluate the EE in the ground state, finding it to scale with the system size in the thermodynamic limit. Contradicting a preliminary expectation suggesting an increase in EE with exchange interaction strength [158, 160], the EE density is found to be a universal constant depending only on system dimensionality. This universality could offer useful benchmarking in analyzing quantum ground states. Examining its dependence on an applied magnetic field, we find that EE remains unchanged on approaching the spin-flop transition, where various classical response functions diverge [146, 163]. Our analytic results are found to agree well with a density matrix renormalization group (DMRG) analysis of a 1D AFM. Further, our evaluated EE per bond provides an analytic shortcut to evaluating EE for the widely employed system partitioning into two spatially separated regions [153].

6.2 MODEL

We consider a d -dimensional uniaxial AFM in an external magnetic field along the z -axis with N spins in each direction on sublattice A (B) pointing along the $(-)$ z -axis described by the Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{J}{\hbar^2} \sum_{i,\delta} \mathbf{S}_A(\mathbf{r}_i) \cdot \mathbf{S}_B(\mathbf{r}_i + \delta) \\ & - \sum_{\alpha \in \{A,B\}} \sum_i \left(\frac{K}{\hbar^2} \left(S_\alpha^{(z)}(\mathbf{r}_{i_\alpha}) \right)^2 + \gamma H S_\alpha^{(z)}(\mathbf{r}_{i_\alpha}) \right), \end{aligned} \quad (6.1)$$

with J the exchange coupling, K the anisotropy energy, δ the vectors to nearest neighbors, $\gamma < 0$ the gyromagnetic ratio and H the external magnetic field. Our final expression for EE does not depend on the lattice or spin S considered, as long as $\sum_\delta \delta = \mathbf{0}$. Thus, for concreteness and without loss of generality, we consider a square lattice. We closely follow the derivation of Ref. [158] in this section and expand upon their results by considering all magnon modes.

Figure 6.1 depicts a 2-dimensional system with $N = 5$ spins per sublattice per dimension. Assuming the magnetic field is below the spin-flop transition [i.e. $|\gamma|\hbar H < 2S\sqrt{K(J+K)}$], we apply a Holstein-Primakoff transformation to express the Hamiltonian in local bosonic operators a_i (b_j) which annihilate a spin flip on the i -th (j -th) site of the A (B) sublattice and satisfy the canonical bosonic commutation relations [145]. We assume periodic boundary conditions. After a Fourier transform we obtain the Hamiltonian up to second order in these ladder operators assuming high S :

$$\mathcal{H} = \sum_{\mathbf{k}} A_+ a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + A_- b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + C_{\mathbf{k}} a_{\mathbf{k}} b_{-\mathbf{k}} + C_{\mathbf{k}}^* b_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger; \quad (6.2)$$

$$A_\pm = JS c + 2KS \pm \gamma \hbar H; \quad C_{\mathbf{k}} = JS \sum_{\delta} e^{i\mathbf{k} \cdot \delta}. \quad (6.3)$$

Here, c is the coordination number and S the total spin per site. The sum over $\mathbf{k}$ runs over the Brillouin zone, i.e. $\mathbf{k} \cdot \hat{\mathbf{e}}_i \in \{-\pi/a, \dots, \pi(N-2)/L\}$, with $L = aN$ and a the lattice spacing. It is known [164] that the eigenstate depends only on the sum $A_+ + A_- \equiv 2A$, hence the magnetic field does not affect the entanglement entropy within the Néel state approximation.

We diagonalize the Hamiltonian by applying a Bogoliubov transformation $\mathcal{H} = \sum_{\mathbf{k}} E_{\mathbf{k}}^- \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + E_{\mathbf{k}}^+ \beta_{\mathbf{k}}^\dagger \beta_{\mathbf{k}}$ to obtain the eigen energy $E_{\mathbf{k}}^\pm = \pm |\gamma|\hbar H + E_{\mathbf{k}}^0$, with $E_{\mathbf{k}}^0 = \sqrt{A^2 - C_{\mathbf{k}}^2}$ and $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are squeezed sublattice magnons as given in Ref. [158]. Thus the ground state is squeezed [165]: Whereas a and b operate only on one sublattice, α and β operate on both. So we expect a finite entanglement in the ground state.

Figure 6.1: The studied system for $d = 2$, $N = 5$ close to a Néel state. Sublattice A (blue) and B (red) both have an equal amount of spins. The exchange interaction (orange) is restricted to nearest neighbors and is identical for each pair of neighbors. The external magnetic field (purple) is along the z direction.

In Fock notation we write a state in terms of $\otimes_{\mathbf{k}} |N_{b_{\mathbf{k}}}, N_{a_{\mathbf{k}}}\rangle_{sub, \mathbf{k}}$ in the sublattice basis, or in terms of $\otimes_{\mathbf{k}} |N_{\beta_{\mathbf{k}}}, N_{\alpha_{\mathbf{k}}}\rangle_{sq, \mathbf{k}}$ in the squeezed basis. The product over $\mathbf{k}$ ranges over all N^d allowed wave modes. The ground state is the squeezed vacuum $\otimes_{\mathbf{k}} |0, 0\rangle_{sq, \mathbf{k}}$. We use the two-mode squeezing operator $S(r_{\mathbf{k}}) = \exp r_{\mathbf{k}} (a_{\mathbf{k}} b_{\mathbf{k}} - a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger)$, with $r_{\mathbf{k}} = \text{arccosh} \sqrt{(A + E_{\mathbf{k}}^0)/2E_{\mathbf{k}}^0}$, and exploit the Baker-Hausdorff lemma [166] to express the ground state in terms of the sublattice basis

$$\begin{aligned} |G\rangle &= \otimes_{\mathbf{k}} |0, 0\rangle_{sq, \mathbf{k}} = \prod_{\mathbf{k}} S(r_{\mathbf{k}}) \otimes_{\mathbf{k}} |0, 0\rangle_{sub, \mathbf{k}} \\ &= \otimes_{\mathbf{k}} \frac{1}{\cosh r_{\mathbf{k}}} \sum_{l=0}^{\infty} (-\tanh r_{\mathbf{k}})^l |l, l\rangle_{sub, \mathbf{k}}. \end{aligned} \quad (6.4)$$

So, for each mode $\mathbf{k}$ we have a sum over l ranging over all occupation numbers of this mode.

6.3 ENTANGLEMENT ENTROPY

Using the Schmidt decomposition we derive the reduced density matrix

$$\begin{aligned} \rho_A &= \text{Tr}_B \rho = \sum_{\mathbf{n}} {}_B \langle \mathbf{n} | G \rangle \langle G | \mathbf{n} \rangle_B = \sum_{\mathbf{l}} {}_B \langle \mathbf{l} | G \rangle \langle G | \mathbf{l} \rangle_B \\ &= \otimes_{\mathbf{k}} \left(\sum_l \frac{\tanh^{2l} r_{\mathbf{k}}}{\cosh^2 r_{\mathbf{k}}} |l\rangle_{A, \mathbf{k}} {}_A \langle l| \right), \end{aligned} \quad (6.5)$$

where $\mathbf{n} = (n_1, n_2, \dots, n_{N^d})$ is a vector of N^d integers used to determine a pure Fock state $|\mathbf{n}\rangle_B$ on the B sublattice in the position basis, i.e. each integer n_i gives

Figure 6.2: The entanglement entropy per wavenumber $s(\mathbf{k})$ in $d = 1$ (a) and $d = 2$, (b) with $K/J = 10^{-4}$. Note that near the Brillouin zone boundaries, the EE vanishes. Whereas in the low wavenumber regime, the EE depends approximately only on the radius of the wavevector. Thus, a smaller contribution from the high- k modes allows a low- k continuum approximation to analytically obtain the total EE.

the occupancy of the i -th position on the B sublattice. Similarly, $\mathbf{l}$ is a vector used to determine a pure Fock state on the B sublattice in the momentum basis.

The eigenvalues of the reduced density matrix follow immediately

$$\rho_A |\mathbf{n}\rangle_A = \lambda_{\mathbf{n}} |\mathbf{n}\rangle_A; \quad \lambda_{\mathbf{n}} = \prod_{i=1}^{N^2} \frac{\tanh^{2n_i} r_{k_i}}{\cosh^2 r_{k_i}}. \quad (6.6)$$

This yields an analytic expression for the EE

$$\begin{aligned} S_{EE} &= - \sum_{\mathbf{n}} \lambda_{\mathbf{n}} \log \lambda_{\mathbf{n}} \\ &= \sum_{\mathbf{k}} 2 \left(\log \cosh r_{\mathbf{k}} - \sinh^2 r_{\mathbf{k}} \log \tanh r_{\mathbf{k}} \right). \end{aligned} \quad (6.7)$$

Note that in the sum over $\mathbf{k}$ all components k_i range from $-\pi/a$ to $\pi(N-2)/L$ in N steps of $2\pi/L$. The squeezing parameter $r_{\mathbf{k}}$ (and thereby the EE) does not depend on the applied external magnetic field when the magnetic field is below the spin-flop strength. Furthermore, there also is no S dependence. This result nicely extends the $k = 0$ entanglement found in Ref. [158].

For brevity we denote the term in the sum as $s(\mathbf{k})$ to investigate scaling behavior in the large N limit

$$S_{EE} = \sum_{\mathbf{k}} s(\mathbf{k}) \approx L^d \int \frac{d^d \mathbf{k}}{(2\pi)^d} s(\mathbf{k}). \quad (6.8)$$

From equation (6.3) we see that the only $\mathbf{k}$ dependence lies in $C_{\mathbf{k}}$. As figure 6.2 demonstrates for the one- and two-dimensional system, for $\mathbf{k}$ approaching the Brillouin zone boundary the contribution to the EE vanishes since $r_{\mathbf{k}} \rightarrow 0$. Furthermore, note that for small $\mathbf{k}$ the EE density depends mostly on the norm of $\mathbf{k}$. This warrants us to consider the small $\mathbf{k}$ limit and expand

$$\frac{|C_{\mathbf{k}}|^2}{A^2} \approx \frac{1}{(1 + \frac{2K}{cJ})^2} \left(1 - \frac{a^2 k^2}{4} \right), \quad (6.9)$$

with $k^2 = \sum_{i=1}^d k_i^2$ and for the square lattice $c = 2^d$, $\delta \cdot \hat{\mathbf{e}}_i = \pm a/2$ and $C_{\mathbf{k}}$ is real. This is the only point in our derivation where details of the lattice structure enter. For instance, for a honeycomb lattice we would have $c = 3$. This demonstrates that for $K \ll J$ our result is universal with respect to the lattice structure. Transforming the integral to spherical coordinates, leaving only the integral over the radial component k , we obtain

$$S_{EE} \approx 2^{\frac{d}{2}+1} N^d \frac{(1 + \frac{K}{2^{d-1}J})^{d/2}}{\pi^{d/2} \Gamma(\frac{d}{2})} I\left(d, \frac{K}{J}\right), \quad (6.10)$$

where Γ is the Euler gamma function and an integral representation for I is given in section 6.A where we derive analytically for $K \ll J$ that $I(d, K/J) \sim d^{-2}$. In figure 6.3 the ratio between the lattice result from equation (6.8) and the continuum limit from equation (6.10) is plotted versus N (a) and the ratio K/J (b). Note that the finite size effects quickly vanish as N increases and that the EE is independent of the ratio K/J in the regime $K/J \ll 1$. The error of the continuum result increases with d as it is caused by the increasing inaccuracy of the small $\mathbf{k}$ approximation: As d increases, so do the degrees of freedom and higher order products of wave numbers become relevant, which are not captured in our approximation.

6.4 DMRG FOR A 1D CHAIN

The DMRG method [167] is a versatile variational numerical tool to find the low energies and corresponding eigenstates of a strongly correlated 1D system in polynomial (in system size) time. The DMRG method is formulated in terms of matrix product states (MPS) [168], making the EE straightforwardly accessible.

The DMRG algorithm approaches the lowest energy state by optimizing the MPS locally, alternating over all sites. It retains only the D most relevant states, selecting them based on the highest singular value/weight s_i . Normalization of the state requires $\sum_i s_i^2 = 1$. The bond dimension $D < D_{\max}$ is set such that the weight of the discarded state is $\sum_{j>D} s_j^2 < 10^{-5}$. From the von Neumann entropy $S_{EE} = -2 \sum_{i=1}^D s_i^2 \log(s_i)$ we see that the minimal required bond dimension is related to the EE. In contrast with the two-sublattice partitioning discussed

Figure 6.3: The lattice entanglement entropy [equation (6.8)] divided by the approximated continuum result [equation (6.10)], plotted as a function of the system size (a) and the ratio K/J for $N = 50$ (b) for $d = 1$ (blue), $d = 2$ (yellow) and $d = 3$ (orange).

Figure 6.4: For the intersublattice entanglement (top) we divide our system into two parts, red and blue, by alternating sites. For the central cut entanglement (CCEE) (bottom) we split the system through the middle. As the DMRG results suggest, the CCEE is a good approximation of the intersublattice entanglement entropy per bond.

above, the DMRG utilizes the bipartition displayed in the bottom of Fig. 6.4. The area law tells us that, for gapped systems, the entanglement and thus the bond dimension is chosen independent of system length [154, 155, 169]. This showcases the great advantage of DMRG, requiring only linear (in N) memory allocation and polynomial computation time. The numerical results presented here are obtained with the TeNPy library [170].

The area law in 1D dictates a linear relation of the EE to the length of the cut: $S_{EE} = S_0 L_{cut}$. We recognize for the central-cut bipartition (bottom Fig. 6.4)

Figure 6.5: Comparison of approximate analytical ground state (6.4) with the full DMRG ground state for $N = 50$. The marks denote the numerical results for spin-3/2 (crosses), spin-2 (triangles) and spin-5/2 (circles) with the analytical result as a dashed line. The black results compare the central-cut entanglement entropy (CCEE) for DMRG to the analytical spatial entanglement density $S_{EE}/2N$. The red results compare the analytical with the numerical ground state energy (given in section 6.C).

$L_{cut} = 1$ and for intersublattice bipartition (top Fig. 6.4) $L_{cut} = 2N$. This allows us to compare the analytic intersublattice EE with the numerical CCEE. The black data in Fig. 6.5 confirm that the numerical CCEE matches the intersublattice EE density (no spin dependence) very well for spin $S = 3/2, 2, 5/2$ and a large range of anisotropy (K/J), moreover confirming numerically the independence of EE with respect to S . As discussed in section 6.B, the 1D spin-1 and spin-1/2 cases are not captured by the magnon approximation (for small K/J) due to dominating quantum fluctuations. We attribute the deviation in the CCEE at small $K/J < 5 \cdot 10^{-2}$ to the existence of low-energy modes. In this region there is either a gapless transition of a gapless phase, increasing correlations and entanglement. Specifically, for half-integer spin S the model at $K = H = 0$ is gapless by virtue of the Lieb-Schultz-Mattis theorem [171, 172]. On the other hand, for $S = 2$ a phase transition was identified at [173] $K/J \approx 0.0046$. In both cases, low-energy states are present, leading to a logarithmic dependence of the CCEE on the system size [154, 169].

Besides the good agreement in EE, Fig. 6.5 shows in red that the analytical and numerical ground state energies match, giving yet another hint that the squeezed state provides a good representation of the low energy physical behavior.

6.5 DISCUSSION

Some key features of and a comparison between our analytic and DMRG methods should be noted. In the former approach, the two-sublattice partitioning allowed us to express the total EE as a sum over $\mathbf{k}$ [equation (6.8)]. This further allowed demonstrating its scaling with system volume and obtaining analytic results [equation (6.10)] via the continuum approximation. As the EE becomes small for $\mathbf{k}$ close to the Brillouin zone boundary, the dependence of EE on the microscopic lattice is expected to be weak, as motivated in figure 6.2b. Thus we expect the result equation (6.8) to be valid for bipartite lattices in general, and in this sense to be universal.

Crucial to these simplifications has been our unconventional choice of the partitioning which admits translation invariance, see figure 6.1. Due to the equivalence between all nearest neighbour exchange “bonds”, the total EE per bond becomes a well-defined quantity. Thus, the area law of EE obtained with more commonly employed partitions dividing the AFM into two parts [134, 153], is understood as the EE per bond times the number of bonds that the partition intersects. This is illustrated in figure 6.4. In this manner, our finding of EE being an extensive property is consistent with the area law and our choice of the partition [153, 154].

Further, we find that EE does not depend on the applied magnetic field and remains constant [see equation (6.7)], as the system approaches the spin-flop transition from below. At this value of applied magnetic field, one of the magnon modes becomes gapless resulting in a divergence in various response function [146, 147, 163], such as the high-frequency susceptibility. Nevertheless, the EE remains well behaved and unperturbed as it is a property of the ground state wavefunction, which remains unaltered on approaching this transition from below. The DMRG analysis corroborates this independence of magnetic field for the EE, see Appendix 6.D.

Our 1D DMRG based results show a good agreement with respect to the energy and EE of the analytic squeezed ground state as displayed in figure 6.5. While the agreement between the energies is excellent, a small deviation in the EE demonstrates that it is more sensitive when comparing the quantum ground states. Quantum Monte Carlo[174] and Tensor Network[175] evidence for 2D antiferromagnetic Heisenberg model, suggests that for higher dimensions the magnon approximation shows less deviation for low K/J . Furthermore, a good agreement between our analytic EE per bond and the central-cut EE for long chains suggests a shortcut in evaluating the latter, disregarding the finite size effects.

Our methodology can be generalized to describe frustrated systems [133] starting from classically ordered states via multisublattice (three-sublattice for a triangular spin lattice [133]) models. Since finite temperature and dissipation effects encourage ordering, this approach offers a natural path towards

accounting for these nonidealities. Besides its relevance for quantum ground states, our work clarifies the presence and nature of entanglement in a broad range of AFMs [137, 144, 147] adequately described via magnon theory from low to high temperatures. Our work clarifies how spins on opposite sublattices of an AFM are entangled much more strongly due to exchange, as compared to spins on the same antiferromagnetic sublattice or in a ferromagnet that derive entanglement from anisotropy [159, 164]. Hence, to transfer this entanglement to external qubits, coupling them to opposite sublattices is preferred.

6.6 SUMMARY

We have investigated the entanglement entropy in the ground state of an ordered antiferromagnet using a two-sublattice partitioning. The translational invariance associated with the latter enabled us to obtain analytic results, consistent with numerics, providing insights and shortcuts in characterizing the entanglement content. We obtained and demonstrate universal behavior of the entropy with respect to several external parameters, such as the specific lattice structure, anisotropy or magnetic field. This finding helps benchmark numerically evaluated quantum ground states and guide the development of ordered antiferromagnets for useful quantum information protocols.

APPENDIX

6.A CONTINUUM RESULT INTEGRAL

The analytical continuum result for the entanglement entropy involves changing the sum to an integral over the d -dimensional volume. After the small- k approximation we are left with some prefactors, given in the main text, times a one dimensional integral over the normalized variable x given by:

$$I(d, x) = \int_0^{\pi/2 \sqrt{2(1+2^{1-d}x)}} dk k^{d-1} [g_k^+(d, x) \log g_k^+(d, x) - g_k^-(d, x) \log g_k^-(d, x)];$$

$$g_k^\pm(d, x) = \frac{1}{2} \left[\left(1 - \left(\frac{1}{1+2^{1-d}x} - k^2 \right)^2 \right)^{-1/2} \pm 1 \right]. \quad (6.11)$$

For $x = K/J = 0$ and small k we approximate the integrand by $-k^{d-1} \log(k)$ and $\pi/2\sqrt{2} \approx 1$ to obtain the proportionality $I(d, 0) \sim d^{-2}$.

6.B SPIN-1 FOR 1D

The magnon approach works particularly well in the S large and/or $D > 1$ limit. Here we present the results for $S = 1$ in 1D. Despite the good correspondence

Figure 6.6: (a) Comparison of approximate analytical ground state (equation 4 in the main text) with the full DMRG ground state for $N = 50$. The square marks denote the numerical results for spin-1 with the analytical result as a dashed line. The black results compare the central-cut entanglement entropy (CCEE) for DMRG to the analytical spatial entanglement entropy density $S_{EE}/2N$. The red results compare the analytical with the numerical ground state energy. (b) Comparison of approximate analytical long range order (G/S^2) (6.12) with the full DMRG ground state for $N = 50$. The marks denote the numerical results for spin-3/2 (circles), spin-2 (triangles) and spin-5/2 (crosses) with the analytical result as a dashed line.

for the ground state energy in Figure 6.6a, the CCEE (also in Figure 6.6a) clearly shows a discrepancy between the full DMRG (squares) and analytical squeeze state (dashed line) results. For $K \gg 1$ the Néel ordered phase is recovered. However, for $K \approx 1/2$ there is an Ising transition ($c = 1/2$, not directly observable in Figure 6.6a) into a Haldane phase (lacking long range order) [176]. The EE of 0.855 for small K is confirmed by other numerical studies [177]. The correspondence of the CCEE between the DMRG and analytical results is assumed to be a coincidence.

For spin $1/2$ the anisotropy term becomes trivial: $(S^{(z)})^2 \propto I$. The remaining model is simply the Heisenberg chain. This integrable model is critical with $c = 1$ and shows no resemblance to a magnon approximation [178]. However, this is to be expected in small S and 1D. For 2D it is known that both spin-1/2 and spin-1 exhibit Néel ordering even in the absence of anisotropy ($K = 0$) [174, 175].

6.B.1 Analytical long range order

The long range correlation function, signaling (Néel) order for the squeezed ground state is

$$\begin{aligned}
 G &= \frac{1}{2N} \sum_{j=1}^N \left\langle S_{A,j}^z S_{A,j+N/2}^z + S_{B,j}^z S_{B,j+N/2}^z \middle| S_{A,j}^z S_{A,j+N/2}^z + S_{B,j}^z S_{B,j+N/2}^z \right\rangle \\
 &= \left[S - \left(\frac{1}{4\pi} \int_0^{2\pi} dk \frac{1}{\sqrt{1 - \frac{1}{(1+K/J)^2} \cos(k)^2}} - \frac{1}{2} \right) \right]^2
 \end{aligned} \tag{6.12}$$

This compares qualitatively to the DMRG results, as we see in Figure 6.6b. The discrepancy is to be expected, because the squeezed states are built on the ansatz of high long range order. The low K/J numerical behavior is explained by the vicinity of a transition/gapless region.

6.C ANALYTICAL GROUNDSTATE ENERGY

The ground state energy of the analytical squeezed ground state is:

$$E_{GS} = -\frac{JS^2 Nc}{4} - KNS^2 - \frac{(JSc + 2K)N}{4} \left[1 - G \left(\frac{JSc}{JSc + 2K} \right) \right], \tag{6.13}$$

with elliptical integral

$$G(a) = \int_0^{2\pi} \frac{dq}{2\pi} \sqrt{1 - a^2 \cos q^2}. \tag{6.14}$$

This is the energy used in Fig. 5 in the main text to compare the DMRG results with.

6.D MAGNETIC FIELD INDEPENDENCE IN 1D

With DMRG we investigate the magnetic field dependence explicitly. As an example we look at spin-3/2 with and without magnetic field γH , as defined in (6.1). In Fig. 6.7 we show the CCEE for $\gamma H = 0$ and $\gamma H = 0.05J$. This value is chosen such that the system is below the spin-flop transition for all K plotted here. For $K > 0.3J$ there is no magnetic field dependence, as expected from the magnon approximation. In Section 6.4 we concluded that for small $K < 0.3J$, the system shows critical behavior and is not described by the magnon phase. The energy dependence for small K is another indicator that we have left the magnon regime.

Figure 6.7: The central cut entanglement entropy for $S = 3/2$, $N = 50$ and a range of K , with magnetic field (crosses $\gamma H = 0.05J$) and without (squares). For $K < 0.3J$, as described in Section 6.4, the system is not in the magnon phase, but enters a critical regime. For the former we see no dependence on magnetic field, while for the latter there is a deviation when introducing γH .

ENTANGLEMENT FOR LIFSHITZ FERMIONS

We investigate fermions with Lifshitz scaling symmetry and study their entanglement entropy in 1+1 dimensions as a function of the scaling exponent z . Remarkably, in the ground state the entanglement entropy vanishes for even values of z , whereas for odd values it is independent of z and equal to the relativistic case with $z = 1$. We show this using the correlation method on the lattice, and also using a holographic cMERA approach. The entanglement entropy in a thermal state is a more detailed function of z and T which we plot using the lattice correlation method. The dependence on the even- or oddness of z still shows for small temperatures, but is washed out for large temperatures or large values of z .

7.1 INTRODUCTION

In this chapter, we study entanglement properties of Dirac-Lifshitz fermions, with dispersion relations of the form:

$$\omega_k^2 = \alpha^2 k^{2z} + m^2, \quad (7.1)$$

with ω, k and m related to frequency, momentum and mass, with units specified in the next section. Furthermore, α is a dimensionful constant and z is a parameter, and we mostly consider cases where z is an integer in order to avoid issues with branch cuts (e.g. when $z = 1/4$, negative k would yield two branches). For $z = 1$, equation (7.1) yields the standard dispersion relation for a Dirac fermion with $\alpha = c$, the speed of light. We call z the Lifshitz exponent, and for $m = 0$, the theory has Lifshitz scaling symmetry acting as:

$$\omega \rightarrow \Lambda^z \omega, \quad \vec{k} \rightarrow \Lambda \vec{k}. \quad (7.2)$$

For this reason, they are called Lifshitz fermions. Besides the scale symmetry, there is rotation and translation symmetry and together with the scale symmetry they form the Lifshitz symmetry algebra. There are however no boost symmetries for $z \neq 1$, further discussions on symmetries can be found in e.g. [179–181]. Some earlier papers considered Lifshitz fermions with $z = 2$ and $z = 3$, see e.g. [182–184] in the context of the chiral anomaly, and [185, 186] where theories

with four-fermi interactions are included. Experimentally, larger than expected dynamical exponents can be seen in heavy fermion systems [187, 188].

It is interesting to study properties of Lifshitz fermions as a function of the dynamical exponent z , and in this chapter we will focus on correlation functions and entanglement entropy (EE), and in particular at the EE at the scale invariant point where $m = 0$. There is extensive literature on EE for free quantum field theories and lattice models with fermions. Various methods can be used, such as the correlation method in real space, the replica method, and Multi-scale Entanglement Renormalisation Ansatz (MERA). For a review see e.g. [189]. The strongest results exist for two-dimensional (1+1) relativistic conformal field theories, starting with the celebrated works of [53, 154]. For this reason, we focus on two dimensions in this chapter as well, to see how the known results from relativistic conformal field theories (CFTs) change when changing the value of z away from one. The holomorphic properties of relativistic CFTs do not, however, apply for $z \neq 1$, and the techniques therefore have to be adapted. We will use two techniques: the correlation method on the lattice [190–192], and the holographic cMERA approach [193].

On the lattice z denotes the range of the interactions: $z = 1$ is nearest neighbor, $z = 2$ next-to-nearest and large values of z imply long range interactions as illustrated in figure 7.1. The lattice spacing breaks conformal invariance, but our numerics are accurate enough to be close to the continuum limit. Furthermore, on the lattice, one can study how the EE changes in the presence of long-range interactions.

Entanglement entropy for Lifshitz bosons also have been studied, such as in the quantum Lifshitz model with $z = 2$ in 2+1 dimensions (see e.g. [194–201] for a partial list of references), and more generally for $z = d + 1$ in [202–204]. More recently studies for generic z were carried out in e.g. [57, 205, 206], see also [207–209] for further references on related topics. The results for bosons compared to fermions differ quite a lot. For even values of z , the EE for massless fermions turns out to vanish in the ground state, whereas for bosons, it is nonzero. For odd values, the EE is independent of z , i.e. all odd values for z give the same result as for $z = 1$. Again, this is very different from Lifshitz bosons, where the EE grows with z as expected from the lattice approach, since higher values of z indicate longer range correlators across the entanglement regions. For fermions, however, these correlations seem to cancel out in the EE. The distinction between even and odd values of z is quite striking for fermions, and indicate that one cannot simply extrapolate to continuous values of z , at least not in an obvious way. This picture is also confirmed by the holographic cMERA approach [193], which nicely reproduces our results obtained from the lattice correlation method. The use of the holographic cMERA approach is therefore of independent interest, as was illustrated for Lifshitz scalar fields in [206].

At finite temperature, we generate EE also for even values of z . We study both the small and large temperature regimes on the lattice, and we show that

the parity of z (even or odd) does not play an important role anymore at high temperature.

This chapter is organized as follows. In section 7.2 we introduce the basics, present the Lagrangian for free Lifshitz fermions and we determine the two-point correlator. We also review the exact results known for $z = 1$, and we make an ansatz for the EE for $z > 1$ using Lifshitz scale invariance at $m = 0$. In section 7.3, we discretize the model and compute the correlators on the lattice. We use the correlator method to compute the EE on the lattice and present various cases. In section 7.4, we rederive the zero temperature results using the cMERA approach for fermions. We end with some conclusions.

7.2 LIFSHITZ FERMIONS IN 1+1 DIMENSIONS

The Lagrangian for a two-component Lifshitz free fermion in two spacetime dimensions with coordinates $\{x^0, x^1\} = \{t, x\}$, is given by

$$\mathcal{L} = \bar{\psi}(\hbar\gamma^0 i\partial_0 + \hbar\alpha\gamma^1(i\partial_1)^z - \mu\alpha^2)\psi, \quad (7.3)$$

with $\bar{\psi} \equiv \psi^\dagger\gamma^0$, and Dirac matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{2\times 2}$. The path integral is then weighted with the standard factor $\exp(iS/\hbar)$ with $S = \int dt dx \mathcal{L}$. Here, α has SI-units m^z/s and is the speed of light for $z = 1$, and μ has units kg/m^{2z-2} and is the mass for $z = 1$. The units of ψ are $m^{-1/2}$ and for $z = 1$ we recover the relativistic Dirac Lagrangian. The Lifshitz scale transformations reads

$$t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x, \quad \psi \rightarrow \lambda^{-1/2} \psi \quad (7.4)$$

and is only a symmetry of the Lagrangian for $\mu = 0$. We will mostly consider the massless case in this chapter. Notice that the scaling weight for a fermion is independent of z (in any number of dimensions!), in contrast with a free boson, whose scaling weight is $(z-1)/2$. This fact has consequences for the EE which we discuss extensively in this chapter.

Space time translation symmetry, together with the Lifshitz scale symmetry generate the Lifshitz algebra in 1+1 spacetime dimension. There is no boost symmetry for generic $z \neq 1$, but there is a $U(1)$ symmetry acting as an overall phase on ψ . In the massless case, there is also chiral symmetry.

Our conventions are as follows. With the $(1+1)$ -dimensional metric $\eta = \text{diag}(+1, -1)$ we choose our Clifford algebra to be¹

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (7.5)$$

¹ In our basis, the charge conjugation matrix is chosen $\mathcal{C} = i\gamma^1$ satisfying $\mathcal{C}^\dagger = \mathcal{C}$, $\mathcal{C}^\dagger\mathcal{C} = 1$, and $\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -(\gamma^\mu)^T$. If we would impose the Majorana condition $\psi^\dagger\gamma^0 = \psi^T\mathcal{C}$, then it implies for the spinor components, $\psi_\pm^* = \mp i\psi_\pm$. The chiral Majorana components are not real, but this is because we are not in a basis with purely imaginary gamma matrices. The reality condition does respect the chiralities however, so $\psi_\pm$ each are Majorana-Weyl spinors.

Figure 7.1: The continuum system (a) of length L is partitioned in a segment of length l and its complement. The lattice system (b and c) has N sites and a lattice spacing ϵ . The interactions are depicted for $z = 1$ (b) and $z = 2$ (c).

Furthermore, we can define chiral components using

$$P_{\pm} = \frac{1 \pm \gamma^5}{2}, \quad \gamma^5 \equiv \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (7.6)$$

The Lagrangian then becomes

$$\mathcal{L} = \hbar \psi_+^\dagger (i\partial_0 + \alpha(i\partial_1)^z) \psi_+ + \hbar \psi_-^\dagger (i\partial_0 - \alpha(i\partial_1)^z) \psi_- - \mu \alpha^2 (\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+), \quad (7.7)$$

and only has chiral symmetry in the massless case, where ψ_+ and ψ_- transform with opposite phases. One can easily check that the action is real upon partial integration. The equation of motion is

$$(i\mathcal{D}_z - m)\psi \equiv \left(\gamma^0 i\partial_0 + \alpha \gamma^1 (i\partial_1)^z - m \right) \psi = 0, \quad (7.8)$$

with $m \equiv \frac{\mu \alpha^2}{\hbar}$. Contrary to the $z = 1$ case, for $z \neq 1$, these chiralities do not correspond to left or right movers which is why all holomorphic CFT techniques no longer apply. With the plane wave ansatz

$$\psi(t, x) = \int d\omega dk \psi(\omega, k) e^{-i\omega t - ikx}, \quad (7.9)$$

one derives the Lifshitz dispersion relation (7.1). After Fourier transformation, the action becomes

$$S = (2\pi)^2 \int d\omega dk \psi^\dagger(\omega, k) \left(\hbar \omega + \hbar \alpha \gamma^5 k^z - \mu \alpha^2 \gamma^0 \right) \psi(\omega, k), \quad (7.10)$$

and the two-point correlator, for $t \equiv t_1 - t_2$ and $x \equiv x_1 - x_2$, is

$$\begin{aligned}
 G_F(t, x) &\equiv \langle \psi^{\dagger\alpha}(t_1, x_1) \psi_\beta(t_2, x_2) \rangle \\
 &= i \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{(\omega - \alpha\gamma^5 k^z + m\gamma^0)^\alpha{}_\beta}{\omega^2 - \alpha^2 k^{2z} - m^2 + i\epsilon} e^{-i\omega t - ikx} \\
 &= \gamma^0 \left((i\mathcal{D}_z)^\dagger + m \right) G_B(t, x) .
 \end{aligned} \tag{7.11}$$

Here, the Lifshitz-scalar Green's function is given by

$$\begin{aligned}
 G_B(t, x) &= i \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{e^{-i\omega t - ikx}}{\omega^2 - \alpha^2 k^{2z} - m^2 + i\epsilon} \\
 &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{-ikx}}{2\omega_k} \left(e^{i\omega_k t} \theta(-t) + e^{-i\omega_k t} \theta(t) \right) ,
 \end{aligned} \tag{7.12}$$

with $\omega_k = \sqrt{\alpha^2 k^{2z} + m^2}$ the positive root. Notice the usual relation with the propagator of a scalar field, this time a scalar field with a Lifshitz dispersion relation (7.1). The propagator $G_F(t, x)$ satisfies the Lifshitz-Dirac equation with a delta function source because of the identity

$$\left(i\mathcal{D}_z - m \right) \gamma^0 \left(i\mathcal{D}_z^\dagger + m \right) = -\gamma^0 \left(\partial_0^2 + \alpha^2 (i\partial_1)^{2z} + m^2 \right) , \tag{7.13}$$

and because the scalar field propagator satisfies the Lifshitz-Klein-Gordon equation for the Green's function. The γ^0 appears because we are considering the propagator $\langle \psi^\dagger \psi \rangle$ instead of $\langle \bar{\psi} \psi \rangle$.

It is interesting to look at the case of a free massless scalar field with Lifshitz scaling. For the equal time correlator, we get

$$\langle \phi(x_1) \phi(x_2) \rangle = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{-ikx}}{2\omega_k} = \frac{1}{2\pi\alpha} 2^{-z} \sqrt{\pi} \frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} |x|^{z-1} . \tag{7.14}$$

Notice that this is consistent with the scaling weight $(z-1)/2$ for a scalar field in 1+1 dimensions. The result for this Fourier transform is formally valid for all values of z by analytic continuation of the Gamma function. If we restrict to integer values, we notice a difference between even and odd values of z , since $\Gamma\left(\frac{1-z}{2}\right)$ for even $z = 2n$ produces a factor $\Gamma\left(\frac{1}{2} - n\right) = \frac{(-4)^n n!}{(2n)!} \sqrt{\pi}$, whereas for odd $z = 2n + 1$, we get $\Gamma(-n)$ which diverges as $\Gamma(z)$ has a simple pole at $z = -n$. In higher dimensions, a similar phenomena happens, as the higher dimensional Fourier transform produces factors of $\Gamma\left(\frac{d-z}{2}\right)$. This divergence

needs to be regularized but we will not go further into this since it does not occur for fermions as we see now.

Similarly to the bosons, the fermionic two-point correlator is,

$$G_F(t, x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{-ikx}}{2\omega_k} \left[e^{-i\omega_k t} \left(\omega_k - \alpha k^z \gamma^5 + m\gamma^0 \right) \theta(t) - e^{i\omega_k t} \left(\omega_k + \alpha k^z \gamma^5 - m\gamma^0 \right) \theta(-t) \right]. \quad (7.15)$$

We now focus on the massless case with $\omega_k = \alpha|k|^z$ where the chiral components decouple, and take the equal time correlator obtained from the limit $t \rightarrow 0^+$, to get

$$\langle \psi_{\pm}^{\dagger}(x_1) \psi_{\pm}(x_2) \rangle = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikx} \left(1 \mp \text{sgn}(k)^z \right). \quad (7.16)$$

The result for the integral depends again on the even- or oddness of z :

$$z \text{ even} : \quad \langle \psi_{+}^{\dagger}(x_1) \psi_{+}(x_2) \rangle = 0, \quad \langle \psi_{-}^{\dagger}(x_1) \psi_{-}(x_2) \rangle = \delta(x_1 - x_2). \quad (7.17)$$

For odd values of z , we have $\text{sgn}(k)^z = \text{sgn}(k)$ and find

$$z \text{ odd} : \quad \langle \psi_{\pm}^{\dagger}(x_1) \psi_{\pm}(x_2) \rangle = \frac{1}{2} \left[\delta(x_1 - x_2) \mp \frac{1}{\pi} \frac{1}{(x_1 - x_2)} \right], \quad (7.18)$$

independent of z . This independence of z is consistent with the fact that the Lifshitz scaling weight for a fermion is independent of z and equal to $-1/2$ in $1+1$ dimensions. The expressions for the correlators are the two possibilities consistent with the Lifshitz symmetries with the correct scaling weight, as $\delta(\lambda x) = |\lambda|^{-1} \delta(x)$.

7.2.1 Entanglement entropy and relation to known results

What we learn from the analysis above in the continuum, is that at zero temperature and zero mass, the two-point function differs for even and odd values of z . In both classes, the correlator does not depend on z . So for odd z , the EE is the same as for the relativistic case with $z = 1$. In that case, the result for the vacuum EE in a subinterval of length l on the real infinite line is well known from conformal field theory, namely [52, 53, 154]

$$S = \frac{c}{3} \log \left(\frac{l}{\varepsilon} \right), \quad (7.19)$$

with $c = 1/2$ for a Weyl fermion and ε the UV cutoff which is the lattice spacing in the next section. For even values of z , the spatial correlators produce zero or delta functions, and this will not produce any entanglement. We show this explicitly using the lattice model and the cMERA approach in subsequent sections.

We can consider finite size effects, and for a relativistic CFT on a line of total length L and with periodic boundary conditions (see figure 7.1), we have [169]

$$S = \frac{c}{3} \log \left(\frac{L}{\pi \varepsilon} \sin \left(\frac{\pi l}{L} \right) \right), \quad (7.20)$$

up to some non-universal additive constant. This expression still obeys Lifshitz scaling properties, so it is a possible candidate for the Lifshitz EE for general values of z , but again only odd values. We show on the lattice that for odd values of z , the finite size effects do not depend on z , so we use the known results for $z = 1$. On a lattice with N sites and a subsystem of N_A sites, (7.20) becomes

$$S = \frac{c}{3} \log \left(\frac{N}{\pi} \sin \left(\frac{\pi N_A}{N} \right) \right). \quad (7.21)$$

For even values of z , finite size effects won't affect the spatial correlators as we show in the next section, so the EE still vanishes. Notice also the symmetry $N_A \rightarrow N - N_A$ which reflects one of the properties of EE in a pure state.

We now add temperature, still keeping $m = 0$ and $L \rightarrow \infty$. The result for the EE should still obey Lifshitz scale invariance, provided we scale the temperature appropriately, $T \rightarrow \lambda^{-z} T$. The only scale invariant and dimensionless quantities are

$$\frac{l}{\varepsilon}, \quad \frac{\varepsilon^z k_B T}{\alpha \hbar} \equiv \frac{\varepsilon^z}{\beta}, \quad (7.22)$$

and combinations thereof such as the cutoff independent quantity $l\beta^{-1/z}$. For $z = 1$ the result for the EE is known [169] and is given by

$$S = \frac{c}{3} \log \left(\frac{\beta}{\pi \varepsilon} \sinh \left(\frac{\pi l}{\beta} \right) \right). \quad (7.23)$$

This result holds when the system is infinitely long and in a thermal state.

At low temperatures, we obtain from (7.23),

$$S = \frac{c}{3} \left[\log \left(\frac{l}{\varepsilon} \right) + \frac{\pi^2 l^2}{6 \beta^2} + \mathcal{O}(l^4 / \beta^4) \right], \quad l \ll \beta, \quad (7.24)$$

consistent with the scaling properties for $z = 1$, for which l/β is scale invariant. Notice that a linear term proportional to l/β is absent in this Taylor expansion. Such a term would produce a volume law, which is what we expect at high

temperatures. Indeed, the high temperature regime computed from (7.23) yields

$$S = \frac{c}{3} \left(\pi \frac{l}{\beta} - \log \left(\frac{l}{\beta} \right) + \log \frac{l}{2\pi\epsilon} + \mathcal{O}(e^{-2\pi l/\beta}) \right), \quad l \gg \beta, \quad (7.25)$$

and we see a volume law linear in l appearing as the leading term.

These temperature corrections however no longer have the right scaling behavior when $z \neq 1$, but we use the scale invariant and dimensionless combinations (7.22) to make an ansatz for the temperature corrections. At small temperatures, we make an ansatz generalizing (7.24):

$$S = \frac{c}{3} \log \left(\frac{l}{\epsilon} \right) + f_2(z) \frac{l^2}{\beta^{2/z}} + \mathcal{O}(l^4/\beta^{4/z}), \quad l \ll \beta^{1/z}, \quad z \text{ odd}, \quad (7.26)$$

for some function $f_2(z)$ independent of any scale with $f_2(1) = c\pi^2/18$. This expansion only holds for odd values of z , because the leading term (the “area” term at zero temperature) for even values of z is absent. Notice again the absence of a linear term in l . This time, there is no a priori reason for it, but our lattice results will establish it. It in fact establishes that, for odd z , there are no odd powers of $l\beta^{-1/z}$ for small temperatures.

For even values of z , the lattice results show that all powers of $l\beta^{-1/z}$ appear, and we can make a low temperature expansion

$$S = f_1(z) \frac{l}{\beta^{1/z}} + f_2(z) \frac{l^2}{\beta^{2/z}} + \mathcal{O}(l^3/\beta^{3/z}), \quad l \ll \beta^{1/z}, \quad z \text{ even}, \quad (7.27)$$

for some functions $f_{1,2}$. The leading term in this expansion is already a volume law.

Similarly, at large temperatures, we generalize the $z = 1$ result to

$$S = \frac{l}{\beta^{1/z}} \left(g(z) + \frac{S_{\text{off}}(z)}{l\beta^{-1/z}} + \mathcal{O}(\beta^{2/z}) \right), \quad l \gg \beta^{1/z}, \quad (7.28)$$

for some function $g(z)$ with $g(1) = c\pi/3$ and a constant offset correction to the expansion $S_{\text{off}}(z)$. It is a non-trivial result that this is the leading term if we don’t assume that a volume law should come out at large temperature, as any higher power of $l/\beta^{1/z}$ would be dominant. There can be subleading terms similar as for $z = 1$, such as logarithmic terms, and we include them in the next section. Again, the lattice approach supports the ansatz (7.28) for both even and odd values of z , and in the next section, we give numerical values for $g(z)$ and $S_{\text{off}}(z)$.

7.3 LATTICE RESULTS

In this section we study the entanglement of Lifshitz fermions on a finite lattice with N lattice sites and lattice spacing ϵ . We discretize and rescale the localized

wave functions $\psi_j \equiv \varepsilon^{1/2} \psi(j\varepsilon)$ to make them dimensionless, and make the plane wave ansatz $\psi_j = c_k(\omega) e^{i(j\tilde{k} + \omega t)}$. Then we discretize the spatial derivative by using the centered difference (to preserve hermiticity of the Lagrangian) limit definition:

$$\partial_1 \psi_j = \frac{\psi_{j+1} - \psi_{j-1}}{2\varepsilon} = \frac{1}{2\varepsilon} (e^{ik\varepsilon} - e^{-ik\varepsilon}) \psi_j = \frac{i}{\varepsilon} \sin(k\varepsilon) \psi_j. \quad (7.29)$$

Hence

$$(i\partial_1)^z \psi_j = (-\tilde{k}(k))^z \psi_j, \quad \text{where} \quad \tilde{k}(k) = \frac{1}{\varepsilon} \sin(k\varepsilon). \quad (7.30)$$

Note that this has the right continuum limit $\tilde{k} \rightarrow k$ when $\varepsilon \rightarrow 0$. The equations of motion in equation (7.8) yield the dispersion relation

$$\omega_k^2 = \alpha^2 \tilde{k}(k)^{2z} + m^2. \quad (7.31)$$

Notice that when $m = 0$, $\omega_k = \alpha |\tilde{k}|^z$. Furthermore, because of the discretization, the dispersion relation is no longer a monotonic function of k , which means that there are in general two modes associated with a given energy. This phenomenon is known as fermion-doubling and results in a central charge $c = 2$ for the lattice Dirac fermions that is a factor 2 larger than the central charge in the continuum system.

A general solution to the equations of motion is a superposition of plane waves which satisfy boundary conditions with a phase shift: $\psi_N = e^{2\pi\theta i} \psi_0$. This restricts the values of k to

$$k = \frac{2\pi(\theta + \kappa)}{L}, \quad \text{with } \kappa \in \{0, 1, \dots, N-1\}, \quad \text{and } L = N\varepsilon. \quad (7.32)$$

As we are interested in the large N limit, whilst keeping L fixed, the value of θ becomes irrelevant. Without loss of generality we consider periodic boundary conditions. We obtain

$$\psi_{\pm,j} = \frac{1}{\sqrt{2N}} \sum_{\kappa=0}^{N-1} \frac{1}{\sqrt{\omega_k}} e^{ij\kappa \frac{2\pi}{N}} \left(\pm a_k^\dagger e^{i\omega_k t} \sqrt{\omega_k \pm \alpha(-\tilde{k})^z} + b_k e^{-i\omega_k t} \sqrt{\omega_k \mp \alpha(-\tilde{k})^z} \right). \quad (7.33)$$

We have introduced the annihilation operators a_k and b_k , which satisfy the usual equal time anti-commutation relations $\{a_p, a_k^\dagger\} = \delta_{p,k} = \{b_p, b_k^\dagger\}$, which follow from the anti-commutation relations of ψ and the Kronecker delta $\delta_{ij} = \frac{1}{N} \sum_{n=1}^N e^{i\frac{2\pi n}{N} n(i-j)}$. This reduces the Hamiltonian to $H = \sum_{\kappa=0}^{N-1} \omega_k (a_k^\dagger a_k + b_k^\dagger b_k -$

1). For the case where $m = 0$, the dispersion relation is gapless and the term $\omega_k \pm \alpha(-\tilde{k})^z$ vanishes depending on the sign of $\tilde{k}$ and the parity of z :

$$\begin{aligned} \psi_{+,j} &= \begin{cases} \frac{1}{\sqrt{2N}} \sum_{\kappa=0}^{N-1} e^{ijk\frac{2\pi}{N}} \left(a_k^\dagger e^{i\omega_k t} \sqrt{1 - \text{sgn}(\tilde{k})} + b_k e^{-i\omega_k t} \sqrt{1 + \text{sgn}(\tilde{k})} \right); & z \text{ odd}, \\ \frac{1}{\sqrt{N}} \sum_{\kappa=0}^{N-1} e^{i(j\kappa\frac{2\pi}{N} + \omega_k t)} a_k^\dagger; & z \text{ even}. \end{cases} \\ \psi_{-,j} &= \begin{cases} \frac{1}{\sqrt{2N}} \sum_{\kappa=0}^{N-1} e^{ijk\frac{2\pi}{N}} \left(-a_k^\dagger e^{i\omega_k t} \sqrt{1 + \text{sgn}(\tilde{k})} + b_k e^{-i\omega_k t} \sqrt{1 - \text{sgn}(\tilde{k})} \right); & z \text{ odd}, \\ \frac{1}{\sqrt{N}} \sum_{\kappa=0}^{N-1} e^{i(j\kappa\frac{2\pi}{N} - \omega_k t)} b_k; & z \text{ even}. \end{cases} \end{aligned} \quad (7.34)$$

Inverting these relations, we express the a and b operators in terms of the spinor operators. Then for even z one easily verifies that the ground state is equal to the direct product of an occupied $+$ -spinor state and empty $-$ -spinor over all sites. As a consequence the EE must vanish for even z . For $m \neq 0$ this argument no longer holds.

We distill the EE from the two point correlation functions [189, 190]. The EE is given by

$$S = - \sum_{n=1}^{2N} (1 - c_n) \log(1 - c_n) + c_n \log c_n, \quad (7.35)$$

where c_n is the n -th eigenvalue of the correlation matrix restricted to our subsystem, i.e. the matrix constructed by all correlations between the local spinor components. The general equal time two point correlation functions of the spinor components are given by

$$\begin{aligned} \langle \psi_{\pm,i}^\dagger \psi_{\pm,j} \rangle &= \frac{1}{2N} \sum_{\kappa=0}^{N-1} e^{i\kappa(j-i)\frac{2\pi}{N}} \left[(1 - \langle N_{a,k} \rangle) (1 \pm \frac{\alpha(-\tilde{k})^z}{\omega_k}) + \langle N_{b,k} \rangle (1 \mp \frac{\alpha(-\tilde{k})^z}{\omega_k}) \right]; \\ \langle \psi_{+,i}^\dagger \psi_{-,j} \rangle &= \langle \psi_{-,i}^\dagger \psi_{+,j} \rangle = \frac{1}{2N} \sum_{\kappa=0}^{N-1} \frac{m}{\omega_k} e^{i\kappa(j-i)\frac{2\pi}{N}} (\langle N_{a,k} \rangle + \langle N_{b,k} \rangle - 1), \end{aligned} \quad (7.36)$$

where we introduced the fermion number operators $N_{a,k} = a_k^\dagger a_k$ and $N_{b,k} = b_k^\dagger b_k$. Note that we are not computing propagators here, i.e. we are not considering a time ordered product. Of particular interest is the massless groundstate of the system, where the above correlators reduce to

$$\langle \psi_{\pm,i}^\dagger \psi_{\pm,j} \rangle = \begin{cases} \frac{1}{2} (1 \pm 1) \delta_{i,j}, & \text{for } z \text{ even;} \\ \frac{1}{2N} \sum_{\kappa=0}^{N-1} e^{i\kappa(j-i)\frac{2\pi}{N}} (1 \mp \text{sign}(\tilde{k})), & \text{for } z \text{ odd.} \end{cases} \quad (7.37)$$

Similar to the continuous case, we see that when $m = 0$ all explicit z dependence drops out in these correlation functions, but the correlators still depend

heavily on the parity of z . In the case that z is even, the EE vanishes. This is due to the fact that the plus spinor correlation sums over all holes but no particles, which yields a Kronecker delta function. The minus spinor correlation sums over all particles, which are not present in the ground state. That is, $c_n = 0$ or $c_n = 1$, which both yield zero EE from equation (7.35). Note furthermore that we have not yet specified the partitioning of our system. Hence, for even z any partitioning will have vanishing entanglement, whereas for odd z , regardless of the partitioning, the entanglement will be independent on the value of z . This is a robust consequence of the scaling symmetry of ψ , given in equation (7.4), being independent of z .

To connect the result for the $i \neq j$ to the large N limit, we express

$$\begin{aligned} \frac{1}{2L} \sum_{\kappa=0}^{N-1} e^{2\pi i \frac{x_j - x_i}{L} \kappa} \text{sign}(\tilde{k}) &\stackrel[N \rightarrow \infty]{\substack{\varepsilon \rightarrow 0 \\ N\varepsilon = L}} = \frac{1}{2L} \sum_{\kappa=0}^{\infty} e^{2\pi i \frac{x_j - x_i}{L} \kappa} \\ &= \frac{1}{2} \frac{1}{1 - e^{2\pi i \frac{x_j - x_i}{L}}} \stackrel{L \rightarrow \infty}{=} \frac{-i}{4\pi(x_i - x_j)}, \end{aligned} \quad (7.38)$$

for the continuous system of finite fixed size L , followed by the large L limit.

Recall that there is a factor of ε missing compared to the continuum result, because in this section we made the wavefunction dimensionless. A second expectation to check is the area law result for conformal field theory [154] which is validated in figure 7.2a. The central charge is 2: Two times the central charge of a continuous Dirac fermion, which is a consequence of the fermion doubling mentioned earlier. The Dirac fermion has a central charge of 1 since it is composed of two Weyl fermions with central charge $1/2$.

Instead of considering the groundstate, one could also consider a thermal state. The expectation value of the number operators then is given by the Fermi-Dirac distribution, which reduces equation (7.36) to

$$\begin{aligned} \langle \psi_{\pm,i}^\dagger \psi_{\pm,j} \rangle &= \frac{1}{2} \delta_{i,j} \pm \frac{1}{2N} \sum_{k=0}^{N-1} e^{ik(j-i) \frac{2\pi}{N}} \frac{\alpha(-\tilde{k})^z}{\omega_k} \tanh\left(\frac{\beta\omega_k}{2\alpha}\right), \\ \langle \psi_{\pm,i}^\dagger \psi_{\mp,j} \rangle &= -\frac{1}{2N} \sum_{k=0}^{N-1} e^{ik(j-i) \frac{2\pi}{N}} \frac{m}{\omega_k} \tanh\left(\frac{\beta\omega_k}{2\alpha}\right). \end{aligned} \quad (7.39)$$

Note that as $T \rightarrow \infty$ the correlation matrix becomes diagonal with maximally degenerate eigenvalue $1/2$. From equation (7.35) it follows that this maximizes the entropy to its upper bound $2N_A \log 2$, yielding a volume law. In figure 7.2b the EE is plotted as a function of z for different temperatures and zero mass. For low z the reminiscences of the parity dependence on z (which we explored in the zero temperature regime) are still visible, but they blur out as z increases and the entropy approaches its maximal value. This also follows from equation (7.39): since $|\tilde{k}| < 1$, we have $\omega_k \rightarrow 0$ as $z \rightarrow \infty$.

Figure 7.2: (a) The EE for odd z and zero mass as a function of the segment size l (dots) of the total circular lattice, with $N = 100$ sites, follows the area law (solid line). The value of the central charge is $c = 2$ which is a consequence of the fermion doubling phenomenon. (b) The EE as a function of Lifshitz scaling parameter z for finite temperatures. Note that turning on the temperature brings an explicit z dependence. For large temperatures, as z increases, the EE approaches its maximum value $S_{max} = 2N_A \log 2$ (dashed black line). For small temperature and small z the reminiscences of the zero temperature results are still visible; the parity of z plays an important role (inset). The results are obtained with $N = 1000$ and $N_A = 50$.

Furthermore, we study the temperature corrections to the area law as a function of z in the high and low temperature regime as suggested in equations (7.25) to (7.27) by numerically computing the EE as a function of temperature in both regimes and making fits for each value of z . The results are given in figures 7.3 and 7.4a. The results again show a strong distinction between even and odd z : For even z a linear dependence on $l\beta^{-1/z}$ appears. The high temperature regime is poorly accessible for low z as a consequence of computational power, due to the upper bound of the EE for finite systems (see figure 7.4b).

7.4 HOLOGRAPHIC ENTANGLEMENT ENTROPY

In this section we use a method of producing the EE through a combination of tensor networks and holographic methods. First, we introduce briefly the continuous Multi-scale Entanglement Renormalisation Ansatz (cMERA) which produces the elements necessary to calculate the EE via holographic techniques. We note here that this method is only one candidate for producing emergent spaces from field theories, another more recent approach comes from path integral optimization, see e.g. [210, 211]. It would be interesting to test the

Figure 7.3: Fits to the functions $S(\beta) = S(\beta = \infty) + f_1(z)l\beta^{-1/z} + f_2(z)l^2\beta^{-2/z}$ in the low temperature regime, and $S(\beta) = S_{off}(z) + g(z)l\beta^{-1/z} + h(z)\log(\epsilon^z/\beta)$ in the high temperature regime. The results for the fit parameters are in figure 7.4a. The system size is $N = 10000$ and the subsystem size is $N_A = 100$. For low values of z the high temperature regime is inaccessible due to the finite system size, which bounds the EE as illustrated in figure 7.4b

Figure 7.4: (a) Fit parameters as a function of z . Note that for even z the leading term in the low temperature regime (blue dots) is linear in $l\beta^{-1/z}$ whereas for odd z it is quadratic, i.e. $f_1 \sim 0$. The high temperature regime (red dots) is poorly accessible for low z as illustrated in (b): For our results there are two relevant bounds when considering the fit to equations (7.25) to (7.27): The temperature scale characterizing high and low temperatures $T_c = (\epsilon N_A)^{-z}$ and the EE saturation limit $S_{max} = 2N_A \log 2$ which is a finite size effect. Since $N_A \leq 100$ is limited by computational capacity, the high temperature regime is poorly accessible for low z (left figure). However, when z increases, T_c decreases such that for high z there is a well accessible regime to fit.

compatibility of the results that follow with these methods, however that is beyond the scope of this work.

Essentially, one produces a metric element for Anti-de Sitter space from information extracted from the Lifshitz field theory under the cMERA transformation. Using this metric we calculate the area of a minimal surface which in the (1+1)d case is the length of a geodesic on a fixed time slice. The EE of the field theory is then proportional to the size or “area” of this minimal surface by the Ryu-Takayanagi conjecture [212].

7.4.1 Review of (c)MERA

At this point we introduce the ideas involved in bringing MERA into the continuum. This section follows closely the presentation of the introductory work [213] and the subsequent work which is relevant to the calculation of EE in this framework [193]. Before introducing the continuous MERA method it should be made clear which view of the MERA we are taking, which is the perspective of the MERA as a quantum circuit. In this context the MERA is viewed in a “top-down” manner. Starting from an initial unentangled state the state is acted upon by a local unitary operator

$$U_1 = \bigotimes_{j=1}^{N/2} u_{2j-1,2j}, \quad (7.40)$$

which entangles adjacent sites. In this example local means that the full unitary operator is comprised of 2-site unitary gates or operators. This is followed by a scale transformation so that the lattice spacing and number of spins/qubits/sites are unchanged. We denote this operation by $\mathcal{R}$. It is equivalent to the coarse-graining/isometry step seen in the “bottom-up” picture [214] but modified to be a unitary operation using auxillary qubits. If the depth of the MERA is $\tau = T = \log_2(N)$, as would be the case for a binary MERA scheme, then the output of the circuit is the state

$$|\Psi_{\text{MERA}}\rangle = U_T \mathcal{R} U_{T-1} \mathcal{R} \dots \mathcal{R} U_1 |\Omega\rangle. \quad (7.41)$$

The question at this point is how to translate the scale transformation, entangling operation and fiducial state to continuum analogues. In translating to the continuum it is necessary to enforce an ultra-violet cut-off for the field theory, which we denote by: $\Lambda = \varepsilon^{-1}$, where as before ε is the lattice constant. The Hilbert space defined by the fields with such a cut-off is denoted by $\mathcal{H}_\Lambda$ such that $|\Psi(u)\rangle \in \mathcal{H}_\Lambda$, where u parametrizes the fields and represents the length/energy scale of interest. This parameter is taken such that the momentum k is effectively cut-off as $|k| \leq \Lambda e^u$. In connection to the discrete case, u effectively corresponds to the layer index τ of the tensor network. By convention

we have that u runs over $(-\infty, 0]$, such that the ultraviolet (UV) and infrared (IR) limits are given by: $u_{UV} = 0, u_{IR} \rightarrow -\infty$. The states given at these limits are denoted as

$$|\Psi(u_{IR})\rangle \equiv |\Omega\rangle, \quad |\Psi(u_{UV})\rangle \equiv |\Psi\rangle, \quad (7.42)$$

such that $|\Omega\rangle$ corresponds to an unentangled reference state and $|\Psi\rangle$ is the ground state in which we compute the EE. Now, as in the lattice implementation we relate a state at any layer or length scale of the MERA to the reference state by a unitary transformation as

$$|\Psi(u)\rangle = U(u, u_{IR}) |\Omega\rangle. \quad (7.43)$$

Likewise an operator, $\mathcal{O}$, can be defined at any scale u as

$$\mathcal{O}(u) \equiv U(0, u)^{-1} \cdot \mathcal{O} \cdot U(0, u), \quad (7.44)$$

in particular, later, we define the Hamiltonian at different length scales by this action. The form of this unitary operator [193, 213] is

$$U(u_1, u_2) = P \exp \left[-i \int_{u_2}^{u_1} (\mathcal{K}(u) + \mathcal{L}) du \right], \quad (7.45)$$

where $\mathcal{K}(u)$ and $\mathcal{L}$ are the continuum analogues of the entangling and scaling operations respectively. P denotes a path ordering such that operators are ordered from large to small values of u . The scale transformation acting on the IR state leaves it invariant since by definition the IR state is unentangled so each spatial point is uncorrelated with any other point. The entangling operator, $\mathcal{K}(u)$, is designed to generate entanglement but only for modes with wave vectors $|k| \leq \Lambda e^u$. This entanglement generation up to a cut-off is achieved through a function $g(k, u)$ which contains an appropriate cut-off function and the variational parameters, $g(u)$. Generically, $g(k, u)$ is a complex valued function but in this setting it will be real valued. Aside from this, the entangling operator is a quadratic functional of the fields. The following form is taken for the entangling operator [213]

$$\mathcal{K}(u) = i \int dk \left[g(k, u) \psi_+^\dagger(k) \psi_-(k) + g^*(k, u) \psi_+(k) \psi_-^\dagger(k) \right]. \quad (7.46)$$

It will be useful in the following discussion to utilize the interaction picture for these unitary operators. This amounts to using

$$U(u_1, u_2) = e^{-iu_1 \mathcal{L}} \cdot P \exp \left(-i \int_{u_2}^{u_1} \tilde{\mathcal{K}}(u) du \right) \cdot e^{iu_2 \mathcal{L}}, \quad (7.47)$$

where $\tilde{\mathcal{K}}(u) \equiv e^{iu \mathcal{L}} \cdot \mathcal{K}(u) \cdot e^{-iu \mathcal{L}}$. The action of $\mathcal{K}(u)$ is essentially a generalized Bogoliubov transformation of the fields.

While comparisons have been made [215, 216] between Anti-de Sitter space and the structure of a MERA network, it has been proposed [193] that by applying a continuous MERA prescription to free field theories one can determine a holographic metric of a space dual to the field theory. In this context, the metric element is given by

$$ds^2 = g_{uu}(u)du^2 + \frac{e^{2u}}{\epsilon^2}d\vec{x}^2 + g_{tt}(u)dt^2. \quad (7.48)$$

If we consider the ground state of the free field theory then the metric element corresponding to the holographic direction, g_{uu} , is related to the variational parameters of the cMERA procedure, $g(u)$, by

$$g_{uu}(u) = \frac{1}{3}g^2(u), \quad (7.49)$$

for fermionic theories, in the bosonic case $g^2(u)$ appears [193] without a factor of $1/3$. The method of determining these parameters is different in both cases. For bosons the variational function $g(u)$ is directly determined from the dispersion relation of the theory. We detail the relation for fermions in the next section. Regardless of this detail, by determining the variational function $g(u)$ using appropriate cMERA methods one may determine a dual metric. Moreover, in the holographic context we compute the EE via the Ryu-Takayanagi proposal [212, 217] meaning we do not require information of the time component of the metric here as we calculate on a fixed time slice of the space.

Once we have obtained this metric element we are able to determine the functional form of the EE by calculating the geodesic length for a subsystem A of length l on the boundary provided that one can determine the correct geodesic for the resulting space.

7.4.2 *AdS/cMERA Method*

Here, we apply the continuous MERA procedure to a free fermionic theory with Lifshitz scaling in $(1+1)$ -dimensions. We proceed in a similar fashion to extant literature [193, 205, 206, 213] with the relativistic ($z = 1$) case having appeared in [213]. For this approach we require the Fourier transformed Hamiltonian of the theory. The Hamiltonian here is obtained from the Dirac-Lifshitz Lagrangian equation (7.3), and has the form:

$$H = - \int dk \left[\hbar \alpha (-k)^z \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \right) - \mu \alpha^2 \left(\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+ \right) \right], \quad (7.50)$$

where the fields are now functions of the momentum. The procedure [213] to find the EE is as follows: firstly an infrared state, $|\Omega\rangle$, is defined by the action of the spinor components on the state. Next the cMERA operator is applied to the

Hamiltonian which manifests as a transformation of the fields. Following this, one extremizes the energy functional using the definition of $|\Omega\rangle$ with respect to the variational function $g(u)$ which appears in the definition of the angle that the field transformation depends on. This determines the angle, φ_k , associated to the true ground state. Having determined φ_k we then determine the metric element $g_{uu}(u)$ which depends on the variational function $g(u)$. The final step is to calculate the geodesic length using the metric element found for a particular subsystem.

The reference state $|\Omega\rangle$ is chosen such that

$$\psi_+(x) |\Omega\rangle = 0 = \psi_-^\dagger(x) |\Omega\rangle. \quad (7.51)$$

The cMERA operation on the Hamiltonian amounts to replacing the fields in the Fourier transformed Hamiltonian with the transformed fields such that

$$\tilde{\Psi}(k) = M_k(u) \Psi(e^{-u}k) = e^{-\frac{u}{2}} \begin{pmatrix} \cos(\varphi_k(u)) & -\sin(\varphi_k(u)) \\ \sin(\varphi_k(u)) & \cos(\varphi_k(u)) \end{pmatrix} \Psi(e^{-u}k), \quad (7.52)$$

where (see App. v1 [213]) the angle is defined as

$$\varphi_k \equiv \lim_{u_{IR} \rightarrow -\infty} \int_0^{u_{IR}} du g(e^{-u}k, u) = \lim_{u_{IR} \rightarrow -\infty} \int_0^{u_{IR}} du g(u) \frac{k}{\Lambda} \Gamma\left(\frac{|k|}{\Lambda}\right). \quad (7.53)$$

where $\Gamma(|k|/\Lambda)$ implements the momentum cut-off and can be taken to be a Heavyside step function, $\Theta(1 - |k|/\Lambda)$. Moreover, by inverting this relation using the Leibniz integral rule we find an expression for $g(u)$ using the form of $g(k, u)$ shown above, the steps involved are presented in [213] which we rederive in section 7.A, but the result is that

$$g(u) = \frac{|k|^2}{\Lambda} \partial_{|k|} \left(\frac{\Lambda}{k} \varphi_k \right) \Big|_{|k|=\Lambda e^u} = -\varphi_k + |k| \partial_{|k|} \varphi_k \Big|_{|k|=\Lambda e^u}. \quad (7.54)$$

After the transformation of the fields, the massive Hamiltonian is given by

$$\begin{aligned} H = & - \int dk e^{-u} \{ [\hbar\alpha(-k)^z \cos(2\varphi_k(u)) - \mu\alpha^2 \sin(2\varphi_k(u))] \\ & \times [\psi_+^\dagger(\tilde{k})\psi_+(\tilde{k}) - \psi_-^\dagger(\tilde{k})\psi_-(\tilde{k})] \\ & - [\hbar\alpha(-k)^z \sin(2\varphi_k(u)) + \mu\alpha^2 \cos(2\varphi_k(u))] \\ & \times [\psi_+^\dagger(\tilde{k})\psi_-(\tilde{k}) - \psi_-^\dagger(\tilde{k})\psi_+(\tilde{k})] \}, \end{aligned} \quad (7.55)$$

where $\tilde{k} = ke^{-u}$. Now we determine the energy functional, $E[g]$, by evaluating the inner product $\langle \Omega | H | \Omega \rangle$ in the infrared limit. We then obtain the energy functional

$$E[g] = \int dx \int \frac{dk}{2\pi} [\hbar\alpha(-k)^z \cos(2\varphi_k) - \mu\alpha^2 \sin(2\varphi_k)]. \quad (7.56)$$

Subsequently, after taking the functional derivative with respect to the metric function $g(u)$ one finds the condition which minimizes the energy to be

$$\tan(2\varphi_k) = -\frac{m}{(-k)^z}. \quad (7.57)$$

It should be noted here that this expression is valid for the range of scales $u \in (-\infty, 0]$ and as a result the resulting expression for the angle is valid up to the momentum cut-off, $|k| < \Lambda$. This should not be really thought of as a restriction since the cut-off Λ should be taken to infinity in the end. As a result, we have the following expression after use of trigonometric identities:

$$\varphi_k(u) = \frac{1}{2} \arcsin \left[\frac{k^z}{\sqrt{k^{2z} + m^2}} \right] - (-1)^z \frac{\pi}{4} \Big|_{k=\Lambda e^u}. \quad (7.58)$$

One should keep in mind that here the momentum is set according to $k \rightarrow \Lambda e^u$ to obtain the angle and should in this context be seen as a positive quantity. However, as a verification of the lattice result, we look at the massless case here. By taking $m = 0$ at this point the angle becomes a constant, differing only with respect to the parity of z and as such the function $g(u)$ is equal to the angle, φ_k , up to an overall sign:

$$g(u) = \frac{\pi}{4} ((-1)^z - 1). \quad (7.59)$$

Given the constant φ_k value, the entropy calculation becomes rather direct which we produce now. Essentially, for the massless case and z -odd, the entropy is found by the calculating the geodesic length using the metric

$$ds^2 = \frac{g^2}{3} du^2 + \frac{e^{2u}}{\varepsilon^2} dx^2, \quad (7.60)$$

Then, using the reparametrization $1/r = e^u/\varepsilon$ and rescaling the x direction by $x \rightarrow (\sqrt{3}/g)x \equiv \tilde{x}$, this is a pure AdS metric for $(2+1)$ -dimensions on a fixed time-slice

$$ds^2 = \frac{(g^2/3)}{r^2} (dr^2 + d\tilde{x}^2). \quad (7.61)$$

The geodesic length is determined using the following parametrization of the curve

$$\gamma = \{\tilde{x}(t) = \frac{l}{2} \cos(\pi t), r(t) = \frac{l}{2} \sin(\pi t) | t \in [0, 1]\}. \quad (7.62)$$

The length of such a curve is then obtained by calculating

$$|\gamma| = \int_0^1 dt \sqrt{g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu} = \frac{2\pi g}{\sqrt{3}} \int_0^{1/2} dt \frac{1}{\sin(\pi t)}, \quad (7.63)$$

$$= -\frac{2g}{\sqrt{3}} [-\log(\cos(\pi t/2)) + \log(\sin(\pi t/2))]_{t=0}^{t=1/2}, \quad (7.64)$$

$$= -\frac{2g}{\sqrt{3}} \log\left(\sin\left(\frac{\pi\alpha}{2}\right)\right). \quad (7.65)$$

A cut-off needs to be inserted of $\alpha \ll 1$ on the lower limit of the integration to prevent the integral diverging. The upper limit yields zero, leaving only the result. Considering the parametrization of r , we have that for small t near the boundary of the space: $r(t) \sim \frac{l\pi\alpha}{2}$. However, we also have the UV cut-off $r \sim \varepsilon$ meaning that, $\alpha \rightarrow 2\varepsilon/\pi l$, and thus we have the result

$$|\gamma| \sim \frac{2g}{\sqrt{3}} \log\left(\frac{l}{\varepsilon}\right). \quad (7.66)$$

Hence, the EE is given by

$$S_z \propto \frac{2g}{\sqrt{3}} \log\left(\frac{l}{\varepsilon}\right), \quad (7.67)$$

meaning that:

$$S_z \propto \begin{cases} \frac{\pi}{\sqrt{3}} \log\left(\frac{l}{\varepsilon}\right), & z \text{ odd, where } g = \pi/2, \\ 0, & z \text{ even, where } g = 0. \end{cases} \quad (7.68)$$

This is in agreement with the results from our correlation function based calculations up to a multiplicative factor. Here that constant would be $c/\pi\sqrt{3}$ which we can determine from comparison to the $z = 1$ known result. Inserting such a factor yields the two cases, distinguished by the parity of z :

$$S_z = \begin{cases} \frac{c}{3} \log\left(\frac{l}{\varepsilon}\right), & z \text{ odd,} \\ 0, & z \text{ even.} \end{cases} \quad (7.69)$$

These two cases are confirmed by the prior results found by calculation using correlation function methods. One should note that the cMERA technology requires additional information to determine the entropy and as yet produces only the functional form of the entropy. In other words, the constant of proportionality in question is not manifestly determined in the cMERA framework.

7.5 CONCLUSION

We have studied the EE between fermions with a Lifshitz scaling symmetry in both continuous and discrete models. The results are quite different from the results for Lifshitz bosons [57, 205, 206]. In the ground state, the most striking difference is that for fermions, there is a strong dependence on the parity of the scaling exponent z . For even z and zero mass, the ground-state becomes a pure product state in the spatial spinor representation. Hence, there is no EE. This is reaffirmed by results from the holographic cMERA approach. Other than its parity, the value of z does not affect the EE of the massless ground-state. This independence on the value of z is a robust consequence of the scaling symmetry

of the system and hence extends to any partitioning. Considering the single interval partitioning, we find for odd z that the area law is reproduced (see figure 7.2a) with a central charge that is twice the value of continuous Dirac fermions due to the fermion doubling on the lattice.

In the thermal state a more explicit dependence on z emerges. However, the parity of z remains a distinguishing factor for low values of z and low temperatures. The low temperature power series expansion of the EE in the scale invariant quantity $l\beta^{-1/z}$ does not contain odd powers for odd z , corresponding to the known relativistic result for $z = 1$.

It would be interesting to have better analytic control of the continuum limit, and to extend the analysis to non-integer, continuous values of z . Even for the free case that we consider here, we expect this to be a nontrivial extension due to branch cuts in the Lifshitz dispersion relation.

There are various further extensions one can consider, such as the mass deformed case where Lifshitz scale symmetry and chiral symmetry is broken. Also, the presence of interactions and extension to higher dimension are useful. For strongly interacting fermions, one can make contact with Lifshitz holography, for which there are known answers for the EE from the Ryu-Takayanagi formula. We leave this for further study.

APPENDIX

7.A EXPRESSION FOR $g(u)$ IN TERMS OF φ_k

Recall the definition of the exact Bogoliubov angle φ_k (written as $f(k)$ in the original work [213])

$$\varphi_k \equiv \lim_{u_{IR} \rightarrow -\infty} \int_0^{u_{IR}} g(e^{-u}k, u) du. \quad (7.70)$$

To isolate the relevant function $g(u)$ which comprises the variational part of $g(k, u)$ we need to note the form chosen for $g(k, u)$ in this setting

$$g(k/\Lambda, u) = g(u) \frac{k}{\Lambda} \Gamma\left(\frac{|k|}{\Lambda}\right), \quad (7.71)$$

where the cut-off function, Γ , is taken to be the Heavyside step function, Θ^2 . This form of the function is chosen so that the k -dependence of $g(k, u)$ is s-wave meaning that it only depends on $|k|$. If instead the (dis)entangler were to depend on the vector k^i then on the holographic side this would correspond to excitations of higher spin fields in the dual higher spin gravity theory. This point

² This choice of cut-off function is for ease of the calculations, although a smooth function such as $\exp(-|k|^2/\Lambda^2)$ could be chosen to ensure that the entangler $\mathcal{K}(u)$ is local. However, for the purposes of this derivation it is not necessary.

is commented on in [193]. Such a dependence on k in $g(k, u)$ would therefore constitute a generalization of current work. One such situation where this may be necessary would be if one were to consider multiple copies of the fermion field thus describing a higher spin theory.

The first step inverting the definition of φ_k is to use the change of variables: $z = e^{-u}$ to give

$$\begin{aligned}
 \varphi_k &= \lim_{u_{IR} \rightarrow -\infty} \int_1^{e^{-u_{IR}}} g(zk, -\ln(z)) \frac{-1}{z} dz, \\
 &= - \int_1^\infty g(-\ln(z)) \frac{zk}{\Lambda} \Gamma\left(\frac{z|k|}{\Lambda}\right) \frac{1}{z} dz, \\
 &= - \frac{k}{\Lambda} \int_1^\infty g(-\ln(z)) \Theta(1 - z|k|/\Lambda) dz, \\
 &= - \frac{k}{\Lambda} \int_1^{\frac{\Lambda}{|k|}} g(-\ln(z)) dz, \\
 \implies -\frac{\Lambda}{k} \varphi_k &= \int_1^{\frac{\Lambda}{|k|}} g(-\ln(z)) dz.
 \end{aligned}$$

Next, with this relation we differentiate both sides with respect to $|k|$ to remove the integral using the Leibniz integral rule

$$\frac{d}{dx} \left(\int_{f_1(x)}^{f_2(x)} h(y) dy \right) = h(f_2(x)) f_2'(x) - h(f_1(x)) f_1'(x). \quad (7.72)$$

Note that our lower limit is independent of $|k|$ so we have a relatively simple result

$$\frac{d}{d|k|} \left(\int_1^{\frac{\Lambda}{|k|}} g(-\ln(z)) dz \right) = g\left(-\ln\left(\frac{\Lambda}{|k|}\right)\right) \frac{d}{d|k|} \left(\frac{\Lambda}{|k|} \right). \quad (7.73)$$

Combining this rule with our relation to φ_k yields

$$\begin{aligned}
 \frac{d}{d|k|} \left(-\frac{\Lambda}{k} \varphi_k \right) &= g\left(-\ln\left(\frac{\Lambda}{|k|}\right)\right) \left(\frac{-\Lambda}{|k|^2} \right), \\
 g\left(-\ln\left(\frac{\Lambda}{|k|}\right)\right) &= \frac{|k|^2}{\Lambda} \frac{d}{d|k|} \left(\frac{\Lambda}{k} \varphi_k \right).
 \end{aligned}$$

The final step is to express this relation in terms of u again which amounts to the replacement, $|k| \rightarrow \Lambda e^u$

$$g(u) = \frac{|k|^2}{\Lambda} \frac{d}{d|k|} \left(\frac{\Lambda}{k} \varphi_k \right) \Big|_{|k|=\Lambda e^u}. \quad (7.74)$$

For the version used in the text we expand the RHS

$$\begin{aligned}
 g(u) &= \frac{|k|^2}{\Lambda} \left(\frac{d}{d|k|} \left(\frac{\Lambda}{k} \right) \varphi_k + \frac{\Lambda}{k} \frac{d\varphi_k}{d|k|} \right) \Big|_{|k|=\Lambda e^u}, \\
 &= \frac{|k|^2}{\Lambda} \left(\left(\frac{-\Lambda}{|k|^2} \right) \text{sign}(k) \varphi_k + \frac{\Lambda}{k} \frac{d\varphi_k}{d|k|} \right) \Big|_{|k|=\Lambda e^u}, \\
 &= -\text{sign}(k) \varphi_k + k \frac{d\varphi_k}{d|k|} \Big|_{|k|=\Lambda e^u}.
 \end{aligned}$$

For the purpose of determining the metric element, $g_{uu}(u) = g^2(u)/3$, the sign of k is irrelevant as k is set to the positive quantity Λe^u in the end and the total expression of $g(u)$ appears as a squared quantity. As such we write

$$g(u) = -\varphi_k + |k| \frac{d\varphi_k}{d|k|} \Big|_{|k|=\Lambda e^u}. \quad (7.75)$$

CONCLUSION AND OUTLOOK

This thesis aims to contribute to the understanding of spin information theory and technology. Crucial to the development of spintronic devices - which employ the magnetic properties of materials, rather than electronic - is a profound understanding of micromagnetic mechanisms and how these affect observables. For instance, such understanding will provide aid in material studies searching for the best materials for a given application.

An essential ingredient for the racetrack memory [1] is for example a mechanism that fixes the chirality of domain walls. Such a mechanism is provided by the Dzyaloshinskii-Moriya interaction, hence the search for materials with this interaction bloomed. In chapter 3 we develop and test a comprehensive model to determine the strength of this interaction from the easily observable domain wall velocity. We show that the domain wall motion is still well described in terms of creep theory, endowed with a richer domain wall elasticity and find good agreement with our experiments, demonstrated in figure 3.3.

Next, we have shifted our focus to spin waves. In chapter 4 we show that the local injection of angular momentum into a magnet configures the magnetic texture into that of a virtual domain wall, i.e. the domain wall position always lies outside the magnet. This texture exhibits threshold behavior as a function of the injected angular momentum shown in figures 4.3 and 4.4 which opens the horizon for novel devices that are switched by this injection amount. We propose two such devices: the non-local magnon transport switch and the spin-Hall-electro-resistance.

Injected angular momentum can also be carried by spin waves, as we model in chapter 5. The domain wall serves as a wave guide for ungapped magnons and facilitates non-local magnon transport between two leads. Our efforts arrived at a Landauer formula given in equation (5.19), confirming our intuitive understanding of the process. Contrary to models that require $U(1)$ symmetry to derive non-local transport in a spin-superfluid state [72], our model only requires a twofold degenerate ground state. With an already well established control over domain walls, the implications for reconfigurable magnonic devices are many and promising.

In the emerging field of quantum magnonics, magnons are used in quantum information technologies [218], which rely on entanglement to outperform classical computers. In chapter 6 we derive the entanglement between the two sublattices of an antiferromagnet analytically and find good agreement with our numerical results. We find that the degree of entanglement is universal with respect to details of the material such as the lattice structure, exchange interaction and anisotropy as indicated by equation (6.10). As a result, antiferromagnets may be useful for quantum information protocols. Future research into the effects of dephasing on the entanglement are required to further develop such devices.

Delving further into entanglement and leaving magnetism behind, in chapter 7 we investigate the entanglement entropy of Lifshitz fermions and demonstrate remarkable differences with bosons. The main result entails the strong dependence on the parity of the Lifshitz exponent: for even values the entanglement vanishes in the ground state, whereas for odd values there is no explicit dependence on the scaling exponent and the entanglement coincides with known results for the relativistic system. In the thermal state, reminiscences of this parity dependence remain, but a more explicit dependence on the scaling exponent prevails as it increases as illustrated in figure 7.2b. The effect of a non-integer scaling exponent is still an open and interesting topic for future research.

Since their abstract conception some spintronic and quantum information devices have quickly been realized, such as the giant magneto resistance or quantum computers. With this thesis we aspire to contribute in the effort of such developments. We have devised theoretical models that interpret, describe and predict remarkable phenomena to be used for this development. Moreover, our research is motivated by a general interest and curiosity in a fundamental understanding of nature and how to capture its opulent phenomena in mathematical models.

What is next are studies to experimentally test the validity of our models, search for appropriate materials and investigate the realizability of our proposed devices. Furthermore, many theoretical efforts are to be made to investigate how our results extend to a broader range of systems, for instance extending our domain wall related models to anti-ferromagnets, which will widen the scope of the aforementioned experimental studies. We hope that this thesis will inspire future theoretical and experimental studies in the field of spin information.

SAMENVATTING

In mijn onderzoek heb ik magneten onder de loep genomen, hetzelfde soort magneten als die je aan je koelkast hebt zitten. Ik bestudeer specifieke vormen en configuraties van magnetische en non-magnetische materialen en ik poog het gedrag hiervan als reactie op een gegeven input te modelleren.

Het doel van deze theoretische modellen is voornamelijk om mij van de straat te houden. Ik vind het proces van puzzelen en ontdekken fascinerend en erg leuk. Daarnaast is het mogelijk dat de inzichten uit mijn onderzoek kunnen leiden tot nieuwe technologieën. Magnetisme speelt al een grote rol in veel computer apparaten en wordt verwacht de grenzen van wat mogelijk is verder op te rekken. Tot slot streef ik de ontwikkeling van kwantumcomputers en kwantum sensoren na door in magnetische en elementaire systemen het fenomeen van verstrengeling te onderzoeken.

9.1 MAGNETISME EN RELATIVITEIT

Waarom is die magneet op jouw koelkast eigenlijk magnetisch? Onze huidige interpretatie van dit fenomeen leunt op zowel speciale relativiteitstheorie als kwantummechanica. In deze sectie wijden we uit over het relativistische gedeelte en reserveren het kwantum gedeelte voor de academische inleiding in hoofdstuk 2.

Om magnetisme te begrijpen zoomen we in op het elektron. Elektronen zijn zeer lichtgewicht, negatief geladen elementaire deeltjes die je over het algemeen rondom de nucleus van een atoom kunt vinden. Als een geladen deeltje beweegt, genereert het een magneetveld. Dit volgt uit de combinatie van elektrostatische kracht en lengtecontractie (en consequentie van de speciale relativiteitstheorie). Het eerste ingrediënt, elektrostatica, ken je misschien nog van natuurkunde op de middelbare school: Tegenovergestelde ladingen trekken elkaar aan en gelijke ladingen stoten elkaar af.

Om het tweede ingrediënt, lengtecontractie, te begrijpen beginnen we door na te denken over hoe we afstand meten van een object dat beweegt. Stel we hebben een balk die beweegt met snelheid v . Ik kan de lengte dan gaan meten door een klok te starten op het moment dat de voorkant een gekozen punt x

Figure 9.1: De tijd die ik meet voor jouw balk om x te passeren (a) verschilt van jouw meting van hoelang het duurt voor x om jouw balk te passeren (b). Een bewegende lading naast een draad met een elektrische stroom voelt geen netto elektrostatische kracht omdat de ladingen in de draad elkaar opheffen (c). Maar in het referentiekader van de bewegende lading (d) zijn de elektronen dichter op elkaar vanwege de lengtecontractie, dus is er wel een kracht op q .

passeert. Ik stop de klok weer zodra de achterkant van de balk hetzelfde punt x passeert, zie figuur 9.1a. Met de tijd t die ik heb gemeten bereken ik de lengte van de balk $L = vt$. Stel nu dat jij op de balk staat met een klok. Omdat je met de balk mee beweegt, zie je het punt x en mij met een snelheid v op je afkomen. Als jij nu ook de tijd t' meet die x nodig heeft om de balk te passeren kan je ook de lengte in jouw bewegende stelsel bepalen, zie figuur 9.1b. De belangrijkste conclusie uit speciale relativiteitstheorie is dat onze gemeten tijden niet overeenkomen (dit wordt tijdsdilatatie genoemd en wordt beschreven in sectie 1.5) en dat we dus ook verschillende lengtes van de balk meten! Jouw meting duurt langer, dus in mijn stelsel is de balk korter.

Nu kunnen we de twee ingrediënten samenvoegen. Langs een draad waar een elektrische stroom doorheen loopt schiet ik een lading q met snelheid v parallel aan de draad, zie figuur 9.1c. In de draad is geen netto lading omdat er evenveel positief als negatief geladen deeltjes zijn. Dus er is ook geen elektrostatische kracht tussen de draad en de lading q . Als we nu in het referentiekader van q kijken door met q mee te bewegen, zien we de stroom in de draad sneller lopen en als gevolg van de lengtecontractie zitten de negatieve ladingen dichter op elkaar, zie figuur 9.1d. Dus in dit stelsel heeft de draad een netto negatieve lading en oefent daarmee een kracht uit op q . Terug in het lab stelsel waar we q zien bewegen, zien we niet de netto lading van de draad, maar wel de kracht. Deze kracht noemen we magnetisch en kunnen we effectief beschrijven met een magnetisch veld zodat we niet steeds hoeven te wisselen van referentiekader om deze relativistische elektrostatische effecten te bepalen.

Dat is in een notendop waarom bewegende elektrische lading een magnetisch veld genereert en magnetische krachten ondervindt. Hoewel die magneet aan je koelkast volledig stil staat, is het toch magnetisch omdat de elektronen in dat materiaal druk rond bewegen. Deze beweging bestaat niet alleen uit een baan om de positief geladen kern van de atomen, maar ook uit het tollen om hun eigen as, wat het meeste bijdraagt aan het magnetisch veld. Dit tollen noemen we

spin en de details en eigenschappen van elektron spin vallen in het regime van kwantummechanica, wat wordt uitgelegd in hoofdstuk 2.

Een tweede bijdrage van de kwantummechanica aan ons begrip van magnetisme is het mechanisme dat ervoor zorgt dat alle elektronen om een as tollen die parallel is aan de as van zijn buurman. Daardoor tellen alle kleine bijdragen aan het magneetveld op. Dit mechanisme heet exchange interactie en wordt ook uiteengezet in hoofdstuk 2. Zonder dit mechanisme zouden alle kleine bijdragen aan het magneetveld willekeurig georiënteerd zijn en optellen tot niets.

9.2 KRUIPENDE DOMEIN WANDEN

Toch zijn er magnetische materialen, zoals ijzer, die geen netto magneetveld hebben. Als we inzoomen zien we dat het materiaal verdeeld is in vele domeinen waarbinnen de spins parallel zijn. Maar de domeinen onderling zijn niet uitgelijnd met hun burens. Je kan het materiaal dan toch magnetisch maken door alle domeinen uit te lijnen met een extern magneetveld, wat je misschien wel eens hebt zien gebeuren met een paperclip.

Technologisch gezien zijn de verschillende magnetische domeinen interessant omdat deze gebruikt kunnen worden om data op te slaan. Een erg dun materiaal heeft een loodrechte magnetische anisotropie: Een mechanisme dat een voorkeur geeft voor de spins om zich loodrecht op het oppervlak te oriënteren. Dit laat nog maar twee toegestane domeinen over, omhoog en omlaag, die dan corresponderen met 0 en 1 als data. Dit principe wordt gebruikt in bijvoorbeeld de harde schijf en magnetic random access memory (MRAM).

Tussen twee domeinen is een klein gebied waar de oriëntatie geleidelijk zijn oriëntatie omdraait van omhoog naar omlaag, zie figuur 9.2. Dit gebied heet een domein wand (DW) en zijn lengte wordt bepaald door twee concurrerende effecten: De exchange interactie, die wil dat spins gelijk georiënteerd zijn aan hun burens en dus een zeer breed gestrekte DW wil; En de anisotropie die spins het liefst omhoog of omlaag wil, maar niets ertussenin en dus een voorkeur heeft voor een abrupte transitie. Het profiel van de DW wordt verder bestudeerd in hoofdstuk 2 en speelt een cruciale rol in mijn recente onderzoek, gepresenteerd in hoofdstukken 3, 4 en 5.

Nieuwe ideeën voor dataopslag hebben geleid tot een toenemende interesse in de controle over de beweging van DWs. In een typisch magnetische dataopslag

Figure 9.2: De twee types van domein wanden: Bloch (links) en Néel (rechts). Ieder profiel tussen deze twee uitersten is mogelijk net als de variant tegen de klok in. De manier waarop de oriëntatie omdraait wordt chiraliteit genoemd.

corresponderen de domeinen met bits van de data en worden geschreven en uitgelezen door een elektrische stroom in een nabijgelegen draad. In MRAM heeft iedere bit zijn eigen schrijf en uitlees apparaatje waardoor je heel snel toegang hebt tot de data, maar minder efficiënt gebruik van ruimte. Een harde schijf heeft maar één schrijf en uitlees kop en de domeinen worden onder die kop rond gedraaid, waardoor je meer data kan opslaan per volume, maar daar minder snel toegang tot hebt.

Het idee van de racetrack memory is vergelijkbaar met de harde schijf, maar in plaats van dat de magneet fysiek ronddraait, racen de domeinen zelf over een lange baan terwijl het materiaal stilstaat [1]. Dit heeft aanzienlijke voordelen omdat domeinen heel snel kunnen bewegen en de baan gebruik kan maken van drie dimensionale ruimte (in tegenstelling tot de harde schijf, die plat moet zijn).

Los van deze voordelen zijn er nog wat uitdagingen die we moeten overkomen om een goed functionerende dataopslag te hebben. Zeer belangrijk is bijvoorbeeld dat alle DWs in dezelfde richting bewegen en met dezelfde snelheid, zodat de data behouden blijft tijdens het racen. De snelheid van een DW hangt sterk af van de chiraliteit (zie figuur 9.2) en die is niet noodzakelijk hetzelfde voor alle DWs op de racetrack. Gelukkig is er een effect genaamd de Dyzalooshinskii-Moriya-interactie (DMI) die ervoor zorgt dat de chiraliteit voor alle DWs gelijk wil zijn.

Dus de zoektocht is nu gaande naar materialen met een geschikte DMI. Helaas kunnen we niet direct aan een materiaal zien hoe sterk de DMI is. Daarom doen we experimenten waaruit we kunnen afleiden wat de sterkte is. Een dergelijk experiment meet de DW snelheid als functie van een extern magnetisch veld dat loodrecht staat op de oriëntatie van de magnetische domeinen, waardoor het voornamelijk de chiraliteit beïnvloedt. De snelheid vs. extern magnetisch veld grafiek wordt sterk gekarakteriseerd door de sterkte van de DMI.

Deze precieze karakterisatie is het onderwerp van hoofdstuk 3. In dat hoofdstuk leid ik een theoretisch model af voor de DW snelheid als functie van de DMI en externe magneetvelden. Door deze parameters in het model af te stemmen zodat de grafieken aansluiten op data van het experiment kwantiseren we de sterkte van de DMI voor verschillende materialen. Het model beschrijft de DW als een lijn die vastgepind is door onzuiverheden in het materiaal. Een klein duwtje zou de DW niet kunnen verplaatsen omdat de pinning kracht sterker is. Echter, vanwege de temperatuur fluctueert de lijn willekeurig tussen verschillende gepinde configuraties en dat kleine duwtje zorgt nu voor een netto kruipende beweging. De elasticiteit van de lijn beïnvloedt de kruipsnelheid aangezien een flexibele DW gemakkelijker herconfigureert. Dit is precies waar de DMI en het externe magnetische veld een rol spelen: De chiraliteit van de DW beïnvloedt de elasticiteit.

Het theoretische model van deze kruipbeweging is toepasbaar op meerdere systemen. Je hebt het waarschijnlijk al in actie gezien zonder het te weten. Bijvoorbeeld barsten in steen of scheuren in papier kruipen door het materiaal,

net als het front van water dat omhoog beweegt als het geabsorbeerd wordt in een poreus materiaal zoals papier.

9.3 SPIN GOLVEN

Als je puur kijkt naar het transport van informatie is het racetrack memory nog redelijk energie inefficiënt omdat het bewegen van de domeinen een hoop heen en weer flippen van de spins met zich meebrengt. De racetrack is daarom minder geschikt om een signaal over te sturen. In onze huidige computers wordt elektriciteit gebruikt om signalen te versturen: Een geleidende draad transporteert het signaal "aan" als er een voltage van 5 Volts op staat en "uit" als er geen voltage op staat. Echter, met de toenemende vraag naar computercapaciteit worden deze elektrische draden steeds kleiner. Eén van de uitdagingen die dit met zich meebrengt is dat deze signalen de computerchip meer verhitten, omdat een smallere draad meer weerstand met zich meebrengt (dit fenomeen herken je misschien van het knijpen in je rietje tijdens het drinken) en deze weerstand zet Volts om in hitte.

Spin golven zijn een geode kandidaat voor een alternatieve manier van signalen transporteren. In plaats van dat we de spin volledig omdraaien, brengen we nu een kleine verstoring aan op de oriëntatie. Vanwege de exchange interactie (die wil dat spins gelijk georiënteerd zijn aan hun burens) wordt die verstoring ook doorgegeven van buur spin tot buur spin, wat een collectieve golf veroorzaakt door de magneet heen. Je kan dit proces vergelijken met de "wave" in een voetbalstadion. De golf zelf beweegt over de tribune, maar de voetbalfans blijven op hun plek. Je kan je wel voorstellen dat informatie verzenden via de "wave" veel efficiënter is, en bovendien sneller, dan om iedereen te vragen op te staan en een rondje te gaan rennen. In hoofdstukken 4, 5 en 6 bestudeer ik deze spin golven in verschillende systemen en theoretische modellen.

In het stadion ontstaat een "wave" wanneer de fans genoeg zijn opgewonden. Een "wave" kan ook worden gestart door het lokaal te forceren. Zo kan een magneet ook opgewonden worden om spin golven te maken. Echter, een dergelijke lokale forcering oefent ook een kracht uit op de grondtoestand van de spins, waardoor de as in de buurt van de forcering gedraaid wordt. Dit beïnvloedt weer hoe efficiënt spin golven in het medium worden geïnjecteerd. Het effect van hoe het magnetische oriëntatie profiel wordt beïnvloed door deze forcering is het onderwerp van hoofdstuk 4. Het blijkt dat het magnetische profiel overeenkomt met een deel van het DW profiel. Als we het DW profiel extrapoleren zien we dat het centrum van de DW buiten de magneet valt, aan de andere kant van de plek waar de spin golf geïnjecteerd wordt. Daarom noemen we het profiel een virtuele DW.

Om apparaten te maken die spin golven gebruiken, hebben we magnetische draden nodig om de signalen te geleiden. Echter, DWs bieden een alternatief

om spin golven te geleiden: Als we een magnetische plaat hebben met een DW door het midden dan kunnen we gemakkelijk laag energetische spin golven aanslaan die op de DW blijven. Zo kunnen we de stroom van informatie sturen door de configuratie van de DW aan te passen. Dit pad van solid-state herconfigureerbare apparaten volg ik in hoofdstuk 5.

9.4 VERSTRENGELING

Spin golven worden verder bestudeerd in hoofdstuk 6 waar ze bewegen over de spins van een antiferromagnetisch rooster. Een antiferromagneet bestaat ook uit een spin rooster, maar nu zorgt de exchange interactie er juist voor dat spin antiparallel zijn aan hun burens. Dus als je langs het rooster loopt zie je om en om spin omhoog en omlaag langskomen. Een dergelijk rooster kan je opsplitsen in twee sub-roosters, één met alle spins omhoog en één met alle spins omlaag. Wat opmerkelijk is, is dat spingolven op deze sub-roosters niet als onafhankelijk beschouwd kunnen worden: ze zijn verstrengeld. Daarmee bedoel ik, als ik een model van het systeem zou willen maken, moet ik de spin golven op beide sub-roosters als een geheel beschouwen. Als ik informatie uit het systeem wil halen over de spingolven op het ene sub-rooster, dan moet ik het ontwarren van het andere, waarmee ik de toestand van beide verander. Misschien heb je al gehoord van het kwantummechanische principe dat observatie het geobserveerde beïnvloed (laat Schrödingers kat een belletje rinkelen?).

Om het idee van verstrengeling op te helderen kunnen we het volgende gedachten experiment doen: Stel ik heb een paar handschoenen en een machine waar ik het paar in kan doen en de machine stopt vervolgens willekeurig iedere handschoen in een afzonderlijke afgesloten doos (dezelfde soort doos waar Schrödingers kat in zat gevangen, maar dit gedachten experiment is proefdiervrij). De handschoenen zijn in een kwantummechanische toestand van zijnde zowel links- als rechtshandig. Zelfs als ik alle details weet van deze toestand, kan ik niet met volledige zekerheid voorspellen welke handschoen ik zal aantreffen als ik een doos open maak.

Nu geef ik jou een doos en stuur ik je met een eerste klas ticket naar de maan 1.3 licht seconden van ons af. Als ik vervolgens mijn doos open en zie dat er een rechtshandige handschoen in zit, weet ik meteen dat jij een linkshandige handschoen in jouw doos hebt. Tot dusver niets raars aan de hand en we zouden dit experiment ook gemakkelijk in het echt kunnen doen (behalve dat reisje naar de maan). Maar er is iets heel spannends gebeurd: op het moment dat ik mijn observatie in de doos deed, werd de inhoud van jouw doos ook vast gelegd. Dit gebeurt instantaan en niet 1.3 seconden later, anders zou jij namelijk nog een kans hebben om ook een rechtshandige handschoen waar te nemen als je ook snel in jouw doos keek. Dus de informatie van mijn observatie reist veel sneller

dan de snelheid van het licht. Dit is precies waar Einstein moeite mee had, dus hij beargumenteerde dat de informatie al voor de observatie bekend was en rustig met jou mee rijst naar de maan. Zelfs als de machine werkelijk willekeurig werkt, dan zou het universum toch wat verborgen variabelen meesturen die de uitkomst bepalen: Er was nooit een kans dat ik de linkshandige handschoen zou waarnemen. De kwantummechanische beschrijving is slechts een effectief model om enigszins om te gaan met ons gebrek aan volledige informatie.

Deze theorie van verborgen variabelen houdt echter geen stand in de Bell experimenten, die ontworpen zijn om het onderscheid te maken tussen kwantum en verborgen variabelen willekeurigheid [2]. Dit laat Einsteins paradox van causaliteit nog steeds onbeantwoord en verschillende interpretaties worden nog steeds nagejaagd, inclusief theorieën over parallelle universa.

De verstrengeling van spin golven in een antiferromagneet kan worden benut in kwantum informatieapplicaties zoals kwantumcomputers, echter, de verstrengeling die ik onderzoek in hoofdstuk 7 in op een fundamenteeler niveau, betreffende twee intervallen van elementaire deeltjes. Zulke elementaire systemen van microscopische deeltjes zijn theoretisch identiek aan modellen van zwaartekracht in gekromde ruimtetijd. Dit heeft als gevolg dat resultaten van het ene theoretische model direct vertaald kunnen worden naar implicaties voor het andere. Bovendien is de mate van verstrengeling een goede indicator voor transities van de toestand van materialen.

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CURRICULUM VITAE

Dion Hartmann is geboren op 20 juli 1992 te Veldhoven. In 2010 behaalde hij zijn gymnasiumdiploma aan het Lorentz Casimir Lyceum in Eindhoven. Vervolgens begon hij in Utrecht aan de bacheloropleidingen Liberal Arts and Sciences, Natuur- en Sterrenkunde en Wiskunde aan de Universiteit Utrecht. Deze heeft Dion in 2014 cum laude voltooid tezamen met de honneurs programma's Humanities college, Descartes college en Experimental physics. Zijn bachelor thesis ging over asymmetrische magnetische bubble expansie in samenwerking met een onderzoeksgroep aan de Technische Universiteit van Eindhoven waar Dion ook experimenten heeft uitgevoerd voor zijn thesis. Daarna begon hij aan de masteropleidingen Theoretical physics en Mathematical sciences. In 2017 rondde hij beide masters af met theses over entanglement entropie van fermionen en over quantum intuitionistische logica, waarna hij begon aan zijn promotieonderzoek onder de begeleiding en supervisie van prof. dr. Rembert Duine. Dat onderzoek heeft geleid tot dit proefschrift.

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