FIXED-JT AVERAGES OF ELECTROMAGNETIC OPERATORS

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Abstract: A method is presented to express fixed-JT traces in terms of fixed- $J_z T_z$ traces for an arbitrary tensor operator. The method is used to evaluate the coefficients of the linear, secular energy variation of the electric quadrupole moment and the gyromagnetic ratio for a number of nuclei in the sd shell. Upper limits for the electromagnetic moments are used to estimate the fluctuations about this secular variation.

1. Introduction

In order to obtain the electromagnetic moments of nuclear states in a shell-model calculation, one has to evaluate the electromagnetic transition operator taken between two identical eigenstates. The required eigenfunctions can be derived from a diagonalization of the hamiltonian in an appropriate configuration space. In most cases this space is determined by the size of the matrices that can be diagonalized by a computer. In order to perform calculations in larger spaces one has to look for different methods. One of these methods is statistical spectroscopy^{1–3}). The idea behind this method is that the relevant nuclear properties are obtained by the calculation of averages of operators in a given configuration space. The averages are given, essentially, by the traces of those operators which can be evaluated directly.

In the case of electromagnetic transition amplitudes it was shown by Draayer *et al.*^{7,15}) that the amplitudes could be described locally in the space of the initial and final energies by a gaussian distribution (Porter–Thomas assumption). The variance of this distribution showed a secular behaviour (as a function of the initial and final energies) that could be well described by low-order polynomials with coefficients that could be expressed in terms of moments of the hamiltonian and the electromagnetic operator.

In this paper we study the case of electromagnetic moments where the situation is more complicated. In general the electromagnetic moments do not fluctuate about zero, but show a nontrivial secular variation which depends on the angular momentum and isospin of the states involved. It is no use to describe the fluctuations by means of the Porter–Thomas assumption because the variance of the diagonal

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elements of an operator cannot be expressed in terms of the moments of the hamiltonian and the electromagnetic operator. Hence they cannot be calculated without the explicit use of shell-model wave functions.

In sect. 2 we present some shell-model results which clearly show the nature of the fluctuations. In sect. 3 we evaluate the properties of the electromagnetic moments without using explicit shell-model wave functions.

As is discussed in sect. 3 the secular behaviour can be derived from fixed- JT traces of a tensor operator. Let us disregard isospin for the moment. The fixed- J traces can be expressed in traces for a fixed value of the angular-momentum projection J_z . The latter traces can be evaluated by taking particle partitions over sets of single-particle states all of which have the same angular-momentum projection^{4,5}). The required propagation polynomials can be expressed in terms of the number of states for fixed angular-momentum projection⁶). Although for a scalar operator it is almost trivial to express traces for fixed J in terms of the traces for fixed J_z , in the case of an arbitrary tensor operator the direct application of the usual relations leads to a lengthy calculation. A concise method to derive traces for fixed J from the fixed- J_z traces of an arbitrary tensor operator is developed in appendix A. Sect. 3 concludes with a discussion of the numerical results for a few nuclei in the sd shell.

In sect. 4 the magnitude of the fluctuations is estimated by the calculation of upper limits. Both scalar and J -dependent upper limits are studied and compared with the shell-model results. Better estimates for the largest moments are obtained with statistical assumptions for random-matrix ensembles. Concluding remarks are made in sect. 5.

2. Shell-model results

The electric quadrupole moments and magnetic dipole moments⁹) of the nuclides ^{20}Ne , ^{21}Ne , ^{24}Mg and ^{28}Si have been calculated with the shell model for the complete sd shell. The surface delta interaction was used. The single-particle energies were taken to be $\epsilon(0d_{5/2}) = -4.15$ MeV, $\epsilon(1s_{1/2}) = -3.28$ MeV and $\epsilon(0d_{3/2}) = 0.88$ MeV.

Some typical examples of the calculated results for the electric quadrupole moments and the magnetic dipole moments are shown in figs. 1a, 1b, 2a and 2b. The secular behaviour will be assumed to be linear. Owing to the large fluctuations one can do no better. In general one can say that the quadrupole moments fluctuate about zero for low-spin values. The magnetic moments show, when plotted versus the energy, a slope that is almost independent of J for each nuclide. The average of the calculated values turns out to be very close to the Schmidt value.

The fluctuations about the secular behaviour are rather large, i.e. of the same order of magnitude as the secular variation over the whole spectrum. In particular in the ground-state region one can find large fluctuations for the isoscalar quadrupole moment.

Fig. 1. Exact shell-model results for the isoscalar quadrupole moment Q^0 (a) and the isoscalar magnetic moment μ^0 (b) as a function of the excitation energy of ^{20}Ne , $T=0$ with angular-momentum values as indicated. The straight line represents the linear secular behaviour with coefficients given in figs. 3 and 4.

Fig. 2. Exact shell-model results for the isoscalar quadrupole moment Q^0 (a) and isovector quadrupole moment Q^1 (b) as a function of the excitation energy of ^{21}Ne , $J = \frac{3}{2}$, $T = \frac{1}{2}$. The straight line represents the linear secular behaviour with coefficients given in figs. 3 and 4.

Concluding one can say that only for large J -values can a clear secular behaviour be observed. In other cases the electromagnetic moments show a purely stochastic behaviour (possibly with the exception of one or two low-lying states) with the magnitude of the fluctuations as the most relevant quantity. The wave functions do not seem to contain any specific information and the use of an explicit calculation might be questioned.

In the following sections it will be shown that the secular behaviour and the magnitude of the fluctuations can also be evaluated without explicit calculation of the shell-model wave function.

3. Evaluation of the coefficients of the secular behaviour

The energy dependence of the expectation values of an electromagnetic operator O^λ can be obtained from an expansion ⁷⁾ of the δ -function $\delta(H - E)$ in polynomials $P_\nu(H)$ and $P_\nu(E)$ which constitute an orthonormal system of functions associated with the eigenvalue density as a weight function, i.e.

$$O^\lambda(E, n, J) = \langle O^\lambda \delta(H - E) \rangle_{nJ} = \sum_\nu \langle O^\lambda P_\nu(H) \rangle_{nJ} P_\nu(E) \quad (3.1)$$

with the expectation value

$$\langle A \rangle_{nJ} = \frac{1}{d_{nJ}} \text{Tr}_{nJ} (A). \quad (3.2)$$

Here d_{nJ} denotes the dimension of the n -particle space of spin J . It is assumed that the central-limit theorem operates such that the eigenvalues E_i of H and also $E_i + \varepsilon O^\lambda(E_i)$ (ε small) possess a gaussian distribution, i.e. the higher-order cumulants are zero for the set $\{E_i\}$ and zero up to first order in ε for the set $\{E_i + \varepsilon O^\lambda(E_i)\}$ [refs. ^{7,8)}]. The two distributions must be identical to within a scale change and hence $O^\lambda(E)$ must show a linear dependence on E . If in eq. (3.1) only the terms $\nu = 0$ and $\nu = 1$ are retained, the energy dependence is given by

$$O^\lambda(E, n, J) = \langle O^\lambda \rangle_{nJ} + \frac{\langle O^\lambda (H - E_{nJ}) \rangle_{nJ} (E - E_{nJ})}{\sigma_{nJ}^2}, \quad (3.3)$$

where E_{nJ} and σ_{nJ}^2 are the centroid and variance of the considered eigenvalue distribution of H , respectively.

In appendix A it is shown that for fixed values of J the expectation value of the tensor operator in eq. (3.3) can be expressed, as was discussed in the introduction, in terms of traces for fixed values of J_z . The latter can be evaluated by the use of a contraction procedure that leads to a diagrammatic expansion ⁶⁾ which describes the way the operators have been contracted. In the present paper the neutron-proton formalism is used rather than the isospin formalism. Thus one obtains for the traces

relating to eq. (3.3) the expansions

$$\text{Tr}_{n_n n_p J_z} (O_0^\lambda) = \sum_{\omega} \text{diagram}, \quad (3.4)$$

$$\text{Tr}_{n_n n_p J_z} (O_0^\lambda H_2) = \sum_{\omega_1 \omega_2 \omega_3} \text{diagram} + \sum_{\omega_1 \omega_2 \omega_3} \text{diagram}, \quad (3.5)$$

$$\text{Tr}_{n_n n_p J_z} (O_0^\lambda H_1) = \sum_{\omega_1 \omega_2} \text{diagram} + \sum_{\omega_1 \omega_2} \text{diagram}, \quad (3.6)$$

where H_1 and H_2 are the one- and two-body parts of the hamiltonian, respectively. The index ω_i tells whether the contraction applies to a proton (π) or a neutron (ν) orbit. As an example an explicit expression for the following diagram is given:

$$\begin{aligned} \text{diagram} &= \sum_{\substack{j_1 j_2 j_3 \\ m \ m_3}} \langle j_1 m_{11}^\pi | O_{00}^\lambda | j_3 m_{33}^\pi \rangle W_{j_2 m_2 j_3 m}^\pi W_{j_2 m_2 j_1 m}^\nu \\ &\times \sum_{\substack{n_1 + n_2 + n_3 = n_n - 1 \\ p_1 + p_2 = n_p - 1}} (-1)^{n_1 + n_2 + p_1} \Omega_{n_3 p_2} (M - (p_1 + 1)m_2 - (n_2 + 1)m - n_1 m), \end{aligned} \quad (3.7)$$

Fig. 3. The average values of the isoscalar ($\tau=0$) and isovector ($\tau=1$) quadrupole moment $\langle Q^\tau \rangle_{JT}$ (a) and the slope $\langle Q^\tau (H - E_{JT}) \rangle_{JT} / \sigma_{JT}$ (b) as a function of the angular momentum J for ^{20}Ne , ^{24}Mg , ^{28}Si ($T=0$) and for ^{21}Ne ($T=\frac{1}{2}$).

Fig. 3.—(contd.)

where $W_{\alpha\beta\gamma\delta}$ represents the antisymmetrized two-body matrix elements and $\Omega_{np}(M)$ the number of states for n neutrons and p protons for given angular-momentum projection M .

The angular-momentum dependence of the average quadrupole moment $\langle Q^\tau \rangle_{JT}$ and of the correlation between the quadrupole moment and the hamiltonian $\langle Q^\tau(H - E_{JT}) \rangle_{JT} / \sigma_{JT}$ are given in figs. 3a and 3b. The centroid and the width of the eigenvalue distribution of the hamiltonian for fixed angular momentum J and isospin T are denoted by E_{JT} and σ_{JT} . In the case of ^{28}Si the average quadrupole moment is zero, which is a consequence of the minus sign under particle-hole transformation for even-order single-particle tensor operators^{10,11}). For the upper part of the sd shell, one finds an increasing value of the quadrupole moment as a function of the angular momentum. The angular-momentum dependence of the correlation $\langle Q^\tau(H - E_{JT}) \rangle_{JT} / \sigma_{JT}$ shows a relatively irregular pattern. In general one finds larger values of the correlation for larger angular momenta.

The matrix element of the z -component of the isoscalar M1 operator can be written as

$$\mu^0 \equiv \langle JJ_z = JT | \mu_z^0 | JJ_z = JT \rangle = \frac{1}{2}J + \frac{1}{2}(g^s - 1) \langle JJ_z = JT | S_z | JJ_z = JT \rangle, \quad (3.8)$$

with g^s the isoscalar spin g -factor and S_z the z -component of the total intrinsic spin. We shall consider the energy dependence of μ^0/J . The J -dependence of the average isoscalar magnetic momentum $\langle \mu_z^0 \rangle_{JT}$ divided by J and of the correlation $\langle \mu_z^0(H - E_{JT}) \rangle_{JT} / \sigma_{JT}$ divided by J are given in figs. 4a and 4b, respectively. The quotient of the average isoscalar moment and J appears to be almost independent of the angular momentum J and is slightly smaller than the $j = \frac{5}{2}$ single-shell value of 0.576 n.m. The correlation of the average isoscalar magnetic moment and the

Fig. 4. The average values of the isoscalar magnetic moment ratio $\langle \mu_z^0 \rangle_{JT} / J$ (a) and the slope $\langle \mu_z^0(H - E_{JT}) \rangle_{JT} / J \sigma_{JT}$ (b) as a function of the angular momentum J for ^{20}Ne , ^{24}Mg , ^{28}Si ($T=0$) and ^{21}Ne ($T=\frac{1}{2}$).

hamiltonian, divided by J , does not depend strongly on the angular momentum. In the case of ^{24}Mg the correlation is remarkably close to zero. This suggests that one can estimate the J -dependence by means of scalar moments of the hamiltonian and the magnetic dipole operator. The correlations of the electric quadrupole operator and the hamiltonian, which show an irregular behaviour as a function of J , can only be obtained [in contrast to e.g. ref. ¹⁶⁾ where scalar moments are used] by exact, fixed- JT averaging.

It is seen that the conclusions that were made in sect. 2 from a few shell-model calculations are now confirmed for many more cases by the use of a moment method. These more general results are thus obtained in an efficient way, but the fluctuations about the averages cannot be reproduced.

In the next section the magnitude of the fluctuations will be considered and upper limits will be presented.

4. Upper limits for the magnitude of the fluctuations

A crude estimate of an upper limit of electromagnetic moments ($\omega_{\max}$) in a space with a definite value of angular momentum and isospin can be obtained from the square root of the total strength and the appropriate geometrical factor,

$$\omega_{\max} = \left[\sum_{\alpha\alpha'} \langle JTT_z\alpha \| O_{00}^{\lambda\tau} \| JTT_z\alpha' \rangle^2 \begin{pmatrix} J & \lambda & J \\ -J & 0 & J \end{pmatrix}^2 \right]^{1/2}. \quad (4.1)$$

This upper limit is interaction independent and depends on the size of the model space. For the isoscalar magnetic moment an upper limit is calculated for the spin operator (cf. eq. (3.8)).

The total strength can be written as a fixed- JT trace ¹²⁾ and subsequently evaluated as indicated above.

In tables 1 and 2 results are shown for quadrupole moments and isoscalar magnetic moments of ^{20}Ne , ^{21}Ne , ^{24}Mg and ^{28}Si in the sd shell. Comparison with the experimental value shows that the calculated upper limit for the quadrupole moment of the $J=2$, $T=0$ states of ^{20}Ne is rather close.

An alternative way to obtain an upper limit is to calculate for the hermitean one-body operator $O_{00}^{\lambda\tau}$ the eigenvalue $\Omega_{\max}$ with maximum absolute value. This can be performed by taking the n th root of the trace of the n th power (n even) of $O_{00}^{\lambda\tau}$ in a space with n_n neutrons and n_p protons:

$$|\Omega_{\max}| = \lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} [\text{Tr}_{n_n n_p} (O_{00}^{\lambda\tau})^n]^{1/n} \leq [\text{Tr}_{n_n n_p} (O_{00}^{\lambda\tau})^n]^{1/n} \equiv \omega_{\max}^{(n)}, \quad (4.2)$$

where the inequality applies for any finite (even) value of n . It will be used to estimate $\Omega_{\max}$. In the case of an isoscalar operator the traces are evaluated in a space of fixed isospin and in the case of an isovector operator the traces are evaluated

TABLE 1

The dimension d , the centroid E (MeV), the variance σ^2 (MeV²), the skewness γ_1 , the excess γ_2 and the upper limits $\omega_{\max}^{(8)}$, $\omega_{\max}^{(12)}$ and $\omega_{\max}^{\text{st}}$ related to the quadrupole moment operator Q^τ ($\tau=0$ or $\tau=1$)

Nucleus	d	E	σ^2	γ_1	γ_2	$\omega_{\max}^{(8)}$	$\omega_{\max}^{(12)}$	$\omega_{\max}^{\text{st}}$	Sum rule	$\omega_{\max}^J$	$\omega_{\max}^{\tau J}$	Exp
$^{20}\text{Ne}, Q^0$	1 716	0	13.29	0.29	-0.27	15.8	13.9	12.6	17.4 ($J=2$)	5.7	3.0	-21
$^{21}\text{Ne}, Q^0$	8 580	0	15.61	0.22	-0.26	20.9	17.0	15.3	38.6 ($J=\frac{3}{2}$)	7.8	3.9	
$^{21}\text{Ne}, Q^1$	14 520	0	14.73	0.001	-0.24	21.3	16.4	15.2	25.0 ($J=\frac{3}{2}$)	5.1	2.6	
$^{24}\text{Mg}, Q^0$	70 785	0	22.57	0.11	-0.20	32.7	24.1	20.6	97.6 ($J=2$)	9.4	4.2	-21
$^{28}\text{Si}, Q^0$	226 512	0	26.82	0.00	-0.20	41.1	28.8	22.3	173.2 ($J=2$)	10.9	4.4	14

The effective charge is equal to zero and the harmonic-oscillator size parameter is taken as $b = 1.0 A^{1/3}$ fm, with A the atomic mass number. Under 'Sum rule' the upper limit $\omega_{\max}$ according to eq. (4.1) is given and under $\omega_{\max}^J$ the estimated upper limit $\omega_{\max}^J / \sqrt{\frac{1}{2}(N_\phi + 1)}$ (cf. eq. (4.8)). Under 'Exp' one finds the experimental values of the quadrupole moment for the lowest state of the indicated angular momentum. Each row of the latter four columns refers to one and the same J -value. All quadrupole moments and upper limits are given in $e \cdot \text{fm}^2$.

TABLE 2
The same quantities as presented in table 1, now calculated for the magnetic dipole moments $\mu^0 - \frac{1}{2}J$ (isoscalar part) and μ^1 (isovector part)

Nucleus	d	E	σ^2	γ_1	γ_2	$\omega_{\max}^{(8)}$	$\omega_{\max}^{(12)}$	$\omega_{\max}^{\text{st}}$	Sum rule	$\omega_{\max}^J$	$\omega_{\max}^{rJ}$	ω_{exact}	Exp
$^{20}\text{Ne}, \mu^0 - \frac{1}{2}J$	1 716	0	0.142	0	-0.45	1.66	1.19	1.30	2.46 ($J=2$)	0.81	0.57	0.76	0.08
$^{21}\text{Ne}, \mu^0 - \frac{1}{2}J$	8 580	0	0.165	0	-0.31	2.05	1.58	1.57	6.14 ($J=\frac{3}{2}$)	1.28	0.81	0.95	0.11
$^{21}\text{Ne}, \mu^1$	14 520	0	25.44	0	-0.27	27.68	21.16	20.1	54.6 ($J=\frac{3}{2}$)	11.1	7.0		-1.5
$^{24}\text{Mg}, \mu^0 - \frac{1}{2}J$	70 785	0	0.228	0	-0.13	3.33	2.45	2.07	14.0 ($J=2$)	1.39	0.80	1.52	0.02
$^{28}\text{Si}, \mu^0 - \frac{1}{2}J$	226 512	0	0.257	0	-0.09	4.16	2.97	2.33	24.5 ($J=2$)	1.58	0.79	2.28	0.12

The upper limit under ω_{exact} is obtained from the maximum eigenvalue of the z -component of the spin operator (cf. eq. (3.8)). Each row of the five columns on the right refers to one and the same J -value.

in a space of fixed isospin projection. In this way upper limits are obtained which must hold for all J -values. As the value of the electromagnetic moments tends to increase for large J -values (see sect. 3) one will thus obtain too large upper limits for the lower J -values.

The parameters of the distribution function $\rho(\omega)$ of the eigenvalues of $O_{00}^{\lambda\tau}$ are also determined by $\text{Tr}_{n_n n_p} ((O_{00}^{\lambda\tau})^p)$. The third and fourth moments show the deviation from a gaussian distribution. An estimate of the extreme eigenvalue $\omega_{\max}$ can be derived from the condition

$$\int_{\omega_{\max}^{\text{st}}}^{\infty} \rho(\omega) d\omega = \frac{1}{2}. \quad (4.3)$$

When the centroid of $\rho(\omega)$ is positive or zero (which is true in all cases that have been considered) this value of $\omega_{\max}^{\text{st}}$ can be used to check the convergence in eq. (4.2) for any finite (even) value of n .

The moments for n_n neutrons and n_p protons in the sd shell can be calculated by an extension of Ginocchio's single-orbit result [ref. ¹³], eq. (55b)].

Calculations have been performed for the magnetic dipole moments and the electric quadrupole moments of ^{20}Ne , ^{21}Ne , ^{24}Mg and ^{28}S in the sd shell. The results of the first five, the 8th and the 12th moments are shown in tables 1 and 2. The centroid E , the width σ , the skewness γ_1 and the excess γ_2 have been derived from the first four moments. The upper limits $\omega_{\max}^{(8)}$ and $\omega_{\max}^{(12)}$ follow from the 8th and 12th moments (cf. eq. (4.2)). The small values of γ_1 and γ_2 show that the distribution function $\rho(\omega)$ of the eigenvalues of $O_{00}^{\lambda\tau}$ is very close to gaussian. All odd moments of the magnetic moment operator are zero because the operator is the sum of the operators l_z and s_z . By using particle-hole symmetry it can be shown that the odd moments of the quadrupole moment operator for ^{28}Si are zero. The centroid $\text{Tr}_{n_n n_p} (O_{00}^{\lambda\tau})$ is always equal to zero for $\lambda \neq 0$. This is a consequence of the vanishing single-particle average,

$$\sum_m \langle jm | O_{00}^{\lambda\tau} | jm \rangle = \sum_m (-1)^{j-m} \begin{pmatrix} j & \lambda & j \\ -m & 0 & m \end{pmatrix} \langle j || O_0^{\lambda\tau} || j \rangle = 0 \text{ for } \lambda \neq 0. \quad (4.4)$$

Comparison of the values for the upper limits obtained from eq. (4.2) shows that for $n = 12$ the convergence is rather good. In case of the isoscalar magnetic moment an exact upper limit is obtained from the maximum eigenvalue of $\frac{1}{2}(g^s - 1)S_z$ (cf. eq. (3.8)). It appears that the upper limits obtained from eq. (4.2) are only about 50% too large.

Assuming that the distribution of the eigenvalues of the operator $O_{00}^{\lambda\tau}$ is also close to gaussian in a space with well-defined angular momentum, one can estimate the magnitude of the fluctuations from the first and second moments of $O_{00}^{\lambda\tau}$. The first moments are given in figs. 3 and 4. The second moments follow from eq. (4.1) and are equal to $(\omega_{\max})^2$. By the use of condition (4.3) for fixed- J densities $\rho(\omega)$ J -dependent upper limits are obtained which are given in the column $\omega_{\max}^J$ of tables

1 and 2 for the J -values indicated. In fact we find appreciably smaller values, but in some cases they are still much larger than the shell-model values (see e.g. figs. 2a and 2b).

Closer upper limits can be obtained by invoking statistical assumptions. If $\{\phi_\alpha\}$ is a complete set of wave functions in the occupation number basis and $|\Omega_i\rangle$ are eigenfunctions of $O_{00}^{\lambda\tau}$ (ϕ_α and Ω_i both for the number of active particles under consideration), then one has

$$\sum_i \langle \Omega_i | O_{00}^{\lambda\tau} | \Omega_i \rangle^2 = \sum_\alpha \langle \phi_\alpha | O_{00}^{\lambda\tau} | \phi_\alpha \rangle^2 + \sum_{\alpha \neq \beta} \langle \phi_\alpha | O_{00}^{\lambda\tau} | \phi_\beta \rangle^2. \quad (4.5)$$

Because $O_{00}^{\lambda\tau}$ is a one-body operator, there is only a small number N_ϕ of wave functions ϕ_β which yield a nonzero matrix element for a given ϕ_α . In the case of an E2 (M1) operator one finds $N_\phi \approx 6(3)$ for ^{20}Ne and $N_\phi \approx 7(4)$ for ^{21}Ne . For ^{24}Mg and ^{28}Si one obtains the values $N_\phi \approx 9(5)$ and $N_\phi \approx 11(7)$, respectively. The nonzero matrix elements are assumed to be distributed according to the gaussian orthogonal ensemble¹⁸⁾. Then the following relations are valid:

$$\overline{\langle \phi_\alpha | O_{00}^{\lambda\tau} | \phi_\alpha \rangle^2} = 2v^2, \quad (4.6a)$$

$$\overline{\langle \phi_\alpha | O_{00}^{\lambda\tau} | \phi_\beta \rangle^2} = v^2 \text{ for } \alpha \neq \beta. \quad (4.6b)$$

The bar denotes ensemble averaging. From eq. (4.5) the ensemble averaged value of the variance of the eigenvalues can be obtained:

$$\frac{1}{N} \sum_i \langle \Omega_i | O_{00}^{\lambda\tau} | \Omega_i \rangle^2 = 2v^2 (1 + \frac{1}{2}(N_\phi - 1)). \quad (4.7)$$

Hence the eigenvalues of $O_{00}^{\lambda\tau}$ are stretched by a factor $\sqrt{\frac{1}{2}(N_\phi + 1)}$ with respect to the diagonal matrix elements, which is the correct asymptotic value for a band matrix having its nonzero and gaussian-distributed elements in N_ϕ (sub)diagonals¹⁸⁾. The hamiltonian eigenvectors may be assumed to form an arbitrary basis with respect to the operator $O_{00}^{\lambda\tau}$ for which the diagonal elements are not stretched by a factor >1 in a transformation to the occupation number basis. Hence the eigenvalues $\langle \Omega_i | O_{00}^{\lambda\tau} | \Omega_i \rangle$ of $O_{00}^{\lambda\tau}$ are stretched by a factor $\sqrt{\frac{1}{2}(N_\phi + 1)}$ with respect to the value of the electromagnetic moments. Hence the upper limit is given by

$$|\langle \psi_i | O | \psi_i \rangle| \leq \frac{O_{\max}}{\sqrt{\frac{1}{2}(N_\phi + 1)}}, \quad (4.8)$$

where $O_{\max}$ is the maximum absolute eigenvalue of $O_{00}^{\lambda\tau}$ in the space under consideration.

The upper limits obtained from eq. (4.8) and the upper limits $\omega_{\max}^J$ are given in tables 1 and 2 in the column $\omega_{\max}^J$. The upper limits for the quadrupole moments of the states $J=2$ are, even with the inclusion of an effective charge of $\Delta e = 0.5 e$ to correct for the finiteness of the sd-shell-model space [see e.g. refs.^{19,20)}], smaller than the experimental values of the lowest-lying states $J=2$. Apparently these

calculated states do not obey the statistical assumptions; this confirms their collective nature.

Eq. (4.8) with N_ϕ derived from the occupation number basis, is expected to give good results for the low-lying states which in most cases have large components in this basis. In general the matrix elements $\langle \psi_i | O_{00}^{\lambda\tau} | \psi_j \rangle$ (with $\{\psi_i\}$ eigenfunctions of the hamiltonian) are nonzero in a space with definite angular momentum and isospin for all values of i and j , provided, of course, that the angular-momentum selection rules are satisfied.

Making the random-matrix assumptions of eqs. (4.6) then yields a stretching factor of $\sqrt{\frac{1}{2}N}$ with N the dimension of the matrix. Using this factor and the sum rule eq. (4.1) to derive the variance of the fluctuations, one obtains results which compare well with shell-model results for cases in which there is no secular dependence (see figs. 2a and 2b).

5. Conclusions

Shell-model calculations show that electromagnetic moments fluctuate strongly about a linear energy dependence of the secular behaviour. As a consequence it is inevitable to estimate the magnitude of the fluctuations before the use of the secular behaviour (which can be evaluated by means of trace techniques) can be justified for the prediction of electromagnetic moments. The variance of the fluctuations cannot be expressed in terms of traces and therefore their magnitude is estimated by means of upper limits, which can be derived from traces of powers of the electromagnetic operators. In other cases the magnitude of the fluctuations is the most relevant property. The estimation of the upper limits for the fluctuations can be improved if statistical assumptions are made.

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Appendix A

In this appendix a novel method is presented to express fixed- JT traces of an arbitrary tensor operator in terms of fixed- $J_z T_z$ traces.

Let O_μ^γ represent a tensor operator with γ denoting the quantum numbers of angular momentum (λ) and isospin (τ), and μ denoting their projections. The trace of O_μ^γ over states of fixed angular momentum and isospin, Γ , is determined by the traces over states with fixed projections Γ_z . This can be seen with the help of the following identity:

$$\text{Tr}_{\Gamma_z} (O_0^\gamma) = \text{Tr}_{\Gamma=\Gamma_z, \Gamma_z} (O_0^\gamma) + \sum_{\Gamma > \Gamma_z} \text{Tr}_{\Gamma \Gamma_z} (O_0^\gamma). \quad (\text{A.1})$$

The ordering of $\Gamma = (J, T)$ is defined as $\Gamma_1 = (J_1, T_1) > \Gamma_2 = (J_2, T_2)$ if $J_1 \geq J_2$ and $T_1 \geq T_2$ but $\Gamma_1 \neq \Gamma_2$. The traces Tr_{Γ_z} in this section are always taken for a fixed number of particles which may be partitioned over different single-particle orbits. For the maximum value of $\Gamma_z (= \Gamma_{z,\max})$ for the configuration under consideration the second term vanishes and the fixed- $\Gamma\Gamma_z$ trace is equal to the fixed- Γ_z trace. The traces for $\Gamma = \Gamma_{\max}$, $\Gamma_z < \Gamma_{z,\max}$ can now be obtained by application of the Wigner-Eckart theorem. Further application of the Wigner-Eckart theorem and eq. (A.1) yields the traces for $\Gamma < \Gamma_z$. Hence, by starting at the maximum value of Γ_z all fixed- Γ, Γ_z traces can be expressed in terms of fixed- Γ_z traces by using eq. (A.1) in a recurrent way.

However, the maximum values of the angular momentum and isospin can be much larger than the values for the relevant physical states. Hence, it would be far more economical to solve eq. (A.1) by starting at $(\Gamma, \Gamma_z) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$.

According to the Wigner-Eckart theorem one has (for reasons of notational simplicity the isospin will not be considered)

$$\text{Tr}_{J_z}(O_0^\lambda) = \sum_{J \geq J_z} \frac{\langle JJ_z \lambda 0 | JJ_z \rangle}{\sqrt{2J+1}} \sum_{\alpha} \langle J\alpha \| O^\lambda \| J\alpha \rangle. \quad (\text{A.2})$$

The right-hand side terms for $J = 0, 1, \dots, [\frac{1}{2}(\lambda - 1)]$ (with $[\frac{1}{2}(\lambda - 1)]$ denoting the largest integer not larger than $\frac{1}{2}(\lambda - 1)$) vanish because of angular-momentum selection rules. (Only the case of integer values of the angular momentum J is considered; the case with the half-integer J -values can be treated analogously.) The Clebsch-Gordan coefficient in eq. (A.2) is a polynomial of order λ in J_z with parity $(-1)^\lambda$:

$$\frac{\langle JJ_z \lambda 0 | JJ_z \rangle}{\sqrt{2J+1}} = \sum_{k=0}^{[\frac{1}{2}\lambda]} \alpha_k^\lambda(J) J_z^{\lambda-2k}. \quad (\text{A.3})$$

Hence, after substitution of this polynomial expansion, eq. (A.2) for $J_z = (0, 1, \dots, [\frac{1}{2}(\lambda + 1)])$ (for odd values of λ the equation with $J_z = 0$ is omitted) constitutes a set of $[\frac{1}{2}\lambda] + 1$ independent linear equations with $[\frac{1}{2}\lambda] + 1$ unknowns:

$$\beta_k^\lambda \equiv \sum_{J \geq [\frac{1}{2}(\lambda + 1)]} \alpha_k^\lambda(J) \sum_{\alpha} \langle J\alpha \| O^\lambda \| J\alpha \rangle, \quad k = 0, 1, \dots, [\frac{1}{2}\lambda]. \quad (\text{A.4})$$

Because the value of the determinant for this set of linear equations is equal to

$$\prod_{1 \leq i < j \leq [\frac{1}{2}(\lambda + 1)]} (j^2 - i^2) \neq 0, \quad (\text{A.5})$$

this system can be solved uniquely. The solutions for β_k^λ are linear combinations of fixed- J_z traces. Although in eq. (A.3) the Clebsch-Gordan coefficient is defined for values $J_z \leq J$ only, the polynomial expansion on the right-hand side does not require this restriction to be meaningful. For values of J obeying the inequalities $J < [\frac{1}{2}(\lambda + 1)]$ and $J_z \leq J$ the right-hand side of eq. (2.3) is equal to zero. With this observation one can now calculate fixed- JJ_z traces.

Substitution of eq. (A.3) into eq. (A.2) yields

$$\text{Tr}_{J_z}(O_0^\lambda) = \left\{ \sum_{J \geq [\frac{1}{2}(\lambda+1)]} - \sum_{[\frac{1}{2}(\lambda+1)] \leq J < J_z} \right\} \sum_k^{\{\frac{1}{2}\lambda\}} \alpha_k^\lambda(J) J_z^{\lambda-2k} \sum_a \langle J\alpha \| O^\lambda \| J\alpha \rangle. \quad (\text{A.6})$$

This relation can be solved for the (reduced) traces $\sum_\alpha \langle J\alpha \| O^\lambda \| J\alpha \rangle$ successively for increasing values $J_z = [\frac{1}{2}(\lambda+1)] + 1, [\frac{1}{2}(\lambda+1)] + 2, \dots$

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