AN EXACTLY SOLVABLE NON-LINEAR BOLTZMANN EQUATION

M.H. ERNST and E.M. HENDRIKS

Instituut voor Theoretische Fysica der Rijksuniversiteit Utrecht, The Netherlands

Received 29 December 1978

The initial value problem for a model Boltzmann equation of a two dimensional gas with a continuous or discrete energy distribution function and a transition probability $\delta(\epsilon - \epsilon')$ is solved exactly; ϵ and ϵ' are the total energies before and after collision.

There exist very few models for which the non-linear Boltzmann equation (BE) can be solved with arbitrary initial conditions. The only one known to us is a model of McKean [1], which admits only velocities ± 1 . Recently an exact solution for *special* initial conditions has been found [2-4] for a number of Maxwell models. Here we solve the BE for a new model with general initial conditions.

Consider the BE for a distribution function $f(v, t)$, isotropic in velocity space and spatially uniform, which reads in a standard notation [5]

$$\partial_t f = \int dv_1 \int d\Omega g \sigma(g, \Omega) [ff_1 - f'f'_1] \quad (1a)$$

$$= \int dv_1 dv' dv'_1 W(vv_1; v'v'_1) [ff_1 - f'f'_1] \quad (1b)$$

Here σ is the differential scattering cross section, Ω the direction of scattering, and $g = |v - v_1|$. Primes indicate a dependence on post collisional velocities. Eq. (1a) can also be expressed in d -dimensional form with a transition probability $W(vv_1; v'v'_1) = 2^{-d} g^{3-d} \times \sigma(g, \Omega) \delta(v + v_1 - v' - v'_1) \delta(v^2 + v_1^2 - v'^2 - v'^2_1)$. It leads directly to the usual conservation laws and to the H-theorem.

Tjon and Wu [3] have proposed a class of models with $W = a(\epsilon) \delta(\epsilon - \epsilon')$ where $a(\epsilon)$ is a function (yet to be chosen) of the total energy $\epsilon = v^2 + v_1^2$ of the colliding particles. The basic simplification of these models is that momentum conservation has been dropped. The BE for these models can be reduced to a form similar to eq. (1a) with a collision rate $g\sigma \propto$

$\epsilon^{d-1} a(\epsilon)$. The choice $a(\epsilon) = \epsilon^{1-d}$ leads to an energy independent collision rate, typical for Maxwell models; here we choose $a(\epsilon) = \epsilon^{2-d}$, yielding a collision rate proportional to ϵ . For comparison, the collision rate in a repulsive intermolecular potential ($\sim r^{-n}$) is proportional to ϵ^{-k} with $k = n^{-1}(d-1) - \frac{1}{2}$, where ϵ is the relative energy, and $n = \infty$ corresponds to hard spheres. For two dimensions this model can be represented most conveniently in terms of the energy distribution function $F(x, t) \propto f(v, t)$ with $x = \frac{1}{2}v^2$, normalizations $\int_0^\infty dx F(x, t) = \int_0^\infty dx x F(x, t) = 1$ (conservation laws), and a final maxwellian $F(x, \infty) = e^{-x}$. It reads

$$\begin{aligned} \partial_t F(x) &= \int_x^\infty dy \int_0^y dy' [F(y-y')F(y')] \\ &\quad - F(y-x)F(x)] \end{aligned} \quad (2a)$$

$$\begin{aligned} &= -(x+1)F(x) \\ &\quad + \int_x^\infty dy \int_0^y dy' F(y-y')F(y'). \end{aligned} \quad (2b)$$

This 2-dimensional BE can be solved for any given initial distribution $F(x, 0)$ in the following manner: The Laplace transform $G(z, t) = \int_0^\infty dx e^{-xz} F(x, t)$ satisfies

$$(\partial_t - \partial_z + 1)G = z^{-1}(1 - G^2). \quad (3)$$

This partial differential equation can be reduced to an ordinary *first* order Riccati differential equation by

substituting: $z = x$; $z + t = y$ and $G(z, t) = g(x, y)$, satisfying $-g'(x) + g(x) + x^{-1}g^2(x) = x^{-1}$. Later on integration constants will become arbitrary functions of the "hidden" parameter $y = z + t$. For Maxwell-type molecules [3,4] $G(z, t)$ satisfies the *second* order partial differential equation $(\partial_t + 1)\partial_z zG = G^2$, where only special solutions (similarity solutions) have been found. Our differential equation can be solved by quadrature after the substitution $g(x) = (1 + x)^{-1} + u^{-1}(x)$. The result is

$$G(z, t) = \frac{\phi(z + t) + (z - 1)e^{-t}}{(1 + z)\phi(z + t) - e^{-t}}, \quad (4)$$

where the arbitrary function $\phi(y) = \phi(z + t)$ is to be determined from the initial distribution $F(x, 0)$ or its Laplace transform $G(z, 0) \equiv G_0(z)$, yielding $\phi(z) = [G_0(z) + z - 1] \cdot [(1 + z)G_0(z) - 1]^{-1}$. A final Laplace inversion yields the exact solution $F(x, t) = (2\pi i)^{-1} \int_{-\infty}^{i\infty} dz e^{xz} G(z, t)$.

Comments. (i) $G_0(z)$ is of course analytic for $\text{Re } z > 0$. The same should hold for $G(z, t)$ for *all* t , as it is the Laplace transform of a distribution function. Furthermore, $F(x, t)$ should be positive for *all* t . We have verified both requirements in a large number of cases, we have not been able to find a counter example, but no general proof has been found. However, one can give many examples of $F(x, 0)$ (not necessarily normalizable, nor positive), such that after a *sufficiently long* time $F(x, t)$ becomes positive, and its Laplace transform analytic in $\text{Re } z > 0$ (see eq. (5)).

(ii) The *initial growth* of a distribution function with $F(x, 0) = 0$ for $x > a$, and $F(a, 0) \neq 0$ can in principle be obtained by expanding the exact solution (4) in powers of t . A more direct way is to solve eq. (2) by iteration, starting from the given $F(x, 0)$. The *dominant* growth for small times is $\propto t$ in the interval $x \leq 2a$, and $\propto t^n$ in the interval $(na, (n+1)a)$ with $n \geq 2$. This result also implies that *new singularities* develop at $x = na$. More generally, initial singularities in $F(x, 0)$ at x_i ($i = 1, 2, \dots$) create new ones at $n_1 x_1 + n_2 x_2 + \dots$ with $n_i \geq 0$. The nature of these new singularities depends on the singularities in $F(x, 0)$, and on the transition probabilities, i.e. on the model considered; e.g. in our model, $F(x, 0) = \delta(x - 1)$ develops as dominant singularities: a step $\theta(x - 2)$, a kink $(x - 3)\theta(x - 3)$, $(x - 4)^2\theta(x - 4)$ etc. The type of singularity is weaker for larger x -values.

(iii) The *complete time evolution of singularities* can be illustrated best by considering an $F(x, 0)$ containing terms like $\sum_i a_i \delta(x - x_i)$. Its Laplace transform $G_0(z)$ behaves like $\sum_i a_i \exp(-x_i z)$, and hence $|G_0(i\eta)| \simeq \text{constant}$ for $\eta \rightarrow \pm\infty$. Eq. (4) implies $G(z, t) \simeq e^{-t} \times G_0(z + t)$ for $\eta = \text{Im } z \rightarrow \pm\infty$ at fixed t , and consequently $F(x, t)$ contains contributions $F_\delta(x, t) = \sum_i a_i \delta(x - x_i) \exp[-(x_i + 1)t]$. An important point to notice is that an initial δ -peak does not widen, but keeps its shape, while its strength decreases at a rate $(x_i + 1)$. At the same time it develops at wide foot (see (ii)). The new singularities decay similarly; in addition the type of singularity may change as time proceeds.

(iv) For a discussion of the *decay* of the *complete initial distribution* it is convenient to replace the integration path $(-\infty, +\infty)$ in the Laplace inversion formula by closed contours in $\text{Re } z < 0$, as can be done in most cases. The arguments are: suppose that $F(x, 0)$ does not contain any δ -peaks, then $|G_0(i\eta)|$ is smaller than a constant for $\eta = \text{Im } z \rightarrow \pm\infty$, and the same holds for $|G(i\eta, t)|$ at fixed t . If $F(x, t)$ does contain δ -peaks, they should be first subtracted out, and then the same arguments apply. Furthermore, in cases where $G(z, t)$ grows at most like a power of ζ for $\zeta = \text{Re } z \rightarrow -\infty$, the contours can be closed. In cases where $G(z, t)$ behaves exponentially like $\exp(-\alpha\zeta)$ for $\zeta \rightarrow \pm\infty$, the contours can also be closed but their contributions should be multiplied by a factor $\theta(x - \alpha)$. The final approach towards the maxwellian $F(x, \infty) = e^{-x}$ can now be discussed in terms of contributions from contours around poles and branch cuts of $G(z, t)$. For large t the dominant (Maxwell) pole in eq. (4) is located at $z_0 \simeq -1 + p_0(t)$, with $p_0(t) \simeq e^{-t}/\phi(t - 1) \simeq O(e^{-t})$, since $|\phi(t)| > \text{constant}$ for $t > 1$. In addition, there may be moving poles at $z_n = -t + c_n + p_n(t)$, where $\phi(c_n) = 0$ with $n \geq 1$ and $p_n(t) \simeq O(e^{-t})$. Hence, for *large* t the total contribution from the poles, $F_p(x, t)$, becomes

$$F_p(x, t) \simeq [1 + O(e^{-t})] e^{-x} e^{xp_0(t)} + \sum_n O(e^{-t}) e^{-xt} \exp[x(c_n + p_n(t))], \quad (5)$$

where we have only indicated the order of magnitude of the residues. The possible branch points of $G(z, t)$, originating from branch points of $\phi(z)$, are all moving in time. Hence, the contributions from branch cuts are proportional to e^{-xt} (as are those from the poles),

and the dominant long time behaviour is given by the first terms in eq. (5), i.e. $F(x, t)/F(x, \infty) \simeq \exp(xp_0(t))$ with $p_0(t) = O(e^{-t})$. This approach is non-uniform, since for thermal energies, $x \sim 1$, it takes a time $t_0 \simeq 1$ (in units of mean free times), and for very large x it takes a time $t_0 \simeq \log x$ before $xp_0(t)$ becomes < 1 , i.e. before the Maxwell value has been reached. For similar results, see ref. [2]. The detailed form of the long time behaviour of $p_0(t)$ depends on the asymptotics of $\phi(t)$ or $G_0(t)$, and, therefore, on the small x -behaviour of $F(x, 0)$. This introduces at most a few factors of t^{-1} into $p_0(t)$, but does not change qualitatively the dominant dependence of $p_0(t) \simeq O(e^{-t})$. We only note a small tendency of a faster approach to the Maxwellian, the closer $F(x, \infty)$ for small x .

(v) The linearized BE $\partial_t h = \Lambda h$ for $F(x, t) = e^{-x} \times (1 + h(x, t))$ can be obtained in explicit form from eq. (2), and can, of course, be solved exactly as an initial value problem. Of special interest in kinetic theory are the eigenvalues λ and the eigenfunctions $h(x, t) = e^{-\lambda t} h(x)$ of the linearized collision operator Λ , which is symmetric and negative semi-definite, as one can show by introducing a scalar product $(f, g) = \int_0^\infty dx \times e^{-x} f(x)g(x)$. The Hilbert-Enskog canonical form [5] for Λ reads in our case $\Lambda h = Kh - (x+1)h$, where the Hilbert operator $Kh = \int_0^\infty dy K(x, y)h(y)$ has the (symmetrized) kernel $\tilde{K}(x, y) = \exp(-\frac{1}{2}(x-y))K(x, y) = 2 \exp(-\frac{1}{2}|x-y|) - (x+y) \exp(-\frac{1}{2}(x+y))$. Unlike e.g. the hard sphere case [5] the Hilbert operator is here non-compact (the first term has actually a continuous spectrum in the interval (0,8)). The complete operator Λ has a two-fold degenerate eigenvalue $\lambda = 0$, with orthogonal eigenfunctions $h_0(x) = 1$ and $h_1(x) = x - 1$, and a continuous spectrum $\lambda = \mu + 1$ with $\mu > 0$. The eigenfunctions in the orthogonal complement of the null space (h_0, h_1) can be obtained directly from the linearized form of eq. (3), and read $h(x, \mu) = e^{\mu/2} \{ \delta(x - \mu) + \theta(x - \mu)(x - \mu - 2) \}$ with orthogonality relations $(h(\mu), h(\mu')) = \delta(\mu - \mu')$. There are for this model no discrete eigenvalues in the gap (0, 1), as follows from the completeness of the set $\{h_0(x), h_1(x), h(x, \mu) \text{ with } \mu > 0\}$.

(vi) The moments, $M_n(t) = \int_0^\infty dx x^n F(x, t) = (-)^n \times G^{(n)}(0, t)$, where $G^{(n)}$ is the n th derivative of $G(z, t)$ with respect to z , can also be obtained directly from the exact solution (4). The conservation laws prescribe $M_0(t) = M_1(t) = 1$. By comparing the moments $M_n(t)$ for $n \geq 2$ with the corresponding mo-

ments $M_n^{(L)}(t)$, obtained from the linearized BE, one sees that none of them are equal, not even in the leading deviations from $M_n(\infty) = M_n^{(L)}(\infty) = n!$ for long times. This is unlike the case of Maxwell molecules, where $M_n(t) = M_n^{(L)}(t)$ for $n = 2, 3$.

(vii) An exactly solvable model for a binary mixture ($i, j = 1, 2$ labels the components) in two dimensions is found by choosing $W_{ij}(v_1; v'_1) = m_i m_j \delta(\epsilon - \epsilon')$ where $\epsilon = m_i v_1^2 + m_j v_1'^2$ is the total energy in an (i, j) -collision.

(viii) A discrete version of eq. (2), where the distribution function, $F_n(t)$, depends on discrete energies n , reads $\dot{F}_n = \sum \delta_{n+m, n'+m'} (F_{n'} F_{m'} - F_n F_m)$. The sum runs over $m, n', m' = 0, 1, 2, \dots$. This BE obeys the conservation laws $\sum_n F_n = 1$ and $\sum_n n F_n = \epsilon$. The equation for the generating function $G(x, t) = \sum_n x^n F_n(t)$ (compare eq. (3)) can be solved similarly, and we find the general solution

$$G(x, t) = \frac{\phi(xe^{-t}) + (\epsilon - \epsilon x - x)e^{-(\epsilon+1)t}}{(1 + \epsilon - \epsilon x)\phi(xe^{-t}) - xe^{-(\epsilon+1)t}}, \quad (6)$$

where the arbitrary function $\phi(x)$ can be determined from the given initial value $G(x, 0) = G_0(x)$. Many of the previous comments can be repeated. We only note that the linearized collision operator has a discrete spectrum λ_k with $\lambda_k = 0$ for $k = 0, 1$ and $\lambda_k = \epsilon + k - 1$ for $k \geq 2$. The eigenfunctions, which will not be quoted here, are orthogonal polynomials, defined on the integers. Finally, a minor modification of the transition probabilities in this discrete version yields yet another solvable BE.

Details on the reported results will be published elsewhere.

It is a pleasure to thank N.G. van Kampen, J. Tjon and I. de Schepper for stimulating discussions and valuable comments.

References

- [1] H.P. McKean Jr., J. Comb. Theory 2 (1967) 358.
- [2] A.V. Bobylev, Sov. Phys. Dokl. 20 (1976) 822; M. Krook and T.T. Wu, Phys. Fluids 20 (1977) 1589.
- [3] J. Tjon and T.T. Wu, preprint, Univ. of Utrecht (August 1978).
- [4] M.H. Ernst, Phys. Lett. 69A (1979) 390.
- [5] J.D. Foch and G.W. Ford, Studies in statistical mechanics V, eds. J. de Boer and G.E. Uhlenbeck (North-Holland, Amsterdam, 1970) p. 128.