On the Admissible Rules of Intuitionistic Propositional Logic

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Abstract

We present a basis for the admissible rules of intuitionistic propositional logic. Thereby a conjecture by de Jongh and Visser is proved. We also present a proof system for the admissible rules, and give semantic criteria for admissibility.

1 Introduction

The admissible rules of a theory are the rules under which the theory is closed. It is well-known that, in contrast to classical propositional logic, intuitionistic propositional logic IPC, has admissible rules which are not derivable. Probably the first nonderivable admissible rule known for this logic is the rule $\neg A \to (B \vee C)/(\neg A \to B) \vee (\neg A \to C)$ stated by Harrop (1960). Extensions of this rule which are as well admissible but not derivable followed [Mints 76] [Citkin 77] but the question whether there were other admissible rules for IPC than the ones known remained open.

In 1975 Friedman posed the problem whether it is decidable if a rule is an admissible rule for IPC or not. In 1984 this question was answered by Rybakov in the affirmative. Moreover, Rybakov showed that the admissible rules of IPC do not have a finite basis. Informally speaking this means that there is no finite set of admissible rules which in some sense 'generates' all

*The author is supported by the Netherlands Organization for Scientific Research (NWO) under grant PGS 22-262.

the admissible rules of IPC. However, this does not exclude the possibility that there is a representation of the admissible rules via a simple infinite basis or in some other clarifying way.

Some years ago de Jongh and Visser isolated a nice r.e. set of rules which they conjectured to be a basis for the admissible rules of IPC. Here we prove this conjecture. Furthermore we present a proof system for the admissible rules. We also give semantic criteria for admissibility which are rather similar to the ones found by Rybakov (1997). Since Visser (1998) proved that the admissible rules of IPC are the same as the propositional admissible rules of Heyting Arithmetic HA this provides us with a proof system and a basis for the propositional admissible rules of HA as well.

One of the results we use (Proposition 3.7) is not much more than a reformulation of some (very interesting) results by Ghilardi. Therefore, we devote one Section (3.6) to the recapitulation of the theorems from [Ghilardi] that we use in this paper.

I thank Dick de Jongh and Albert Visser for introducing me to the subject and for helpful discussions. I also thank Lev Beklemishev for stimulating discussions we had during my stay at the University of Münster, November 1998. Finally I thank Vladimir Rybakov for his interest in the results and for the related questions he posed to me.

2 Preliminaries

2.1 Admissible rules

In this section we will define the notions studied in this paper. We will define what an *admissible rule* is and what a *basis for the admissible rules* is. Since we will only work in the context of intuitionistic propositional logic we will not define these notions in full generality. So, we will define in fact what a *propositional* admissible rule is and what a basis for the *propositional* admissible rules is. Once these definitions have been given it is easy to see how they can be generalized in many ways. For a general setting and for interesting results about admissible rules in the context of other logics see [Rybakov 97] and [Visser 98].

For the rest of the paper we fix a language for intuitionistic propositional logic, with variables $p_0, p_1, \ldots$ Unless explicitly stated otherwise, formulas are meant to be propositional formulas in this language. The letters A, B, C, D, E, F will always range over formulas and p, q, r, s, t over propositional variables. We write $\vdash$ for derivability in IPC.

An $\mathfrak{L}$-*substitution* σ is a map which assigns to every propositional variable a formula in the language $\mathfrak{L}$. For a propositional formula A, we write $\sigma(A)$ for the result of applying σ to A, i.e. for the result of substituting $\sigma(p_i)$ for p_i in A. When $\mathfrak{L}$ is our fixed language of propositional logic mentioned above, we say 'substitution' instead of '$\mathfrak{L}$-substitution'.

A *rule* is an expression of the form

$$\frac{A_1 \ \dots \ A_n}{B}.$$

We sometimes write $A_1, \dots, A_n/B$ for this expression. We say that an expression

$$\frac{A'_1 \dots A'_n}{B'},$$

is a substitution instance of such a rule when there is a substitution σ such that $\sigma(A_i) = A'_i$ and $\sigma(B) = B'$.

Let T be some theory in a language $\mathfrak{L}$. We say that a rule A/B is an *admissible rule of* T, and write $A \mathrel{|\!\sim}_T B$, if

$$\text{for all } \mathfrak{L}\text{-substitutions } \sigma\text{: if } T \vdash \sigma(A) \text{ then } T \vdash \sigma(B).$$

In this case we also say that A *admissibly derives* B *in* T. We write $\mathrel{|\!\sim}$ for $\mathrel{|\!\sim}_{\mathsf{IPC}}$.

2.1.1 Bases

For a set of rules $\mathcal{R}$ and a set of formulas $\mathcal{A}$, we say that B is *derivable in* T *by the set of rules* $\mathcal{R}$ *from assumptions* $\mathcal{A}$ when there is a sequence of formulas $(B_1, \dots, B_n)$, where $B_n = B$, such that for every $i \leq n$ either $B_i \in \mathcal{A}$ or there are $B_{i_1}, \dots, B_{i_m}$ with $i_j < i$ such that either

$$\vdash_T (B_{i_1} \wedge \dots \wedge B_{i_m}) \to B$$

or

$$\frac{B_{i_1} \dots B_{i_m}}{B_i}$$

is a substitution instance of some rule in $\mathcal{R}$.

We call a set of rules $\mathcal{R}$ a *basis* (in T) for some other set of rules $\mathcal{R}' \supseteq \mathcal{R}$ if for every rule

$$\frac{A_1 \dots A_n}{B}$$

in $\mathcal{R}'$, B is derivable in T by the rules $\mathcal{R}$ from the assumptions $A_1, \dots, A_n$. Given T, we say that a set $\mathcal{R}$ of admissible rules of T is a *basis for the admissible rules of* T when $\mathcal{R}$ is a basis for the set of admissible rules of T.

2.1.2 Subbases

If a theory T has the so-called Disjunction Property;

$$DP \qquad \text{if } T \vdash A \vee B \text{ then } T \vdash A \text{ or } T \vdash B$$

then it follows that if $A \mathrel{|\!\!\sim}_T B$ and $C \mathrel{|\!\!\sim}_T D$, then also $A \vee C \mathrel{|\!\!\sim}_T B \vee D$. However the rule $(A \vee C)/(B \vee D)$ does not have to be derivable from the rules A/B and C/D in T. Therefore, in the context of theories which possess the Disjunction Property, the notion of a basis for the admissible rules seems too restrictive. This accounts for the notion of a *subbasis for the admissible rules*, introduced below. That is, for theories with the Disjunction Property, we think that the right notion of a basis (for the admissible rules), is in fact that what we will call a subbasis here: a set $\mathcal{R}$ of admissible rules of T is a *subbasis for the admissible rules of* T if the following is a basis for the admissible rules of T: the collection of rules of the form

$$\frac{A \vee p}{B \vee p}$$

where the rule A/B is in $\mathcal{R}$ and p does not occur in A or B.

2.2 Kripke models

In this paper we will use Kripke models for intuitionistic propositional logic in many ways. Therefore, we fix some notation and terminology concerning Kripke models in advance. Most of the notions introduced here are standard, so that the reader who is familiar with Kripke models can skip this section and consult it later when necessary. The only exception is the notion of a tight predecessor of a set of nodes, terminology invented to simplify talking about the special kind of Kripke models we will use later on.

A *Kripke model* K is a triple $(W, \preccurlyeq, \Vdash)$, where W is a set, $\preccurlyeq$ is a partial order on W and $\Vdash$ is the so-called forcing relation defined as usual, see for example [Troelstra,Van Dalen 88]. If no confusion is possible we use the same symbols $\preccurlyeq$ and $\Vdash$ for the partial order and forcing relation of different models.

For two nodes w, v we say that w *is below* v when $w \preccurlyeq v$. In this case we also say that v *is above* w. We write $w \prec v$ or $w \succ v$ if $w \neq v$, and $w \preccurlyeq v$

or $w \succcurlyeq v$ respectively. If $w \prec v$ we call v a *successor of* w and we call w a *predecessor of* v. We call a model *rooted* when it contains a node which is below all other nodes in the model.

We say that $K' = (W', \preccurlyeq', \Vdash')$ is a *submodel of* $K = (W, \preccurlyeq, \Vdash)$ if W' is a subset of W, and $\preccurlyeq', \models'$ are the restrictions of the corresponding relations of K to W'. We say that K' is a *finite submodel* when W' is finite. We write K_w for K' if $W' = \{x \in W \mid w \preccurlyeq x\}$. A submodel of the form K_w is called the *submodel generated by* w. Note that submodels are completely characterized by their domain. Therefore, we will from now on notationally confuse a submodel with its domain.

For Kripke models $K_1, \ldots, K_n$ we denote by $(\sum_i K_i)'$ the Kripke model which is the result of attaching one new node, say b, below all nodes in $K_1, \ldots, K_n$, at which no propositional variable is valid.

We repeat from [Ghilardi] the following definitions. We say that two rooted Kripke models are *variants* of each other when they have the same domain and partial order, and their forcing relations only possibly differ at the roots. A class of Kripke models is called *stable* if for every model K in the class and every node w of K, K_w is in the class as well. A class of rooted Kripke models has the *extension property* when for every finite set of Kripke models $K_1, \ldots, K_n$ in this class there is a variant of $(\sum_i K_i)'$ which is in this class as well. When $\mathcal{K}$ is a class of Kripke models we say that A is valid in $\mathcal{K}$, notation $\mathcal{K} \models A$, when A is valid in every model of $\mathcal{K}$.

2.2.1 Tight predecessors

Consider a Kripke model $K = (W, \preccurlyeq, \Vdash)$, some node u in K and a set U of nodes in K. We say that u is a *tight predecessor of* U, if

$$\forall x \in U (u \preccurlyeq x) \wedge \forall x \succ u \exists y \in U (y \preccurlyeq x).$$

In the sequel we will actually only consider tight predecessors of finite sets of nodes. We often write 'a tight predecessor of $u_1, \ldots, u_n$' instead of 'a tight predecessor of $\{u_1, \ldots, u_n\}$'.

Observe that a set does not necessarily have a tight predecessor but that every node in a Kripke model is a tight predecessor of some set, namely, of the set of all its successors.

3 Admissible rules.

The proof of our main theorem (Theorem 3.20) proceeds as follows. In the first subsection we define a proof system, called AR, which derives expressions of the form $A \rhd B$, where A and B are propositional formulas. In Section 3.12 we then show that AR is in fact a proof system for the admissible rules: AR derives $A \rhd B$ iff $A \mathrel{|\!\!\sim} B$. The proof of this fact has two main ingredients: In Section 3.3 we characterize AR in terms of Kripke models. We define what an AR-model is and show that AR derives $A \rhd B$ if and only if B is valid in all AR-models on which A is valid. Note that in the light of Section 3.12 this is a semantical characterization of the admissible rules. In Section 3.6 we derive a semantical characterization (in terms of classes of finite Kripke models) of the admissible rules from results by Ghilardi, from his beautiful paper [Ghilardi]. In Section 3.12 we show that these two characterizations are 'the same', which leads to the result mentioned above. Finally, in the last section we show how this provides us with a basis for the admissible rules.

3.1 The system AR

As said, the system AR is a proof system which derives expressions of the form $A \rhd B$, called *sequents*, where A and B are propositional formulas. To keep the definition of this system readable, we will use the following abbreviation,

$$(A)(B_1, \dots, B_n) \equiv_{def} (A \to B_1) \vee \dots \vee (A \to B_n).$$

Furthermore, we adhere to some reading conventions as to omit parentheses when possible. The negation binds stronger that $\wedge$ and $\vee$, which in turn bind stronger than $\rhd$, which binds stronger than $\to$. So, for example the expression $(A \to B \vee C) \rhd D \to E$ means $((A \to (B \vee C)) \rhd D) \to E$.

Axiom schemes:

$$V \quad (A \to r \vee s) \vee t \rhd (A)(r, s, p_1, \dots, p_n) \vee t \quad \text{for } A = \textstyle\bigwedge_{i=1}^{n}(p_i \to q_i)$$

$$I \quad A \rhd B \qquad \text{where } \mathsf{IPC} \vdash (A \to B)$$

Rules:

$$Conj \ \frac{C \rhd A \quad C \rhd B}{C \rhd A \wedge B} \qquad\qquad Cut \ \frac{A \rhd B \quad B \rhd C}{A \rhd C}$$

Note that V is not a scheme in the strict sense. It consists in fact of the infinitely many schemes V_n which are

$$V_n \quad (\bigwedge_{i=1}^{n}(p_i \to q_i) \to r \vee s) \vee t \rhd (\bigwedge_{i=1}^{n}(p_i \to q_i))(r, s, p_1, \ldots, p_n) \vee t$$

De Jong and Visser observed that the rules corresponding to V_n (see Section 3.18) are admissible and conjectured them to be a basis, see the Introduction.

As noted before, if $A \mathrel{|\!\sim} C$ and $B \mathrel{|\!\sim} C$ then also $A \vee B \mathrel{|\!\sim} C$. This property of the admissible rules is not reflected in the rules of AR. That is, there is no rule

$$Disj \quad \frac{A \rhd C \quad B \rhd C}{A \vee B \rhd C}$$

However, it turns out that AR satisfies this rule. This is the next lemma, which we will need in the completeness proof for AR to come.

Lemma 3.2 If $\mathsf{AR} \vdash A \rhd C$ and $\mathsf{AR} \vdash B \rhd C$ then $\mathsf{AR} \vdash A \vee B \rhd C$.

Proof. It is easy to prove (with an induction to the length of derivation) that $\mathsf{AR} \vdash A \rhd B$ implies $\mathsf{AR} \vdash A \vee C \rhd B \vee C$. Hence $\mathsf{AR} \vdash A \rhd B$ implies $\mathsf{AR} \vdash C \vee A \rhd C \vee B$ too.

Now assume $\mathsf{AR} \vdash A \rhd C$ and $\mathsf{AR} \vdash B \rhd C$. From the previous observation it follows that $\mathsf{AR} \vdash A \vee B \rhd C \vee B$ and $\mathsf{AR} \vdash C \vee B \rhd C \vee C$. Clearly, also $\mathsf{AR} \vdash C \vee C \rhd C$. Applying Cut (twice) gives the desired result. **QED**

3.3 Completeness of AR

We are going to characterize AR in terms of Kripke models. The Kripke models we use have special properties, they are the so-called AR-models defined as follows.

Definition 1 We call a Kripke model K an AR-*model* when it is a rooted model in which every finite set of nodes $\{u_1, \ldots, u_n\}$ has a tight predecessor u, i.e. a node u such that

$u \preccurlyeq u_1, \ldots, u_n \wedge \forall u' \succ u \, (u_i \preccurlyeq u', \text{ for some } i \in \{1, \ldots, n\})$.

(We write '$x \preccurlyeq y_1, \ldots, y_n$' for '$x \preccurlyeq y_1 \wedge x \preccurlyeq y_2 \wedge \ldots \wedge x \preccurlyeq y_n$'.)

We will prove that AR derives $A \rhd B$ if and only if B is valid in every AR-model in which A is valid. The proof uses a lemma which we present separately in advance. Before stating it, let us remind the reader that a set of formulas x is called IPC-*saturated* if it is a consistent set such that for all A and B, if $x \vdash A \vee B$, then $A \in x$ or $B \in x$. In particular, x is closed under deduction in IPC.

Lemma 3.4 Let Θ be some set of formulae. Every IPC-saturated set $x \subseteq \Theta$ can be extended to an IPC-saturated set $y \subseteq \Theta$ such that for no IPC-saturated set y' it holds that $y \subset y' \subseteq \Theta$.

Proof. Let x and Θ be as in the lemma. We construct a sequence $y_0 \subseteq y_1 \subseteq \ldots$, such that for all i, $*(y_i)$ holds, where the property $*(\cdot)$ is defined as

$*(z)$ iff for all n, for all $A_1, \ldots, A_n$: if $z \vdash A_1 \vee \ldots \vee A_n$, then $A_i \in \Theta$ for some $i = 1, \ldots, n$.

We construct the sequence of sets as follows. Let $C_0, C_1, \ldots$ be an enumeration of all formulae in which every formula occurs infinitely often. We put $y_0 = x$. Clearly $*(y_0)$ holds. Suppose y_i is already defined. Then we put

$$y_{i+1} \equiv_{def} \begin{cases} y_i \cup \{C_i\} & \text{if } *(y_i \cup \{C_i\}) \text{ does hold} \\ y_i & \text{if } *(y_i \cup \{C_i\}) \text{ does not hold.} \end{cases}$$

Now we take $y = \bigcup_i y_i$. First, we have to see that this is indeed an IPC-saturated set. And second we have to show that there are no proper supersets of y which are IPC-saturated and are contained in Θ.

To see that y is IPC-saturated, suppose $y \vdash A \vee B$. Hence $y_i \vdash A \vee B$, for some i. There are $i \leq j \leq k$ such that $C_j = A$ and $C_k = B$. If $*(y_j \cup \{C_j\})$ or $*(y_k \cup \{C_k\})$ holds, then clearly A or B is in y. We show that indeed one of $*(y_j \cup \{C_j\})$ and $*(y_k \cup \{C_k\})$ must hold. Arguing by contradiction, assume this is not the case. Thus there are $A_1, \ldots, A_n, B_1, \ldots, B_m$ such that $y_j, C_j \vdash \bigvee_{i=1}^n A_i$ and $y_k, C_k \vdash \bigvee_{i=1}^m B_i$ but none of $A_1, \ldots, A_n, B_1, \ldots, B_m$ is in Θ. Since $y_i \subseteq y_j \subseteq y_k$ and $y_i \vdash C_j \vee C_k$, this implies that $y_k \vdash \bigvee_{i=1}^n A_i \vee \bigvee_{i=1}^m B_i$, which contradicts the fact that $*(y_k)$ holds.

To see that there are no IPC-saturated proper supersets of y which are contained in Θ, consider an IPC-saturated set $y \subseteq y' \subseteq \Theta$. We show that $y = y'$. Consider a formula $A \in y'$, and suppose $C_i = A$. It is easy to see that since $y_i \subseteq y' \subseteq \Theta$ and the fact that y' is saturated, $*(y_i \cup \{C_i\})$ holds. Hence $A \in y$. Therefore $y = y'$. **QED**

Now we are ready to prove the following lemma.

Proposition 3.5 $\mathsf{AR} \vdash A \rhd B$ iff B is valid on all AR-models on which A is valid.

Proof. ($\Rightarrow$) We have to see that if $\mathsf{AR} \vdash A \rhd B$ and A is valid on an AR-model, then B is valid on this model as well. This can be shown by induction to the length of the derivation of $A \rhd B$ in AR. The case that $A \rhd B$ is an instance of the axiom scheme I is easy. In the induction step we have to consider the two rules. All of them are straightforward.

Therefore, we only consider V. We have to show that for any conjunct of implications $A = \bigwedge_{i=1}^{n}(E_i \to F_i)$, if $(A \to B \vee C) \vee D$ is valid on all AR-models, then so is $(A)(B, C, E_1, \ldots, E_n) \vee D$. Therefore, assume that indeed for such a formula A, $(A \to B \vee C) \vee D$ is valid on an AR-model K. Let v be the root of K. We show that $(A)(B, C, E_1, \ldots, E_n) \vee D$ is valid in K at v, whence that $(A)(B, C, E_1, \ldots, E_n) \vee D$ is valid in K.

Arguing by contradiction, assume $(A)(B, C, E_1, \ldots, E_n) \vee D$ is not valid at v. Hence $(A \to B \vee C)$ is valid at v. Moreover, $\neg A$ is not valid at v. Therefore, there is a nonempty set U of nodes, such that

$$\forall x (x \Vdash A \text{ iff for some } u \in U,\ u \preccurlyeq x).$$

Since $(A)(B, C, E_1, \ldots, E_n)$ is not valid at v, there are, for some $m \leq n + 2$, nodes $u_{i_1}, \ldots, u_{i_m} \in U$ such that

$$\forall D \in \{B, C, E_1, \ldots, E_n\} \exists u \in \{u_{i_1}, \ldots, u_{i_m}\}\ u \nVdash D.$$

Since we consider an AR-model the set $\{u_{i_1}, \ldots, u_{i_m}\}$ has a tight predecessor. That means that there is a node u such that

$$u \preccurlyeq u_{i_1}, \ldots, u_{i_m} \wedge \forall u' \succ u (u_{i_j} \preccurlyeq u', \text{ for some } j \leq m).$$

If A is valid at u then B or C has to be valid at u, which contradicts the fact that for both B and C there is a node in $u_{i_1}, \ldots, u_{i_m}$ which does not validate the formula. On the other hand, if A is not valid at u, then since A is valid at all nodes $u' \succ u$, E_j has to be valid at u, for some j. But this is a contradiction as well, since for every $j \in \{1, \ldots, n\}$ there is a node in $u_{i_1}, \ldots, u_{i_m}$ which does not validate E_j.
($\Leftarrow$) Assume $\mathsf{AR} \nvdash A \rhd B$. We construct an AR-model K in which A is valid while B is not.

First we construct an IPC-saturated set of formulas v in such a way that

$$A \in v,\ B \notin v, \text{ for all } C \rhd D\text{: if } \mathsf{AR} \vdash C \rhd D \text{ and } C \in v, \text{ then } D \in v. \quad (1)$$

This v will be the root of the model K we are going to construct. The existence of v is proved in the following Claim.
Claim If $\mathsf{AR} \nvdash A \rhd B$, then there is an IPC-saturated set v such that $A \in v$ and $B \notin v$, which has the property that if for some C, D, $\mathsf{AR} \vdash C \rhd D$ and $C \in v$, then $D \in v$ as well.

Proof of Claim. Assume $\mathsf{AR} \nvdash A \rhd B$. We construct a sequence of finite sets $\{A\} = x_0 \subseteq x_1 \subseteq \ldots$ such that for all i, $\mathsf{AR} \nvdash (\bigwedge x_i) \rhd B$, and if $\mathsf{AR} \vdash (\bigwedge x_i) \rhd C$, then $C \in x_j$ for some j. The set v we look for will be the set $\bigcup x_i$.

Let $C_0, C_1, \ldots$ be an enumeration of all formulas in which every formula occurs infinitely often. Given the set x_i, we show how to construct x_{i+1}.

$$x_{i+1} \equiv_{def} \begin{cases} x_i & \text{if } \mathsf{AR} \nvdash (\bigwedge x_i) \rhd C_i \\ x_i \cup \{C_i\} & \text{if } \mathsf{AR} \vdash (\bigwedge x_i) \rhd C_i,\ C_i \text{ is not a disjunction} \\ x_i \cup \{D_j, C_i\} & \text{if } \mathsf{AR} \vdash (\bigwedge x_i) \rhd C_i,\ C_i = D_1 \vee D_2,\ j = 1, 2 \\ & \text{is the least such that } \mathsf{AR} \nvdash (\bigwedge x_i \wedge D_j) \rhd B. \end{cases}$$

It is easy to see that each of these sets x_i has the desired properties, assuming it is well-defined. Thus it remains to show that they are indeed well-defined, i.e. that given x_i, x_{i+1} exists. Therefore, suppose $\mathsf{AR} \vdash (\bigwedge x_i) \rhd C_i$ and $C_i = (D_1 \vee D_2)$. We have to see that either $\mathsf{AR} \nvdash (\bigwedge x_i \wedge D_1) \rhd B$ or $\mathsf{AR} \nvdash (\bigwedge x_i \wedge D_2) \rhd B$. Arguing by contradiction, assume this is not the case. But then we can derive the contradiction that $\mathsf{AR} \vdash (\bigwedge x_i) \rhd B$ in the following way (we do not state all the rules used, but only the crucial ones).

$$\begin{array}{lll} \mathsf{AR} \vdash & (\bigwedge x_i \wedge D_1) \rhd B & \\ & (\bigwedge x_i \wedge D_2) \rhd B & \\ & (\bigwedge x_i \wedge (D_1 \vee D_2)) \rhd B & \text{(Lemma 3.2)} \\ & (\bigwedge x_i) \rhd (\bigwedge x_i \wedge (D_1 \vee D_2)) & \text{(assumption on } x_i) \\ & (\bigwedge x_i) \rhd B & (\mathit{Cut}) \end{array}$$

Now we take $v = \bigcup_i x_i$. It is easy to see that v has the desired properties. This proves the Claim.

Thus we know that there exists an IPC-saturated set v which satisfies (1). Next we construct our model K as follows. Its domain consists of all IPC-saturated sets which extend v. Its partial order $\preccurlyeq$ is the subset relation $\subseteq$. And the forcing relation is defined via

$$w \Vdash p \text{ iff } p \in w, \text{ for propositional variables } p.$$

It is easy to see that this indeed defines a Kripke model, that the model is rooted, and that A is valid in this model but B is not. Thus it only remains to show that K is an AR-model.

Therefore, consider nodes $u_1, \ldots, u_n \in K$. We have to show that there is a node u such that

$$u \preccurlyeq u_1, \ldots, u_n \wedge \forall u' \succ u (u_i \preccurlyeq u', \text{ for some } i \leq n).$$

First note that $u_1 \cap \ldots \cap u_n$ is not saturated in general. Therefore, although $u_1 \cap \ldots \cap u_n$ contains v, it does not have to be a node in K. Let now

$$\Delta = \{E \to F \mid (E \to F) \in u_1 \cap \ldots \cap u_n \wedge E \notin u_1 \cap \ldots \cap u_n\}.$$

Then we have

Claim The set $\{C \mid v \cup \Delta \vdash C\}$ is IPC-saturated.

Proof of Claim. Suppose $v \cup \Delta \vdash C_1 \vee C_2$. This implies that there is a conjunct $D = \bigwedge_{i=1}^{m}(E_i \to F_i)$ of implications in Δ, such that $v \vdash (D \to C_1 \vee C_2)$. Thus $(D \to C_1 \vee C_2) \in v$, because v is saturated. Since $(D \to C_1 \vee C_2) \rhd (D)(C_1, C_2, E_1, \ldots, E_m)$ is derivable in AR, the way v is constructed implies that then also $(D)(C_1, C_2, E_1, \ldots, E_m) \in v$. And thus one of $(D \to C_1), (D \to C_2), (D \to E_1), \ldots, (D \to E_m)$ is in v. Since no E_i is in $u_1 \cap \ldots \cap u_n$, this implies that v does not contain any of $(D \to E_i)$. Therefore v contains either $(D \to C_1)$ or $(D \to C_2)$. Hence $v \cup \Delta$ derives either C_1 or C_2. This proves the Claim.

By the previous claim and the fact that $v \cup \Delta \subseteq u_1 \cap \ldots \cap u_n$, it follows from Lemma 3.4 that $\{C \mid v \cup \Delta \vdash C\}$ can be extended to an IPC-saturated set $u \subseteq u_1 \cap \ldots \cap u_n$ such that there are no saturated sets u' with $u \subset u' \subseteq u_1 \cap \ldots \cap u_n$. We show that this is the set we look for, i.e. if $u' \succ u$ for some saturated set u', then $u_i \preccurlyeq u'$, for some $i \in \{1, \ldots, n\}$.

Suppose not, that is, let $u \subset u'$ for some saturated set u' and assume that no u_i is contained in u'. We derive a contradiction. For all $i \leq n$, we (can) choose a formula $A_i \in u_i$ outside u'. Then the formula $A_1 \vee \ldots \vee A_n$ is in $u_1 \cap \ldots \cap u_n$ but not in u'. From the construction of u, and the fact that u' is a superset of u, it follows that u' is not contained in $u_1 \cap \ldots \cap u_n$. Thus there is a formula $E \in u'$ which is not in this intersection. Now $(E \to A_1 \vee \ldots \vee A_n)$ is an element of Δ, thus also of u. Hence $A_1 \vee \ldots \vee A_n$ should be in u', a contradiction. This finally proves the proposition. **QED**

3.6 Results by Ghilardi

In the proof of the characterization of the admissible rules in terms of $\rhd$, in the subsection below, we will use, besides the semantical completeness of AR just treated, the following fact which follows from results proved by S. Ghilardi in [Ghilardi]:

Proposition 3.7 If $A \mathrel{|\!\sim} B$, then B is valid in every stable class of finite rooted Kripke models which has the extension property (see Section 2.2) and in which A is valid.

This subsection is devoted to the recapitulation of the results of Ghilardi which lead to this proposition and to its proof. First we have to introduce some terminology.

3.7.1 Terminology

Let $\bar{p}$ be a sequence of propositional variables. We say that a formula A is a *formula in* $\bar{p}$, when all the propositional variables in A are among the variables in the sequence $\bar{p}$. We say that a Kripke model is a Kripke model *over* $\bar{p}$, when the forcing relation of the model is only defined for formulas in $\bar{p}$. If $\bar{p}$ is the sequence of all the propositional variables that occur in A, then $Mod(A)$ denotes all *finite* models of A over $\bar{p}$.

Following Fine [Fine 74] [Fine 85], Ghilardi defines equivalence relations $\sim_n$ and preorders $\leq_n$ between rooted Kripke models. Let K, K' be two rooted Kripke models with roots b and b' respectively.

$$
\begin{array}{lll}
K \sim_0^{\bar{p}} K' & \equiv_{def} & b \Vdash p \text{ iff } b' \Vdash p, \text{ for all atoms } p \text{ in } \bar{p}. \\
K \sim_{n+1}^{\bar{p}} K' & \equiv_{def} & \forall k \in K \exists k' \in K'((K)_k \sim_n (K')_{k'}) \text{ and vice versa.} \\
K \leq_0^{\bar{p}} K' & \equiv_{def} & b' \Vdash p \text{ implies } b \Vdash p, \text{ for all atoms } p \text{ in } \bar{p}. \\
K \leq_{n+1}^{\bar{p}} K' & \equiv_{def} & \forall k \in K \exists k' \in K'((K)_k \sim_n (K')_{k'}).
\end{array}
$$

When it is clear from the context to which sequence of variables we refer we omit this in the notation.

Moreover Ghilardi uses a measure of complexity, $c(\cdot)$, on propositional formulas defined as follows. Put $c(A) = 0$ if A is a propositional variable, $c(A \circ B) = max\{c(A), c(B)\}$, for $\circ = \wedge, \vee$, and $c(A \rightarrow B) = 1 + max\{c(A), c(B)\}$.

3.7.2 The proof of Proposition 3.7

In the proof of Proposition 3.7 we will use four results by Ghilardi which we will state below. The first two are about the relation $\leq_n$.

Proposition 3.8 (Ghilardi) For two finite rooted Kripke models K and K' over $\bar{p}$ it holds that $K \leq_n K'$ iff for all formulas A in $\bar{p}$ with $c(A) \leq n$, $K' \models A$ implies $K \models A$.

Proposition 3.9 (Ghilardi) If a class $\mathcal{K}$ of finite rooted Kripke models over $\bar{p}$ is such that for some n for all Kripke models K over $\bar{p}$

$$\text{if there is a } K' \in \mathcal{K} \text{ with } K \leq_n K' \text{ then } K \in \mathcal{K},$$

then $\mathcal{K}$=$Mod(A)$ for some formula A in $\bar{p}$.

Furthermore, he observes that under certain conditions the closure of a class of models under $\leq_n$ preserves the extension property

Proposition 3.10 (Ghilardi) If a stable class $\mathcal{K}$ of finite rooted Kripke models over $\bar{p}$ has the extension property then so does the class of models

$$\{K \mid K \text{ is a finite rooted model over } \bar{p} \text{ and } \exists K' \in \mathcal{K}(K \leq_n K')\}.$$

The heart of Proposition 3.7 is the following

Theorem 3.11 (Ghilardi) Let A be a formula in $\bar{p}$. If $Mod(A)$ has the extension property then there is a substitution σ such that $\vdash \sigma(A)$ and for all formulas D in $\bar{p}$, $A \vdash D \leftrightarrow \sigma(D)$.

Now the proof of Proposition 3.7 runs as follows.

Proof of Proposition 3.7. Suppose $A \mid\!\sim B$ and let $\mathcal{K}$ be a stable class of finite rooted Kripke models with the extension property in which A is valid. Assume that all the propositional variables in A and B are among $\bar{p}$. Then let $\mathcal{K}'$ be the class of all Kripke models of $\mathcal{K}$, but then considered as Kripke models over $\bar{p}$. Note that $\mathcal{K}'$ is again a stable class of finite rooted Kripke models with the extension property in which A is valid. Let n be some number such that $c(A) \leq n$, and let

$$\mathcal{K}'' = \{K \mid K \text{ is a finite rooted model over } \bar{p} \text{ and } \exists K' \in \mathcal{K}'(K \leq_n K')\}.$$

By Proposition 3.8, A is valid in the class $\mathcal{K}''$ because it is valid in $\mathcal{K}'$. And by Proposition 3.9 we know that $\mathcal{K}'' = Mod(C)$ for some formula C in $\bar{p}$. Since, by Proposition 3.10, we also know that $\mathcal{K}''$ has the extension property, we can apply Theorem 3.11 to conclude that there is a substitution σ such that

$$\mathsf{IPC} \vdash \sigma(C) \text{ and } C \vdash B \leftrightarrow \sigma(B).$$

Clearly, the fact that A is valid in $Mod(C)$ implies that $C \vdash A$. Hence $\mathsf{IPC} \vdash \sigma(A)$. But this implies that $\sigma(B)$ is derivable, because $A \mid\!\sim B$. Thus certainly $C \vdash \sigma(B)$, and whence $C \vdash B$. Therefore, B is valid in $Mod(C)$. It is easy to see that this implies that B is valid in $\mathcal{K}$ as well. **QED**

3.12 Characterizations of admissibility

We are now ready to give the promised characterizations of the admissible rules of IPC. One is in terms of $\rhd$, a proof system for the admissible rules. The other two are in terms of Kripke models. Let us state them before we consider their proofs.

Theorem 3.13 $A \mathrel{|\!\sim} B$ iff $\mathsf{AR} \vdash A \rhd B$.

Corollary 3.14 $A \mathrel{|\!\sim} B$ iff B is valid in every AR-model in which A is valid.

Corollary 3.15 $A \mathrel{|\!\sim} B$ iff B is valid in every stable class of finite rooted Kripke models with the extension property in which A is valid.

The second and third characterization are corollaries of the first one in combination with Proposition 3.5 and Lemma 3.16, the last of which also is needed in the proof of the first one. Lemma 3.16 shows that there is a natural correspondence between AR-models and stable classes of finite rooted Kripke models with the extension property. Therefore, the two corollaries are in some sense the same. We first treat this lemma and then we prove Theorem 3.13.

Lemma 3.16 For all n and all finite sequences of propositional variables $\bar{p}$ we have the following correspondence:
(a) For every AR-model K there is a stable class $\mathcal{K}$ of finite rooted Kripke models with the extension property such that

$$\text{for all } A \text{ in } \bar{p} \text{ with } c(A) \leq n\text{: } K \models A \text{ iff } \mathcal{K} \models A.$$

(b) For every stable class $\mathcal{K}$ of finite rooted Kripke models with the extension property there is an AR-model K such that

$$\text{for all } A\text{: } K \models A \text{ iff } \mathcal{K} \models A.$$

Proof. Let n be some number and let $\bar{p}$ be some finite sequence of propositional variables. First of all, let $\mathcal{A}$ be the set of all formulas A in $\bar{p}$ with $c(A) \leq n$. This set is, modulo provable equivalence, finite.

To show part (a) of the lemma, suppose K is an AR-model. Let $\mathcal{K}$ be the class of all Kripke models K' such that K' is a finite rooted submodel of K, and such that

$$\forall A \in \mathcal{A} \forall x \in K' (K', x \Vdash A \text{ iff } K, x \Vdash A). \tag{2}$$

It is easy to see that $\mathcal{K}$ is stable. We show that $\mathcal{K}$ has the extension property.

Consider models $K_1, \dots, K_n$ in $\mathcal{K}$, with roots $u_1, \dots, u_n$ respectively. Let u be a tight predecessor of $u_1, \dots, u_n$ in K. That means that

$$u \preccurlyeq u_1, \dots, u_n \wedge \forall u' \succ u(u_i \preccurlyeq u', \text{ for some } i \in \{1, \dots, n\}).$$

Let K' be the submodel the domain of which is the union of $\{u\}$ and the domains of $K_1, \dots, K_n$. It is easy to see K' satisfies (2). Hence K' is in $\mathcal{K}$. This shows that $\mathcal{K}$ has the extension property.

It remains to show that

$$\text{for all } A \in \mathcal{A}\text{: } K \models A \text{ iff } \mathcal{K} \models A.$$

The direction from left to right follows from the definition of $\mathcal{K}$. The direction from right to left is shown by contraposition, i.e. by showing that for all $A \in \mathcal{A}$ it holds that whenever $K \not\models A$ there is a $K' \in \mathcal{K}$ such that $K' \not\models A$ (it suffices to show that $\mathcal{K}$ is not empty, but the proof is the same). This again follows from the following standard result. We include the proof for the sake of completeness.

Claim For every Kripke model K, for every node w in K, there is a finite rooted submodel K' of K with root w, such that

$$\forall A \in \mathcal{A} \forall x \in K' (K', x \Vdash A \text{ iff } K, x \Vdash A). \tag{3}$$

Proof of Claim. Let $\mathcal{A}$, $K = (W, \preccurlyeq, \Vdash)$ and w be as in the claim. Now we choose step by step, starting with w, a finite subset of W a copy of which will be the domain W_w of our new model $K' = (W_w, \preccurlyeq_w, \Vdash_w)$. Put $\alpha_{\langle\rangle} = w$. Suppose α_σ is defined. We choose elements $\alpha_{\sigma * \langle B \to C \rangle}$ in W, for all elements $(B \to C) \in \{(D \to E) \in \mathcal{A} \mid K, \alpha_\sigma \nVdash D \to E\}$. The node $\alpha_{\sigma * \langle B \to C \rangle}$ is an element $v \in W$ such that $\alpha_\sigma \preccurlyeq v$, $K, v \Vdash B$ and $K, v \nVdash C$. Note that such elements can always be found.

Now define $W_w = \{\sigma \mid \sigma \text{ is defined }\}$, and define the partial order and the forcing relation on K as

$$\begin{array}{lll} \sigma \preccurlyeq_w \tau & \equiv_{def} & \alpha_\sigma \preccurlyeq \alpha_\tau. \\ \sigma \Vdash_w p & \equiv_{def} & \alpha_\sigma \Vdash p, \text{ for p} \in \bar{p}. \end{array}$$

Clearly, K' is finite, as $\mathcal{A}$ is finite too. It is also easy to infer that (3) is satisfied. This proves the claim, and thereby part (a) of the correspondence.

To show part (b) of the lemma, let $\mathcal{K}$ be a stable class of finite rooted Kripke models with the extension property. The model K we are going to construct will consist of equivalence classes of nodes of models in $\mathcal{K}$.

Replace every model in $\mathcal{K}$ by an isomorphic copy, in such a way that the domains of distinct models are disjoint.

Let us define for nodes $k \in K$ and $k' \in K'$

$$k \cong k' \equiv_{def} (K)_k \text{ and } (K')_{k'} \text{ are isomorphic.}$$

(Remember that K_k is the submodel of K generated by k, see Section 2.2.) We write $k \Vdash A$ when A is valid at k in the unique model in $\mathcal{K}$ to which k belongs.

Now we define the domain of K as the set of all $\cong$-equivalence classes $[k]$ of nodes k of models in $\mathcal{K}$. The partial order and the forcing relation on K are defined via

$$\begin{array}{lll} [k] \preccurlyeq [k'] & \equiv_{def} & \exists l \in [k]\, \exists l' \in [k']\ (l, l' \text{ are nodes in the same model} \\ & & \text{and } l \preccurlyeq l' \text{ holds in this model.}) \\ [k] \Vdash p & \equiv_{def} & k \Vdash p. \end{array}$$

Since every two $\cong$-equivalent nodes force the same propositional variables the notion of forcing is well-defined. We have to see that K is in fact an AR-model and that

$$\text{for all } A\text{: } K \models A \text{ iff } \mathcal{K} \models A. \tag{4}$$

We show that K is an AR-model and leave the proof of (4) to the reader.

Consider nodes $[k_1], \ldots, [k_n]$ in K. Assume k_i is a node in the model $K_i \in \mathcal{K}$. Since $\mathcal{K}$ has the extension property there is (an isomorphic copy of) a variant of $(\sum (K_i)_{k_i})'$ in $\mathcal{K}$. Let b be the root of this variant. It is easy to see that $[b]$ is a tight predecessor of $[k_1], \ldots, [k_n]$ in K. This proves part (b) of the correspondence. **QED**

Corollary 3.17 The following are equivalent

(a) B is valid in every AR-model in which A is valid.

(b) B is valid in every stable class of finite rooted Kripke models with the extension property in which A is valid.

Now we are ready to give the

Proof of Theorem 3.13. ($\Leftarrow$) (De Jongh and Visser) We have to show that for all instances A/B of V and I, A admissibly derives B, and we have to see that the three rules of AR preserve admissibility. That is, when reading $\mid\!\sim$ for $\rhd$, if the assumptions of a rule are valid then so is the conclusion. For the two rules this is trivial. Therefore, it remains to treat the axioms.

For instances A/B of I it clearly is the case that $A \mathrel{|\!\sim} B$. Thus all we have to show is that for every instance A/B of the scheme V it holds that if A is derivable in IPC then so is B.

Therefore, consider such instance A/B of V. Let $X = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and let $A = X \to C \vee D$ and $B = (X)(C, D, E_i, \ldots, E_n)$. Arguing by contradiction, suppose A is derivable but B is not. This implies that none of the formulas $(X \to C), (X \to D), (X \to E_1), \ldots, (X \to E_n)$ is derivable. Thus there are Kripke models $K_1, \ldots, K_{n+2}$ at which X is valid but at which respectively $C, D, E_1, \ldots, E_n$ are not valid. Consider the model $(\sum K_i)'$ and call its root b. Since A is derivable A is valid at b. Note furthermore that none of the formulas $C, D, E_1, \ldots, E_n$ can be valid at b. Therefore, the conjunction X cannot be valid at b. But it cannot be not valid either. For if so, there is some $i \leq n$ for which there is a node above b at which E_i is valid while F_i is not valid. As X is valid at all nodes except b the only possibility for this is the node b itself. Thus one of the formulas $E_1, \ldots, E_n$ would be valid at b, which cannot be.
($\Rightarrow$) Immediate from Proposition 3.7, Corollary 3.17 and Proposition 3.5.

QED

3.18 A basis and a subbasis

Let R_{V_i} denote the rule corresponding to V_i (see Section 3.1), i.e let

$$R_{V_i} \qquad (\bigwedge_{i=1}^{n}(p_i \to q_i) \to r \vee s) \vee t / (\bigwedge_{i=1}^{n}(p_i \to q_i))(p_1, \ldots, p_n, r, s) \vee t.$$

Further, let

$$R^-_{V_i} \qquad (\bigwedge_{i=1}^{n}(p_i \to q_i) \to r \vee s) / (\bigwedge_{i=1}^{n}(p_i \to q_i))(p_1, \ldots, p_n, r, s).$$

We need one more lemma to establish that the sets of rules $\{R_{V_1}, R_{V_2}, \ldots\}$ and $\{R^-_{V_1}, R^-_{V_2}, \ldots\}$ are respectively a basis and a subbasis for the admissible rules of IPC.

Lemma 3.19 If $\mathsf{AR} \vdash A \rhd B$ then the rule A/B is derivable in IPC from the set of rules $\{R_{V_1}, R_{V_2}, \ldots\}$.

Proof. We prove the proposition by induction on the length n of the derivation of $A \rhd B$ in AR. For $n = 0$ there is nothing to prove.

For $n > 0$, suppose the last rule applied in the derivation of $A \rhd B$ is the Conjunction rule. This implies that there are B_1, B_2 such that $B = B_1 \wedge B_2$, and such that $A \rhd B_1$ and $A \rhd B_2$ are derivable, and moreover have derivations of length smaller than n. By the induction hypothesis, A/B_1 and A/B_2 are derivable in IPC from $\{R_{V_1}, R_{V_2}, \dots\}$. And thus $A/B_1 \wedge B_2$ is derivable in IPC from $\{R_{V_1}, R_{V_2}, \dots\}$ as well. The case that the last rule applied in the derivation of $A \rhd B$ is the Cut Rule is completely similar. **QED**

Theorem 3.20 $\{R_{V_1}, R_{V_2}, \dots\}$ is a basis for the admissible rules of IPC.

Proof. Immediate from Lemma 3.19 and Theorem 3.13. **QED**

Corollary 3.21 $\{R^-_{V_1}, R^-_{V_2}, \dots\}$ is a subbasis for the admissible rules of IPC.

Visser (1998) showed that the admissible rules of IPC are the same as the propositional admissible rules of HA. This gives us

Corollary 3.22 $\{R_{V_1}, R_{V_2}, \dots\}$ and $\{R^-_{V_1}, R^-_{V_2}, \dots\}$ are respectively a basis and a subbasis for the propositional admissible rules of HA.

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