$N=2$ supersymmetric $a=4$-Korteweg–de Vries hierarchy derived via Gardner's deformation of Kaup–Boussinesq equation

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We consider the problem of constructing Gardner's deformations for the $N=2$ supersymmetric $a=4$-Korteweg–de Vries (SKdV) equation; such deformations yield recurrence relations between the super-Hamiltonians of the hierarchy. We prove the nonexistence of supersymmetry-invariant deformations that retract to Gardner's formulas for the Korteweg-de Vries (KdV) with equation under the component reduction. At the same time, we propose a two-step scheme for the recursive production of the integrals of motion for the $N=2$, $a=4$-SKdV. First, we find a new Gardner's deformation of the Kaup–Boussinesq equation, which is contained in the bosonic limit of the superhierarchy. This yields the recurrence relation between the Hamiltonians of the limit, whence we determine the bosonic super-Hamiltonians of the full $N=2$, $a=4$-SKdV hierarchy. Our method is applicable toward the solution of Gardner's deformation problems for other supersymmetric KdV-type systems. © *2010 American Institute of Physics.* [doi:10.1063/1.3447731]

I. INTRODUCTION

This paper is devoted to the Korteweg–de Vries (KdV) equation and its generalizations.[1] We consider completely integrable, multi-Hamiltonian evolutionary $N=2$ supersymmetric equations upon a scalar, complex bosonic $N=2$ superfield,

$$\boldsymbol{u}(x,t;\theta_1,\theta_2) = u_0(x,t) + \theta_1 \cdot u_1(x,t) + \theta_2 \cdot u_2(x,t) + \theta_1\theta_2 \cdot u_{12}(x,t), \tag{1}$$

where θ_1 and θ_2 are Grassmann variables satisfying $\theta_1^2=\theta_2^2=\theta_1\theta_2+\theta_2\theta_1=0$. Also, we investigate one- and two-component reductions of such four-component $N=2$ supersystems upon $\boldsymbol{u}$. In particular, we study the bosonic limits, which are obtained by the constraint

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$$u_1 = u_2 \equiv 0. \tag{2}$$

We analyze the structures that are inherited by the limits from the full supersystems and, conversely, recover the integrability properties of the entire $N=2$ hierarchies from their bosonic counterparts.

We address the following problem (see Ref. 2). It is known that the $N=2$ supersymmetric KdV equation (SKdv) with $a=4$,[3,4]

$$\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})_x + \frac{a-1}{2}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \quad \mathcal{D}_i = \frac{\partial}{\partial\theta_i} + \theta_i\cdot\frac{\mathrm{d}}{\mathrm{d}x}, \tag{3}$$

possesses an infinite hierarchy of bosonic Hamiltonian superfunctionals $\mathcal{H}^{(k)}$ whose densities $\boldsymbol{h}^{(k)}$ are integrals of motion. We study whether these super-Hamiltonians can be produced recursively by using those which are already obtained. In particular, this can be done via Gardner's deformations,[1,2] which suggests finding a parametric family of superequations $\mathcal{E}(\epsilon)$ upon the generating superfunction $\tilde{\boldsymbol{u}}(\epsilon)=\sum_{k=0}^{+\infty}\boldsymbol{h}^{(k)}\cdot\epsilon^k$ for the integrals of motion such that initial superequation (3) is $\mathcal{E}(0)$. It is further supposed that, at each ϵ, the evolutionary equation $\mathcal{E}(\epsilon)$ is given in the form of a (super-)conserved current, and there is the Gardner–Miura substitution $\mathfrak{m}_\epsilon:\mathcal{E}(\epsilon)\to\mathcal{E}(0)$. Hence, expanding $\mathfrak{m}_\epsilon$ in ϵ and using the initial condition $\tilde{\boldsymbol{u}}(0)=\boldsymbol{u}$ at $\epsilon=0$, one obtains the differential recurrence relation between the Taylor coefficients $\boldsymbol{h}^{(k)}$ of the generating function $\tilde{\boldsymbol{u}}$ (see Ref. 1 or Refs. 4–8 and references therein, for details and examples).

Let us summarize our main result. Under some natural assumptions, we prove the absence of $N=2$ supersymmetry-invariant Gardner's deformations for the bi-Hamiltonian $N=2$, $a=4$-SKdV. Still, we show that the deformation problem must be addressed in a different way, and then we solve it in two steps. First, in Sec. I we recall that the tri-Hamiltonian hierarchy for the bosonic limit of (3) with $a=4$ contains the Kaup–Boussinesq equation, see Refs. 9–13 in the context of this paper. Then in Sec. III we construct new deformations for the Kaup–Boussinesq equation such that the Miura contraction $\mathfrak{m}_\epsilon$ now incorporates Gardner's map for the KdV equation (Ref. 1, cf. Refs. 5 and 7). Second, extending the Hamiltonians $H^{(k)}$ for the Kaup–Boussinesq hierarchy to the superfunctionals $\mathcal{H}^{(k)}$ in Sec. IV, we reproduce the bosonic conservation laws for (3) with $a=4$. Finally, we describe necessary conditions upon a class of Gardner's deformations for (3) that reproduce its *fermionic* local conserved densities (cf. Ref. 2). All notions and constructions from geometry of differential equations, which are used in this paper, are standard, see Refs. 8 and 14–16.

Remark 1: The recurrence relations between the (super-)Hamiltonians of the hierarchy are much more informative than the usual recursion operators that propagate symmetries. In particular, the symmetries can be used to produce new explicit solutions from known ones, but the integrals of motion help to find those primary solutions.

Let us also note that, within the Lax framework of superpseudodifferential operators, the calculation of the $(n+1)$st residue does not take into account the n residues, which are already known at smaller indices. This is why the method of Gardner's deformations becomes highly preferrable. Indeed, there is no need to multiply any pseudodifferential operators by applying the Leibnitz rule an increasing number of times, and all the previously obtained quantities are used at each inductive step. By this argument, we understand Gardner's deformations as the transformation in the space of the integrals of motion that maps the residues to Taylor coefficients of the generating functions $\tilde{\boldsymbol{u}}(\epsilon)$ and which, therefore, endows this space with the additional structure (that is, with the recurrence relations between the integrals).

Still there is a deep intrinsic relation between the Lax (or, more generally, zero-curvature) representations for integrable systems and Gardner's deformations for them. Namely, both approaches manifest the matrix and vector field representations of the Lie algebras related to such systems, and the deformation parameter ϵ is inverse proportional to the eigenvalue in the linear spectral problem.[17]

II. $N=2$ $a=4$–SKDV AS BI-HAMILTONIAN SUPEREXTENSION OF KAUP–BOUSSINESQ SYSTEM

Let us begin with the KdV equation,

$$u_{12;t} + u_{12;xxx} + 6u_{12}u_{12;x} = 0. \tag{4}$$

Its second Hamiltonian operator, $\hat{A}_2^{\mathrm{KdV}} = \mathrm{d}^3/\mathrm{d}x^3 + 4u_{12}\mathrm{d}/\mathrm{d}x + 2u_{12;x}$, which relates (4) to the functional $H_{\mathrm{KdV}}^{(2)} = -\frac{1}{2}\int u_{12}^2 \mathrm{d}x$, can be extended[18] in the $(2|2)$-graded field setup to the parity-preserving Hamiltonian operator,[3]

$$\hat{P}_2 = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & -u_2 & u_1 & 2u_0\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{0;x} \\ -u_2 & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u_{12} & -2u_0\frac{\mathrm{d}}{\mathrm{d}x} - u_{0;x} & 3u_1\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{1;x} \\ u_1 & 2u_0\frac{\mathrm{d}}{\mathrm{d}x} + u_{0;x} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u_{12} & 3u_2\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{2;x} \\ 2u_0\frac{\mathrm{d}}{\mathrm{d}x} & -3u_1\frac{\mathrm{d}}{\mathrm{d}x} - u_{1;x} & -3u_2\frac{\mathrm{d}}{\mathrm{d}x} - u_{2;x} & \underline{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u_{12}\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{12;x}} \end{pmatrix}. \tag{5}$$

Here the fields u_0 and u_{12} are bosonic and u_1 and u_2 are fermionic together with their derivatives with respect to x. Likewise, the components $\psi_0 \simeq \delta\mathcal{H}/\delta u_0$ and $\psi_{12} \simeq \delta\mathcal{H}/\delta u_{12}$ of the columns $\vec{\psi} = {}^t(\psi_0, \psi_1, \psi_2, \psi_{12})$ are even graded and ψ_1, ψ_2 are odd graded. Operator (5) is unique in the class of Hamiltonian total differential operators that merge to scalar $N=2$ superoperators which are local in $\mathcal{D}_i$ and whose coefficients depend on the superfield $\boldsymbol{u}$ and its superderivatives, see (9) below. Operator (5) determines the $N=2$ classical superconformal algebra.[19] Conversely, the Poisson bracket given by (5) reduces to the second Poisson bracket for (4), whenever one sets equal to zero the fields u_0, u_1, and u_2 both in the coefficients of (5) and in all Hamiltonians; the operator $\hat{A}_2^{\mathrm{KdV}}$ is underlined in (5).

By construction, Mathieu's extensions of KdV equation (4) are determined by operator (5) and the bosonic Hamiltonian functional,

$$\mathcal{H}^{(2)} = \int [u_0u_{0;xx} \underline{- u_{12}^2} + u_1u_{1;x} + u_2u_{2;x} + a\cdot(u_0^2u_{12} - 2u_0u_1u_2)]\mathrm{d}x, \tag{6}$$

which incorporates $H_{\mathrm{KdV}}^{(2)}$ as the underlined term; similar to (9), Hamiltonian (6) will be realized by (8) as the bosonic $N=2$ super-Hamiltonian. Now we have that

$$u_{i;t} = (\hat{P}_2)_{ij}(\delta\mathcal{H}^{(2)}/\delta u_j), \quad i,j \in \{0,1,2,12\}.$$

This yields the system

$$u_{0;t} = -u_{0;xxx} + (au_0^3 - (a+2)u_0u_{12} + (a-1)u_1u_2)_x, \tag{7a}$$

$$u_{1;t} = -u_{1;xxx} + ((a+2)u_0u_{2;x} + (a-1)u_{0;x}u_2 - 3u_1u_{12} + 3au_0^2u_1)_x, \tag{7b}$$

$$u_{2;t} = -u_{2;xxx} + (-(a+2)u_0u_{1;x} - (a-1)u_{0;x}u_1 - 3u_2u_{12} + 3au_0^2u_2)_x, \tag{7c}$$

$$\begin{aligned} \underline{u_{12;t}} &= \underline{-u_{12;xxx} - 6u_{12}u_{12;x}} + 3au_{0;x}u_{0;xx} + (a+2)u_0u_{0;xxx} \\ &\quad + 3u_1u_{1;xx} + 3u_2u_{2;xx} + 3a(u_0^2u_{12} - 2u_0u_1u_2)_x. \end{aligned} \tag{7d}$$

Obviously, it retracts to (4), which we underline in (7), under the reduction $u_0=0$, $u_1=u_2=0$.

At all $a \in \mathbb{R}$, Hamiltonian (6) equals

$$\mathcal{H}^{(2)} = \int \left(\boldsymbol{u} \mathcal{D}_1 \mathcal{D}_2(\boldsymbol{u}) + \frac{a}{3} \boldsymbol{u}^3 \right) \mathrm{d}\boldsymbol{\theta} \mathrm{d}x, \quad \text{where } \mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta_1 \mathrm{d}\theta_2. \tag{8}$$

Likewise, structure (5), which is independent of a, produces the $N=2$ superoperator,

$$\hat{\boldsymbol{P}}_2 = \mathcal{D}_1 \mathcal{D}_2 \frac{\mathrm{d}}{\mathrm{d}x} + 2\boldsymbol{u} \frac{\mathrm{d}}{\mathrm{d}x} - \mathcal{D}_1(\boldsymbol{u})\mathcal{D}_1 - \mathcal{D}_2(\boldsymbol{u})\mathcal{D}_2 + 2\boldsymbol{u}_x. \tag{9}$$

Thus we recover Mathieu's superequations (3),[4] which are Hamiltonian with respect to (9) and functional (8): $\boldsymbol{u}_t = \hat{\boldsymbol{P}}_2((\delta/\delta\boldsymbol{u})(\mathcal{H}_2))$. In component notation, superequations (3) are (7).

The assumption that, for a given a, supersystem (3) admits infinitely many integrals of motion yields the triplet $a \in \{-2, 1, 4\}$, see Ref. 4. The same values of a are exhibited by the Painlevé analysis for $N=2$ superequations (3), see Ref. 20.

Three systems (3) have the common second Poisson structure, which is given by (9), but the three "junior" first Hamiltonian operators $\hat{\boldsymbol{P}}_1$ for them do not coincide.[3,4,21] Moreover, system (3) with $a=4$ is radically different from the other two, both from the Hamiltonian and Lax viewpoints.

Proposition 1: The $N=2$ supersymmetric hierarchy of Mathieu's $a=4$ KdV equation is bi-Hamiltonian with respect to local superoperator (9) and the junior Hamiltonian operator[22] $\hat{\boldsymbol{P}}_1^{a=4} = \mathrm{d}/\mathrm{d}x$, which is obtained from $\hat{\boldsymbol{P}}_2^{a=4}$ by the shift $\boldsymbol{u} \mapsto \boldsymbol{u} + \boldsymbol{\lambda}$ of the superfield $\boldsymbol{u}$, see Refs. 23 and 24,

$$\hat{\boldsymbol{P}}_1^{a=4} = \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{2} \cdot \left. \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}} \right|_{\boldsymbol{\lambda}=0} \hat{\boldsymbol{P}}_2^{a=4}\big|_{\boldsymbol{u}+\boldsymbol{\lambda}}.$$

The two operators are Poisson compatible and generate the tower of *nonlocal* higher structures $\hat{\boldsymbol{P}}_{k+2} = (\hat{\boldsymbol{P}}_2 \circ \hat{\boldsymbol{P}}_1^{-1})^k \circ \hat{\boldsymbol{P}}_2$, $k \geq 1$, for the $N=2$, $a=4$-SKdV hierarchy, see Refs. 25 and 26. Although $\hat{\boldsymbol{P}}_3$ is nonlocal (cf. Ref. 13), its bosonic limits under (2) yield the *local* third Hamiltonian structure $\hat{A}_2$ for the Kaup–Boussinesq equation, which determines the evolution along the second time $t_2 \equiv \xi$ in the bosonic limit of the $N=2$, $a=4$-SKdV hierarchy (see Proposition 2).

Remark 2: The Kaup–Boussinesq system[9] arising here is equivalent to the Kaup–Broer system (the difference amounts to notation). A bi-Hamiltonian $N=2$ superextension of the latter is known from Ref. 11. A tri-Hamiltonian two-fermion $N=1$ superextension of the Kaup–Broer system was constructed in Ref. 12, such that in the bosonic limit the three known Hamiltonian structures for the initial system are recovered. At the same time, a boson-fermion $N=1$ superextension of the Kaup–Broer equation with two local and the nonlocal third Hamiltonian structures was derived in Ref. 13; seemingly, the latter equaled the composition $\hat{\boldsymbol{P}}_2 \circ \hat{\boldsymbol{P}}_1^{-1} \circ \hat{\boldsymbol{P}}_2$, but it remained to prove that the suggested nonlocal superoperator is skew adjoint, that the bracket induced on the space of bosonic super-Hamiltonians does satisfy the Jacobi identity, and that the hierarchy flows produced by the nonlocal operator remain local.

There is a deep reason for the geometry of the $a=4$-SKdV to be exceptionally rich. All the three integrable $N=2$ supersymmetric KdV equations (3) admit the Lax representations $L_{t_3} = [A^{(3)}, L]$, see Refs. 2, 3, 27, and 28. For $a=4$, the four roots of the Lax operator $L_{a=4} = -(\mathcal{D}_1\mathcal{D}_2 + \boldsymbol{u})^2$, which are $\mathcal{L}_{1,\pm} = \pm \mathrm{i}(\mathcal{D}_1\mathcal{D}_2 + \boldsymbol{u})$, $\mathrm{i}^2 = -1$, and the superpseudodifferential operators $\mathcal{L}_{2,\pm} = \pm \mathrm{d}/\mathrm{d}x + \Sigma_{i>0}(\cdots) \cdot (\mathrm{d}/\mathrm{d}x)^{-i}$, generate the odd-index flows of the SKdV hierarchy via $L_{t_{2k+1}} = [(\mathcal{L}_2^{2k+1})_{\geq 0}, L]$. In particular, we have $A_{a=4}^{(3)} = (L^{3/2})_{\geq 0} \bmod (\mathcal{D}_1\mathcal{D}_2 + \boldsymbol{u})^3$. However, the *entire* $a=4$ hierarchy is reproduced in the Lax form via $(\mathcal{L}_1^k \mathcal{L}_2)_{t_\ell} = [(\mathcal{L}_1^\ell \mathcal{L}_2)_{\geq 0}, \mathcal{L}_1^k \mathcal{L}_2]$ for all $k \in \mathbb{N}$, cf. Ref. 29. Hence the superresidues[30] of the operators $\mathcal{L}_1^k \mathcal{L}_2$ are conserved.

Consequently, unlike the other two, superequations (3) with $a=4$ admits twice as many constants of motion as there are for the superequations with $a=-2$ or $a=1$. For convenience, let us recall that superequations (3) are homogeneous with respect to the weights $|\mathrm{d}/\mathrm{d}x| \equiv 1$, $|\boldsymbol{u}| = 1$, $|\mathrm{d}/\mathrm{d}t| = 3$. Hence we conclude that, for each non-negative integer k, there appears the nontrivial conserved density Sres $\mathcal{L}_1^k\mathcal{L}_2$, see above, of weight $k+1$. The even weights also enter the play. Consequently, there are twice as many commuting superflows assigned to the twice as many Hamiltonians.

Example 1: The additional super-Hamiltonian $\mathcal{H}^{(1)}=\frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}\boldsymbol{\theta}\mathrm{d}x$ for (3) with $a=4$, and second structure (9) or, equivalently, the first operator $\hat{\boldsymbol{P}}_1=\mathrm{d}/\mathrm{d}x$ and the Hamiltonian $\mathcal{H}^{(2)}$, or $\hat{\boldsymbol{P}}_3$ and $\mathcal{H}^{(0)}=\int \boldsymbol{u}\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$, see above, generate the $N=2$ supersymmetric equation

$$\boldsymbol{u}_\xi=\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}_x+4\boldsymbol{u}\boldsymbol{u}_x=\hat{\boldsymbol{P}}_3\left(\frac{\delta}{\delta\boldsymbol{u}}(\mathcal{H}^{(0)})\right)=\hat{\boldsymbol{P}}_2\left(\frac{\delta}{\delta\boldsymbol{u}}(\mathcal{H}^{(1)})\right)=\hat{\boldsymbol{P}}_1\left(\frac{\delta}{\delta\boldsymbol{u}}(\mathcal{H}^{(2)})\right),\quad \xi\equiv t_2. \tag{10}$$

Superequation (10) was referred to as the $N=2$ "Burgers" equation in Refs. 15 and 31 due to the recovery of $\boldsymbol{u}_\xi=\boldsymbol{u}_{xx}+4\boldsymbol{u}\boldsymbol{u}_x$ on the diagonal $\theta_1=\theta_2$. On the other hand, the bosonic limit of (10) is the tri-Hamiltonian "minus" Kaup–Boussinesq system (see Refs. 5, 7, 9, and 10 and references therein),

$$u_{0;\xi}=(-u_{12}+2u_0^2)_x,\quad u_{12;\xi}=(u_{0;xx}+4u_0u_{12})_x. \tag{11}$$

System (11) is equivalent to the Kaup–Broer equation via an invertible substitution. In these terms, superequation (10) is a superextension of the Kaup–Boussinesq system.[11–13] In their turn, the first three Poisson structures for (3) with $a=4$ are reduced under (2) to the respective *local* structures for (11), see Proposition 2.

Our interest in the recursive production of the integrals of motion for (3) grew after the discovery, see Ref. 31, of new n-soliton solutions,

$$\boldsymbol{u}=\mathsf{A}(a)\cdot\mathcal{D}_1\mathcal{D}_2\log\left(1+\sum_{i=1}^{n}\alpha_i\exp(k_ix-k_i^3\cdot t\pm \mathrm{i}k_i\cdot\theta_1\theta_2)\right),\quad \mathsf{A}(a)=\begin{cases}1, & a=1\\ \frac{1}{2}, & a=4,\end{cases} \tag{12}$$

for superequations (3) with $a=1$ or $a=4$ (but not $a=-2$ or any other $a\in\mathbb{R}\setminus\{1,4\}$). In formula (12), the wave numbers $k_i\in\mathbb{R}$ are arbitrary, and the phases α_i can be rescaled to $+1$ for nonsingular n-soliton solutions by appropriate shifts of n higher times in the SKdV hierarchy. A spontaneous decay of fast solitons and their transition into the virtual states, on the emerging background of previously invisible, slow solitons, look paradoxal for such KdV-type systems ($a=1$ or $a=4$), since they possess an infinity of the integrals of motion.

New solutions (12) of (3) with $a=1$ or $a=4$ are subject to condition (2) and therefore satisfy the bosonic limits of these $N=2$ supersystems. In the same way, bosonic limit (11) of (10) admits multisoliton solutions in Hirota's form (12), now with the exponents $\eta_i=k_ix\pm \mathrm{i}k_i^2\xi\pm \mathrm{i}k_i\theta_1\theta_2$, see Ref. 31. This makes the role of such two-component bosonic reductions particularly important. We recall that reduction (2) of (3) with $a=1$ yields the Kersten–Krasil'shchik equation, see Refs. 31 and 32 and references therein. In this paper, we consider the bosonic limit of the $N=2$, $a=4$ SKdV equation,

$$u_{0;t}=-u_{0;xxx}+12u_0^2u_{0;x}-6(u_0u_{12})_x, \tag{13a}$$

$$u_{12;t}=-u_{12;xxx}-6u_{12}u_{12;x}+12u_{0;x}u_{0;xx}+6u_0u_{0;xxx}+12(u_0^2u_{12})_x, \tag{13b}$$

which succeeds the Kaup–Boussinesq equation (11) in its tri-Hamiltonian hierarchy. We construct a new Gardner deformation for it (cf. Ref. 7).

In general, system (7) with $a=4$ admits three one-component reductions (except $u_0\not\equiv 0$) and three two-component reductions, which are indicated by the edges that connect the remaining components in the diagram,

$$\begin{array}{ccccc} & & u_0 & & \\ & & \| & & \\ u_1 & = & u_{12} & = & u_2. \end{array}$$

System (7) with $a=4$ has no three-component reductions obtained by setting to zero only one of the four fields in (1). We conclude this paper by presenting a Gardner deformation for the two-component boson-fermion reduction $u_0 \equiv 0$, $u_2 \equiv 0$ of the $N=2$, $a=4$-SKdV system, see (25).

III. DEFORMATION PROBLEM FOR $N=2$, $a=4$-SKDV EQUATION

In this section, we formulate the two-step algorithm for a recursive production of the bosonic super-Hamiltonians $\mathcal{H}^{(k)}[\boldsymbol{u}]$ for the $N=2$ supersymmetric $a=4$-SKdV hierarchy. Essentially, we convert the geometric problem to an explicit computational procedure. Our scheme can be applied to other KdV-type supersystems [in particular, to (3) with $a=-2$ or $a=1$].

By definition, a classical Gardner's deformation for an integrable evolutionary equation $\mathcal{E}$ is the diagram

$$\mathfrak{m}_\epsilon : \mathcal{E}(\epsilon) \to \mathcal{E},$$

where the equation $\mathcal{E}(\epsilon)$ is a parametric extension of the initial system $\mathcal{E}=\mathcal{E}(0)$ and $\mathfrak{m}_\epsilon$ is the Miura contraction.[1,5,8] Under the assumption that $\mathcal{E}(\epsilon)$ be in the form of a (super-)conserved current, the Taylor coefficients $\tilde{\boldsymbol{u}}^{(k)}$ of the formal power series $\tilde{\boldsymbol{u}}=\sum_{k=0}^{+\infty}\tilde{\boldsymbol{u}}^{(k)}\cdot\epsilon^k$ are termwise conserved on $\mathcal{E}(\epsilon)$ and hence on $\mathcal{E}$. Therefore, the contraction $\mathfrak{m}_\epsilon$ yields the recurrence relations, ordered by the powers of ϵ, between these densities $\tilde{\boldsymbol{u}}^{(k)}$, while the equality $\mathcal{E}(0)=\mathcal{E}$ specifies its initial condition.

Example 2: (Reference 1) The contraction,

$$\mathfrak{m}_\epsilon = \{u_{12} = \tilde{u}_{12} \pm \epsilon\tilde{u}_{12;x} - \epsilon^2\tilde{u}_{12}^2\}, \tag{14a}$$

maps solutions $\tilde{u}_{12}(x,t;\epsilon)$ of the extended equation $\mathcal{E}(\epsilon)$,

$$\tilde{u}_{12;t} + (\tilde{u}_{12;xx} + 3\tilde{u}_{12}^2 - 2\epsilon^2\cdot\tilde{u}_{12}^3)_x = 0, \tag{14b}$$

to solutions $u_{12}(x,t)$ of the KdV equation (4). Plugging the series $\tilde{u}_{12}=\sum_{k=0}^{+\infty}u_{12}^{(k)}\cdot\epsilon^k$ in $\mathfrak{m}_\epsilon$ for $\tilde{u}_{12}$, we obtain the chain of equations ordered by the powers of ϵ,

$$u_{12} = \sum_{k=0}^{+\infty} \tilde{u}_{12}^{(k)}\cdot\epsilon^k \pm \tilde{u}_{12;x}^{(k)}\cdot\epsilon^{k+1} - \sum_{\substack{i+j=k\\ i,j\geq 0}} \tilde{u}_{12}^{(i)}\tilde{u}_{12}^{(j)}\cdot\epsilon^{k+2}.$$

Let us fix the plus sign in (14a) by reversing $\epsilon \to -\epsilon$ if necessary. Equating the coefficients of ϵ^k, we obtain the relations

$$u = \tilde{u}_{12}^{(0)}, \quad 0 = \tilde{u}_{12}^{(1)} + \tilde{u}_{12;x}^{(0)}, \quad 0 = \tilde{u}_{12}^{(k)} + \tilde{u}_{12;x}^{(k-1)} - \sum_{\substack{i+j=k-2\\ i,j\geq 0}} \tilde{u}_{12}^{(i)}\tilde{u}_{12}^{(j)}, \quad k\geq 2.$$

Hence, from the initial condition $\tilde{u}_{12}^{(0)}=u_{12}$, we recursively generate the densities

$$\tilde{u}_{12}^{(1)} = -u_{12;x}, \quad \tilde{u}_{12}^{(2)} = u_{12;xx} - u_{12}^2, \quad \tilde{u}_{12}^{(3)} = -u_{12;xxx} + 4u_{12;x}u_{12},$$

$$\tilde{u}_{12}^{(4)} = u_{12;4x} - 6u_{12;xx}u_{12} - 5u_{12;x}^2 + 2u_{12}^3,$$

$$\tilde{u}_{12}^{(5)} = -u_{12;5x} + 8u_{12;xxx}u_{12} + 18u_{12;xx}u_{12;x} - 16u_{12;x}u_{12}^2,$$

$$\tilde{u}_{12}^{(6)} = u_{12;6x} - 10u_{12;4x}u_{12} - 28u_{12;xxx}u_{12;x} - 19u_{12;xx}^2 + 30u_{12;xx}u_{12}^2 + 50u_{12;x}^2u_{12} - 5u_{12}^4,$$

$$\tilde{u}_{12}^{(7)} = -u_{12;7x} + 12u_{12;5x}u_{12} + 40u_{12;4x}u_{12;x} + 68u_{12;xxx}u_{12;xx} - 48u_{12;xxx}u_{12}^2 - 216u_{12;xx}u_{12;x}u_{12} - 60u_{12;x}^3 + 64u_{12;x}u_{12}^3,$$

etc. The conservation $\tilde{u}_{12;t}=(\mathrm{d}/\mathrm{d}x)(\cdot)$ implies that each coefficient $u_{12}^{(k)}$ is conserved on (4).

The densities $u_{12}^{(2k)}=c(k)\cdot u_{12}^k+\cdots$, $c(k)=\text{const}$, determine the Hamiltonians $\mathcal{H}_{12}^{(k)}=\int h_{12}^{(k)}[u_{12}]\mathrm{d}x$ of the renowned KdV hierarchy. Let us show that all of them are nontrivial. Consider the zero-order part $\breve{u}_{12}^{\mathrm{KdV}}$, such that $\tilde{u}_{12}([u_{12}],\epsilon)=\breve{u}_{12}^{\mathrm{KdV}}(u_{12},\epsilon)+\cdots$, where the dots denote summands containing derivatives of u_{12}. Taking the zero-order component of (14a), we conclude that the generating function $\breve{u}_{12}^{\mathrm{KdV}}$ satisfies the algebraic recurrence relation $u_{12}=\breve{u}_{12}^{\mathrm{KdV}}-\epsilon^2(\breve{u}_{12}^{\mathrm{KdV}})^2$. We choose the root by the initial condition $\breve{u}_{12}^{\mathrm{KdV}}|_{\epsilon=0}=u_{12}$, which yields

$$\breve{u}_{12}^{\mathrm{KdV}} = (1-\sqrt{1-4\epsilon^2 u_{12}})/(2\epsilon^2). \tag{15}$$

Moreover, the Taylor coefficients $\breve{u}_{12}^{(k)}(u_{12})$ in $\breve{u}_{12}^{\mathrm{KdV}}=\sum_{k=0}^{+\infty}\breve{u}_{12}^{(k)}\cdot\epsilon^{2k}$ equal $c(k)\cdot u_{12}^{k+1}$, where $c(k)$ are positive and grow with k. This is readily seen by induction over k with the base $\breve{u}_{12}^{(0)}=u_{12}$. Expanding both sides of the equality $u_{12}=\breve{u}_{12}^{\mathrm{KdV}}-\epsilon^2\cdot(\breve{u}_{12}^{\mathrm{KdV}})^2$ in ϵ^2, we notice that

$$\breve{u}_{12}^{(k)} = \sum_{\substack{i+j=k-1,\\ i,j\geq 0}} \breve{u}_{12}^{(i)}\cdot\breve{u}_{12}^{(j)} = \sum_{i+j=k-1} c(i)c(j)\cdot u_{12}^{k+1}.$$

Therefore, the next coefficient, $c(k)=\sum_{i+j=k-1}c(i)\cdot c(j)$, is the sum over $i,j\geq 0$ of products of positive numbers, whence $c(k+1)>c(k)>0$. This proves the claim.

Let us list the densities $h_{\mathrm{KdV}}^{(k)}\sim u_{12}^{(2k)}$ mod im $\mathrm{d}/\mathrm{d}x$ of the first seven Hamiltonians for (4). These will be correlated in Sec. IV with the lowest seven Hamiltonians for (3), see Ref. 4 and (24) below. We have

$$h_{\mathrm{KdV}}^{(1)} = u_{12}^2, \quad h_{\mathrm{KdV}}^{(2)} = 2u_{12}^3 - u_{12;x}^2 + 2u_{12}^3 + u_{12;xx}, \quad h_{\mathrm{KdV}}^{(3)} = 5u_{12}^4 + 5u_{12;xx}u_{12}^2 + u_{12;xx}^2,$$

$$h_{\mathrm{KdV}}^{(4)} = -14u_{12}^5 + 70u_{12}^2u_{12;x}^2 + 14u_{12}u_{12;xxx}u_{12;x} + u_{12;xxx}^2,$$

$$h_{\mathrm{KdV}}^{(5)} = 42u_{12}^6 - 420u_{12}^3u_{12;x}^2 + 9u_{12}^2u_{12;6x} + 126u_{12}^2u_{12;xx}^2 + u_{12;4x}^2 - 7u_{12;xx}^3 - 35u_{12;x}^4,$$

$$h_{\mathrm{KdV}}^{(6)} = 1056u_{12}^7 - 18\,480u_{12}^4u_{12;x}^2 + 7392u_{12}^3u_{12;xx}^2 + 55u_{12}^2u_{12;8x} - 1584u_{12}^2u_{12;xxx}^2 + 66u_{12}u_{12;4x}^2 + 3520u_{12}u_{12;xx}^3 - 6160u_{12}u_{12;x}^4 - 8u_{12;5x}^2 + 3696u_{12;xx}^2u_{12;x}^2,$$

$$h_{\mathrm{KdV}}^{(7)} = 15\,444u_{12}^8 - 432\,432u_{12}^5u_{12;x}^2 + 4004u_{12}^4u_{12;6x} + 216\,216u_{12}^4u_{12;xx}^2 + 2145u_{12}^3u_{12;8x} - 45\,760u_{12}^3u_{12;xxx}^2 + 3861u_{12}^2u_{12;4x}^2 + 133\,848u_{12}^2u_{12;xx}^3 - 360\,360u_{12}^2u_{12;x}^4 - 936u_{12}u_{12;5x}^2 + 36u_{12;6x}^2 + 6552u_{12;4x}^2u_{12;xx} + 72\,072u_{12;xxx}^2u_{12;x}^2 - 28\,314u_{12;xx}^4.$$

At the same time, the densities $u_{12}^{(2k+1)}=(\mathrm{d}/\mathrm{d}x)(\cdot)\sim 0$ are trivial. Indeed, for $\omega_0:=\sum_{k=0}^{+\infty}u_{12}^{(2k)}\cdot\epsilon^{2k}$ and $\omega_1:=\sum_{k=0}^{+\infty}u_{12}^{(2k+1)}\cdot\epsilon^{2k}$, such that $\tilde{u}=\omega_0+\epsilon\cdot\omega_1$, we equate the odd powers of ϵ in (14a) and obtain $\omega_1=(1/2\epsilon^2)(\mathrm{d}/\mathrm{d}x)\log(1-2\epsilon^2\omega_0)$.

In what follows, using deformation (14) of (4), we fix the coefficients of differential monomials in u_{12} within a bigger deformation problem (see Sec. III) for two-component system (13).

We split the Gardner deformation problem for the $N=2$ supersymmetric hierarchy of (3) with $a=4$ in two main and several auxiliary steps.

First, we note that Miura's contraction $\mathfrak{m}_\epsilon:\mathcal{E}(\epsilon)\to\mathcal{E}$, which encodes the recurrence relation between the conserved densities, is common for all equations of the hierarchy. Indeed, the densities (and hence any differential relations between them) are shared by all the equations. Therefore, we pass to the deformation problem for the $N=2$ super-Burgers equation (10). This makes the first simplification of the Gardner deformation problem for the $N=2$, $a=4$ super-KdV hierarchy.

Second, let $\boldsymbol{h}^{(k)}$ be an $N=2$ superconserved density for an evolutionary superequation $\mathcal{E}$, meaning that its velocity with respect to a time τ, $(\mathrm{d}/\mathrm{d}\tau)\boldsymbol{h}^{(k)}=\mathcal{D}_1(\cdots)+\mathcal{D}_2(\cdots)$, is a total divergence on $\mathcal{E}$. By definition of $\mathcal{D}_i$, see (3), the $\theta_1\theta_2$-component $h_{12}^{(k)}$ of such $\boldsymbol{h}^{(k)}=h_0^{(k)}+\theta_1\cdot h_1^{(k)}+\theta_2\cdot h_2^{(k)}+\theta_1\theta_2\cdot h_{12}^{(k)}$ is conserved in the classical sense, $(\mathrm{d}/\mathrm{d}\tau)h_{12}^{(k)}=(\mathrm{d}/\mathrm{d}x)(\cdots)$ on $\mathcal{E}$. Let us consider the correlation between the conservation laws for the full $N=2$ supersystem $\mathcal{E}$ and for its reductions that are obtained by setting certain component(s) of $\boldsymbol{u}$ to zero. In what follows, we study bosonic reduction (2). Other reductions of superequation (3) are discussed in Sec. IV, see (25).

We suppose that the bosonic limit $\lim_B \mathcal{E}$ of the superequation $\mathcal{E}$ exists, which is the case for (3) and (10). By the above, each conserved superdensity $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$ determines the conserved density $h_{12}^{(k)}[u_0,u_{12}]$, which may become trivial. As in Ref. 28, we assume that the supersystem $\mathcal{E}$ does not admit any conserved superdensities that vanish under reduction (2). Then, for such $h_{12}^{(k)}$ that originates from $\boldsymbol{h}^{(k)}$ by construction, the equivalence class $\{\boldsymbol{h}^{(k)} \bmod \operatorname{im}\mathcal{D}_i\}$ is uniquely determined by

$$\int h_{12}^{(k)}[u_0,u_{12}]\mathrm{d}x=\int \boldsymbol{h}^{(k)}[\boldsymbol{u}]\big|_{u_1=u_2=0}\mathrm{d}\boldsymbol{\theta}\mathrm{d}x,\quad \text{here } N=2 \quad \text{and} \quad \mathrm{d}\boldsymbol{\theta}=\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Berezin's definition of a superintegration, $\int\mathrm{d}\theta_i=0$ and $\int\theta_i\mathrm{d}\theta_i=1$, implies that the problem of recursive generation of the $N=2$ super-Hamiltonians $\mathcal{H}^{(k)}=\int\boldsymbol{h}^{(k)}\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$ for the SKdV hierarchy amounts to the generation of the equivalence classes $\int h_{12}^{(k)}\mathrm{d}x$ for the respective $\theta_1\theta_2$-component. We conclude that a solution of Gardner's deformation problem for the supersymmetric system (10) may not be subject to the supersymmetry invariance. This is a key point to further reasonings.

We stress that the equivalence class of such functions $h_{12}^{(k)}[u_0,u_{12}]$ that originate from $\mathcal{H}^{(k)}$ by (2) is, generally, much more narrow than the equivalence class $\{h_{12}^{(k)} \bmod \operatorname{im}\mathrm{d}/\mathrm{d}x\}$ of all conserved densities for the bosonic limit $\lim_B \mathcal{E}$. Obviously, there are differential functions of the form $(\mathrm{d}/\mathrm{d}x)(f[u_0,u_{12}])$ that cannot be obtained[33] as the $\theta_1\theta_2$-component of any $[\mathcal{D}_1(\cdot)+\mathcal{D}_2(\cdot)]\big|_{u_1=u_2=0}$, which is trivial in the supersense. Therefore, let $h_{12}^{(k)}$ be *any* recursively given sequence of integrals of motion for $\lim_B \mathcal{E}$ (e.g., suppose that they are the densities of the Hamiltonians $\mathcal{H}^{(k)}$ for the hierarchy of $\lim_B \mathcal{E}$), and let it be known that each $\mathcal{H}^{(k)}=\int h_{12}^{(k)}\mathrm{d}x$ does correspond to the superanalog $\mathcal{H}^{(k)}=\int\boldsymbol{h}^{(k)}\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$. Then the reconstruction of $\boldsymbol{h}^{(k)}$ requires an intermediate step, which is the elimination of excessive, homologically trivial terms under $\mathrm{d}/\mathrm{d}x$ that preclude a given $h_{12}^{(k)}$ to be extended to the full superdensity in terms of the $N=2$ superfield $\boldsymbol{u}$. This is illustrated in Sec. IV.

Third, the gap between the two types of equivalence for the integrals of motion manifests the distinction between the deformations $(\lim_B \mathcal{E})(\epsilon)$ of bosonic limits and, on the other hand, the bosonic limits $\lim_B \mathcal{E}(\epsilon)$ of $N=2$ superdeformations. The two operations, Gardner's extension of $\mathcal{E}$ to $\mathcal{E}(\epsilon)$ and taking the bosonic limit $\lim_B \mathcal{F}$ of an equation $\mathcal{F}$, are not permutable. The resulting systems can be different. Namely, according to the classical scheme Refs. 1 and 8, *each* equation in the evolutionary system $(\lim_B \mathcal{E})(\epsilon)$ represents a conserved current, whence each Taylor coefficient of the respective field is conserved, see Example 2. At the same time, for $\lim_B \mathcal{E}(\epsilon)$, the conservation is required only for the field $\tilde{u}_{12}(\epsilon)$, which is the $\theta_1\theta_2$-component of the extended superfield $\tilde{\boldsymbol{u}}(\epsilon)$. Other equations in $\lim_B \mathcal{E}(\epsilon)$ can have any form.[34]

In this notation, we strengthen the problem of recursive generation of the super-Hamiltonians for the $N=2$ superequation (10). Namely, in Sec. III we construct true Gardner's deformations for its two-component bosonic limit (11). Moreover, the known deformation (14) for (4) upon the component u_{12} of (1) allows to fix the coefficients of the terms that contain only u_{12} or its derivatives. The solution to the Gardner deformation problem generates the recurrence relation between the nontrivial conserved densities $h_{12}^{(k)}$ which, in the meantime, depend on u_0 and u_{12}. By correlating them with the $\theta_1\theta_2$-components of the superdensities $\boldsymbol{h}^{(k)}$ that depend on $\boldsymbol{u}$, we derive the Hamiltonians $\mathscr{H}^{(k)}$, $k\geq 0$, for the $N=2$ supersymmetric $a=4$-KdV hierarchy, see Sec. IV.

IV. NEW DEFORMATION OF THE KAUP–BOUSSINESQ EQUATION

In this section, we construct a new Gardner's deformation $\mathfrak{m}_\epsilon:(\lim_B \mathcal{E})(\epsilon)\to\lim_B \mathcal{E}$ for the "minus" Kaup–Boussinesq equation (11), which is the bosonic limit of the $N=2$ supersymmetric system (10). We will use the known deformation (14) to fix several coefficients in the Miura contraction $\mathfrak{m}_\epsilon$, which ensures the difference of new solution (16) and (17) from previously known deformations of (11), see Ref. 7. We prove that the new deformation is maximally nontrivial: It yields infinitely many nontrivial conserved densities, and none of the Hamiltonians is lost.

In components, the $N=2$ superequation (10) reads

$$u_{0;\xi}=(-u_{12}+2u_0^2)_x,\quad u_{1;\xi}=(u_{2;x}+4u_0u_1)_x,$$

$$u_{2;\xi}=(-u_{1,x}+4u_0u_2)_x,\quad u_{12;\xi}=(u_{0;xx}+4u_0u_{12}-4u_1u_2)_x.$$

Clearly, it admits reduction (2); moreover, Kaup–Boussinesq system (11) is the only possible limit for (10). Let us summarize its well-known properties.[9,10]

Proposition 2: The completely integrable Kaup–Boussinesq system (11) inherits the local tri-Hamiltonian structure from the two local ($\hat{P}_1$ and $\hat{P}_2$) and the nonlocal $\hat{P}_3=\hat{P}_2\circ\hat{P}_1\circ\hat{P}_2$ operators for the $N=2$, $a=4$-SKdV hierarchy under the bosonic limit (2),

$$\begin{pmatrix}u_0\\ u_{12}\end{pmatrix}_\xi=\hat{A}_1^{12}\begin{pmatrix}\delta/\delta u_0\\ \delta/\delta u_{12}\end{pmatrix}\left(\int\left[2u_0^2u_{12}-\frac{1}{2}u_{12}^2-\frac{1}{2}u_{0;x}^2\right]\mathrm{d}x\right)$$
$$=\hat{A}_1^{0}\begin{pmatrix}\delta/\delta u_0\\ \delta/\delta u_{12}\end{pmatrix}\left(-\int u_0u_{12}\mathrm{d}x\right)=\hat{A}_2\begin{pmatrix}\delta/\delta u_0\\ \delta/\delta u_{12}\end{pmatrix}\left(-\int u_{12}\mathrm{d}x\right).$$

The senior Hamiltonian operator $\hat{A}_2$ is

$$\begin{pmatrix} u_{0;x}+2u_0\frac{\mathrm{d}}{\mathrm{d}x} & u_{12;x}-4u_0u_{0;x}-2u_0^2\frac{\mathrm{d}}{\mathrm{d}x}+2u_{12}\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3\\ u_{12;x}-2u_0^2\frac{\mathrm{d}}{\mathrm{d}x}+2u_{12}\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 & -4u_0u_{12}\frac{\mathrm{d}}{\mathrm{d}x}-4\frac{\mathrm{d}}{\mathrm{d}x}\circ u_0u_{12}-u_0\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3\circ u_0\end{pmatrix}.$$

The junior Hamiltonian operators $\hat{A}_1^0$ and $\hat{A}_1^{12}$ are obtained from $\hat{A}_2$ by the shifts of the respective fields, cf. Refs. 23 and 24,

$$\hat{A}_1^0=\begin{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} & -2u_{0;x}-2u_0\frac{\mathrm{d}}{\mathrm{d}x}\\ -2u_0\frac{\mathrm{d}}{\mathrm{d}x} & -2u_{12;x}-4u_{12}\frac{\mathrm{d}}{\mathrm{d}x}-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3\end{pmatrix}=\frac{1}{2}\cdot\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\hat{A}_2|_{u_0+\lambda}$$

and

$$\hat{A}_1^{12}=\begin{pmatrix}0 & \frac{\mathrm{d}}{\mathrm{d}x}\\ \frac{\mathrm{d}}{\mathrm{d}x} & 0\end{pmatrix}=\frac{1}{2}\cdot\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_{\mu=0}\hat{A}_2|_{u_{12}+\mu}.$$

The three operators $\hat{A}_1^0$, $\hat{A}_1^{12}$, and $\hat{A}_2$ are Poisson compatible.

Kaup–Boussinesq equation (11) admits an infinite sequence of integrals of motion. We will derive them via the Gardner deformation. Unlike in Ref. 7, from now on we always assume that (14a) is recovered under $\tilde{u}_0 \equiv 0$.

We assume that both the extension $\mathcal{E}(\epsilon)$ of (11) and the contraction $\mathfrak{m}_\epsilon : \mathcal{E}(\epsilon) \to \mathcal{E}$ into (11) are homogeneous polynomials in ϵ. From now on, we denote reduction (11) by $\mathcal{E}$.

First, let us estimate the degrees in ϵ for such polynomials $\mathcal{E}(\epsilon)$ and $\mathfrak{m}_\epsilon$, by balancing the powers of ϵ in the left- and right-hand sides of (11) with u_0 and u_{12} replaced by the Miura contraction $\mathfrak{m}_\epsilon = \{u_0 = u_0([\tilde{u}_0, \tilde{u}_{12}], \epsilon), u_{12} = u_{12}([\tilde{u}_0, \tilde{u}_{12}], \epsilon)\}$. The time evolution in the left-hand side, which is of the form $u_\xi = \partial_{\tilde{u}_\xi}(\mathfrak{m}_\epsilon)$ by the chain rule, sums the degrees in ϵ: $\deg u_\xi = \deg \mathfrak{m}_\epsilon + \deg \mathcal{E}(\epsilon)$. At the same time, we notice that system (11) is only quadratic nonlinear. Hence its right-hand side, with $\mathfrak{m}_\epsilon$ substituted for u_0 and u_{12}, gives the degree $2 \times \deg \mathfrak{m}_\epsilon$, irrespective of $\deg \mathcal{E}(\epsilon)$. Consequently, we obtain the balance[35] 1:1 for $\max \deg \mathfrak{m}_\epsilon : \max \deg \mathcal{E}(\epsilon)$. This is in contrast with the balance 1:2 for polynomial deformations of bosonic limit (13) for initial SKdV system (3), which is cubic nonlinear[36] (cf. Ref. 4).

Obviously, a lower degree polynomial extension $\mathcal{E}(\epsilon)$ contains fewer undetermined coefficients. This is the first profit we gain from passing to (10) instead of (3). By the same argument, we conclude that $\mathfrak{m}_\epsilon : \mathcal{E}(\epsilon) \to \mathcal{E}$, viewed as the algebraic system upon these coefficients, is only *quadratic* nonlinear with respect to the coefficients in $\mathfrak{m}_\epsilon$ [and, obviously, *linear* with respect to the coefficients in $\mathcal{E}(\epsilon)$; this is valid for any balance $\deg \mathfrak{m}_\epsilon : \deg \mathcal{E}(\epsilon)$]. Hence the size of this overdetermined algebraic system is further decreased.

Second, we use the unique admissible homogeneity weights for Kaup–Boussinesq system (11),

$$|u_0| = 1, \quad |u_{12}| = 2, \quad |\mathrm{d}/\mathrm{d}\xi| = 2,$$

here $|\mathrm{d}/\mathrm{d}x| \equiv 1$ is the normalization. The Miura contraction $\mathfrak{m}_\epsilon = \{u_0 = \tilde{u}_0 + \epsilon \cdot (\cdots), \quad u_{12} = \tilde{u}_{12} + \epsilon \cdot (\cdots)\}$, which we assume regular at the origin, implies that $|\tilde{u}_0| = 1$ and $|\tilde{u}_{12}| = 2$ as well. We let $|\epsilon| = -1$ be the difference of weights for every two successive Hamiltonians for the $N=2$, $a=4$-SKdV hierarchy, see Ref. 4 and (24) below. In this setup, all functional coefficients of the powers ϵ^k both in $\mathcal{E}(\epsilon)$ and $\mathfrak{m}_\epsilon$ are homogeneous differential polynomials in u_0, u_{12}, and their derivatives with respect to x. It is again important that the time ξ of weight $|\mathrm{d}/\mathrm{d}\xi| = 2$ in (10) precedes the time t with $|\mathrm{d}/\mathrm{d}t| = 3$ in the hierarchy of (3), where $|\theta_i| = -\frac{1}{2}$ and $|\boldsymbol{u}| = 1$. As before, we have further decreased the number of undetermined coefficients.

The polynomial ansatz for Gardner's deformation of (11) is generated by the procedure GENSSPOLY, see Appendix A, which is a new possibility in the analytic software.[16,37] We thus obtain the determining system $\mathfrak{m}_\epsilon : \mathcal{E}(\epsilon) \to \mathcal{E}$. Using SSTOOLS, we split it to the overdetermined system of algebraic equations, which are linear with respect to $\mathcal{E}(\epsilon)$ and quadratic nonlinear with respect to $\mathfrak{m}_\epsilon$. Moreover, we claim that this system is *triangular*. Indeed, it is ordered by the powers of ϵ, since the determining system is identically satisfied at zeroth order and because equations at lower orders of ϵ involve only the coefficients of its lower powers from $\mathfrak{m}_\epsilon$ and $\mathcal{E}(\epsilon)$.

Third, we use deformation (14) of the KdV equation.[1] We recall the following.

- Miura's contraction $\mathfrak{m}_\epsilon$ is common for all two-component systems in the bosonic limit, see (2), of the $N=2$, $a=4$-SKdV hierarchy;
- For any a, the bosonic limit of (3), see (7) and (13), incorporates the KdV equation (4).

Using (14a), we fix those coefficients in $\mathfrak{m}_\epsilon$ which depend only on u_{12} and its derivatives, but not on u_0 or its derivatives. Apparently, we discard the knowledge of such coefficients in the extension of bosonic limit (13), since for us now it is not the object to be deformed. But the minimization of the algebraic system, which we have achieved by passing to (10), is so significant that this temporary loss in inessential. Furthermore, the above reasoning shows that the recovery of the coefficients in the extension $\mathcal{E}(\epsilon)$ amounts to solution of linear equations, while finding the coefficients in $\mathfrak{m}_\epsilon$ would cost us the necessity to solve nonlinear algebraic systems. We managed to fix some of those constants for granted.

We finally remark that the normalization of at least one coefficient in the deformation problem cancels the redundant dilation of the parameter ϵ, which, otherwise, would remain until the end. This is our fourth simplification.[38]

We let the degrees $\deg \mathfrak{m}_\epsilon = \deg \mathcal{E}(\epsilon)$ be equal to four (cf. Ref. 4). Under this assumption, the two-component homogeneous polynomial extension $\mathcal{E}(\epsilon)$ of system (11) contains 160 undetermined coefficients. At the same time, the two components of the Miura contraction $\mathfrak{m}_\epsilon$ depend on 94 coefficients. However, we decrease this number by 9, setting the coefficient of $\tilde{u}_{12;x}$ equal to +1 and, similarly, to −1 for $\tilde{u}_{12}^2$ [see (14a), where the ± sign is absorbed by $\epsilon \mapsto -\epsilon$]. Likewise, we set equal to zero the seven coefficients of $\tilde{u}_{12;xx}$, $\tilde{u}_{12}\tilde{u}_{12;x}$, $\tilde{u}_{12;xxx}$, $\tilde{u}_{12}^3$, $\tilde{u}_{12;x}^2$, $\tilde{u}_{12}\tilde{u}_{12;xx}$, and $\tilde{u}_{12;xxxx}$ in $\mathfrak{m}_\epsilon$.

The resulting algebraic system with the shortened list of unknowns and with the auxiliary list of nine substitutions is handled by SSTOOLS and then solved by using CRACK.[39]

Theorem 3: *Under the above assumptions, the Gardner deformation problem for Kaup–Boussinesq equation (11) has a unique real solution of degree of 4. The Miura contraction $\mathfrak{m}_\epsilon$ is given by*

$$u_0 = \tilde{u}_0 + \epsilon \tilde{u}_{0;x} - 2\epsilon^2 \tilde{u}_{12}\tilde{u}_0, \tag{16a}$$

$$u_{12} = \tilde{u}_{12} + \epsilon(\tilde{u}_{12;x} - 2\tilde{u}_0\tilde{u}_{0;x}) + \epsilon^2(4\tilde{u}_{12}\tilde{u}_0^2 - \tilde{u}_{12}^2 - \tilde{u}_{0;x}^2) + 4\epsilon^3\tilde{u}_{12}\tilde{u}_0\tilde{u}_{0;x} - 4\epsilon^4\tilde{u}_{12}^2\tilde{u}_0^2. \tag{16b}$$

The extension $\mathcal{E}(\epsilon)$ of (11) is

$$\tilde{u}_{0;\xi} = -\tilde{u}_{12;x} + 4u_0\tilde{u}_{0;x} + 2\epsilon(\tilde{u}_0\tilde{u}_{0;x})_x - 4\epsilon^2(\tilde{u}_0^2 u_{12})_x, \tag{17a}$$

$$\tilde{u}_{12;\xi} = \tilde{u}_{0;xxx} + 4(\tilde{u}_0\tilde{u}_{12})_x - 2\epsilon(\tilde{u}_0\tilde{u}_{12;x})_x - 4\epsilon^2(\tilde{u}_0\tilde{u}_{12}^2)_x. \tag{17b}$$

System (17) preserves the first Hamiltonian operator $\hat{A}_1^\epsilon = \begin{pmatrix} 0 & \mathrm{d}/\mathrm{d}x \\ \mathrm{d}/\mathrm{d}x & 0 \end{pmatrix}$ from $\hat{A}_1^{12}$ for (11).

The Miura contraction $\mathfrak{m}_\epsilon$ is shared by all equations in the Kaup–Boussinesq hierarchy. Solving the linear algebraic system, we find the extension $(\lim_B \mathcal{E}_{\mathrm{SKdV}}^{a=4})(\epsilon)$ for bosonic limit (13) of (3) with a=4,

$$\begin{aligned}\tilde{u}_{0;t} = &-\tilde{u}_{0;xxx} - 6(\tilde{u}_0\tilde{u}_{12})_x + 12\tilde{u}_0^2\tilde{u}_{0;x} + 12\epsilon(\tilde{u}_0^2\tilde{u}_{0;x})_x + 6\epsilon^2(\tilde{u}_0\tilde{u}_{12}^2 - 4\tilde{u}_{12}\tilde{u}_0^3 + \tilde{u}_0\tilde{u}_{0;x}^2)_x \\ &+ \epsilon^3((-24)\tilde{u}_{12}\tilde{u}_0^2\tilde{u}_{0;x})_x + \epsilon^4(24\tilde{u}_{12}^2\tilde{u}_0^3)_x,\end{aligned} \tag{18a}$$

$$\begin{aligned}\tilde{u}_{12;t} = &-\tilde{u}_{12;xxx} - 6\tilde{u}_{12}\tilde{u}_{12;x} + 12(\tilde{u}_0^2\tilde{u}_{12})_x + 6\tilde{u}_0\tilde{u}_{0;xxx} + 12\tilde{u}_{0;xx}\tilde{u}_{0;x} \\ &+ 6\epsilon(\tilde{u}_{0;xx}\tilde{u}_{0;x} - 2\tilde{u}_0^2\tilde{u}_{12;x})_x \\ &+ 2\epsilon^2(\tilde{u}_{12}^3 - 18\tilde{u}_{12}^2\tilde{u}_0^2 - 6\tilde{u}_{12}\tilde{u}_0\tilde{u}_{0;xx} - 3\tilde{u}_{12}\tilde{u}_{0;x}^2 - 6\tilde{u}_0\tilde{u}_{12;x}\tilde{u}_{0;x})_x \\ &+ 24\epsilon^3(\tilde{u}_{12}\tilde{u}_0^3\tilde{u}_{12;x})_x + 24\epsilon^4(\tilde{u}_{12}^3\tilde{u}_0^2)_x.\end{aligned} \tag{18b}$$

Now we expand the fields $\tilde{u}_0(\epsilon) = \Sigma_{k=0}^{+\infty}\tilde{u}_0^{(k)} \cdot \epsilon^k$ and $\tilde{u}_{12}(\epsilon) = \Sigma_{k=0}^{+\infty}\tilde{u}_{12}^{(k)} \cdot \epsilon^k$ and plug the formal power series for $\tilde{u}_0$ and $\tilde{u}_{12}$ in $\mathfrak{m}_\epsilon$. Hence we start from $\tilde{u}_0^{(0)} = u_0$ and $\tilde{u}_{12}^{(0)} = u_{12}$, which is standard, and proceed with the recurrence relations between the conserved densities $u_0^{(k)}$ and $u_{12}^{(k)}$,

$$\tilde{u}_0^{(1)} = -u_{0;x}, \quad \tilde{u}_0^{(n)} = -\frac{\mathrm{d}}{\mathrm{d}x}\tilde{u}_0^{(n-1)} + \sum_{j+k=n-2} 2\tilde{u}_{12}^{(k)}\tilde{u}_0^{(j)}, \quad \forall\, n \geq 2,$$

$$\tilde{u}_{12}^{(1)} = 2u_0u_{0;x} - u_{12;x}, \quad \tilde{u}_{12}^{(2)} = u_{12}^2 + u_{12;xx} - 4u_{12}u_0^2 - 3u_{0;x}^2 - 4u_0u_{0;xx},$$

$$\widetilde{u}_{12}^{(3)} = \sum_{j+k=2} 2\widetilde{u}_0^{(j)} \frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_0^{(k)} - \frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_{12}^{(2)} + \sum_{j+k=1}\left(\widetilde{u}_{12}^{(j)}\widetilde{u}_{12}^{(k)} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_0^{(j)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_0^{(k)}\right)\right)$$
$$- \sum_{j+k+l=1} 4\widetilde{u}_{12}^{(j)}\widetilde{u}_0^{(k)}\widetilde{u}_0^{(l)} - 4u_{12}u_0u_{0;x},$$

$$\widetilde{u}_{12}^{(n)} = -\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_{12}^{(n-1)} + \sum_{j+k=n-1} 2\widetilde{u}_0^{(j)}\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_0^{(k)} + \sum_{j+k=n-2}\left(\widetilde{u}_{12}^{(j)}\widetilde{u}_{12}^{(k)} + \frac{\mathrm{d}}{\mathrm{d}x}(\widetilde{u}_0^{(j)})\frac{\mathrm{d}}{\mathrm{d}x}(\widetilde{u}_0^{(k)})\right)$$
$$- \sum_{j+k+l=n-2} 4\widetilde{u}_{12}^{(j)}\widetilde{u}_0^{(k)}\widetilde{u}_0^{(l)} - \sum_{j+k+l=n-3} 4\widetilde{u}_{12}^{(j)}\widetilde{u}_0^{(k)}\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{u}_0^{(l)}$$
$$+ \sum_{j+k+l+m=n-4} 4\widetilde{u}_{12}^{(j)}\widetilde{u}_{12}^{(k)}\widetilde{u}_0^{(l)}\widetilde{u}_0^{(m)}, \quad \forall\, n \geq 4.$$

Example 3: Following this recurrence, let us generate the eight lowest weight nontrivial conserved densities, which start the tower of Hamiltonians for the Kaup–Boussinesq hierarchy.

We begin with $\widetilde{u}_0^{(0)}=u_0$ and $\widetilde{u}_{12}^{(0)}=u_{12}$. Next, we obtain the densities

$$\widetilde{u}_0^{(2)} = u_{0;xx} + 2u_0u_{12}, \quad \widetilde{u}_{12}^{(2)} = -4u_{0;xx}u_0 - 3u_{0;x}^2 + u_{12;xx} - 4u_0^2u_{12} + u_{12}^2,$$

which contribute to the tri-Hamiltonian representation of (11), see Proposition 2. Now we proceed with

$$\widetilde{u}_0^{(4)} = u_{0;4x} - 12u_{0;xx}u_0^2 + 6u_{0;xx}u_{12} - 18u_{0;x}^2u_0 + 10u_{0;x}u_{12;x} + 6u_{12;xx}u_0 - 8u_0^3u_{12} + 6u_0u_{12}^2,$$

$$\widetilde{u}_{12}^{(4)} = -8u_{0;4x}u_0 - 20u_{0;xxx}u_{0;x} - 13u_{0;xx}^2 + 32u_{0;xx}u_0^3 - 48u_{0;xx}u_0u_{12} + 72u_{0;x}^2u_0^2 - 38u_{0;x}^2u_{12}$$
$$- 80u_{0;x}u_{12;x}u_0 + u_{12;4x} - 24u_{12;xx}u_0^2 + 6u_{12;xx}u_{12} + 5u_{12;x}^2 + 16u_0^4u_{12} - 24u_0^2u_{12}^2 + 2u_{12}^3,$$

$$\widetilde{u}_0^{(6)} = u_{0;6x} - 40u_{0;4x}u_0^2 + 10u_{0;4x}u_{12} - 200u_{0;xxx}u_{0;x}u_0 + 28u_{0;xxx}u_{12;x} - 130u_{0;xx}^2u_0$$
$$- 198u_{0;xx}u_{0;x}^2 + 38u_{0;xx}u_{12;xx} + 80u_{0;xx}u_0^4 - 240u_{0;xx}u_0^2u_{12} + 30u_{0;xx}u_{12}^2 + 240u_{0;x}^2u_0^3$$
$$- 380u_{0;x}^2u_0u_{12} + 28u_{0;x}u_{12;xxx} - 400u_{0;x}u_{12;x}u_0^2 + 100u_{0;x}u_{12;x}u_{12} + 10u_{12;4x}u_0$$
$$- 80u_{12;xx}u_0^3 + 60u_{12;xx}u_0u_{12} + 50u_{12;x}^2u_0 + 32u_0^5u_{12} - 80u_0^3u_{12}^2 + 20u_0u_{12}^3,$$

$$\widetilde{u}_{12}^{(6)} = -12u_{0;6x}u_0 - 42u_{0;5x}u_{0;x} - 80u_{0;4x}u_{0;xx} + 160u_{0;4x}u_0^3 - 120u_{0;4x}u_0u_{12} - 49u_{0;xxx}^2$$
$$+ 1200u_{0;xxx}u_{0;x}u_0^2 - 312u_{0;xxx}u_{0;x}u_{12} - 336u_{0;xxx}u_{12;x}u_0 + 780u_{0;xx}^2u_0^2 - 206u_{0;xx}^2u_{12}$$
$$+ 2376u_{0;xx}u_{0;x}^2u_0 - 716u_{0;xx}u_{0;x}u_{12;x} - 456u_{0;xx}u_{12;xx}u_0 - 192u_{0;xx}u_0^5 + 960u_{0;xx}u_0^3u_{12}$$
$$- 360u_{0;xx}u_0u_{12}^2 + 297u_{0;x}^4 - 366u_{0;x}^2u_{12;xx} - 720u_{0;x}^2u_0^4 + 2280u_{0;x}^2u_0^2u_{12} - 290u_{0;x}^2u_{12}^2$$
$$- 336u_{0;x}u_{12;xxx}u_0 + 1600u_{0;x}u_{12;x}u_0^3 - 1200u_{0;x}u_{12;x}u_0u_{12} + u_{12;6x} - 60u_{12;4x}u_0^2$$
$$+ 10u_{12;4x}u_{12} + 28u_{12;xxx}u_{12;x} + 19u_{12;xx}^2 + 240u_{12;xx}u_0^4 - 360u_{12;xx}u_0^2u_{12} + 30u_{12;xx}u_{12}^2$$
$$- 300u_{12;x}^2u_0^2 + 50u_{12;x}^2u_{12} - 64u_0^6u_{12} + 240u_0^4u_{12}^2 - 120u_0^2u_{12}^3 + 5u_{12}^4,$$

etc. We will use these formulas in Sec. IV, where, as an illustration, we rederive the seven super-Hamiltonians of Ref. 4.

Theorem 4: *In the above notation, the following statements hold.*

- *The conserved densities* $\widetilde{u}_0^{(2k)}$ *and* $\widetilde{u}_{12}^{(2k)}$ *of weights* $2k+1$ *and* $2k+2$*, respectively, are nontrivial for all integers* $k\geq 0$.
- *Consider the zero-order components* $\breve{u}_0(u_0,u_{12},\epsilon)$ *and* $\breve{u}_{12}(u_0,u_{12},\epsilon)$ *of the series*

$\tilde{u}_0([u_0,u_{12}],\epsilon)$ and $\tilde{u}_{12}([u_0,u_{12}],\epsilon)$ with differential-polynomial coefficients. Then these generating functions are given by the formulas

$$(\breve{u}_0(u_0,u_{12},\epsilon^2))^2 = \frac{1}{8\epsilon^2}\cdot[4\epsilon^2(u_0^2+u_{12}) - 1 + \sqrt{1+8\epsilon^2(u_0^2-u_{12})+16\epsilon^4(u_0^2+u_{12})^2}], \tag{19a}$$

$$\breve{u}_{12}(u_0,u_{12},\epsilon^2) = \frac{1}{2\epsilon^2}\cdot\left[1 - \sqrt{\frac{1}{2} - 2\epsilon^2(u_{12}+u_0^2) + \frac{1}{2}\sqrt{1+8\epsilon^2(u_0^2-u_{12})+16\epsilon^4(u_0^2+u_{12})^2}}\right]. \tag{19b}$$

- *The generating functions for the odd-index conserved densities $\tilde{u}_0^{(2k+1)}$ and $\tilde{u}_{12}^{(2k+1)}$ are expressed via the even-index densities, see (21) and (22), respectively. We claim that all the odd-index densities are trivial.*

Proof: The densities $\tilde{u}_0^{(k)}$ and $\tilde{u}_{12}^{(k)}$, which are conserved for bosonic limit (13) of the $N=2$, $a=4$-SKdV system (7), retract to the conserved densities for the KdV equation (4) under $u_0\equiv 0$, see Example 2. The corresponding reduction of $\breve{u}_{12}(u_0,u_{12},\epsilon)$ is generating function (15). This implies that $\breve{u}_{12}=\sum_{k=0}^{+\infty}c(k)u_{12}^k\cdot\epsilon^{2k}+\cdots$, whence the densities $\tilde{u}_{12}^{(2k)}$ are nontrivial.

Following the line of reasonings on p. 7, we consider the zero-order terms in Miura's contraction (16), which yields

$$u_0 = \breve{u}_0\cdot(1-2\epsilon^2\breve{u}_{12}), \tag{20a}$$

$$u_{12} = \breve{u}_{12} + \epsilon^2(4\breve{u}_0^2\breve{u}_{12} - \breve{u}_{12}^2) - 4\epsilon^4\breve{u}_0^2\breve{u}_{12}^2. \tag{20b}$$

Therefore,

$$\breve{u}_0 = \frac{u_0}{1-2\epsilon^2\breve{u}_{12}} = \sum_{k=0}^{+\infty} u_0\cdot(2\epsilon^2\breve{u}_{12})^k.$$

Since the coefficients $c(k)$ of $u_{12}^k\cdot\epsilon^{2k}$ in $\breve{u}_{12}$ are positive, so are the coefficients of $u_0u_{12}^k\cdot\epsilon^{2k}$ in $\breve{u}_0$ for all $k\geq 0$. This proves that the conserved densities $\tilde{u}_0^{(2k)}$ are nontrivial as well.

Second, squaring (20a) and adding it to (20b), we obtain the equality $u_0^2+u_{12}=\breve{u}_0^2+\breve{u}_{12}-\epsilon^2\breve{u}_{12}^2$. In agreement with $\breve{u}_0|_{\epsilon=0}=u_0$ and $\breve{u}_{12}|_{\epsilon=0}=u_{12}$, we choose the root $\breve{u}_{12}=[1-\sqrt{1-4\epsilon^2\cdot(u_{12}+u_0^2-\breve{u}_0^2)}]/(2\epsilon^2)$ of this quadratic equation. Hence (20a) yields the biquadratic equation upon $\breve{u}_0$,

$$1-4\epsilon^2(u_{12}+u_0^2-\breve{u}_0^2) = u_0^2/\breve{u}_0^2.$$

As above, the proper choice of its root gives (19a), whence we return to $\breve{u}_{12}$ and finally obtain (19b).

Finally, let us substitute the expansions $\tilde{u}_0=v_0(\epsilon^2)+\epsilon\cdot v_1(\epsilon^2)$ and $\tilde{u}_{12}=\omega_0(\epsilon^2)+\epsilon\cdot\omega_1(\epsilon^2)$ in (16) for $\tilde{u}_0$ and $\tilde{u}_{12}$, see Example 2. By balancing the odd powers of ϵ in (16a), it is then easy to deduce the equality

$$v_1 \equiv \sum_{k=0}^{+\infty}\tilde{u}_0^{(2k+1)}\cdot\epsilon^{2k} = \frac{1}{4\epsilon^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\log(1-4\epsilon^2\cdot v_0), \quad \text{where} \quad v_0\equiv\sum_{\ell=0}^{+\infty}\tilde{u}_0^{(2\ell)}\cdot\epsilon^{2\ell}. \tag{21}$$

The balance of odd powers of ϵ in (16b) yields the algebraic equation upon ω_1, whence, in agreement with the initial condition $\omega_1(0)=\tilde{u}_{12}^{(1)}$, we choose its root

$$\begin{aligned}\omega_1 = [&1 - 2\epsilon^2\omega_0 + 4\epsilon^2 v_0^2 + 4\epsilon^4(v_1^2 - 2\omega_0 v_0^2 + v_0 v_{1;x} + v_1 v_{0;x}) - 8\epsilon^6 v_1^2\omega_0 \\ &- (1 + 4\epsilon^2(2v_0^2 - \omega_0) + 4\epsilon^4(\omega_0^2 + 2v_0 v_{1;x} - 8\omega_0 v_0^2 + 2v_1 v_{0;x} + 2v_1^2 + 4v_0^4) \\ &+ 16\epsilon^6(2\omega_0^2 v_0^2 - 2v_1^2\omega_0 - \omega_0 v_0 v_{1;x} - \omega_0 v_1 v_{0;x} - 2v_0^2 v_1 v_{0;x} + 2v_1 v_0 \omega_{0;x} + 2v_1^2 v_0^2 - 4\omega_0 v_0^4 + 2v_0^3 v_{1;x}) \\ &+ 16\epsilon^8(v_1^4 + 2\omega_0^2 v_1^2 + 4\omega_0^2 v_0^4 - 2v_1^2 v_0 v_{1;x} - 4\omega_0 v_0^3 v_{1;x} + 8v_1^2\omega_0 v_0^2 + 2v_1^3 v_{0;x} \\ &+ v_0^2 v_{1;x}^2 + v_1^2 v_{0;x}^2 + 4\omega_0 v_0^2 v_1 v_{0;x} - 2v_0 v_{1;x} v_1 v_{0;x}) \\ &+ 64\epsilon^{10}(v_0 v_{1;x} v_1^2\omega_0 - 2\omega_0^2 v_0^2 v_1^2 - v_1^3 v_{0;x}\omega_0 - v_1^4\omega_0) + 64\epsilon^{12} v_1^4\omega_0^2)^{1/2}]/(16\epsilon^6 v_1 v_0).\end{aligned} \tag{22}$$

We claim that, using the balance of the even powers of ϵ in (16), the representation $\Sigma_{k=0}^{+\infty}\tilde{u}_{12}^{(2k+1)}\cdot\epsilon^{2k}\in \mathrm{im}(\mathrm{d}/\mathrm{d}x)$ can be deduced, whence $\tilde{u}_{12}^{(2k+1)}\sim 0$. ■

V. SUPER-HAMILTONIANS FOR $N=2$, $a=4$-SKDV HIERARCHY

In this section, we assign the bosonic super-Hamiltonians $\mathcal{H}^{(k)}=\int \boldsymbol{h}^{(k)}[\boldsymbol{u}]\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$ of (3) with $a=4$ to the Hamiltonians $H^{(k)}=\int h_{12}^{(k)}[u_0,u_{12}]\mathrm{d}x$ of its bosonic limit (13). Also, we establish the no-go result on the superfield, $N=2$ supersymmetry-invariant deformations of $a=4$-SKdV that retract to (14) under the respective reduction in superfield (1). At the same time, we initiate the study of Gardner's deformations for reductions of (7) other than (2), and here we find the deformations of two-component fermion-boson limit in it. However, we observe that the new solutions cannot be merged with deformation (18) for the bosonic limit of (7).

From Sec. III, we know the procedure for recursive production of the Hamiltonians $H^{(k)}=\int h^{(k)}\mathrm{d}x$ for bosonic limit (13) of the $N=2$, $a=4$-SKdV equation, here $h^{(2k)}=\tilde{u}_0^{(2k)}$ and $h^{(2k+1)}=\tilde{u}_{12}^{(2k)}$. In Sec. II, we explained why the reconstruction of the densities $\boldsymbol{h}^{(k)}$ for the bosonic super-Hamiltonians $\mathcal{H}^{(k)}$ from $h^{(k)}[u_0,u_{12}]$ requires an intermediate step. Namely, it amounts to the proper choice of the representatives $h_{12}^{(k)}$ within the equivalence class $\{h^{(k)} \bmod \mathrm{im}(\mathrm{d}/\mathrm{d}x)\}$ such that $h_{12}^{(k)}$ can be realized under (2) as the $\theta_1\theta_2$-component of the superdensity $\boldsymbol{h}^{(k)}$. This allows to restore the dependence on the components u_1 and u_2 of (1) and to recover the supersymmetry invariance. The former means that each $\boldsymbol{h}^{(k)}$ is conserved on (7) and the latter implies that $\boldsymbol{h}^{(k)}$ becomes a differential function in $\boldsymbol{u}$.

The correlation between *unknown* bosonic superdifferential polynomials $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$ and the densities $h^{(k)}[u_0,u_{12}]$, which are produced by the recurrence relation, is established as follows. First, we generate the homogeneous superdifferential polynomial ansatz for the bosonic $\boldsymbol{h}^{(k)}$ using GENSSPOLY, see Appendix A. Second, we split the superfield $\boldsymbol{u}$ using the right-hand side of (1) and obtain the $\theta_1\theta_2$- component $h_{12}^{(k)}[u_0,u_1,u_2,u_{12}]$ of the differential function $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$. This is done by the procedure TOCOO, see Appendix A, which now is also available in SSTOOLS.[16,37] Third, we set to zero the components u_1 and u_2 of the superfield $\boldsymbol{u}$. This gives the ansatz $h_{12}^{(k)}[u_0,u_{12}]$ for the representative of the conserved density in the vast equivalence class. By the above, the gap between $h_{12}^{(k)}$ and the known $h^{(k)}$ amounts to $(\mathrm{d}/\mathrm{d}x)(f^{(k)})$, where $f^{(k)}[u_0,u_{12}]$ is a homogeneous differential polynomial. We remark that the choice of f is not unique due to the freedom in the choice of $\boldsymbol{h}^{(k)} \bmod \mathcal{D}_1(\cdots)+\mathcal{D}_2(\cdots)$. We thus arrive at the linear algebraic equation,

$$h_{12}^{(k)} - \frac{\mathrm{d}}{\mathrm{d}x}f^{(k)} = h^{(k)}, \tag{23}$$

which implies the equality of the respective coefficients in the polynomials. The homogeneous polynomial ansatz for $f^{(k)}$ is again generated by GENSSPOLY. Then Eq. (23) is split to the algebraic system by SSTOOLS and solved by CRACK.[39] Hence we obtain the coefficients in $h_{12}^{(k)}$ and $f^{(k)}$. *A posteriori*, the freedom in the choice of $f^{(k)}$ is redundant, and it is convenient to set the surviving *unassigned* coefficients to zero. Indeed, they originate from the choice of a representative from the equivalence class for the superdensity $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$. This concludes the algorithm for the recursive production of homogeneous bosonic $N=2$ supersymmetry-invariant super-Hamiltonians $\mathcal{H}^{(k)}$ for the $N=2$, $a=4$-SKdV hierarchy.

Example 4: Let us reproduce the first seven super-Hamiltonians for (3), which were found in Ref. 4. In contrast with Example 3, we now list the *properly chosen* representatives $h_{12}^{(k)}[u_0,u_{12}]$ for the equivalence classes of conserved densities $\tilde{u}_0^{(2k)}$ and $\tilde{u}_{12}^{(2k)}$, here $k\leq 3$. Then we expose the conserved superdensities $\boldsymbol{h}^{(k)}$, such that the respective expressions $h_{12}^{(k)}$ are obtained from the $\theta_1\theta_2$-components $\int \boldsymbol{h}^{(k)}\mathrm{d}\boldsymbol{\theta}$ by reduction (2),

$$h_{12}^{(0)} = u_0 \sim \tilde{u}_0^{(0)}, \quad \boldsymbol{h}^{(0)} = -\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u}) \sim 0, \tag{24a}$$

$$h_{12}^{(1)} = u_{12} \sim \tilde{u}_{12}^{(0)}, \quad \boldsymbol{h}^{(1)} = \boldsymbol{u}, \tag{24b}$$

$$h_{12}^{(2)} = -2u_{12}u_0 \sim \tilde{u}_0^{(2)}, \quad \boldsymbol{h}^{(2)} = \boldsymbol{u}^2, \tag{24c}$$

$$h_{12}^{(3)} = \tfrac{3}{4}u_{12}^2 - 3u_{12}u_0^2 + \tfrac{3}{4}u_{0;x}^2 \sim \tilde{u}_{12}^{(2)}, \quad \boldsymbol{h}^{(3)} = \boldsymbol{u}^3 - \tfrac{3}{4}\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u}), \tag{24d}$$

$$\begin{gathered} h_{12}^{(4)} = 3u_{12}^2u_0 - 4u_{12}u_0^3 - \tfrac{3}{2}u_0^2u_{0;xx} - u_{12;x}u_{0;x} \sim \tilde{u}_0^{(4)}, \\ \boldsymbol{h}^{(4)} = \boldsymbol{u}^4 - \tfrac{1}{2}\boldsymbol{u}\boldsymbol{u}_{xx} - \tfrac{3}{2}\boldsymbol{u}^2\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u}), \end{gathered} \tag{24e}$$

$$\begin{aligned} h_{12}^{(5)} = {} & -\tfrac{5}{4}u_{12}^3 + \tfrac{15}{2}u_{12}^2u_0^2 - 5u_{12}u_0^4 + 5u_{12}u_0u_{0;xx} + \tfrac{15}{8}u_{12}u_{0;x}^2 + \tfrac{15}{2}u_0^2u_{0;x}^2 + \tfrac{5}{16}u_{12;x}^2 \\ & + \tfrac{5}{16}u_{0;xx}^2 \sim \tilde{u}_{12}^{(4)}, \quad \boldsymbol{h}^{(5)} = \boldsymbol{u}^5 - \tfrac{15}{16}\boldsymbol{u}^2\boldsymbol{u}_{xx} + \tfrac{5}{8}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2\boldsymbol{u} - \tfrac{5}{2}\boldsymbol{u}^3\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}, \end{aligned} \tag{24f}$$

$$\begin{aligned} h_{12}^{(6)} = {} & -\tfrac{15}{4}u_{12}^3u_0 + 15u_{12}^2u_0^3 - \tfrac{15}{8}u_{12}^2u_{0;xx} - 6u_{12}u_0^5 - \tfrac{75}{4}u_{12}u_0u_{0;x}^2 - \tfrac{3}{8}u_{12}u_{0;xxxx} \\ & + 5u_0^3u_{12;xx} + 15u_0^3u_{0;x}^2 + \tfrac{15}{8}u_0u_{12;x}^2 + \tfrac{15}{8}u_0u_{0;xx}^2 \sim \tilde{u}_0^{(6)}, \\ \boldsymbol{h}^{(6)} = {} & \boldsymbol{u}^6 - \tfrac{15}{8}\boldsymbol{u}^3\boldsymbol{u}_{xx} + \tfrac{3}{16}\boldsymbol{u}\boldsymbol{u}_{4x} + \tfrac{15}{8}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2 - \tfrac{15}{4}\boldsymbol{u}^4\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{15}{8}\boldsymbol{u}_{xx}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} - \\ & + \tfrac{5}{8}\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u})\mathcal{D}_1(\boldsymbol{u})\mathcal{D}_1(\boldsymbol{u}_x), \end{aligned} \tag{24g}$$

$$\begin{aligned} h_{12}^{(7)} = {} & -\tfrac{21}{8}u_{0;4x}u_0u_{12} + \tfrac{7}{64}u_{0;xxx}^2 + \tfrac{105}{16}u_{0;xx}^2u_0^2 + \tfrac{35}{32}u_{0;xx}^2u_{12} - \tfrac{105}{8}u_{0;xx}u_0u_{12}^2 - \tfrac{105}{64}u_{0;4x}^4 \\ & - \tfrac{35}{16}u_{0;x}^2u_{12;xx} + \tfrac{105}{4}u_{0;x}^2u_0^4 - \tfrac{525}{8}u_{0;x}^2u_0^2u_{12} - \tfrac{175}{32}u_{0;x}^2u_{12}^2 + \tfrac{7}{64}u_{12;xx}^2 + \tfrac{35}{4}u_{12;xx}u_0^4 \\ & + \tfrac{105}{16}u_{12;x}^2u_0^2 - \tfrac{35}{32}u_{12;x}^2u_{12} - 7u_0^6u_{12} + \tfrac{105}{4}u_0^4u_{12}^2 - \tfrac{105}{8}u_0^2u_{12}^3 + \tfrac{35}{64}u_{12}^4 \sim \tilde{u}_{12}^{(6)}, \\ \boldsymbol{h}^{(7)} = {} & \boldsymbol{u}^7 - \tfrac{105}{32}\boldsymbol{u}^3\boldsymbol{u}_{xx} + \tfrac{7}{32}\boldsymbol{u}^2\boldsymbol{u}_{4x} - \tfrac{35}{64}\boldsymbol{u}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^3 + \tfrac{35}{8}\boldsymbol{u}^3(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2 - \tfrac{35}{64}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2\boldsymbol{u}_{xx} \\ & - \tfrac{21}{4}\boldsymbol{u}^5\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{105}{16}\boldsymbol{u}^2\boldsymbol{u}_{xx}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{315}{64}\boldsymbol{u}\boldsymbol{u}_x^2\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{35}{16}\boldsymbol{u}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})(\mathcal{D}_1\boldsymbol{u})(\mathcal{D}_1\boldsymbol{u}_x) \\ & - \tfrac{7}{64}\boldsymbol{u}_{4x}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} - \tfrac{7}{8}\boldsymbol{u}(\mathcal{D}_1\boldsymbol{u}_{xx})(\mathcal{D}_1\boldsymbol{u}_x). \end{aligned} \tag{24h}$$

Of course, our superdensities $\boldsymbol{h}^{(k)}$ are equivalent to those in Ref. 4 up to adding trivial terms $\mathcal{D}_1(\cdots)+\mathcal{D}_2(\cdots)$.

Remark 3: Until now, we have not yet reported any attempt of construction of Gardner's *superfield* deformation for (3), which means that the ansatz for $\mathfrak{m}_\epsilon$ and $\mathcal{E}(\epsilon)$ is written in superfunctions of $\boldsymbol{u}$ (cf. Ref. 4). This would yield the super-Hamiltonians $\mathscr{H}^{(k)}$ at once, and the intermediate deformation (18) of a reduction (2) for (3) would not be necessary. At the same time, the knowledge of Gardner's deformations for the reductions allows to inherit a part of the coefficients in the superfield ansatz by fixing them in the component expansions [e.g., see (14), (16), and (18)].

Unfortunately, this cut-through does not work for the $N=2$, $a=4$-SKdV equation.

Theorem 5: *($N=2$, $a=4$ "no go") Under the assumptions that $N=2$ supersymmetry-invariant*

Gardner's deformation $\mathfrak{m}_\epsilon : \mathcal{E}(\epsilon) \to \mathcal{E}$ of (3) with $a=4$ is regular at $\epsilon=0$, is scaling homogeneous and retracts to (14) under the reduction $u_0=0$, $u_1=u_2=0$ in the superfield (1), there is no such deformation.

This rigidity statement, although under a principally different set of initial hypotheses, is contained in Ref. 4. In particular, there it was supposed that $\deg \mathfrak{m}_\epsilon = \deg \mathcal{E}(\epsilon)=2$, which turns to be on the obstruction threshold, see below. We reveal the general nature of this "no go" result.

Proof: Suppose there is the superfield Miura contraction $\mathfrak{m}_\epsilon$,

$$\boldsymbol{u} = \tilde{\boldsymbol{u}} + \epsilon(p_3\tilde{\boldsymbol{u}}^2 - p_1\mathcal{D}_1\mathcal{D}_2\tilde{\boldsymbol{u}} + p_2\tilde{\boldsymbol{u}}_x) + \epsilon^2(p_{15}\tilde{\boldsymbol{u}}^3 + p_{13}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}_x + p_{10}\mathcal{D}_2(\tilde{\boldsymbol{u}})\mathcal{D}_1(\tilde{\boldsymbol{u}}) - p_{12}\mathcal{D}_1\mathcal{D}_2(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}} - p_{11}\mathcal{D}_1\mathcal{D}_2(\tilde{\boldsymbol{u}}_x) + p_{14}\tilde{\boldsymbol{u}}_{xx}) + \cdots .$$

To recover deformation (14) upon u_{12} in $\boldsymbol{u}$, we split $\mathfrak{m}_\epsilon$ in components and fix the coefficients of $\epsilon\tilde{u}_{12;x}$ and $\epsilon^2\tilde{u}_{12}^2$, see (14a). By this argument, the expansion of $\tilde{\boldsymbol{u}}_x$ yields $p_2=1$, while the equality $-p_{12}\mathcal{D}_1\mathcal{D}_2(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}} + p_{10}\mathcal{D}_2(\tilde{\boldsymbol{u}})\mathcal{D}_1(\tilde{\boldsymbol{u}}) = (p_{12}-p_{10})\theta_1\theta_2 u_{12}^2 + \cdots$ implies that $p_{12}=p_{10}-1$. Next, we generate the homogeneous ansatz for $\mathcal{E}(\epsilon)$, which contains $\tilde{\boldsymbol{u}}_t = \cdots + \epsilon^2 \cdot (\mathrm{d}/\mathrm{d}x)(q_{17}(\mathcal{D}_2\boldsymbol{u})(\mathcal{D}_1\boldsymbol{u})\boldsymbol{u} + \cdots) + \cdots$ in the right-hand side (the coefficient q_{17} will appear in the obstruction). We stress that now both $\mathfrak{m}_\epsilon$ and $\mathcal{E}(\epsilon)$ can be formal power series in ϵ without any finite-degree polynomial truncation.

Now we split the determining equation $\mathfrak{m}_\epsilon : \mathcal{E}(\epsilon) \to \mathcal{E}$ to the sequence of superdifferential polynomial equalities ordered by the powers of ϵ. By the regularity assumption, the coefficients of higher powers of ϵ never contribute to the equations that arise at its lower degrees. Consequently, every contradiction obtained at a finite order in the algebraic system is universal and precludes the existence of a solution. (Of course, we assume that the contradiction is not created artificially by an excessively low order polynomial truncation of the expansions in ϵ.)

This is the case for the $N=2$, $a=4$-SKdV. Using CRACK,[39] we solve all but two algebraic equations in the quadratic approximation. The remaining system is

$$q_{17} = -p_{10}, \quad p_{10} + q_{17} + 1 = 0.$$

This contradiction concludes the proof. ■

Remark 4: In Theorem 5 for (3) with $a=4$, we state the nonexistence of the Gardner deformation in a class of differential superpolynomials in $\boldsymbol{u}$, that is, of $N=2$ supersymmetry-invariant solutions that incorporate (14). Still, we do *not* claim the nonexistence of local regular Gardner's deformations for four-component system (7) in the class of differential functions of u_0, u_1, u_2, and u_{12}.

Consequently, it is worthy to deform the reductions of (7) other than (2). Clearly, if there is a deformation for the entire system, then such partial solutions contribute to it by fixing the parts of the coefficients.

Example 5: Let us consider the reduction $u_0=0$, $u_2=0$ in (7) with $a=4$. This is the two-component boson-fermion system,

$$u_{1;t} = -u_{1;xxx} - 3(u_1u_{12})_x, \quad u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x} + 3u_1u_{1;xx}. \tag{25}$$

Notice that system (25) is *quadratic* nonlinear in both fields, hence the balance $\deg \mathfrak{m}_\epsilon : \deg \mathcal{E}(\epsilon)$ for its polynomial Gardner's deformations remains 1:1.

We found a unique Gardner's deformation of degree ≤ 4 for (25): the Miura contraction $\mathfrak{m}_\epsilon$ is cubic in ϵ,

$$u_1 = \tilde{u}_1, \quad u_{12} = \tilde{u}_{12} - \tfrac{1}{9}\epsilon^3\tilde{u}_1\tilde{u}_{1;xx}, \tag{26a}$$

and the extension $\mathcal{E}(\epsilon)$ is given by the formulas

$$\tilde{u}_{1;t} = -\tilde{u}_{1;xxx} - 3(\tilde{u}_1\tilde{u}_{12})_x,$$

$$\tilde{u}_{12;t} = -\tilde{u}_{12;xxx} - 6\tilde{u}_{12}\tilde{u}_{12;x} + 3\tilde{u}_1\tilde{u}_{1;xx} + \tfrac{1}{3}\epsilon^3(u_1u_{1;xx}u_{12} - 3u_1u_{1;x}u_{12;x} + u_{1;x}u_{1;xxx})_x. \quad (26b)$$

However, we observe, first, that contraction (14a) is not recovered[40] by (26a) under $u_1 \equiv 0$. Hence deformation (26) and its mirror copy under $u_1 \leftrightarrow -u_2$ cannot be merged with (16) and (18) to become parts of the deformation for (7).

Second, we recall that the fields u_1 and u_2 are, seemingly, the only local fermionic conserved densities for (7) with a=4. Consequently, either the velocities $\tilde{u}_{1;t}$ and $\tilde{u}_{2;t}$ in Gardner's extensions $\mathcal{E}(\epsilon)$ of (7) are not expressed in the form of conserved currents (although this is indeed so at ϵ =0) or the components $u_i = u_i([\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_{12}], \epsilon)$ of the Miura contractions $\mathfrak{m}_\epsilon$ are the identity mappings $u_i = \tilde{u}_i$, here $i=1,2$, whence either the Taylor coefficients $\tilde{u}_i^{(k)}$ of $\tilde{u}_i$ are not termwise conserved on (7) or there appear no recurrence relations at all. This will be the object of another paper.

VI. CONCLUSION

We obtained the no-go statement for regular, scaling-homogeneous polynomial Gardner's deformations of the N=2, a=4-SKdV equation under the assumption that the solutions retract to original formulas (14) by Gardner.[1] At the same time, we found a new deformation (16) and (17) of Kaup–Boussinesq equation (11) that specifies the second flow in the bosonic limit of the superhierarchy. We emphasize that other known nontrivial deformations for the Kaup–Boussinesq equation[7] can be used for this purpose with equal success.

We exposed the two-step procedure for recursive production of the bosonic super-Hamiltonians $\mathcal{H}^{(k)}$. We formulated the entire algorithm in full detail such that, with elementary modifications, it is applicable to other supersymmetric KdV-type systems.

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APPENDIX A: NEW EXTENSIONS IN THE SOFTWARE SSTOOLS

The syntaxis of the homogeneous differential polynomial generator is as follows.
`GENSSPOLY(N,wglist,cname,mode)`,
where

- `N` is the number of Grassmann variables $\theta_1, \ldots, \theta_N$;
- `wglist` is the list of lists {`afwlist, abwlist, wgt`}, each containing the list `afwlist` of weights for the fermionic super- fields and the list `abwlist` of weights for the bosonic super-fields; here `wgt` is the weight of the polynomial to be constructed;
- `cname` is the prefix for the names of arising undetermined coefficients (e.g., `p` produces $p_1, p_2, \ldots$);
- `mode` is the list of flags, which can be `fonly`, whence only fermionic polynomials are generated, or `bonly`, which yields the bosonic output.

The splitting to components for differential polynomials in superfields is performed by the call `TOCOO(N,nf,nb,ex)`,
where

- `N` is the number of Grassmann variables $\theta_1, \ldots, \theta_N$;
- `nf` is the number of fermionic super-fields `f(1),…,f(nf)`;
- `nb` is the number of bosonic super-fields `b(1),…,b(nb)`;
- `ex` is the super-field expression to be split in components.

For $N=2$, we have `f(i)=f(i,0,0)+b(i,1,0)*th(1)+b(i,0,1)*th(2)+f(i,1,1)*th(1)*th(2)`, `b(i)=b(i,0,0)+f(i,1,0)*th(1)+f(i,0,1)*th(2)+b(i,1,1)*th(1)*th(2)` as the splitting convention. The reduction (2) is achieved by setting `b(i,0,1)`, `b(i,1,0)`, `f(j,0,1)`, and `f(j,1,0)` to zero for all $i \in [1,\mathtt{nb}]$ and $j \in [1,\mathtt{nf}]$.

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[18] Likewise, we will extend Gardner's deformation (14) of (4) to deformation (18) of two-component bosonic limit (13) for (3) with $a=4$. Hence we reproduce the conservation laws for (13) and, again, extend them to the bosonic super-Hamiltonians of full system (3).

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[22] The nonzero entries of the (4×4)-matrix representation $\hat{P}_1$ for the Hamiltonian superoperator $\hat{\boldsymbol{P}}_1^{a=4}$ are $(\hat{P}_1)_{0,12} = (\hat{P}_1)_{2,1} = (\hat{P}_1)_{12,0} = -(\hat{P}_1)_{1,2} = \mathrm{d}/\mathrm{d}x$.

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[30] We recall that the $N=2$ super-residue Sres M of a superpseudodifferential operator M is the coefficient of $\mathcal{D}_1\mathcal{D}_2 \circ (\mathrm{d}/\mathrm{d}x)^{-1}$ in M.

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[33] Under the assumption of weight homogeneity, the freedom in the choice of such $f[u_0, u_{12}]$ is decreased, but the gap still remains.
[34] Still, the four components of the original $N=2$ supersymmetric equations within the hierarchy of (3) *are* written in the form of conserved currents. A helpful counterexample, Gardner's extension of the $N=1$ super-KdV equation, is discussed in Refs. 4 and 41.
[35] This estimate is rough and can be improved by operating separately with the components of $\mathfrak{m}_\epsilon$ and $\mathcal{E}(\epsilon)$ since, in particular, Kaup–Boussinesq system (11) is *linear* in u_{12}.
[36] Reductions other than (2) can produce quadratic-nonlinear subsystems of the cubic-nonlinear system (3), e.g., if one sets $u_0=0$ and $u_2=0$, see (25).

[37] T. Wolf (2005, 2009) The interactive use of SSTOOLS. Online tutorial: http://lie.math.brocku.ca/crack/susy/.

[38] There is one more possibility to reduce the size of the algebraic system: this can be achieved by a thorough balance of

the *differential orders* of $\mathfrak{m}_\epsilon$ and $\mathcal{E}(\epsilon)$.

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[40] Surprisingly, quadratic approximation (14a) in the deformation problem for (7) is very restrictive and leads to a unique solution (16a), (16b), (17a), (17b), (18a), and (18b) for (13). Relaxing this constraint and thus permitting the coefficient of $\epsilon^2\tilde{u}_{12}^2$ in $\mathfrak{m}_\epsilon$ be arbitrary, we obtain two other real and two pairs of complex conjugate solutions for the deformation problem. They constitute the real and the complex orbit, respectively, under the action of the discrete symmetry $u_0 \mapsto -u_0$, $\xi \mapsto -\xi$ of (11).

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